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An unconditionally energy-stable linear Crank-Nicolson scheme for the Swift-Hohenberg equation ☆

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ABSTRACT

In this work, we propose and analyze a stabilized linear Crank-Nicolson scheme for the Swift-Hohenberg equation. More precisely, we treat the nonlinear term explicitly while two second-order stabilization terms are added to improve the stability of the scheme. We show that our scheme satisfies the discrete energy dissipation. We prove rigorously that our scheme is second-order accurate in time. Moreover, we adopt a spectral-Galerkin approximation for the spacial variables and establish error estimates for the fully discrete scheme. Numerical experiments are presented to show the accuracy and energy stability of our scheme.

1. Introduction

The Swift-Hohenberg (SH) equation is a widely applied phase-field model and it was originally derived by Swift and Hohenberg [24] to describe Rayleigh-Bénard convection. Related applications can be found in complex pattern formation [3,25], complex fluids and biological tissues [9,10,20]. The SH equation is derived from the following free energy functional

$$E(u) = \int_{\Omega} \left(\frac{1}{2}(\Delta u)^2 - |\nabla u|^2 + F(u) \right) \mathrm{d}x,$$

where $\Omega = [0, L_x] \times [0, L_y]$ (L_x and L_y are two positive constants) is a domain in $\mathbb{R}^2$, u is the density field. A classic example for the SH equation is convection of a thin layer of fluid heated from below for which we can think of the scalar quantity u as representing the temperature of the fluid in the mid plane. $F(u) = \frac{1}{4}u^4 + \frac{1-\epsilon}{2}u^2$, $0 < \epsilon < 1$ is a constant with physical significance and Δ is the Laplacian operator. The SH equation is given by

$$u_t = -\frac{\delta E}{\delta u} = -(\Delta^2 u + 2\Delta u + f(u)), \tag{1.1}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative, $f(u) = F'(u) = u^3 + (1-\epsilon)u$, $f'(u) = 3u^2 + 1 - \epsilon$. The parameter ϵ measures how far the temperature is above the minimum temperature required for convection: for $\epsilon < 0$, the heating is too small to cause convection, while for $\epsilon > 0$, convection occurs. The free energy is nonincreasing in time. Here we study the numerical scheme of SH equation with periodic boundary condition, that is,

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$$u(x+L_x,y,t)=u(x,y,t),\ \ u(x,y+L_y,t)=u(x,y,t),\ \ \forall\,(x,y)\in\Omega,\ t>0.$$

As a nonlinear fourth-order partial differential equation, the SH equation is difficult to be solved analytically. Hence, various numerical schemes have been proposed in recent years. To design a numerical scheme satisfying energy dissipation law, the linear terms are generally treated implicitly and the nonlinear terms are treated by different approaches. A very efficient approach is the convex splitting method [4]. Based on the convex splitting method, Lee presented a non-iterative scheme for the SH equation with quadratic-cubic nonlinearity without convergence analysis in [14]. Zhang and Ma constructed and analyzed a large time-stepping scheme for the SH equation in [30]. In [18], the authors proposed a second-order energy-stable numerical scheme for the SH equation and presented an optimal error estimate for the scheme. By applying the Crank-Nicolson scheme, a semi-implicit second-order method for the SH equation was given in [8]. The convex splitting method is unconditionally energy stable and uniquely solvable. However, to solve the fully discrete nonlinear systems, these methods generally require the use of a iteration. Hence, the computational costs are often high and the implementations are usually complicated. Another efficient approach is the stabilization method. By introducing artificial stabilization terms, one can alleviate the time step restriction and balance the explicit treatment of the nonlinear term, see [21]. Efficiency and simplicity are the main advantages of the stabilization method. The operator splitting method is also a very powerful approach for solving the phase-field models. In [12], based on the operator splitting scheme, the first- and second-order Fourier spectral methods were presented for the SH equation. In [13], a new conservative SH equation was introduced and its first-order and second-order mass conservative operator splitting schemes were proposed. In [28], a fast explicit high-order operator splitting scheme was presented for the SH equation with a nonlocal nonlinearity. There are also various interesting linear approaches that attract the attention of many scholars, such as the invariant energy quadratization (IEQ) scheme [29] and the scalar auxiliary variable (SAV) scheme [23]. These approaches provide linear numerical schemes and satisfy unconditional energy stability based on a modified energy functional.

The main goal of this work is to design an unconditionally energy-stable linear second-order Crank-Nicolson scheme for the SH equation based on the stabilization method. The unconditional energy stability means the numerical scheme preserves the energy dissipation law at the discrete level without any constraints on the time step size. The stabilization method has been applied to a variety of gradient flow systems [7,21,27,18,19,15] and two-phase flow problems [1,22]. We treat the nonlinear term explicitly, while two second-order stabilization terms are added to improve the stability of the scheme. Similar treatment for the Cahn-Hilliard equation has been presented in [26]. Moreover, we prove rigorously that our scheme is second-order accurate in time. To obtain the fully discrete scheme, we adopt a spectral-Galerkin approximation for the spacial variables, some applications of spectral methods can be found in reference [5,6,16]. We also establish error estimates for the fully discrete scheme. Numerical results are presented to validate our theoretical analysis and show that the proposed scheme is easy to implement and is energy-stable with different time step sizes, the energy decay is robust with respect to the stabilization constants.

The rest of the paper is organized as follows. In Section 2, we construct the stabilized linear Crank-Nicolson scheme and prove our scheme is unconditionally energy-stable. In Section 3, we present the convergence analysis for our scheme, which shows that the proposed scheme is second-order accurate in time. Moreover, we adopt a spectral-Galerkin approximation for the spacial variables and establish error estimates for the fully discrete linear Crank-Nicolson scheme. In Section 4, several numerical experiments are provided to illustrate the accuracy, robustness and energy stability of the proposed scheme. Finally, some conclusions are given in Section 5.

2. The stabilized linear Crank-Nicolson scheme

We first introduce some notations which will be used in the analysis. We use $\|\cdot\|_{m,p}$ to denote the standard norm of the Sobolev space $W^{m,p}(\Omega)$. In particular, we use $\|\cdot\|_{L^p}$ to denote the norm of $W^{0,p}(\Omega)=L^p(\Omega)$; $\|\cdot\|_m$ to denote the norm of $W^{m,2}(\Omega)=H^m(\Omega)$, and $\|\cdot\|$ to denote the norm of $W^{0,2}(\Omega)=L^2(\Omega)$. Let $(\cdot,\cdot)$ represent the L^2 inner product. Let K be any positive integer, T be the final time, $\tau=T/K$ be the time step size, $t^n=n\tau\ \ (n=0,1,2,\cdots,K)$ be the time mesh points, u^n be the numerical approximation of $u(t^n)$. The stabilized linear Crank-Nicolson scheme is as follows.

Scheme Given u^n and u^{n-1}, we can calculate u^{n+1} as follows:

$$\frac{u^{n+1}-u^n}{\tau}+\Delta^2\left(\frac{u^{n+1}+u^n}{2}\right)+2\Delta\left(\frac{u^{n+1}+u^n}{2}\right)$$
$$+f(\hat{u}^{n+1/2})-A\tau\Delta(u^{n+1}-u^n)+B(u^{n+1}-2u^n+u^{n-1})=0, \tag{2.1}$$

where A and B are given stabilization constants. As we shall see below, the scheme is energy-stable if $A>0$ and $B\geq L$. $\hat{u}^{n+1/2}=\frac{3}{2}u^n-\frac{1}{2}u^{n-1}$, which is a second-order approximation to $u(t^{n+1/2})$.

