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Error analysis of first- and second-order linear, unconditionally energy-stable schemes for the Swift-Hohenberg equation ☆

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A B S T R A C T

In this work, we present first- and second-order energy-stable linear schemes for the Swift-Hohenberg equation based on first-order backward Euler and Crank-Nicolson schemes, respectively. We prove rigorously that the schemes satisfy the energy dissipation property. We also derive the error analysis for our schemes. Moreover, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete second-order Crank-Nicolson scheme. Numerical results are presented to validate our theoretical analysis.

1. Introduction

The Swift-Hohenberg (SH) equation is a widely applied phase-field model and it was originally derived by Swift and Hohenberg [20] to describe Rayleigh-Bénard convection. Related applications can be found in complex pattern formation [2,21], complex fluids and biological tissues [6,7,17]. The SH equation is derived from the following free energy functional

$$E(u)=\int_\Omega \left(\frac{1}{2}u(1+\Delta)^2u+\frac{\beta}{2}u^2+F(u)\right)\mathrm{d}x, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^d$ $(d=1,2,3)$, u is the density field, $F(u)=\frac{1}{4}u^4-\frac{\epsilon+\beta}{2}u^2$, $0<\epsilon<1$ is a constant with physical significance and Δ is the Laplace operator. The SH equation is given by

$$u_t=-\frac{\delta E}{\delta u}=-((1+\Delta)^2u+\beta u+f(u)), \tag{1.2}$$

with the periodic boundary conditions and initial conditions $u|_{t=0}=u^0$, where $\frac{\delta}{\delta u}$ denotes the variational derivative, $u_t=\frac{\partial u}{\partial t}$, $f(u)=F'(u)=u^3-(\epsilon+\beta)u$. A classic example for the SH equation is convection of a thin layer of fluid heated from below for which we can think of the scalar quantity u as representing the temperature of the fluid in the mid plane. The parameter ϵ measures how far the temperature is above the minimum temperature required for convection: for $\epsilon<0$, the heating is too small to cause convection, while for $\epsilon>0$, convection occurs. The free energy is nonincreasing in time:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\int_\Omega \frac{\delta E}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x=-\int_\Omega (u_t)^2\mathrm{d}x\le 0.$$

Here we study the numerical scheme of SH equation with periodic boundary conditions since that is used very frequently in numerical or analytical works of SH equation. If we choose other physical boundary conditions like Neumann type, the analysis is also true.

As a nonlinear fourth-order partial differential equation, the SH equation is difficult to be solved analytically. Hence, various numerical schemes have been proposed in recent years. To design a numerical scheme satisfying energy dissipation law, the linear terms are generally treated implicitly and the nonlinear terms are treated by different approaches. A very efficient approach is the convex splitting method [3]. Based on the convex

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splitting method, Lee presented a non-iterative scheme for the SH equation with quadratic-cubic nonlinearity without convergence analysis in [10]. Zhang and Ma constructed and analyzed a large time-stepping scheme for the SH equation in [29]. In [13], the authors proposed a second-order energy-stable numerical scheme for the SH equation and presented an optimal error estimate for the scheme. By applying the Crank-Nicolson scheme, a semi-implicit second-order method for the SH equation was given in [4]. The convex splitting method is unconditionally energy stable and uniquely solvable. However, to solve the fully discrete nonlinear systems, these methods generally require the use of a iteration. Hence, the computational costs are often high and the implementations are usually complicated. Another efficient approach is the stabilization method. By introducing artificial stabilization terms, one can alleviate the time step restriction and balance the explicit treatment of the nonlinear term, see [18]. Efficiency and simplicity are the main advantages of the stabilization method. In [15], the authors proposed a stabilized linear predictor-corrector scheme for the SH equation, they also proved rigorously that the scheme satisfies the energy dissipation law and is second-order accurate. In [14], a stabilized linear Crank-Nicolson scheme for the SH equation was proposed and analyzed. The operator splitting method is also a very powerful approach for solving the phase-field models. In [9], based on the operator splitting scheme, the first- and second-order Fourier spectral methods were presented for the SH equation. In [5], a new conservative SH equation was introduced and its first-order and second-order mass conservative operator splitting schemes were proposed. In [23], a fast explicit high-order operator splitting scheme was presented for the SH equation with a nonlocal nonlinearity. There are also various interesting linear approaches that attract the attention of many scholars, such as invariant energy quadratization (IEQ) scheme [25] and scalar auxiliary variable (SAV) scheme [19]. These approaches provide linear numerical schemes and satisfy unconditional energy stability based on a modified energy functional and related applications can be found in [16,28,27,8,22].

