Math. Nachr. **288,** No. 7, 784–797 (2015) / **DOI** 10.1002/mana.201200191

An operator theoretical approach to Riemann-Liouville fractional Cauchy problem

Zhan-Dong Mei[*1], **Ji-Gen Peng**[**1], and **Yang Zhang**[***2]

[1] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
[2] Xi'an Technological University North Institute of Information Engineering, Xi'an, P. R. China

Received 27 June 2012, revised 5 January 2015, accepted 12 January 2015
Published online 25 February 2015

Key words Riemann-Liouville fractional Cauchy problem, Riemann-Liouville fractional resolvent
MSC (2010) Primary: 45N05; Secondary: 47D99

This paper is concerned with developing an operator theory for Riemann-Liouville fractional Cauchy problem. Two notions, named Riemann-Liouville fractional resolvent and solution operator, are developed. Some of their properties are deduced. Moreover, the obtained results are applied to Riemann-Liouville fractional Cauchy problem.

1 Introduction and preliminaries

Consider the following Riemann-Liouville α-order abstract Cauchy problem

$$(FAC)_{x,y}\begin{cases} D_t^\alpha u(t) = Au(t), & t > 0,\\ (g_{2-\alpha} * u)(0) = x, & (g_{2-\alpha} * u)'(0) = y,\end{cases}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a closed densely defined linear operator, $(X, \|\cdot\|)$ is a Banach space, $D(A)$ is the domain of A endowed with the graph norm $\|x\|_{D(A)} = \|x\| + \|Ax\|$, $g_{2-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$ for $t > 0$ and $g_{2-\alpha}(t) = 0$ for $t \le 0$, $x, y \in X$, $*$ is the convolution of functions $g_{2-\alpha}(t) * u(t) = (g_{2-\alpha} * u)(t) = \int_0^t g_{2-\alpha}(t-\sigma)u(\sigma)\,d\sigma$, D_t^α is the α-order Riemann-Liouville fractional derivative operator which is defined as follows.

Definition 1.1 ([12]) Let $\alpha > 0$, $m = [\alpha]$ denotes the smallest integer greater than or equal to α. The *α-order Riemann-Liouville fractional derivative* is defined by

$$D_t^\alpha u(t) = \frac{d^m}{dt^m}(g_{m-\alpha} * u)(t) = \frac{d^m}{dt^m} J_t^{m-\alpha} u(t), \tag{1.1}$$

where $u \in L^1((0,T); X)$, $g_{m-\alpha} * u \in W^{m,1}((0,T); X)$, J_t^α is the α-order Riemann-Liouville fractional integral defined by

$$J_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\,d\tau. \tag{1.2}$$

When $\alpha = m,\ m \in N$, define $D_t^m = \frac{d^m}{dt^m}$.

Set $J_t^0 u(t) = u(t)$. The fractional integral operators $\{J_t^\alpha\}_{\alpha \ge 0}$ satisfy the semigroup property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \qquad \alpha, \beta \ge 0. \tag{1.3}$$

A strong motivation for studying and investigating the solution and the properties for Riemann-Liouville fractional Cauchy problems comes from the fact that they can be used to model efficiently anomalous diffusion

* Corresponding author: e-mail: zhdmei@mail.xjtu.edu.cn, Phone: +86 029 82663158
** e-mail: jgpeng@mail.xjtu.edu.cn
*** e-mail: uuu85@sina.com

on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1], [10] and the references therein), fractional random walk, etc. Moreover, the observation of a universal response of electromagnetic, acoustic, and mechanical influence shows that the existence of some transfer processes in a medium, which are not described by a usual diffusion equation. In [7]–[9], [11], the authors generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α:

$$D_t^\alpha u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le \alpha \le 2,$$

where D_t^α is the Riemann-Liouville fractional derivative operator of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The case $\alpha = 1$ corresponds to an parabolic equation, and $\alpha = 2$ is the case of a hyperbolic one.

It is reasonable that the initial condition of $(FAC)_{x,y}$ is expressed in terms of Riemann-Liouville fractional derivatives: on some examples from the field of viscoelasticity, Heymans and Podlubny [5] demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives, and it is possible to obtain initial values for such initial conditions by appropriate measurements; when Riemann-Liouville fractional derivative is applied to describe the constitutive equations of viscoelastic materials, Nutting's Law is

$$\sigma(t) = \nu D_t^\alpha \varepsilon(t),$$

where σ is stress, ε is strain, ν is model constant, Du and Wang pointed out in [3] that in order to describe the physical properties of real viscoelastic materials, initial condition in terms Riemann-Liouville fractional derivatives should be used.

Recently, Peng, Li and Jia [6] were concerned with the special case of α-order abstract Cauchy problem

$$(FAC)_{0,y} \begin{cases} D_t^\alpha u(t) = Au(t), & t > 0, \\ (g_{2-\alpha} * u)(0) = 0, \ (g_{2-\alpha} * u)'(0) = y. \end{cases}$$

They introduced the notion of strongly continuous fractional resolvent as follows.

Definition 1.2 ([6]) Let $1 < \alpha < 2$. A family $\{S_\alpha(t)\}_{t\ge 0}$ of bounded linear operators of X is called a *strongly continuous fractional resolvent of order* α (or α*-order fractional resolvent*, for short) if it satisfies the following assumptions:

(a) $S_\alpha(t)$ is strongly continuous on R^+ and

$$\lim_{t\to 0^+} \frac{S_\alpha(t)}{t^{\alpha-1}} x = \frac{1}{\Gamma(\alpha)} x, \quad x \in X,$$

(b) $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t), \ s,t \ge 0,$

(c) the equality holds:

$$S_\alpha(s) J_t^\alpha S_\alpha(t) - J_2^\alpha S_\alpha(s) S_\alpha(t) = \frac{s^{\alpha-1}}{\Gamma(\alpha)} J_t^\alpha S_\alpha(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^\alpha S_\alpha(s), \quad t,s \ge 0.$$

Using the above notion, the authors of [6] proved that if the operator A generates an α-order fractional resolvent, then $FAC)_{0,y}$ has a unique strong solution for $y \in D(A)$ and has a unique weak solution for $y \in X$.