The existence and uniqueness of solution of the SH equation are beyond the scope of this paper, the reader may refer to [17,11]. We assume that the solution u of equation (1.1) exists and satisfies

$$\|u\|_{L^\infty(0,T;H^1(\Omega))}+\|u_t\|_{L^\infty(0,T;H^2(\Omega))}+\|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+\|u_{tt}\|_{L^\infty(0,T;H^4(\Omega))}+\|u_{ttt}\|_{L^2(0,T;L^2(\Omega))}\leq C_*. \tag{2.2}$$

Denote $L = 3C_*^2 + (1-\epsilon)|\Omega|$, where $|\Omega|$ is the measure of the domain Ω. Since

$$\|f'(u)\| = \|3u^2 + 1 - \epsilon\| \le \|3u^2\| + \|1-\epsilon\| \le 3\|u\|^2_{L^\infty(0,T;H^1(\Omega))} + (1-\epsilon)|\Omega| \le L,$$

we have

$$\|f(u) - f(v)\| \le \|f'(\xi)\|\|u - v\| \le L\|u - v\|, \tag{2.3}$$

where ξ is a number between u and v.

Theorem 2.1. *Under the condition $B \ge L$, the scheme (2.1) satisfies the following energy dissipation law*

$$\tilde{E}^{n+1} \le \tilde{E}^n + (L-B)\|u^{n+1} - u^n\|^2 + \left(\frac{L}{2} - B\right)\|u^n - u^{n-1}\|^2, \quad for\ n \ge 1,$$

where

$$\tilde{E}^{n+1} = E(u^{n+1}) + \left(\frac{B}{2} + \frac{L}{4}\right)\|u^{n+1} - u^n\|^2.$$

Proof. Taking the inner product of (2.1) with $u^{n+1} - u^n$, we derive

$$\begin{aligned}&\frac{1}{\tau}\|u^{n+1} - u^n\|^2 + \frac{1}{2}(\Delta^2(u^{n+1} + u^n), u^{n+1} - u^n) + (\Delta(u^{n+1} + u^n), u^{n+1} - u^n) + (f(\hat{u}^{n+1/2}), u^{n+1} - u^n)\\ &- A\tau(\Delta(u^{n+1} - u^n), u^{n+1} - u^n) + B(u^{n+1} - 2u^n + u^{n-1}, u^{n+1} - u^n) = 0.\end{aligned} \tag{2.4}$$

Note that

$$\begin{aligned}\frac{1}{2}(\Delta^2(u^{n+1} + u^n), u^{n+1} - u^n) &= -\frac{1}{2}(\nabla\Delta(u^{n+1} + u^n), \nabla(u^{n+1} - u^n))\\ &= \frac{1}{2}(\Delta(u^{n+1} + u^n), \Delta(u^{n+1} - u^n))\\ &= \frac{1}{2}(\Delta u^{n+1} + \Delta u^n, \Delta u^{n+1} - \Delta u^n)\\ &= \frac{1}{2}\|\Delta u^{n+1}\|^2 - \frac{1}{2}\|\Delta u^n\|^2,\end{aligned} \tag{2.5}$$

$$\begin{aligned}(\Delta(u^{n+1} + u^n), u^{n+1} - u^n) &= -(\nabla(u^{n+1} + u^n), \nabla(u^{n+1} - u^n))\\ &= -(\nabla u^{n+1} + \nabla u^n, \nabla u^{n+1} - \nabla u^n)\\ &= -(\|\nabla u^{n+1}\|^2 - \|\nabla u^n\|^2),\end{aligned} \tag{2.6}$$

$$\begin{aligned}B(u^{n+1} - 2u^n + u^{n-1}, u^{n+1} - u^n) &= B((u^{n+1} - u^n) - (u^n - u^{n-1}), u^{n+1} - u^n)\\ &= \frac{B}{2}\|u^{n+1} - u^n\|^2 - \frac{B}{2}\|u^n - u^{n-1}\|^2 + \frac{B}{2}\|u^{n+1} - 2u^n + u^{n-1}\|^2,\end{aligned} \tag{2.7}$$

$$\begin{aligned}-A\tau(\Delta(u^{n+1} - u^n), u^{n+1} - u^n) &= A\tau(\nabla(u^{n+1} - u^n), \nabla(u^{n+1} - u^n))\\ &= A\tau\|\nabla(u^{n+1} - u^n)\|^2.\end{aligned} \tag{2.8}$$

To deal with the nonlinear term, we expand $F(u^{n+1})$ and $F(u^n)$ at $\hat{u}^{n+1/2}$ as

$$F(u^{n+1}) = F(\hat{u}^{n+1/2}) + f(\hat{u}^{n+1/2})(u^{n+1} - \hat{u}^{n+1/2}) + \frac{1}{2}f'(\xi_1^n)(u^{n+1} - \hat{u}^{n+1/2})^2, \tag{2.9}$$

$$F(u^n) = F(\hat{u}^{n+1/2}) + f(\hat{u}^{n+1/2})(u^n - \hat{u}^{n+1/2}) + \frac{1}{2}f'(\xi_2^n)(u^n - \hat{u}^{n+1/2})^2, \tag{2.10}$$

where ξ_1^n is a number between u^{n+1} and $\hat{u}^{n+1/2}$, ξ_2^n is a number between u^n and $\hat{u}^{n+1/2}$. Subtracting (2.10) from (2.9), we get

$$\begin{aligned}&F(u^{n+1}) - F(u^n) - f(\hat{u}^{n+1/2})(u^{n+1} - u^n)\\ &= \frac{1}{2}f'(\xi_1^n)[(u^{n+1} - \hat{u}^{n+1/2})^2 - (u^n - \hat{u}^{n+1/2})^2] - \frac{1}{2}(f'(\xi_2^n) - f'(\xi_1^n))(u^n - \hat{u}^{n+1/2})^2\\ &= \frac{1}{2}f'(\xi_1^n)(u^{n+1} - u^n)(u^{n+1} - 2u^n + u^{n-1}) - \frac{1}{8}(f'(\xi_2^n) - f'(\xi_1^n))(u^n - u^{n-1})^2\\ &\le \frac{L}{4}(|u^{n+1} - u^n|^2 + |u^{n+1} - 2u^n + u^{n-1}|^2) + \frac{L}{4}|u^n - u^{n-1}|^2.\end{aligned}$$

Taking integration, we obtain

$$\begin{aligned}&(F(u^{n+1})-F(u^n)-f(\hat{u}^{n+1/2})(u^{n+1}-u^n),1)\\ \le\ &\frac{L}{4}(\|u^{n+1}-u^n\|^2+\|u^{n+1}-2u^n+u^{n-1}\|^2+\|u^n-u^{n-1}\|^2).\end{aligned}\tag{2.11}$$

Combining (2.4), (2.5), (2.6), (2.7), (2.8) and (2.11), dropping some nonnegative terms, we have

$$\begin{aligned}&\frac{1}{2}\|\Delta u^{n+1}\|^2-\frac{1}{2}\|\Delta u^n\|^2-(\|\nabla u^{n+1}\|^2-\|\nabla u^n\|^2)+(F(u^{n+1})-F(u^n),1)\\ &+\frac{B}{2}\|u^{n+1}-u^n\|^2-\frac{B}{2}\|u^n-u^{n-1}\|^2+\frac{L}{4}\|u^{n+1}-u^n\|^2-\frac{L}{4}\|u^n-u^{n-1}\|^2\\ \le\ &\frac{L}{2}\|u^{n+1}-u^n\|^2+\left(\frac{L}{4}-\frac{B}{2}\right)\|u^{n+1}-2u^n+u^{n-1}\|^2\\ \le\ &\frac{L}{2}\|u^{n+1}-u^n\|^2+\left(\frac{L}{2}-B\right)(\|u^{n+1}-u^n\|^2+\|u^n-u^{n-1}\|^2)\\ =\ &(L-B)\|u^{n+1}-u^n\|^2+\left(\frac{L}{2}-B\right)\|u^n-u^{n-1}\|^2,\end{aligned}$$

which implies the desired result provided $B\ge L$. □

3. Convergence analysis

Denote $e^n=u^n-u(t^n)$ and $t^{n+1/2}=(t^n+t^{n+1})/2$. Let $u\lesssim v$ denote there is a positive constant C that is independent on τ and n such that $u\le Cv$. We now derive the error analysis of the proposed scheme, which shows the second-order convergence in time.