In this work, we design and analyze first- and second-order unconditionally energy-stable linear schemes combined with IEQ approach for the SH equation. Although there exist some works about IEQ type schemes for the SH equation, such as [11,12,24], almost all works only focus on the unconditional energy stability. In view of the absence of error analysis, the main goal of this paper is to derive the error analysis for the first- and second-order IEQ schemes for the SH equation. In [26], Yang and Zhang gave the convergence analysis for the IEQ schemes for solving the Cahn-Hilliard and Allen-Cahn equations with general nonlinear potential, but the authors considered only time discrete schemes in their study and Remark 4.1 was given in their article to indicate, for the fully discrete IEQ scheme of the Cahn-Hilliard equations, their were not clear on how to derive the corresponding error analysis using Galerkin type approximations and this was a challenging work. In this work, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete IEQ scheme, which is not studied in [26]. In order to get optimal error estimates, some reasonable conditions about continuity and boundedness for the nonlinear terms are given. Unconditional energy stability and unique solvability are also rigorously proved. Numerical tests are presented to support our theoretical results.

The rest of the paper is organized as follows. In Section 2, we design the first-order linear energy-stable scheme and prove the scheme satisfies the energy dissipation property. Then we derive the error estimate. In Section 3, we construct the second-order linear energy-stable scheme and prove the scheme satisfies the energy dissipation property. Then the error estimate is derived. In Section 4, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete second-order Crank-Nicolson scheme. In Section 5, numerical tests are provided to illustrate the accuracy and energy stability of the proposed schemes. In the end, some conclusions are presented in Section 6.

We introduce some notations which will be used in the analysis. We denote the spaces $L^p(\Omega)$ associated with the L^p norm $\|u\|_{L^p} := (\int_\Omega |u(x)|^p \mathrm{d}x)^{1/p}$. We also introduce the space $L^\infty(\Omega)$ with $\|v\|_{L^\infty} = \sup_{x\in\Omega}|v(x)|$. $W^{k,p}(\Omega)$ stands for the standard Sobolev spaces equipped with the standard Sobolev norms $\|\cdot\|_{k,p}$. For $p=2$, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and the corresponding norm is $\|\cdot\|_k$. The space $L^p(0,T;V)$ represents the L^p space on the interval $(0,T)$ with values in the function space V. We denote by $(\cdot,\cdot)$ the inner product in L^2 and $\|\cdot\|$ the norm in L^2. Let $x \lesssim y$ denote there is a positive constant C that is independent on time step size τ and n such that $x \le Cy$. Let $K>0$ be any positive integer, T be the final time and set

$$\tau = T/K,\ t_n = n\tau,\ \text{for } n \le K,$$

let u^n be the numerical approximation of $u(t_n)$.

2. The first-order semi-discrete linear energy-stable scheme

In this section, we develop a first-order semi-discrete time-stepping numerical scheme to solve the SH equation based on the IEQ method. For this purpose, we introduce following auxiliary variable

$$W = \sqrt{F(u)+D},$$

D is a positive constant to make $F(u)+D>0$ and ensure $W=\sqrt{F(u)+D}$ is well-defined for any $u\in\mathbb{R}$. Since

$$\frac{1}{4}u^4 - \frac{\epsilon+\beta}{2}u^2 + \frac{(\epsilon+\beta)^2}{4} = \left(\frac{1}{2}u^2 - \frac{\epsilon+\beta}{2}\right)^2 \ge 0$$

and $F(u) = \frac{1}{4}u^4 - \frac{\epsilon+\beta}{2}u^2 \ge -\frac{(\epsilon+\beta)^2}{4}$, $F(u)$ is bounded from below. We are able to choose a positive constant D such that $D > \frac{(\epsilon+\beta)^2}{4}$. Thus, the energy functional (1.1) becomes

$$\hat{E}(u,W) = \int_\Omega \left(\frac{1}{2}u(1+\Delta)^2 u + \frac{\beta}{2}u^2 + W^2 - D\right)\mathrm{d}x. \tag{2.1}$$