The purpose of this paper is to develop an operator theory to study Riemann-Liouville fractional Cauchy problems $(FAC)_{x,y}$. The arrangement of this paper is as follows. In Section 2, we introduce the notion of Riemann-Liouville fractional resolvent and obtain some of its properties. In Section 3, the solution of fractional abstract Cauchy problem $(FAC)_{x,y}$ is considered. The related notion of solution operator for $(FAC)_{x,0}$ is developed. It is proved that a solution operator is essentially a Riemann-Liouville fractional resolvent. Moreover, an illustrative example, fractional diffuse problem is presented.

Throughout this paper, we always assume $1 < \alpha < 2$, without any additional statement.

2 Riemann-Liouville fractional resolvent

In this section, we introduce the notion of Riemann-Liouville fractional resolvent, and obtain some of its properties.

Definition 2.1 A family $\{T_\alpha(t)\}_{t>0}$ of bounded linear operators is called *Riemann-Liouviille α-order fractional resolvent* if it satisfies the following assumptions:

(a) For any $x \in X$, $T_\alpha(\cdot)x \in C((0,\infty), X)$, and

$$\lim_{t\to 0^+} \Gamma(\alpha-1)t^{2-\alpha}T_\alpha(t)x = x \quad \text{for all} \quad x \in X; \tag{2.1}$$

(b) $T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s)$ for all $t, s > 0$;

(c) for all $t, s > 0$, there holds

$$T_\alpha(s)J_t^\alpha T_\alpha(t) - J_s^\alpha T_\alpha(s)T_\alpha(t) = \frac{s^{\alpha-2}}{\Gamma(\alpha-1)} J_t^\alpha T_\alpha(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} J_s^\alpha T_\alpha(s). \tag{2.2}$$

Remark 2.2 The reason why we don't consider the value of $T_\alpha(\cdot)$ at 0 is that the function $t \mapsto \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ exists singularity at 0.

Remark 2.3 By (a) of Definition 2.1, it follows that $T_\alpha(\cdot)x \in L^1_{loc}(R^+, X)$, for any $x \in X$. In fact, (2.1) implies that there exists $\delta > 0$, such that

$$\|T_\alpha(t)x\| \le \frac{3}{2}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\|x\|, \quad \forall\, 0 < t < \delta.$$

Thus, for any $\tau_0 > 0$, we have that

$$\int_0^{\tau_0} \|T_\alpha(t)x\|\,dt \le \int_0^{\tau_0} \frac{3}{2}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\,dt\|x\| = \frac{3}{2}\frac{\tau_0^{\alpha-1}}{\Gamma(\alpha)}\|x\|, \text{ if } \tau_0 \le \delta,$$

and

$$\begin{aligned}\int_0^{\tau_0} \|T_\alpha(t)x\|\,dt &= \int_0^{\delta} \|T_\alpha(t)x\|\,dt + \int_\delta^{\tau_0} \|T_\alpha(t)x\|\,dt \\ &\le \int_0^{\delta} \frac{3}{2}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\,dt\|x\| + \int_\delta^{\tau_0} \|T_\alpha(t)x\|\,dt \\ &\le \frac{3}{2}\frac{\delta^{\alpha-1}}{\Gamma(\alpha)}\|x\| + \int_\delta^{\tau_0} \|T_\alpha(t)x\|\,dt, \text{ if } \tau_0 > \delta.\end{aligned}$$

Therefore, $T_\alpha(\cdot)x \in L^1_{loc}(R^+, X)$.

Remark 2.4 Define PT_α: for any $x \in X$,

$$(PT_\alpha)(t)x \triangleq \begin{cases} \Gamma(\alpha-1)t^{2-\alpha}T_\alpha(t)x, & t > 0, \\ x, & t = 0. \end{cases}$$

Then, (a) of Definition 2.1 implies that $(PT_\alpha)(\cdot)$ is strongly continuous.

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\to 0+} \frac{t^{2-\alpha}T_\alpha(t)x - \frac{1}{\Gamma(\alpha-1)}x}{t^\alpha} \text{ exists} \right\}$$

and

$$Ax = \Gamma(2\alpha-1)\lim_{t\to 0+} \frac{t^{2-\alpha}T_\alpha(t)x - \frac{1}{\Gamma(\alpha-1)}x}{t^\alpha} \quad \text{for} \quad x \in D(A)$$

is the generator of the Riemann-Liouville α-order fractional resolvent $T_\alpha(t)$, $D(A)$ is the domain of A.

Definition 2.5 The Riemann-Liouville α-order fractional resolvent $\{T_\alpha(t)\}_{t>0}$ is called *exponentially bounded* if there exist constants $M \geq 1$, $\omega \geq 0$ such that

$$\|T_\alpha(t)\| \leq Me^{\omega t}, \quad t > 0. \tag{2.3}$$

An operator A is said to belong $\mathcal{C}^\alpha(M,\omega)$ if A generates an α-order fractional resolvent $T_\alpha(t)$ satisfying (2.3). Denote $\mathcal{C}^\alpha(\omega) = \bigcup\{\mathcal{C}^\alpha(M,\omega); M \geq 1\}$.

Proposition 2.6 *Let $\{T_\alpha(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional resolvent such that $\|T_\alpha(t)\| \leq Me^{\omega t}$, $t > 0$ for some $M \geq 1$, $\omega > 0$. Let $R(\lambda) = \int_0^\infty e^{-\lambda t} T_\alpha(t)\,dt$, $\lambda > \omega$. Then for $\lambda,\ \mu > \omega$,*

$$\lambda R(\mu) - \mu R(\lambda) = (\lambda^\alpha - \mu^\alpha) R(\mu) R(\lambda). \tag{2.4}$$

Proof. The exponential boundedness implies that $T(\cdot)$ is Laplace transformable. The proof is derived directly by taking Laplace transform with respect to t and s on both sides of (2.2). □

Proposition 2.7 *Let $\{T_\alpha(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional resolvent on X, and A its generator. Then*

(a) *$T_\alpha(t)D(A) \subset D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for each $x \in D(A)$.*
(b) *For each $x \in X$, $t > 0$,*

$$T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + AJ_t^\alpha T_\alpha(t)x. \tag{2.5}$$

(c) *For each $x \in D(A)$, $t > 0$,*

$$T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^\alpha T_\alpha(t)Ax, \tag{2.6}$$

and

$$T_\alpha(\cdot)x \in C^1((0,\infty);X). \tag{2.7}$$

(d) *A is closed and densely defined.*

Proof.