Theorem 3.1. *Assuming the analytical solution of (1.1) satisfies the regularity condition (2.2), we have the following error estimate*

$$\|u^k-u(t^k)\|\lesssim\tau^2,\ \ 0\le k\le K.$$

Proof. At time level $t^{n+1/2}$, the equation (1.1) becomes

$$u_t(t^{n+1/2})+\Delta^2u(t^{n+1/2})+2\Delta u(t^{n+1/2})+f(u(t^{n+1/2}))=0.\tag{3.1}$$

Subtracting (3.1) from (2.1), we have

$$\begin{aligned}&\frac{e^{n+1}-e^n}{\tau}+\Delta^2\left(\frac{e^{n+1}+e^n}{2}\right)+2\Delta\left(\frac{e^{n+1}+e^n}{2}\right)+f(\hat{u}^{n+1/2})-f(u(t^{n+1/2}))-A\tau\Delta(e^{n+1}-e^n)\\ &+B(e^{n+1}-2e^n+e^{n-1})+G_1^n+G_2^n=0,\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}G_1^n&=\frac{u(t^{n+1})-u(t^n)}{\tau}-u_t(t^{n+1/2}),\\ G_2^n&=(\Delta^2+2\Delta)\left(\frac{u(t^{n+1})+u(t^n)}{2}-u(t^{n+1/2})\right)-A\tau\Delta(u(t^{n+1})-u(t^n))\\ &\quad+B(u(t^{n+1})-2u(t^n)+u(t^{n-1})).\end{aligned}$$

By applying the Taylor expansion and the regularity condition (2.2), we have

$$\|G_1^n\|^2\le C_*^2\tau^4,\ \ \|G_2^n\|^2\le(A+B+1)^2C_*^2\tau^4.$$

Taking the inner product of (3.2) with $e^{n+1}+e^n$, we obtain

$$\begin{aligned}&\frac{1}{\tau}(\|e^{n+1}\|^2-\|e^n\|^2)+\frac{1}{2}\|\Delta(e^{n+1}+e^n)\|^2\\ =\ &-(\Delta(e^{n+1}+e^n),e^{n+1}+e^n)-(f(\hat{u}^{n+1/2})-f(u(t^{n+1/2})),e^{n+1}+e^n)\\ &-B(e^{n+1}-2e^n+e^{n-1},e^{n+1}+e^n)-(G_1^n,e^{n+1}+e^n)-(G_2^n,e^{n+1}+e^n)\\ &+A\tau(\Delta(e^{n+1}-e^n),e^{n+1}+e^n).\end{aligned}\tag{3.3}$$

We now estimate each terms at the right of (3.3). Using the Young inequality $(u, v) \le \frac{\alpha}{2}\|u\|^2 + \frac{1}{2\alpha}\|v\|^2$ and setting $\alpha = 2\nu$, we have

$$\begin{aligned}
&-(\Delta(e^{n+1}+e^n), e^{n+1}+e^n) \le \frac{1}{\nu}\|\Delta(e^{n+1}+e^n)\|^2 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2. \qquad (3.4)\\
&\quad -(f(\hat{u}^{n+1/2}) - f(u(t^{n+1/2})), e^{n+1}+e^n)\\
&\le (|f(\hat{u}^{n+1/2}) - f(u(t^{n+1/2}))|, |e^{n+1}+e^n|)\\
&\le (L|\hat{u}^{n+1/2} - u(t^{n+1/2})|, |e^{n+1}+e^n|)\\
&\le L(|\frac{3}{2}e^n - \frac{1}{2}e^{n-1}|, |e^{n+1}+e^n|) + L(|\frac{3}{2}u(t^n) - \frac{1}{2}u(t^{n-1}) - u(t^{n+1/2})|, |e^{n+1}+e^n|)\\
&\le \frac{L^2}{\nu}\|\frac{3}{2}e^n - \frac{1}{2}e^{n-1}\|^2 + \frac{L^2}{\nu}\|G_3^n\|^2 + \frac{\nu}{2}\|e^{n+1}+e^n\|^2,
\end{aligned}$$

where

$$G_3^n = \frac{3}{2}u(t^n) - \frac{1}{2}u(t^{n-1}) - u(t^{n+1/2}).$$

Similarly, by applying the Taylor expansion and the regularity condition (2.2), we have

$$\|G_3^n\|^2 \le C_*^2\tau^4.$$

Thus, we have

$$-(f(\hat{u}^{n+1/2}) - f(u(t^{n+1/2})), e^{n+1}+e^n) \le \frac{L^2}{\nu}\|\frac{3}{2}e^n - \frac{1}{2}e^{n-1}\|^2 + \frac{\nu}{2}\|e^{n+1}+e^n\|^2 + \frac{L^2C_*^2}{\nu}\tau^4. \qquad (3.5)$$

$$\begin{aligned}
-(G_1^n, e^{n+1}+e^n) &\le |(G_1^n, e^{n+1}+e^n)|\\
&\le \frac{1}{\nu}\|G_1^n\|^2 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2\\
&\le \frac{C_*^2}{\nu}\tau^4 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2. \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
-(G_2^n, e^{n+1}+e^n) &\le |(G_2^n, e^{n+1}+e^n)|\\
&\le \frac{1}{\nu}\|G_2^n\|^2 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2\\
&\le \frac{1}{\nu}(A+B+1)^2C_*^2\tau^4 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2. \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
A\tau(\Delta(e^{n+1}-e^n), e^{n+1}+e^n) &= A\tau(e^{n+1}-e^n, \Delta(e^{n+1}+e^n))\\
&\le \frac{1}{2\nu}\|e^{n+1}-e^n\|^2 + \frac{\nu}{2}A^2\tau^2\|\Delta(e^{n+1}+e^n)\|^2. \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
-B(e^{n+1}-2e^n+e^{n-1}, e^{n+1}+e^n) &\le B|(e^{n+1}-e^n-(e^n-e^{n-1}), e^{n+1}+e^n)|\\
&\le B(\|e^{n+1}\|^2 - \|e^n\|^2) + B|(e^n-e^{n-1}, e^{n+1}+e^n)|\\
&\le B(\|e^{n+1}\|^2 + \|e^n\|^2) + \frac{B^2}{\nu}\|e^n-e^{n-1}\|^2 + \frac{\nu}{4}\|e^{n+1}+e^n\|^2. \qquad (3.9)
\end{aligned}$$

Combining (3.3)-(3.9), we have

$$\begin{aligned}
&\frac{1}{\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) + (\frac{1}{2} - \frac{1}{\nu} - \frac{\nu}{2}A^2\tau^2)\|\Delta(e^{n+1}+e^n)\|^2\\
&\le (4+B)\|e^{n+1}\|^2 + (4\nu + \frac{9L^2+4B^2}{2\nu} + B)\|e^n\|^2 + \frac{L^2+4B^2}{2\nu}\|e^{n-1}\|^2 + C_1C_*^2\tau^4, \qquad (3.10)
\end{aligned}$$

where

$$C_1 = \frac{L^2+1+(A+B+1)^2}{\nu}.$$

If $\nu > 2$ and A is set small enough, it holds that

$$\sqrt{\frac{1-2/\nu}{\nu A^2}} \ge \tau \ \text{ and } \ \frac{1}{2} - \frac{1}{\nu} - \frac{\nu}{2}A^2\tau^2 \ge 0.$$