We denote $H(u) = 2\frac{\mathrm{d}}{\mathrm{d}u}W(u) = \frac{f(u)}{\sqrt{F(u)+D}}$ and we have the following equivalent PDE system:

$$u_t + (1+\Delta)^2 u + \beta u + H(u)W = 0, \tag{2.2}$$

$$W_t = \frac{1}{2}H(u)u_t, \tag{2.3}$$

with the periodic boundary conditions and initial conditions

$$u|_{t=0} = u^0,\quad W|_{t=0} = \sqrt{F(u^0)+D}.$$

The equivalent PDE system still satisfies the unconditional energy stability. By taking the L^2 inner product of (2.2) with u_t and taking the L^2 inner product of (2.3) with $2W$, summing up the resulting equations, we have the following unconditional energy stability of the equivalent PDE system (2.2)-(2.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(u,W) = -\|u_t\|^2 \le 0.$$

We design the first-order semi-discrete scheme based on the backward Euler method as follows,

$$\frac{u^{n+1}-u^n}{\tau} + (1+\Delta)^2 u^{n+1} + \beta u^{n+1} + H(u^n)W^{n+1} = 0, \tag{2.4}$$

$$W^{n+1} - W^n = \frac{1}{2}H(u^n)(u^{n+1}-u^n). \tag{2.5}$$

Because we tackle the nonlinear coefficient $H(u)$ of the variable W explicitly, we can write the equation (2.5) as follows:

$$W^{n+1} = \frac{1}{2}H(u^n)u^{n+1} + e_1^n, \tag{2.6}$$

with $e_1^n = W^n - \frac{1}{2}H(u^n)u^n$. In turn, (2.4) can be written as a linear system:

$$\gamma^* u^{n+1} = -G_1(u^{n+1}) + e_3^n - e_4^n, \tag{2.7}$$

with $\gamma^* = \frac{1}{\tau}$, $G_1(u^{n+1}) = \frac{1}{2}H(u^n)H(u^n)u^{n+1} + (1+\Delta)^2 u^{n+1} + \beta u^{n+1}$, $e_3^n = \frac{1}{\tau}u^n$, $e_4^n = H(u^n)e_1^n$. Hence, we can get the solution u^{n+1} directly from (2.7). After we obtain u^{n+1}, W^{n+1} is naturally obtained from (2.6). Moreover, we note that

$$(G_1(u),v) = \frac{1}{2}(H(u^n)u, H(u^n)v) + ((1+\Delta)u,(1+\Delta)v) + \frac{\beta}{2}(u,v), \tag{2.8}$$

provided v satisfies the identical boundary conditions as u. Hence, the linear operator $G_1(\cdot)$ is symmetric. Furthermore, for each u, we notice that

$$(G_1(u),u) = \frac{1}{2}\|H(u^n)u\|^2 + \|(1+\Delta)u\|^2 + \frac{\beta}{2}\|u\|^2 \ge 0, \tag{2.9}$$

where "$=$" is valid if and only if $u \equiv 0$.

We now prove the well-posedness of the system (2.4)-(2.5) (or (2.7)) as follows.

Theorem 2.1. *The linear system (2.7) can be solved uniquely, and the linear operator is a symmetric positive definite operator.*

Proof. From (2.7), it is obvious that u^{n+1} solves the following system with unknown u,

$$\gamma^* u + G_1(u) = e_3^n - e_4^n. \tag{2.10}$$

Let us denote the above linear system (2.10) by $\mathbb{T}u = y$.