(a) Let $x \in D(A)$, for $t, s > 0$, it follows from $T_\alpha(t)T_\alpha(s) = T_\alpha(s)T_\alpha(t)$ that

$$\frac{s^{2-\alpha}T_\alpha(s)T_\alpha(t)x - \frac{1}{\Gamma(\alpha-1)}T_\alpha(t)x}{s^\alpha} = \frac{T_\alpha(t)\left(s^{2-\alpha}T_\alpha(s)x - \frac{1}{\Gamma(\alpha-1)}x\right)}{s^\alpha},$$

since $T_\alpha(t)$ is bounded linear operator, then

$$\Gamma(2\alpha-1)\lim_{s\to 0+}\frac{s^{2-\alpha}T_\alpha(s)T_\alpha(t)x - \frac{1}{\Gamma(\alpha-1)}T_\alpha(t)x}{s^\alpha} = T_\alpha(t)Ax.$$

That is, $T_\alpha(t)x \in D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$.

(b) For each $x \in X$, we have

$$\begin{aligned}
&\left\|\Gamma(2\alpha-1)\frac{J_s^\alpha T_\alpha(s)x}{s^{2\alpha-2}} - x\right\| \\
&\quad = \left\|\frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)}\int_0^s s^{2-2\alpha}(s-\tau)^{\alpha-1}T_\alpha(\tau)x\,d\tau - x\right\| \\
&\quad = \left\|\frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)}\int_0^1 s^{2-\alpha}(1-\tau)^{\alpha-1}T_\alpha(s\tau)x\,d\tau - x\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\Gamma(\alpha-1)} \int_0^1 s^{2-\alpha}\Gamma(\alpha-1)(1-\tau)^{\alpha-1} T_\alpha(s\tau)x\,d\tau - x \right\| \\
&= \left\| \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-2}\Gamma(\alpha-1)\frac{T_\alpha(s\tau)}{(s\tau)^{\alpha-2}}x\,d\tau \right. \\
&\quad \left. - \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-2}x\,d\tau \right\| \\
&\le \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-1}\tau^{\alpha-2}d\tau \sup_{\tau\in[0,1]} \left\| \Gamma(\alpha-1)(s\tau)^{2-\alpha}T_\alpha(s\tau)x - x \right\| \\
&\le \sup_{\tau\in[0,1]} \left\| \Gamma(\alpha-1)(s\tau)^{2-\alpha}T_\alpha(s\tau)x - x \right\|. \qquad (2.8)
\end{aligned}
$$

The combination of (2.8) and (a) of Definition 2.1 implies that

$$\lim_{s\to 0^+} \Gamma(2\alpha-1)\frac{J_s^\alpha T_\alpha(s)x}{s^{2\alpha-2}} = x. \qquad (2.9)$$

By (c) of Definition 2.1, it follows that

$$\left(T_\alpha(s) - \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}\right) J_t^\alpha T_\alpha(t)x = J_s^\alpha T_\alpha(s)\left(T_\alpha(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\right). \qquad (2.10)$$

Thus,

$$
\begin{aligned}
AJ_t^\alpha T_\alpha(t)x &= \Gamma(2\alpha-1)\lim_{s\to 0+} \frac{\left(s^{2-\alpha}T_\alpha(s) - \frac{1}{\Gamma(\alpha-1)}\right)J_t^\alpha T_\alpha(t)x}{s^\alpha} \\
&= \Gamma(2\alpha-1)\lim_{s\to 0+} \frac{J_s^\alpha T_\alpha(s)\left(T_\alpha(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\right)}{s^{2\alpha-2}} \\
&= T_\alpha(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x, \qquad (2.11)
\end{aligned}
$$

therefore (b) holds.

(c) For any $x \in D(A)$, the limit

$$\lim_{s\to 0+} \frac{s^{2-\alpha}T_\alpha(s)x - \frac{1}{\Gamma(\alpha-1)}x}{s^\alpha}$$

exists, then the function

$$g(s) = \frac{s^{2-\alpha}T_\alpha(s)x - \frac{1}{\Gamma(\alpha-1)}x}{s^\alpha}$$

is bounded for sufficiently small $s > 0$. For $t > 0$, by the dominated convergence theorem, we derive

$$
\begin{aligned}
&T_\alpha(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x \\
&= AJ_t^\alpha T_\alpha(t)x \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)}\lim_{s\to 0+} \frac{s^{2-\alpha}T_\alpha(s) - \frac{1}{\Gamma(\alpha-1)}}{s^\alpha}\int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)x\,d\tau \\
&= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)}\lim_{s\to 0+} \int_0^t (t-\tau)^{\alpha-1}T_\alpha(\tau)\frac{s^{2-\alpha}T_\alpha(s) - \frac{1}{\Gamma(\alpha-1)}}{s^\alpha}x\,d\tau
\end{aligned}
$$

$$= \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau) \lim_{s\to 0+} \frac{s^{2-\alpha}T_\alpha(s) - \frac{1}{\Gamma(\alpha-1)}}{s^\alpha} x\, d\tau$$

$$= J_t^\alpha T_\alpha(t)Ax. \tag{2.12}$$

From (2.12) it follows that $T_\alpha(t)x$ is differentiable on $(0,\infty)$ for all $x \in D(A)$ and

$$\begin{aligned}
\frac{d}{dt}T_\alpha(t)x &= \frac{d}{dt}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^\alpha T_\alpha(t)Ax\right) \\
&= -\frac{(2-\alpha)t^{\alpha-3}}{\Gamma(\alpha-1)}x + \frac{d}{dt}J_t^\alpha T_\alpha(t)Ax \\
&= -\frac{(2-\alpha)t^{\alpha-3}}{\Gamma(\alpha-1)}x + J_t^{\alpha-1} T_\alpha(t)Ax \\
&= -\frac{(2-\alpha)t^{\alpha-3}}{\Gamma(\alpha-1)}x + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2} T_\alpha(\tau)Ax\, d\tau \\
&= -\frac{(2-\alpha)t^{\alpha-3}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} T_\alpha(t\tau)Ax\, d\tau.
\end{aligned} \tag{2.13}$$