Multiplying (3.10) by τ and ignoring some nonnegative terms, summing up for n from 1 to $N-1$ and noting that $e^0=0$, we have

$$\|e^N\|^2 \le \tau \sum_{n=1}^{N} 3C_2\|e^n\|^2 + C_1C_*^2T\tau^4 + \|e^1\|^2,$$

where

$$C_2 = \max\left\{4\nu + B, 4\nu + \frac{9L^2+4B^2}{2\nu} + B, \frac{L^2+4B^2}{2\nu}\right\}.$$

Applying the discrete Grönwall inequality, we obtain the desired result. □

Next, we adopt a spectral-Galerkin approximation for the spacial variables and establish error estimates for the fully discrete linear Crank-Nicolson scheme. We denote by V_N the space of polynomials of degree $\le N$ in each direction, and for any $\varphi \in H^m(\Omega)$, we define a projection $\Pi_N : H^m(\Omega) \to V_N$ by

$$(\nabla(\Pi_N\varphi - \varphi), \nabla\psi_N) = 0, \ \ (\Pi_N\varphi - \varphi, 1) = 0, \ \forall\, \psi_N \in V_N. \tag{3.11}$$

It is well known that the following estimates hold [2]:

$$\|\varphi - \Pi_N\varphi\|_s \lesssim N^{s-m}\|\varphi\|_m, \ s = 0, 1, \ \forall\, \varphi \in H^m(\Omega), \ m \ge 1. \tag{3.12}$$

Let $L_0^2(\Omega) = \{v \in L^2(\Omega) : (v, 1) = 0\}$. The discrete Laplacian $\Delta_N : V_N \cap L_0^2(\Omega) \to V_N \cap L_0^2(\Omega)$ is defined as follows: for any $\psi_N \in V_N \cap L_0^2(\Omega)$, let $\Delta_N\psi_N$ be the unique solution to

$$(\Delta_N\psi_N, \chi_N) = -(\nabla\psi_N, \nabla\chi_N), \ \forall\, \chi_N \in V_N. \tag{3.13}$$

Let $w_N^{n+1/2} = -\Delta_N\left(\frac{u_N^{n+1}+u_N^n}{2}\right)$, the fully discrete mixed form of (2.1) is:

$$\frac{u_N^{n+1}-u_N^n}{\tau} - \Delta_N w_N^{n+1/2} + 2\Delta_N\left(\frac{u_N^{n+1}+u_N^n}{2}\right) + f(\hat{u}_N^{n+1/2}) - A\tau\Delta_N(u_N^{n+1}-u_N^n) + B(u_N^{n+1} - 2u_N^n + u_N^{n-1}) = 0,$$
$$w_N^{n+1/2} = -\Delta_N\left(\frac{u_N^{n+1}+u_N^n}{2}\right).$$

The spectral-Galerkin method for the linear Crank-Nicolson scheme (2.1) reads: Given $u_N^0 = \Pi_N u_0$, find $u_N^{n+1} \in V_N$ such that

$$\left(\frac{u_N^{n+1}-u_N^n}{\tau}, v_N\right) + (\nabla w_N^{n+1/2}, \nabla v_N) - 2\left(\nabla\frac{u_N^{n+1}+u_N^n}{2}, \nabla v_N\right) + (f(\hat{u}_N^{n+1/2}), \nabla v_N)$$
$$+ A\tau(\nabla(u_N^{n+1}-u_N^n), \nabla v_N) + B(u_N^{n+1} - 2u_N^n + u_N^{n-1}, v_N) = 0, \ \forall\, v_N \in V_N \tag{3.14}$$
$$(w_N^{n+1/2}, \psi_N) = \left(\nabla\frac{u_N^{n+1}+u_N^n}{2}, \nabla\psi_N\right), \ \forall\, \psi_N \in V_N \tag{3.15}$$

Let us denote

$$\rho_u^n = \Pi_N u(t^n) - u(t^n), \ \ \sigma_u^n = u_N^n - \Pi_N u(t^n),$$
$$\rho_w^{n+1/2} = \Pi_N w(t^{n+1/2}) - w(t^{n+1/2}), \ \ \sigma_w^{n+1/2} = w_N^{n+1/2} - \Pi_N w(t^{n+1/2}),$$

thus,

$$e_u^n = u_N^n - u(t^n) = \rho_u^n + \sigma_u^n, \ \ e_w^{n+1/2} = w_N^{n+1/2} - w(t^{n+1/2}) = \rho_w^{n+1/2} + \sigma_w^{n+1/2}. \tag{3.16}$$

By the definition of the projection Π_N,

$$(\nabla\rho_u^n, \nabla\psi_N) = 0, \ \ (\nabla\rho_w^{n+1/2}, \nabla\psi_N) = 0, \ \forall\, \psi_N \in V_N. \tag{3.17}$$

We also denote

$$R_1^{n+1/2} = \frac{u(t^{n+1}) - u(t^n)}{\tau} - u_t(t^{n+1/2}), \quad R_2^{n+1/2} = \frac{u(t^{n+1}) + u(t^n)}{2} - u(t^{n+1/2}),$$
$$R_3^{n+1} = \tau(u(t^{n+1}) - u(t^n)), \quad R_4^n = u(t^{n+1}) - 2u(t^n) + u(t^{n-1}),$$
$$R_5^{n+1/2} = \frac{3}{2}u(t^n) - \frac{1}{2}u(t^{n-1}) - u(t^{n+1/2}).$$

By using the Taylor expansion, we can easily derive the following estimates

$$\|R_1^{n+1/2}\| \le \frac{1}{24}\|u_{ttt}\|_{L^\infty(0,T;L^2(\Omega))}\tau^2, \quad \|R_2^{n+1/2}\| \le \frac{1}{8}\|u_{tt}\|_{L^\infty(0,T;H^2(\Omega))}\tau^2, \tag{3.18}$$
$$\|R_3^{n+1}\| \le \|u_t\|_{L^\infty(0,T;L^2(\Omega))}\tau^2, \quad \|R_4^n\| \le \|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))}\tau^2, \tag{3.19}$$
$$\|R_5^{n+1}\| \le \frac{9}{8}\|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))}\tau^2. \tag{3.20}$$

Theorem 3.2. *Assuming that $u \in L^2(0,T;H^m(\Omega))$, $u_t \in L^\infty(0,T;L^2(\Omega))$, $u_{tt} \in L^\infty(0,T;H^2(\Omega))$ and $u_{ttt} \in L^\infty(0,T;L^2(\Omega))$, then we have the following error estimate*

$$\|u(t^n) - u_N^n\| \lesssim N^{-m} + \tau^2, \quad 0 \le n \le T/\tau.$$

Proof. Let $w = -\Delta u$, the mixed weak form of (1.1) is

$$(u_t, v_N) + (\nabla w, \nabla v_N) - 2(\nabla u, \nabla v_N) + (f(u), v_N) = 0, \quad \forall\, v_N \in V_N, \tag{3.21}$$
$$(w, \psi_N) = (\nabla u, \nabla \psi_N), \quad \forall\, \psi_N \in V_N. \tag{3.22}$$