Now use the dominated convergence theorem to (2.13) to conclude that

$$T_\alpha(\cdot)x \in C^1((0,\infty);X),\ x \in D(A).$$

(d) Assume that $t > 0$. Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From part (c), we have

$$\begin{aligned}
T_\alpha(t)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x &= \lim_{n\to\infty}\left(T_\alpha(t)x_n - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x_n\right) \\
&= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)Ax_n\, d\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)y\, d\tau \\
&= J_t^\alpha T_\alpha(t)y.
\end{aligned} \tag{2.14}$$

Using (2.9), we have

$$\begin{aligned}
Ax &= \Gamma(2\alpha-1)\lim_{t\to 0+} \tfrac{t^{2-\alpha}T_\alpha(t)x - \frac{1}{\Gamma(\alpha-1)}x}{t^\alpha} \\
&= \Gamma(2\alpha-1)\lim_{t\to 0+} \tfrac{J_t^\alpha T_\alpha(t)y}{t^{2\alpha-2}} \\
&= y.
\end{aligned} \tag{2.15}$$

The closeness of A is proved.

For every $x \in X$, set $x_t = J_t^\alpha T_\alpha(t)x$, from (2.11) it follows that $x_t \in D(A)$, and by (2.9) we have $\Gamma(2\alpha-1)t^{2-2\alpha}x_t \to x$ as $t \to 0+$. Thus $\overline{D(A)} = X$. □

Theorem 2.8 *Let $\{T_\alpha(t)\}_{t>0}$ and $\{S_\alpha(t)\}_{t>0}$ be Riemann-Liouville α-order fractional resolvents with generators A and B respectively. If $A = B$ then $T_\alpha(t) = S_\alpha(t)$ for $t > 0$.*

P r o o f . For $x \in D(A)$, by properties (c), (a) of Proposition 3.7, we obtain

$$\frac{t^{\alpha-2}}{\Gamma(\alpha-2)} * S_\alpha(t)x = (T_\alpha(t) - J_t^\alpha A T_\alpha(t)) * S_\alpha(t)x$$

$$
\begin{aligned}
&= T_\alpha(t) * S_\alpha(t)x - g_\alpha(t) * AT_\alpha(t) * S_\alpha(t)x \\
&= T_\alpha(t) * S_\alpha(t)x - g_\alpha(t) * T_\alpha(t) * AS_\alpha(t)x \\
&= T_\alpha(t) * (S_\alpha(t)x - g_\alpha(t) * AS_\alpha(t)x) \\
&= T_\alpha(t) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x \\
&= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * T_\alpha(t)x,
\end{aligned}
$$

by Titchmarsh's theorem, $T_\alpha(t)x = S_\alpha(t)x$ for every $x \in D(A)$, $t \geq 0$, from density of $D(A)$, it follows that $T_\alpha(t) = S_\alpha(t)$. □

Theorem 2.9 *$A \in \mathcal{C}^\alpha(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there is a family $\{T_\alpha(t)\}_{t>0}$ of bounded linear operators satisfying*

1) *for any $x \in X$, $T_\alpha(\cdot)x \in C((0,\infty), X)$, and*

$$\lim_{t\to 0^+} \Gamma(\alpha-1)t^{2-\alpha}T_\alpha(t)x = x \quad \text{for all} \quad x \in X; \tag{2.16}$$

2) $\|T_\alpha(t)\| \leq Me^{\omega t}$, $M \geq 1$, $t > 0$.
3) *there holds*

$$\lambda R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}T_\alpha(t)x\,dt, \quad \lambda > \omega,\ x \in X. \tag{2.17}$$

If this is the case, $T_\alpha(t)$ is the Riemann-Liouville α-order fractional resolvent generated by A.

P r o o f . (Necessity) Assume A to be a generator of $\{T_\alpha(t)\}_{t>0}$ such that

$$\|T_\alpha(t)\| \leq Me^{\omega t},\ t > 0.$$

Hence $T_\alpha(\cdot)$ is Laplace transformable. Let

$$R(\lambda) = \int_0^\infty e^{-\lambda t}T_\alpha(t)\,dt,\ \ \lambda > \omega > 0. \tag{2.18}$$

Then, by (c) of Proposition 2.7,

$$T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^\alpha T_\alpha(t)Ax, \quad \forall x \in D(A). \tag{2.19}$$

Taking Laplace transform on both sides of (2.19), we obtain

$$R(\lambda)x = \lambda^{-\alpha+1}x + \lambda^{-\alpha}R(\lambda)Ax = \lambda^{-\alpha+1}x + \lambda^{-\alpha}AR(\lambda)x, \quad \forall x \in D(A), \tag{2.20}$$

where the commutativity of A and $T(\cdot)$ is used. By (d) of Proposition 2.7, A is dense in X. This implies that

$$R(\lambda) = \lambda^{-\alpha+1} + \lambda^{-\alpha}AR(\lambda) \quad \text{on} \quad X. \tag{2.21}$$

Hence

$$\frac{R(\lambda)}{\lambda}(\lambda^\alpha - A)x = x$$

and

$$(\lambda^\alpha - A)\frac{R(\lambda)}{\lambda} = I \quad \text{on} \quad X.$$

This implies that $(\omega^\alpha, \infty) \in \rho(A)$ and

$$R(\lambda) = \lambda(\lambda^\alpha - A)^{-1}. \tag{2.22}$$

(Sufficiency) By (2.17), we derive that

$$R(\lambda^\alpha, A) = \frac{R(\lambda)}{\lambda}.$$

Using resolvent identify related to operator A as follows

$$R(\lambda^\alpha, A) - R(\mu^\alpha, A) = (\mu^\alpha - \lambda^\alpha)R(\lambda^\alpha, A) - R(\mu^\alpha, A),$$

we obtain that

$$\frac{R(\lambda)}{\lambda} - \frac{R(\mu)}{\mu} = (\mu^\alpha - \lambda^\alpha)\frac{R(\lambda)}{\lambda}\frac{R(\mu)}{\mu}.$$

By virtue of Laplace transform, we obtain

$$T_\alpha(s)J_t^\alpha T_\alpha(t) - J_s^\alpha T_\alpha(s)T_\alpha(t) = \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}J_t^\alpha T_\alpha(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}J_s^\alpha T_\alpha(s) \tag{2.23}$$

for all $t, s > 0$. □

Below we shall relate Riemann-Liouville α-order fractional resolvent to α-order fractional resolvent defined by Li, Peng and Jia [6]. To this end, we first study the relationship of Riemann-Liouville α-order fractional resolvent and solution operator of the following fractional differential equation

$$\begin{cases} D_t^\alpha[u(t) - x] = Au(t), & t > 0, \\ u(0) = x, \ \ u'(0) = 0. \end{cases} \tag{2.24}$$

Here the solution operator of the above system is defined by

Definition 2.10 ([2]) A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators of X is called a *solution operator for system* (2.24) if the following three conditions are satisfied:

(a) $T(t)$ is strongly continuous for $t \geq 0$ and $T(0) = I$,
(b) $T(t)D(A) \subset D(A)$ and $AT(t)x = T(t)Ax$ for all $x \in D(A)$ and $t \geq 0$,
(c) for any $x \in D(A)$, there holds

$$T(t)x = x + J_t^\alpha T(t)Ax, \quad t \geq 0.$$

Theorem 2.11 *Assume that A generates a Riemann-Liouville α-order fractional resolvent $\{T_\alpha(t)\}_{t>0}$ on X. Then, there exists a solution operator $\{S_\alpha(t)\}_{t\geq 0}$ for* (2.24).

Proof. Define $S_\alpha(t) = J_t^{2-\alpha}T_\alpha(t)$, $t > 0$. For any $y \in X$,

$$\begin{aligned} \lim_{t\to 0^+} S_\alpha(t)y &= \lim_{t\to 0^+} J_t^{2-\alpha}T_\alpha(t)y \\ &= \frac{1}{\Gamma(2-\alpha)}\lim_{t\to 0^+}\int_0^t (t-\sigma)^{1-\alpha}T_\alpha(\sigma)y\, d\sigma \\ &= \frac{1}{\Gamma(2-\alpha)}\lim_{t\to 0^+}\int_0^t (t-\sigma)^{1-\alpha}T_\alpha(\sigma)y\, d\sigma \\ &= \frac{1}{\Gamma(2-\alpha)}\lim_{t\to 0^+}\int_0^1 t^{2-\alpha}(1-\sigma)^{1-\alpha}T_\alpha(t\sigma)y\, d\sigma \\ &= \frac{1}{\Gamma(2-\alpha)}\lim_{t\to 0^+}\int_0^1 \sigma^{\alpha-2}(1-\sigma)^{1-\alpha}(t\sigma)^{2-\alpha}T_\alpha(t\sigma)y\, d\sigma. \end{aligned}$$

By dominated convergence theorem, it follows that

$$\begin{aligned} &\lim_{t\to 0^+} S_\alpha(t)y \\ &\quad = \lim_{t\to 0^+} J_t^{2-\alpha}T_\alpha(t)y \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^1 \sigma^{\alpha-2}(1-\sigma)^{1-\alpha} \lim_{t\to 0^+} \Gamma(\alpha-1)(t\sigma)^{2-\alpha} T_\alpha(t\sigma) y \, d\sigma \\
&= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^1 \sigma^{\alpha-2}(1-\sigma)^{1-\alpha} y \, d\sigma \\
&= y.
\end{aligned} \tag{2.25}
$$

This means that we can define $S_\alpha(0) = I$. Then, $\{S_\alpha(t)\}_{t\geq 0}$ is strongly continuous with $S_\alpha(0) = I$.

For any $x \in D(A)$, by (a) of Proposition 2.7, $T_\alpha(\cdot)x \in C((0,\infty), D(A))$

$$
T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x + J_t^\alpha T_\alpha(t)Ax. \tag{2.26}
$$

Taking $2-\alpha$ times integral on both sides of (2.26), we obtain

$$
\begin{aligned}
J_t^{2-\alpha} T_\alpha(t)x &= (g_{2-\alpha} * T_\alpha(\cdot)x)(t) \\
&= J_t^{2-\alpha} \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x + J_t^{2-\alpha} J_t^\alpha T_\alpha(t)Ax \\
&= x + J_t^\alpha J_t^{2-\alpha} T_\alpha(t)Ax,
\end{aligned}
$$

that is,

$$
S_\alpha(t)x = x + J_t^\alpha S_\alpha(t)Ax, \quad t > 0. \tag{2.27}
$$

By the definition of $S_\alpha(0)$, we derive that (2.27) holds indeed for $t \geq 0$. Therefore, by Definition 2.10, $\{S_\alpha(t)\}_{t\geq 0}$ is an α-resolvent operator function generated by A. The proof is completed. □

Combining Theorem 2.11 with Proposition 3.14 of [6], we derive the following theorem.

Theorem 2.12 *Assume that A generates a Riemann-Liouville α-order fractional resolvent $\{T_\alpha(t)\}_{t>0}$ on X. Then, A is the generator of an α-order fractional resolvent defined by* [6].

3 Fractional Cauchy problem

This section is set to discussing the solution of fractional abstract Cauchy problem $(FAC)_{x,y}$. A related notion, named solution operator for $(FAC)_{x,0}$ is developed. It is proved that a solution operator is essentially a Riemann-Liouville fractional resolvent. Moreover, an illustrative example, fractional diffuse problem is presented.