Subtracting (3.21)-(3.22) from (3.14)-(3.15) at $t^{n+1/2}$, we get

$$\begin{aligned}
&\left(\frac{e_u^{n+1} - e_u^n}{\tau}, v_N\right) + (R_1^{n+1/2}, v_N) + (\nabla(e_w^{n+1/2}), \nabla v_N) - 2\left(\nabla\frac{e_u^{n+1} + e_u^n}{2}, \nabla v_N\right) - 2(\nabla R_2^{n+1/2}, \nabla v_N)\\
&+ (f(\hat{u}_N^{n+1/2}) - f(u(t^{n+1/2})), v_N) + A\tau(\nabla(e_u^{n+1} - e_u^n), \nabla v_N) + A(\nabla R_3^{n+1}, \nabla v_N) + B(e_u^{n+1} - 2e_u^n + e_u^{n-1}, v_N)\\
&+ B(R_4^n, v_N) = 0,\\
&(e_w^{n+1/2}, \psi_N) = \left(\nabla\frac{e_u^{n+1} + e_u^n}{2}, \nabla\psi_N\right) + (\nabla R_2^{n+1/2}, \nabla\psi_N).
\end{aligned}$$

Using (3.16) and (3.17), we have

$$\begin{aligned}
&\left(\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}, v_N\right) + \left(\frac{\rho_u^{n+1} - \rho_u^n}{\tau}, v_N\right) + (R_1^{n+1/2}, v_N) + (\nabla\sigma_w^{n+1/2}, \nabla v_N) - 2\left(\nabla\frac{\sigma_u^{n+1} + \sigma_u^n}{2}, \nabla v_N\right)\\
&- 2(\nabla R_2^{n+1/2}, \nabla v_N) + (f(\hat{u}_N^{n+1/2}) - f(u(t^{n+1/2})), v_N) + A\tau(\nabla(\sigma_u^{n+1} - \sigma_u^n), \nabla v_N) + A\tau(\nabla R_3^{n+1}, \nabla v_N)\\
&+ B(\rho_u^{n+1} - 2\rho_u^n + \rho_u^{n-1}, v_N) + B(\sigma_u^{n+1} - 2\sigma_u^n + \sigma_u^{n-1}, v_N) + B(R_4^n, v_N) = 0,
\end{aligned} \tag{3.23}$$
$$(\rho_w^{n+1/2}, \psi_N) + (\sigma_w^{n+1/2}, \psi_N) = \left(\nabla\frac{\sigma_u^{n+1} + \sigma_u^n}{2}, \nabla\psi_N\right) + (\nabla R_2^{n+1/2}, \nabla\psi_N). \tag{3.24}$$

Taking $\psi_N = \Delta_N \frac{\sigma_u^{n+1} + \sigma_u^n}{2}$ in (3.24) and using (3.13) and (3.17), we have

$$\left(\nabla\sigma_w^{n+1/2}, \nabla\frac{\sigma_u^{n+1} + \sigma_u^n}{2}\right) = \frac{1}{4}\|\Delta_N(\sigma_u^{n+1} + \sigma_u^n)\|^2 + \left(\Delta_N R_2^{n+1/2}, \Delta_N\frac{\sigma_u^{n+1} + \sigma_u^n}{2}\right). \tag{3.25}$$

Taking $v_N = \frac{\sigma_u^{n+1} + \sigma_u^n}{2}$ in (3.23) and using (3.13) and (3.17), multiplying both sides of the resulting equation by 2τ, we have

$$\begin{aligned}
&\|\sigma_u^{n+1}\|^2 - \|\sigma_u^n\|^2 + \frac{1}{2}\tau\|\Delta_N(\sigma_u^{n+1} + \sigma_u^n)\|^2 + A\tau^2(\nabla(\sigma_u^{n+1} - \sigma_u^n), \nabla(\sigma_u^{n+1} + \sigma_u^n))\\
= &-(\rho_u^{n+1} - \rho_u^n, \sigma_u^{n+1} + \sigma_u^n) - \tau(R_1^{n+1/2}, \sigma_u^{n+1} + \sigma_u^n) - \tau(\Delta_N R_2^{n+1/2}, \Delta_N(\sigma_u^{n+1/2} + \sigma_u^n))\\
&+ 2\tau(\nabla R_2^{n+1/2}, \nabla(\sigma_u^{n+1} + \sigma_u^n)) - \tau(f(\hat{u}_N^{n+1/2}) - f(u(t^{n+1/2})), \sigma_u^{n+1} + \sigma_u^n)\\
&- A\tau^2(\nabla R_3^{n+1}, \nabla(\sigma_u^{n+1} + \sigma_u^n)) - B\tau(\rho_u^{n+1} - 2\rho_u^n + \rho_u^{n-1}, \sigma_u^{n+1} + \sigma_u^n)\\
&- B\tau(\sigma_u^{n+1} - 2\sigma_u^n + \sigma_u^{n-1}, \sigma_u^{n+1} + \sigma_u^n) - B\tau(R_4^n, \sigma_u^{n+1} + \sigma_u^n)\\
&+ \frac{1}{2}(\nabla(\sigma_u^{n+1} + \sigma_u^n), \nabla(\sigma_u^{n+1} + \sigma_u^n)).
\end{aligned} \tag{3.26}$$

First,

$$A\tau^2(\nabla(\sigma_u^{n+1}-\sigma_u^n),\nabla(\sigma_u^{n+1}+\sigma_u^n)) = A\tau^2(\|\nabla\sigma_u^{n+1}\|^2-\|\nabla\sigma_u^n\|^2). \tag{3.27}$$

Next, we estimate each terms at the right of (3.26),

$$\begin{aligned}
-(\rho_u^{n+1}-\rho_u^n,\sigma_u^{n+1}+\sigma_u^n) &\le \frac{1}{2\tau}\|\rho_u^{n+1}-\rho_u^n\|^2+\frac{\tau}{2}\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&\le (\|\rho_u^{n+1}\|^2+\|\rho_u^n\|^2)+\tau(\|\sigma_u^{n+1}\|^2+\|\sigma_u^n\|^2),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
-\tau(R_1^{n+1/2},\sigma_u^{n+1}+\sigma_u^n) &\le \frac{\tau}{2}\|R_1^{n+1/2}\|^2+\frac{\tau}{2}\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&\le \frac{\tau}{2}\|R_1^{n+1/2}\|^2+\tau\|\sigma_u^{n+1}\|^2+\tau\|\sigma_u^n\|^2.
\end{aligned} \tag{3.29}$$

$$-\tau(\Delta_N R_2^{n+1/2},\Delta_N(\sigma_u^{n+1}+\sigma_u^n)) \le \frac{1}{2\eta}\tau\|\Delta_N R_2^{n+1/2}\|^2+\frac{\eta}{2}\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2, \tag{3.30}$$

$$\begin{aligned}
2\tau(\nabla R_2^{n+1/2},\nabla(\sigma_u^{n+1}+\sigma_u^n)) &\le 2\tau|(R_2^{n+1/2},\Delta_N(\sigma_u^{n+1}+\sigma_u^n))|\\
&\le \frac{1}{\eta}\tau\|R_2^{n+1/2}\|^2+\eta\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
-A\tau^2(\nabla R_3^{n+1},\nabla(\sigma_u^{n+1}+\sigma_u^n)) &\le A\tau^2|(R_3^{n+1},\Delta_N(\sigma_u^{n+1}+\sigma_u^n))|\\
&\le \frac{1}{2\eta}\tau^3A^2\|R_3^{n+1}\|^2+\frac{\eta}{2}\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2.
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
&-B\tau(\rho_u^{n+1}-2\rho_u^n+\rho_u^{n-1},\sigma_u^{n+1}+\sigma_u^n) \le \frac{B\tau}{2}\|\rho_u^{n+1}-2\rho_u^n+\rho_u^{n-1}\|^2+\frac{B\tau}{2}\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&=\frac{B\tau}{2}\|(\rho_u^{n+1}-\rho_u^n)-(\rho_u^n-\rho_u^{n-1})\|^2+\frac{B\tau}{2}\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&\le B\tau\|\rho_u^{n+1}-\rho_u^n\|^2+B\tau\|\rho_u^n-\rho_u^{n-1}\|^2+B\tau\|\sigma_u^{n+1}\|^2+B\tau\|\sigma_u^n\|^2\\
&\le 2B\tau\|\rho_u^{n+1}\|^2+4B\tau\|\rho_u^n\|^2+2B\tau\|\rho_u^{n-1}\|^2+B\tau\|\sigma_u^{n+1}\|^2+B\tau\|\sigma_u^n\|^2.
\end{aligned} \tag{3.33}$$