Definition 3.1 A family $\{T_\alpha(t)\}_{t>0}$ of bounded linear operators of X is called a *solution operator* for $(FAC)_{x,0}$ if the following three conditions are satisfied:

(a) for any $x \in X$, $T_\alpha(\cdot)x \in C((0,\infty), X)$, and

$$
\lim_{t\to 0^+} \Gamma(\alpha-1)t^{2-\alpha} T_\alpha(t)x = x \quad \text{for all} \quad x \in X; \tag{3.1}
$$

(b) $T_\alpha(t)D(A) \subset D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$ and $t > 0$,

(c) for any $x \in D(A)$, there holds

$$
T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} x + J_t^\alpha T_\alpha(t)Ax, \quad t > 0.
$$

By Proposition 2.7, it follows that a Riemann-Liouville α-order fractional resolvent is a solution operator for $(FAC)_{x,0}$. On the other hand, assume that $\{T_\alpha(t)\}_{t>0}$ is a solution operator for $(FAC)_{x,0}$. For $x \in D(A)$, we have

$$
\frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * [T_\alpha(s)T_\alpha(t)x] = (T_\alpha(t) - J_t^\alpha AT_\alpha(t)) * [T_\alpha(s)T_\alpha(t)x]
$$

$$= T_\alpha(t) * [T_\alpha(s)T_\alpha(t)x - J_t^\alpha AT_\alpha(s)T_\alpha(t)x]$$
$$= T_\alpha(t) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}T_\alpha(s)x$$
$$= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} * [T_\alpha(t)T_\alpha(s)x], \quad t, s > 0.$$

By Titchmarsh's theorem, $T_\alpha(s)T_\alpha(t)x = T_\alpha(t)T_\alpha(s)x$ for every $x \in D(A)$, $t, s \geq 0$. By the densely-definedness of $D(A)$, it follows that $T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s)$.

Since A is closed densely defined, we derive that for any $x \in X$,

$$T_\alpha(s)x = \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}x + AJ_s^\alpha T_\alpha(s)x, \quad t > 0. \tag{3.2}$$

Replace x with $J_t^\alpha T_\alpha(t)x$, we obtain

$$\begin{aligned} &T_\alpha(s)J_t^\alpha T_\alpha(t)x \\ &= \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}J_t^\alpha T_\alpha(t)x + AJ_s^\alpha T_\alpha(s)J_t^\alpha T_\alpha(t)x \\ &= \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}J_t^\alpha T_\alpha(t)x + AJ_t^\alpha T_\alpha(t)J_s^\alpha T_\alpha(s)x \\ &= \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}J_t^\alpha T_\alpha(t)x + T_\alpha(t)J_s^\alpha T_\alpha(s)x - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}J_s^\alpha T_\alpha(s)x, \quad t, s > 0, \end{aligned}$$

which indicates that (2.2) holds. Hence, $\{T_\alpha(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional resolvent. So we have

Theorem 3.2 *A family $\{T_\alpha(t)\}_{t>0}$ of bounded linear operators is a solution operator for $(FAC)_{x,0}$ if and only if it is a Riemann-Liouville α-order fractional resolvent.*

Definition 3.3 A function $u \in C((0,\infty), X)$ is called a *strong solution* of $(FAC)_{x,0}$ if $u \in C((0,\infty), D(A))$, $J^{2-\alpha}u \in C^2((0,\infty), X)$ and $(FAC)_{x,0}$ holds.

Definition 3.4 A function $u \in C((0,\infty), X)$ is called a *mild solution* of $(FAC)_{x,0}$ if $J_t^\alpha u(t) \in D(A)$, and $u(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + AJ_t^\alpha u(t)$, $t > 0$.

Theorem 3.5 *Assume that $\{T_\alpha(t)\}_{t>0}$ is the Riemann-Liouville α-order fractional resolvent generated by A, then*

i) *For every $x \in D(A)$, $T_\alpha(\cdot)x$ is the strong solution of $(FAC)_{x,0}$;*
ii) *for every $x \in X$, $T_\alpha(\cdot)x$ is the mild solution of $(FAC)_{x,0}$.*

Proof. For any $x \in D(A)$, by (a) of Proposition 2.7, $T_\alpha(\cdot)x \in C((0,\infty), D(A))$

$$T_\alpha(t)x = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^\alpha T_\alpha(t)Ax. \tag{3.3}$$

Taking $2-\alpha$ times integral on both sides of (3.3) and observe that

$$\lim_{t\to 0^+} J_t^{2-\alpha}T_\alpha(t) = 0,$$

we obtain

$$\begin{aligned} J_t^{2-\alpha}T_\alpha(t)x &= (g_{2-\alpha} * T_\alpha(\cdot)x)(t) \\ &= J_t^{2-\alpha}\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + J_t^{2-\alpha}J_t^\alpha T_\alpha(t)Ax \end{aligned}$$

$$= x + J_t^{\alpha} J_t^{2-\alpha} T_{\alpha}(t)Ax$$
$$= x + J_{\alpha}^{2} T_{\alpha}(t)Ax.$$

Hence, $(g_{2-\alpha} * T_{\alpha}(\cdot)x)(0) = x$, $J_t^{2-\alpha} T_{\alpha}(\cdot)x \in C^2((0,\infty), X)$,

$$\begin{aligned}\frac{d}{dt}(g_{2-\alpha} * T_{\alpha}(\cdot)x)(t)\Big|_{t=0} &= \frac{d}{dt}(x + J_{\alpha}^{2} T_{\alpha}(t)Ax)\Big|_{t=0} \\ &= \int_0^t T_{\alpha}(\sigma)Ax\ d\sigma\Big|_{t=0} \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}D_t^{\alpha} T_{\alpha}(t)x &= D_t^{\alpha}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x\right) + D_t^{\alpha} J_t^{\alpha} T_{\alpha}(t)Ax \\ &= \frac{d^2}{dt^2} J_t^{2-\alpha} \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + T_{\alpha}(t)Ax \\ &= T_{\alpha}(t)Ax \\ &= AT_{\alpha}(t)x.\end{aligned}$$

This completes the proof of i). ii) can be obtained directly from the definition and Theorem 3.2. □

Definition 3.6 A function $u \in C((0,\infty), X)$ is called a *strong solution* of $(FAC)_{x,y}$ if $u \in C((0,\infty), D(A))$, $J^{2-\alpha}u \in C^2((0,\infty), X)$ and $(FAC)_{x,y}$ holds.