$$-B\tau(\sigma_u^{n+1}-2\sigma_u^n+\sigma_u^{n-1},\sigma_u^{n+1}+\sigma_u^n) \le 3B\tau\|\sigma_u^{n+1}\|^2+5B\tau\|\sigma_u^n\|^2+2B\tau\|\sigma_u^{n-1}\|^2. \tag{3.34}$$

$$\begin{aligned}
-B\tau(R_4^{n+1},\sigma_u^{n+1}+\sigma_u^n) &\le \frac{1}{2}B\tau\|R_4^{n+1}\|^2+\frac{1}{2}B\tau\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&\le \frac{1}{2}B\tau\|R_4^n\|^2+B\tau\|\sigma_u^{n+1}\|^2+B\tau\|\sigma_u^n\|^2.
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\frac{1}{2}(\nabla(\sigma_u^{n+1}+\sigma_u^n),\nabla(\sigma_u^{n+1}+\sigma_u^n)) &\le \frac{1}{2}|(\sigma_u^{n+1}+\sigma_u^n,\Delta_N(\sigma_u^{n+1}+\sigma_u^n))|\\
&\le \frac{1}{4\eta}\|\sigma_u^{n+1}+\sigma_u^n\|^2+\frac{\eta}{4}\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2\\
&\le \frac{1}{2\eta}(\|\sigma_u^{n+1}\|^2+\|\sigma_u^n\|^2)+\frac{\eta}{4}\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2.
\end{aligned} \tag{3.36}$$

Since

$$\begin{aligned}
\|f(\hat{u}_N^{n+1/2})-f(u(t^{n+1/2}))\| &\le L\|\hat{u}_N^{n+1/2}-u(t^{n+1/2})\|\\
&=\|\frac{3}{2}u_N^n-\frac{1}{2}u_N^{n-1}-u(t^{n+1/2})\|\\
&=\|\frac{3}{2}u_N^n-\frac{1}{2}u_N^{n-1}-\frac{3}{2}u(t^n)+\frac{1}{2}u(t^{n-1})+\frac{3}{2}u(t^n)-\frac{1}{2}u(t^{n-1})-u(t^{n+1/2})\|\\
&\le\|\frac{3}{2}e_u^n-\frac{1}{2}e_u^{n-1}\|+\|\frac{3}{2}u(t^n)-\frac{1}{2}u(t^{n-1})-u(t^{n+1/2})\|\\
&\le\|\frac{3}{2}\rho_u^n-\frac{1}{2}\rho_u^{n-1}\|+\|\frac{3}{2}\sigma_u^n-\frac{1}{2}\sigma_u^{n-1}\|+\|R_5^{n+1/2}\|,
\end{aligned}$$

we have

$$
\begin{aligned}
&-\tau(f(\hat{u}_N^{n+1/2})-f(u(t^{n+1/2})),\sigma_u^{n+1}+\sigma_u^n)\le\|f(\hat{u}_N^{n+1/2})-f(u(t^{n+1/2}))\|\|\sigma_u^{n+1}+\sigma_u^n\|\\
&\le\tau\|\frac{3}{2}\rho_u^n-\frac{1}{2}\rho_u^{n-1}\|\|\sigma_u^{n+1}+\sigma_u^n\|+\tau\|\frac{3}{2}\sigma_u^n-\frac{1}{2}\sigma_u^{n-1}\|\|\sigma_u^{n+1}+\sigma_u^n\|+\tau\|R_5^{n+1/2}\|\|\sigma_u^{n+1}+\sigma_u^n\|\\
&\le\frac{1}{2}\tau\|\frac{3}{2}\rho_u^n-\frac{1}{2}\rho_u^{n-1}\|^2+\frac{1}{2}\tau\|\frac{3}{2}\sigma_u^n-\frac{1}{2}\sigma_u^{n-1}\|^2+\frac{1}{2}\tau\|R_5^{n+1/2}\|^2+\frac{3}{2}\tau\|\sigma_u^{n+1}+\sigma_u^n\|^2\\
&\le\frac{9}{4}\tau\|\rho_u^n\|^2+\frac{1}{4}\tau\|\rho_u^{n-1}\|^2+3\tau\|\sigma_u^{n+1}\|^2+\frac{21}{4}\tau\|\sigma_u^n\|^2+\frac{1}{4}\tau\|\sigma_u^{n-1}\|^2+\frac{1}{2}\tau\|R_5^{n+1/2}\|^2.
\end{aligned}
\tag{3.37}
$$

Combining (3.26)-(3.37), we derive

$$
\begin{aligned}
&\|\sigma_u^{n+1}\|^2-\|\sigma_u^n\|^2+\frac{1}{2}\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2+A\tau^2(\|\nabla\sigma_u^{n+1}\|^2-\|\nabla\sigma_u^n\|^2)\\
&\le(\frac{1}{\tau}+2B\tau)\|\rho_u^{n+1}\|^2+(\frac{1}{\tau}+\frac{9}{4}\tau+4B\tau)\|\rho_u^n\|^2+(\frac{1}{4}\tau+2B\tau)\|\rho_u^{n-1}\|^2\\
&\quad+(5\tau+5B\tau)\|\sigma_u^{n+1}\|^2+(\frac{29}{4}\tau+7B\tau)\|\sigma_u^n\|^2+(\frac{1}{4}\tau+2B\tau)\|\sigma_u^{n-1}\|^2\\
&\quad+\frac{\tau}{2}\|R_1^{n+1/2}\|^2+\frac{1}{2\eta}\tau\|\Delta_N R_2^{n+1/2}\|^2+\frac{1}{\eta}\tau\|R_2^{n+1/2}\|^2+\frac{1}{2\eta}A^2\tau^3\|R_3^{n+1}\|^2+\frac{1}{2}B\tau\|R_4^{n+1}\|^2\\
&\quad+\frac{1}{2}\tau\|R_5^{n+1/2}\|^2+\frac{9\eta}{4}\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2.
\end{aligned}
$$

Choosing $\eta=\frac{1}{9}$ such that $\frac{9}{4}\eta=\frac{1}{4}$ and using (3.12), (3.18)-(3.20), we have

$$
\begin{aligned}
&\|\sigma_u^{n+1}\|^2-\|\sigma_u^n\|^2+\frac{1}{4}\tau\|\Delta_N(\sigma_u^{n+1}+\sigma_u^n)\|^2+A\tau^2(\|\nabla\sigma_u^{n+1}\|^2-\|\nabla\sigma_u^n\|^2)\\
&\lesssim\tau(\|\sigma_u^{n+1}\|^2+\|\sigma_u^n\|^2+\|\sigma_u^{n-1}\|^2)+\tau^5+N^{-2m}.
\end{aligned}
\tag{3.38}
$$

Summing over n $(n=0,1,2,\cdots,k-1)$ and using the Grönwall's inequality, we have

$$\|\sigma_u^k\|\lesssim\tau^2+N^{-m}.$$

In addition, because $\|u(t^k)-u_N^k\|\le\|\rho_u^k\|+\|\sigma_u^k\|$ and (3.12), we get

$$\|u(t^k)-u_N^k\|\lesssim\tau^2+N^{-m}.\quad\square$$

4. Numerical experiments

In this section, we give several numerical experiments for the SH equation to verify the accuracy and energy stability of the proposed scheme. We apply the spectral-Galerkin method for spacial discretization to solve the SH equation with the periodic boundary condition.