Definition 3.7 A function $u \in C((0,\infty), X)$ is called a *mild solution* of $(FAC)_{x,y}$ if $J_t^{\alpha}u(t) \in D(A)$, and $u(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y + AJ_t^{\alpha}u(t)$, $t > 0$.

Theorem 3.8 *Assume that $\{T_{\alpha}(t)\}_{t>0}$ is the Riemann-Liouville α-order fractional resolvent generated by A, then*

i) *For every $x, y \in D(A)$, $T_{\alpha}(\cdot)x + \int_0^{\cdot} T_{\alpha}(\sigma)y\ d\sigma$ is the strong solution of $(FAC)_{x,y}$;*
ii) *for every $x, y \in X$, $T_{\alpha}(\cdot)x + \int_0^{\cdot} T_{\alpha}(\sigma)y\ d\sigma$ is the mild solution of $(FAC)_{x,y}$.*

Proof.

i) For any $x \in D(A)$, by (a) of Proposition 2.7, $T_{\alpha}(\cdot)x \in C((0,\infty), D(A))$. By (b) of Proposition 2.7, it follows that,

$$\begin{aligned}&T_{\alpha}(t)x + \int_0^t T_{\alpha}(\sigma)y\ d\sigma \\ &= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y + J_t^{\alpha}\left(T_{\alpha}(t)Ax + \int_0^t T_{\alpha}(\sigma)\ d\sigma\, Ay\right).\end{aligned} \tag{3.4}$$

Taking $2-\alpha$ times integral on both sides of (3.4), we obtain

$$\begin{aligned}&J_t^{2-\alpha}\left(T_{\alpha}(t)x + \int_0^t T_{\alpha}(\sigma)y\ d\sigma\right) \\ &= \left(g_{2-\alpha} * \left(T_{\alpha}(\cdot)x + \int_0^{\cdot} T_{\alpha}(\sigma)y\ d\sigma\right)\right)(t) \\ &= J_t^{2-\alpha}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y\right) + J_t^{2-\alpha} J_t^{\alpha}\left(T_{\alpha}(t)Ax + \int_0^t T_{\alpha}(\sigma)\ d\sigma\, Ay\right)\end{aligned}$$

$$= J_t^{2-\alpha}\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y\right) + J_t^{\alpha}J_t^{2-\alpha}\left(T_\alpha(t)Ax + \int_0^t T_\alpha(\sigma)\, d\sigma\, Ay\right)$$

$$= x + ty + J_t^2 T_\alpha(t)Ax + J_t^3 T_\alpha(t)Ay.$$

Observe that by (2.25), $\lim_{t\to 0^+} J_t^{2-\alpha}T_\alpha(t) = I$. Hence, $\left(g_{2-\alpha} * \left(T_\alpha(\cdot)x + \int_0^{\cdot} T_\alpha(\sigma)y\, d\sigma\right)\right)(0) = x$, $J_t^{2-\alpha}\left(T_\alpha(\cdot)x + \int_0^{\cdot} T_\alpha(\sigma)\, d\sigma\right) \in C^2((0,\infty), X)$,

$$\frac{d}{dt}\left(g_{2-\alpha} * \left(T_\alpha(\cdot)x + \int_0^{\cdot} T_\alpha(\sigma)y\, d\sigma\right)\right)(t)\Big|_{t=0}$$

$$= \frac{d}{dt}[x + ty + J_t^2 T_\alpha(t)Ax + J_t^3 T_\alpha(t)Ay]\Big|_{t=0}$$

$$= y + \int_0^t T_\alpha(\sigma)Ax\, d\sigma\Big|_{t=0} + \int_0^t (t-\sigma)T_\alpha(\sigma)Ay\, d\sigma\Big|_{t=0}$$

$$= y,$$

and

$$D_t^\alpha\left(T_\alpha(t)x + \int_0^t T_\alpha(\sigma)y\, d\sigma\right)$$

$$= D_t^\alpha\left(\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y\right) + D_t^\alpha J_t^\alpha T_\alpha(t)Ax + D_t^\alpha J_t^\alpha \int_0^t T_\alpha(\sigma)\, d\sigma\, Ay$$

$$= \frac{d^2}{dt^2}\left[x + ty + J_t^2 T_\alpha(t)Ax + J_t^3 T_\alpha(t)Ay\right]$$

$$= T_\alpha(t)Ax + \int_0^t T_\alpha(\sigma)\, d\sigma\, Ay$$

$$= A\left(T_\alpha(t)x + \int_0^t T_\alpha(\sigma)y\, d\sigma\right).$$

ii) sine A is closed and densely defined, for any $x, y \in X$, by (3.4), it follows that

$$T_\alpha(t)x + \int_0^t T_\alpha(\sigma)y\, d\sigma$$

$$= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}x + \frac{t^{\alpha-1}}{\Gamma(\alpha)}y + AJ_t^\alpha\left(T_\alpha(t)x + \int_0^t T_\alpha(\sigma)\, d\sigma y\right).$$

□

Example 3.9 Let $\alpha \in (1, 2)$. Consider the following fractional diffuse problem

$$(FDE)_{f,g}\begin{cases} D_t^\alpha u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x), \quad 0 < x < 1,\ t > 0, \\ u(t,0) = u(t,1) = 0, \\ (g_{2-\alpha}(t) * u(t,x))|_{t=0} = f(x),\ \ \dfrac{\partial u}{\partial t}(g_{2-\alpha}(t) * u(t,x))|_{t=0} = g(x). \end{cases}$$

Take $X = L^2(0,1)$, $A = \frac{\partial^2}{\partial x^2}$ with domain $D(A) = \left\{\varphi \in W^{2,2}(0,1), \varphi(0) = \varphi(1) = 0\right\}$. It is easy to see that A has eigenvalues $-n^2\pi^2$ with eigenfunctions $\sin n\pi x$, $n \in N$. For any $f \in X$, there exists $\{f_n\}_{n\in N}$ such that $f(x) = \sum_{n=1}^{\infty} f_n \sin n\pi x$. We define the family $\{T_\alpha(t)\}_{t>0}$ by

$$(T_\alpha(t)f)(x) = \sum_{n=1}^{\infty} t^{\alpha-2}E_{\alpha,\alpha-1}\left(-n^2\pi^2 t^\alpha\right) f_n \sin n\pi x,$$

where $E_{\alpha,\alpha-1}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\alpha-1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha - 1)}, \quad z \in \mathbb{C}.$$