To start the second-order scheme, we use the following first-order scheme to calculate u^1,

$$\frac{u^1-u^0}{\tau}+\Delta^2u^1+2\Delta u^0+(u^0)^3+(1-\epsilon)u^1=0.$$

4.1. Accuracy and energy stability

We first test the temporal convergence rate of our scheme with the initial condition

$$u(x,y)=\sin(\frac{\pi x}{16})\cos(\frac{\pi y}{16}),$$

on the domain $\Omega=[0,32]\times[0,32]$, namely, $L_x=L_y=32$. We take $N=256$ so that the spacial discretization errors are negligible compared with the temporal discretization errors. We take $\epsilon=0.5$, $A=0.5$, $B=0.5$. There's nothing special about this set of parameters, we can choose other parameters, such as $\epsilon=0.01$, $A=1$ and $B=1$ and it will provide similar numerical results. The errors are calculated by comparison with the reference solution with $\tau=2^{-14}$. The notation u_τ denotes the numerical approximate solution with the time step size τ, $\bar{u}$ denotes the reference solution. The l^2 error can be defined as

$$\|u_N^K-\bar{u}_N^K\|_{l^2}=\sqrt{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}h^2|u_{i,j}^K-\bar{u}_{i,j}^K|^2},$$

Table 1
The errors, rates of convergence and CPU time at $T = 1$ for the phase variable u with different time step sizes. The physical parameter is $\epsilon = 0.5$ and the stabilized constant is $A = B = 0.5$.

τ	l^2 error	Rate	CPU-time (s)
1/8	1.9639e-03	-	0.0312
1/16	4.4382e-04	2.15	0.0780
1/32	1.1348e-04	1.97	0.1716
1/64	2.9095e-05	1.96	0.2808
1/128	7.3857e-06	1.98	0.4524
1/256	1.8613e-06	1.99	1.0140
1/512	4.6697e-07	1.99	2.1996
1/1024	1.1665e-07	2.00	4.0560

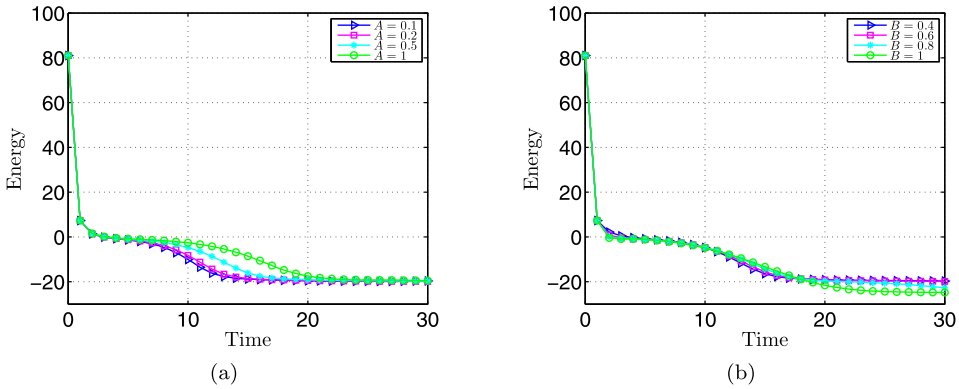

Fig. 1. Evolution of the energy with different stabilization constants A and B.

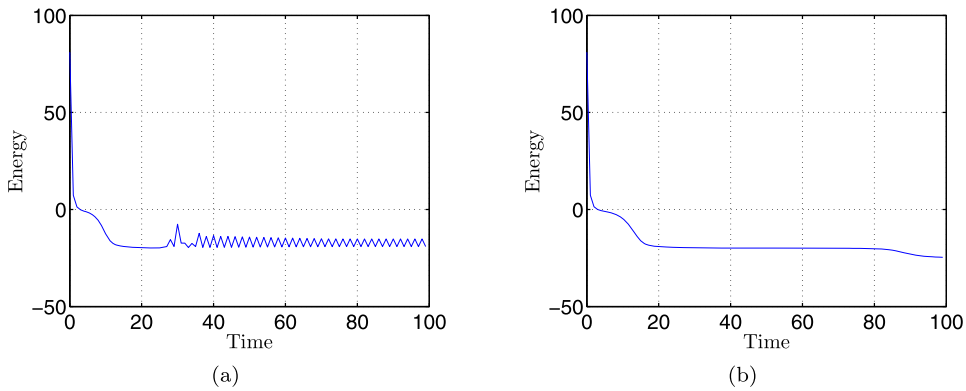

Fig. 2. Evolution of the energy with $A = 0$, $B = 0.5$, $\tau = 1$ (a) and $A = 0.5$, $B = 0.5$, $\tau = 1$ (b).

where $h = L_x/N = L_y/N$. The temporal convergence rate can be computed by the following formula

$$\text{rate} = \frac{\log(\|u_\tau - \bar{u}\|_{l^2}/\|u_{\tau/2} - \bar{u}\|_{l^2})}{\log(2)}.$$

In Table 1, we show the l^2 errors of the phase variable with different time step sizes at $T = 1$ and we can observe that our scheme gives desired rate of accuracy in time. Fig. 1 (a) shows that the energy decay with respect to different stabilization constant A. Fig. 1 (b) shows that the energy decay with respect to different stabilization constant B. Fig. 2 shows the energy evolution with $A = 0$, $B = 0.5$, $\tau = 1$ and $A = 0.5$, $B = 0.5$, $\tau = 1$, which implies that the term $-A\tau\Delta(u^{n+1} - u^n)$ is necessary for the scheme to be energy-stable. Fig. 3 shows the energy evolution with $A = -1$, $B = -1$, $\tau = 1$ and $A = -1$, $B = 1$, $\tau = 1$. We observe that the energy quickly blows up, which implies that parameters A and B can not be less than zero. If the exact solution is given by

$$u(x, y, t) = \sin(\frac{\pi x}{16})\cos(\frac{\pi y}{16})\exp(-t),$$

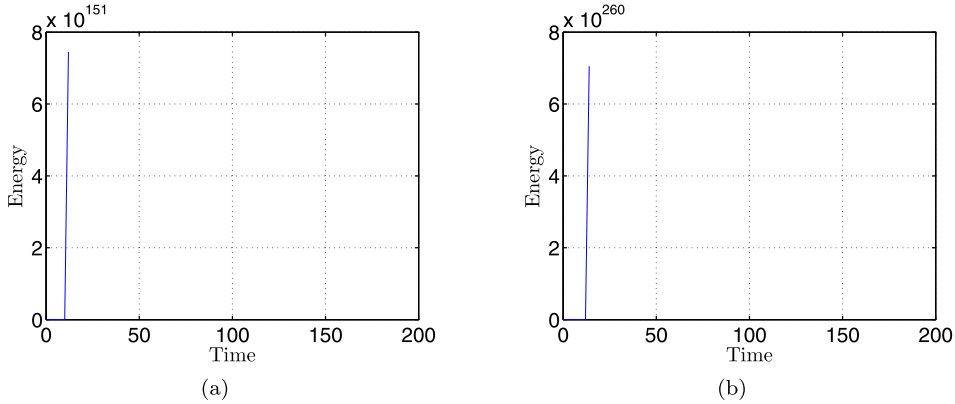

Fig. 3. Evolution of the energy with $A = -1$, $B = -1$, $\tau = 1$ (a) and $A = -1$, $B = 1$, $\tau = 1$ (b).