The Mittag-Leffler function $E_{\alpha,\alpha-1}$ is related to the Laplace integral

$$\int_0^{\infty} e^{-\lambda t} t^{\alpha-2} E_{\alpha,\alpha-1}(\omega t^{\alpha})\, dt = \frac{\lambda}{\lambda^{\alpha} - \omega}, \quad Re\lambda > |\omega|^{1/\alpha}.$$

For more information about Mittag-Leffler function, we refer to [4], [12]. We shall show that $\{T(t)\}_{t>0}$ is a Riemann-Liouville fractional resolvent.

a) Obversely, $T(\cdot)f \in C((0,\infty), X)$. Moreover,

$$\Gamma(\alpha-1)t^{2-\alpha}T_{\alpha}(t)f = \sum_{n=1}^{\infty} \Gamma(\alpha-1)E_{\alpha,\alpha-1}(-n^2\pi^2 t^{\alpha})c_n \sin(n\pi\cdot)$$
$$\to f (\text{as } t \to 0^+) \text{ in } X.$$

b) The commutativity of $T_{\alpha}(t)$ and $T_{\alpha}(s)$, $t, s > 0$ is obtained directly by the definition of $\{T_{\alpha}(t)\}_{t>0}$.

c) Through simple calculate, we obtain the Laplace transform of $\hat{T}_{\alpha}(\lambda)$ by

$$\hat{T}_{\alpha}(\lambda)f = \sum_{n=1}^{\infty} \frac{\lambda^{\alpha}}{\lambda} + n^2\pi^2 f_n sin(n\pi\cdot).$$

Assume that $f \in D(A)$ such that

$$\lambda f - Af = g, \quad \lambda \in \rho(A).$$

Then, $f = \sum_{n=0}^{\infty} f_n sin(n\pi\cdot)$, and

$$\sum_{n=0}^{\infty} (\lambda + n^2\pi^2) f_n sin(n\pi\cdot) = \sum_{n=0}^{\infty} g_n sin(n\pi\cdot), \quad \lambda \in \rho(A).$$

This indicates that

$$R(\lambda, A)g = \sum_{n=0}^{\infty} \frac{g_n}{\lambda + n^2\pi^2} sin(n\cdot), \quad \lambda \in \rho(A)$$

and

$$\lambda R(\lambda^{\alpha}, A)g = \sum_{n=0}^{\infty} \frac{\lambda g_n}{\lambda^{\alpha} + n^2\pi^2} sin(n\pi\cdot), \tag{3.5}$$

for any $\lambda^{\alpha} \in \rho(A),\ x \in X$. Hence

$$\lambda R(\lambda^{\alpha}, A)g = \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) g\, dt, \quad \lambda > \omega,\ g \in X.$$

Observe that A is closed and densely defined. By Theorem 2.9, it follows that $\{T_{\alpha}(t)\}_{t>0}$ is a Riemann-Liouville fractional resolvent generated by A. Theorem 3.8 implies that

$$T_{\alpha}(t)f + \int_0^t T_{\alpha}(\sigma)g\, d\sigma$$
$$= \sum_{n=1}^{\infty} \left[t^{\alpha-2}E_{\alpha,\alpha-1}\left(-n^2\pi^2t^{\alpha}\right) f_n + t^{\alpha-1}E_{\alpha,\alpha}\left(-n^2\pi^2t^{\alpha}\right) g_n\right] \sin(n\pi\cdot)$$

is the solution of $(FDE)_{f,g}$.

Acknowledgements This work was supported by the Natural Science Foundation of China (grant nos. 11301412 and 11131006), Research Fund for the Doctoral Program of Higher Education of China (grant no. 20130201120053), Shaanxi Province Natural Science Foundation of China (grant no. 2014JQ1017), Project funded by China Postdoctoral Science Foundation (grant no. 2014M550482), the Fundamental Research Funds for the Central Universities (grant no. 2012jdhz52). The authors wish to thank the anonymous reviewers for helpful and insightful comments and suggestions.

References

[1] V. V. Anh and N. N. Leonenko, Spectral analysis of fractional kinetic equations with random data, J. Stat. Phys. **104**, 1349–1387 (2001).
[2] E. Bazhlekova, Fractional Evolution Equations in Banach Spaces (University Press Facilities, Eindhoven University of Technology, 2001).
[3] L. M. Du and Z. H. Wang, Initialized fractional differential equations with Riemann-Liouville fractional-order derivative, Eur. Phys. J. Special Topics **193**, 49–60 (2011).
[4] A. Erdélyi, Higher Transcendental Functions Vol. 3 (McGraw-Hill, New York, 1955).
[5] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheol Acta. **45**, 765–771 (2006).
[6] K. Li, J. Peng, and J. Jia, Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives, J. Funct. Anal. **263**, 476–510 (2012).
[7] F. Mainardi, Some basic problem in continuum and statistical mechanics, in: Fractals and Fractional Calculus in Continuum Mechanics, edited by A. Carpinteri and F. Mainardi, CISM Courses and Lecture, Vol. 378 (Springer-Verlag, Wien, 1997), pp. 291C348.
[8] F. Mainardi and P. Paradisi, Model of diffusion waves in viscoelasticity based on fractal calculus, in: Proceedings of IEEE Conference of Decision and Control Vol. 5, edited by O. R. Gonzales (IEEE, New York, 1997), pp. 4961–4966.
[9] F. Mainardi and G. Pagnini, The wright functions as solutions of time-fractional diffusion equation, Appl. Math. Comput. **141**, 51–62 (2003).
[10] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. **339**, 1–77 (2000).
[11] R. R. Nigmatullin, To the theoretical explanation of the "universal response", Phys. Sta. Sol. (b) **123**, 739–745 (1984).
[12] I. Podlubny, Fractional Differential Equations (Academic Press, New York, 1999).