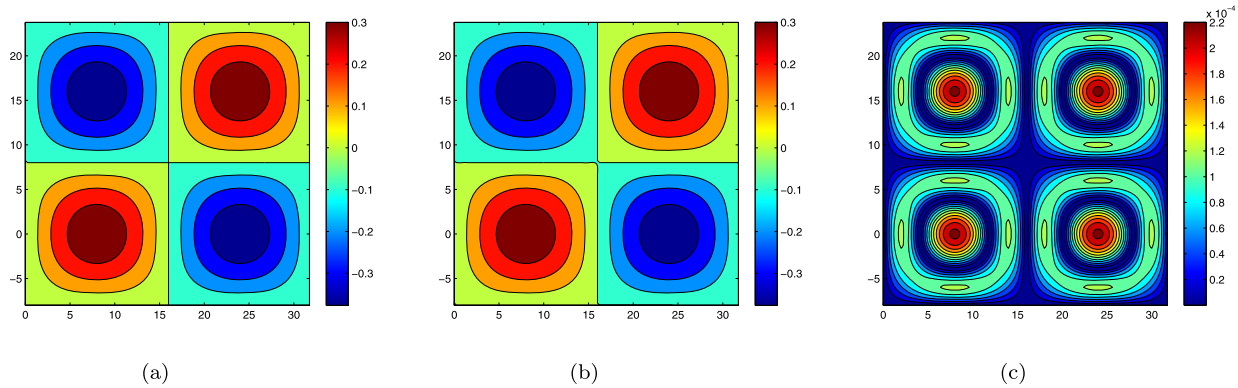

Fig. 4. The profiles at $T = 1$ of the exact solution (a) and the numerical solution (b) and the absolute error between the exact solution and the numerical solution (c). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

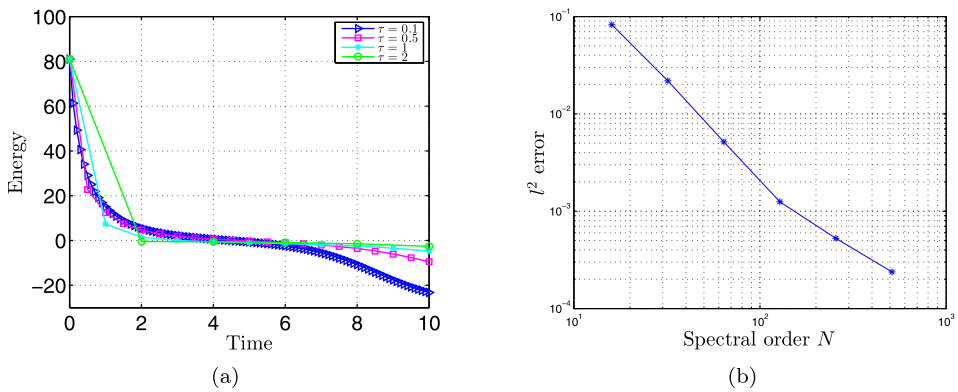

Fig. 5. Evolution of the energy with distinct time step sizes τ (a) and the spacial l^2 errors at $T = 1$ (b).

and we add an artificial, time-dependent forcing term at the right hand side of (1.1). Profiles at $T = 1$ of the exact solution and the numerical solution are given in Fig. 4 (a) and Fig. 4 (b), respectively. The profile of absolute error between the exact solution and the numerical solution is plotted in Fig. 4 (c). Evolution of the energy with distinct time step size τ is plotted in Fig. 5 (a) and spatial l^2 errors at $T = 1$ are plotted in Fig. 5 (b).

4.2. Phase transition behaviors

We apply our scheme to check the evolution from a randomly perturbed nonequilibrium state to a steady state. The initial condition is $u(x, y) = 0.4 + \text{rand}(x, y)$, where $\text{rand}(x, y)$ is the random number between -0.02 and 0.02 at the grid points. We set $\epsilon = 0.5$, $A = B = 0.5$, $N = 128$, $\tau = 1$ and $T = 100$. Fig. 6 shows the time evolution of the phase transition

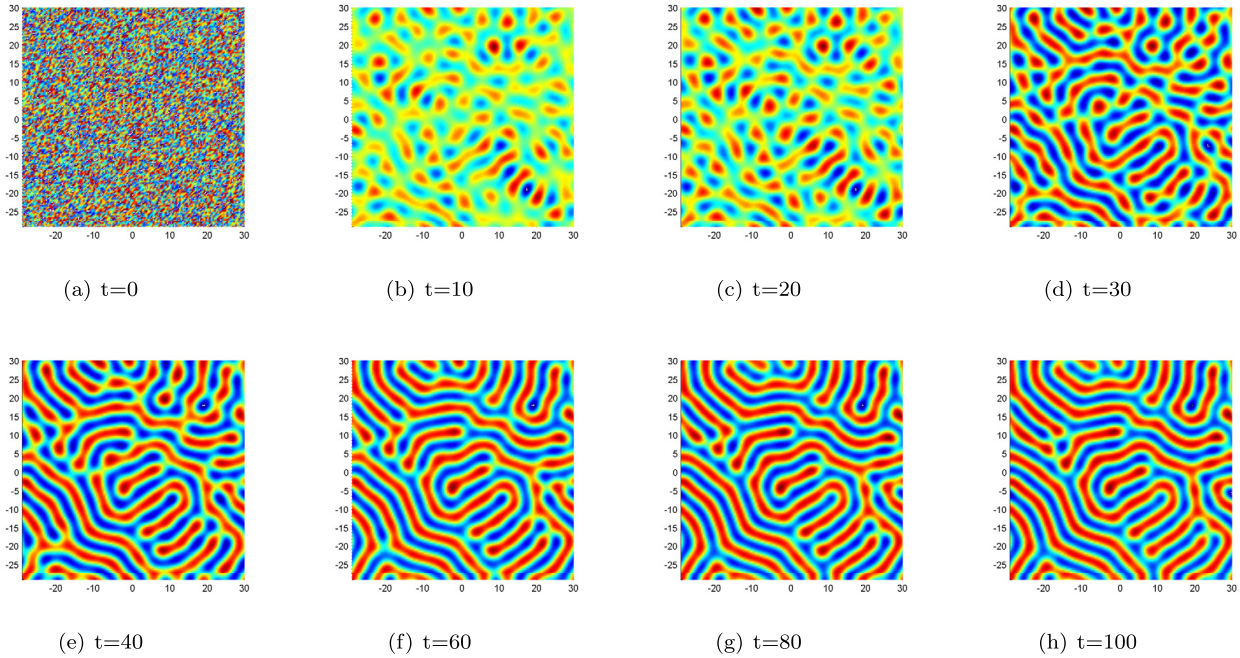

Fig. 6. The evolution of the phase transition behavior. Snapshots of the numerical approximation of the phase variable u are taken at t =0, 10, 20, 30, 40, 60, 80, 100. The computational domain is $[-30, 30] \times [-30, 30]$. The parameters are $\epsilon = 0.5$, $A = B = 0.5$, $N = 128$, $\tau = 1$, $T = 100$.

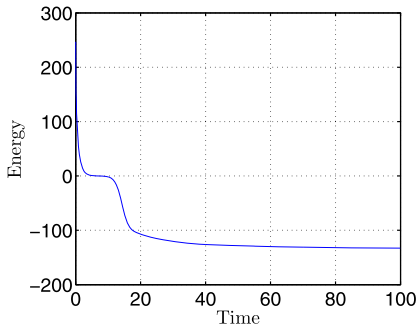

Fig. 7. Evolution of the energy with the random initial condition.

behavior, which validates that our scheme does lead to the expected states. Fig. 7 shows the energy evolution with the random initial condition.

5. Conclusions

In the work, we design a stabilized linear Crank-Nicolson scheme for the SH equation. Rigorous results about convergence and error estimates are derived, which show the second-order convergence in time of our proposed scheme. Numerical tests show our scheme is energy-stable with a large enough time step size and the energy decay is robust with the stabilization constants.

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