1. For each u_1 and u_2 in $H^2(\Omega)$, applying integration by parts, we obtain

$$\begin{aligned}(\mathbb{T}(u_1),u_2) &= \gamma^*(u_1,u_2) + (G_1(u_1),u_2)\\ &\le C_1(\|u_1\|\|u_2\| + \|\nabla u_1\|\|\nabla u_2\| + \|\Delta u_1\|\|\Delta u_2\|)\\ &\le C_1\|u_1\|_2\|u_2\|_2.\end{aligned} \tag{2.11}$$

Hence, the boundedness of the bilinear form $(\mathbb{T}(\cdot),\cdot)$ is proved.

2. For each u in $H^2(\Omega)$, it is not hard to obtain that

$$(\mathbb{T}(u),u) = \gamma^*\|u\|^2 + \frac{1}{2}\|H(u^n)u\|^2 + \|(1+\Delta)u\|^2 + \frac{\beta}{2}\|u\|^2 \ge C_2\|u\|_2^2. \tag{2.12}$$

Consequently, the coercivity of bilinear form $(\mathbb{T}(\cdot),\cdot)$ is proved. Here, C_1 and C_2 are positive constants.

In this way, the well-posedness of the system $\mathbb{T}u = y$ is obtained from the Lax-Milgram theorem, that is, the linear system (2.10) has a unique solution in $H^2(\Omega)$. Moreover, we can easily obtain

$$(\mathbb{T}(u_1),u_2) = (u_1,\mathbb{T}(u_2)). \tag{2.13}$$

From this, $\mathbb{T}$ is symmetric. At the same time, the positive definiteness of $\mathbb{T}$ comes from coercivity in (2.12). Hence, $\mathbb{T}$ is a symmetric positive definite operator. □

Theorem 2.2. *The scheme (2.4)-(2.5) (or (2.7)) satisfies the discrete energy dissipation law as follows*

$$E_{1st}^{n+1} - E_{1st}^n \le -\frac{1}{\tau}\|u^{n+1}-u^n\|^2 \le 0, \tag{2.14}$$

where

$$E_{1st}^n = \|W^n\|^2 + \frac{1}{2}\|(1+\Delta)u^n\|^2 + \frac{\beta}{2}\|u^n\|^2 - D|\Omega|. \tag{2.15}$$

Proof. First of all, by taking the L^2 inner product of $u^{n+1}-u^n$ with (2.4) and using the following identities

$$2(x-y,x)=\|x\|^2-\|y\|^2+\|x-y\|^2, \tag{2.16}$$

we derive

$$-\frac{1}{\tau}\|u^{n+1}-u^n\|^2=(H(u^n)W^{n+1},u^{n+1}-u^n)+\frac{1}{2}(\|(1+\Delta)u^{n+1}\|^2-\|(1+\Delta)u^n\|^2 \\ +\|(1+\Delta)(u^{n+1}-u^n)\|^2)+\frac{\beta}{2}(\|u^{n+1}\|^2-\|u^n\|^2+\|u^{n+1}-u^n\|^2). \tag{2.17}$$

Secondly, by taking the L^2 inner product of $2W^{n+1}$ with (2.5), we get

$$\|W^{n+1}\|^2-\|W^n\|^2+\|W^{n+1}-W^n\|^2=(H(u^n)(u^{n+1}-u^n),W^{n+1}). \tag{2.18}$$

In the end, combining (2.17) and (2.18) together, we derive

$$\|W^{n+1}\|^2-\|W^n\|^2+\frac{1}{2}\left(\|(1+\Delta)u^{n+1}\|^2-\|(1+\Delta)u^n\|^2\right)+\frac{\beta}{2}(\|u^{n+1}\|^2-\|u^n\|^2) \\ +\|W^{n+1}-W^n\|^2+\frac{1}{2}\|(1+\Delta)(u^{n+1}-u^n)\|^2+\frac{\beta}{2}\|u^{n+1}-u^n\|^2=-\frac{1}{\tau}\|u^{n+1}-u^n\|^2, \tag{2.19}$$

after deleting some positive terms, we conclude the result (2.14). □

We now give the error analysis for the first-order scheme (2.4)-(2.5). First, we formulate a truncation form for the SH system (2.4)-(2.5) as follows:

$$\frac{u(t_{n+1})-u(t_n)}{\tau}+(1+\Delta)^2u(t_{n+1})+\beta u(t_{n+1})+H(u(t_n))W(t_{n+1})=G_u^{n+1}, \tag{2.20}$$

$$W(t_{n+1})-W(t_n)=\frac{1}{2}H(u(t_n))(u(t_{n+1})-u(t_n))+\tau G_W^{n+1}, \tag{2.21}$$

where

$$G_u^{n+1}=\frac{u(t_{n+1})-u(t_n)}{\tau}-u_t(t_{n+1})-H(u(t_{n+1}))W(t_{n+1})+H(u(t_n))W(t_{n+1}),$$

$$G_W^{n+1}=\frac{W(t_{n+1})-W(t_n)}{\tau}-W_t(t_{n+1})+\frac{1}{2}H(u(t_{n+1}))u_t(t_{n+1})-\frac{1}{2}H(u(t_n))\frac{u(t_{n+1})-u(t_n)}{\tau}.$$

To derive the error estimate, we assume that the analytic solution of the system (2.2)-(2.3) satisfies the following regularity conditions

$$u\in L^\infty(0,T;H^4(\Omega)),\quad W\in L^\infty(0,T;L^\infty(\Omega)), \tag{2.22}$$

$$u_t\in L^\infty(0,T;L^\infty(\Omega)),\quad W_{tt},u_{tt}\in L^2(0,T;L^2(\Omega)). \tag{2.23}$$

We define the error functions for $n=0,1,2,\cdots,K$ as

$$e_u^n=u(t_n)-u^n,\ e_H^n=H(u(t_n))-H(u^n),\ e_W^n=W(t_n)-W^n.$$

Subtracting (2.4)-(2.5) from (2.20)-(2.21), respectively, we get the following error equations for $n\ge 0$,

$$\frac{e_u^{n+1}-e_u^n}{\tau}+(1+\Delta)^2e_u^{n+1}+\beta e_u^{n+1}+e_H^nW(t_{n+1})+H(u^n)e_W^{n+1}=G_u^{n+1}, \tag{2.24}$$

$$e_W^{n+1}-e_W^n=\frac{1}{2}e_H^n(u(t_{n+1})-u(t_n))+\frac{1}{2}H(u^n)(e_u^{n+1}-e_u^n)+\tau G_W^{n+1}. \tag{2.25}$$

Before further investigation, we introduce the following lemma.

Lemma 2.1. *[26] Suppose (i) $F(x)$ is uniformly bounded from below: $F(x)>-D$ for any $x\in\mathbb{R}$; (ii) $F(x)\in C^3(\mathbb{R})$; and (iii) there exists a positive constant D_0 such that*

$$\max_{0\le n\le M}(\|u(t_n)\|_{L^\infty},\|\nabla u(t_n)\|_{L^3},\|u^n\|_{L^\infty})\le D_0,$$

then we have

$$\|H(u(t_n))-H(u^n)\|\le\hat{C}_0\|u(t_n)-u^n\|,$$

$$\|\nabla H(u(t_n))-\nabla H(u^n)\|\le\hat{D}_0(\|u(t_n)-u^n\|+\|\nabla(u(t_n)-u^n)\|),$$

for $n\le M$, where $\hat{C}_0$ and $\hat{D}_0$ are positive constants dependent on Ω, D_0 and D.

Using Lemma 2.1, we can easily derive the following estimate for the truncation errors.

Lemma 2.2. *Under the regularity conditions (2.22)-(2.23), the truncation errors satisfy*

$$\tau\sum_{n=0}^{K-1}(\|G_u^{n+1}\|^2+\|G_W^{n+1}\|^2)\lesssim\tau^2.$$

Proof. Since the proof is rather straight forward, we omit the details. □

Let $\nu = \max_{0\le t\le T} \|u(t)\|_{L^\infty} + 1$, we now prove the L^∞ stability of solution u^n.

Lemma 2.3. *Under the regularity conditions (2.22)-(2.23), there exists a positive constant r (which is given in the proof), such that when $\tau \le r$, the numerical solution u^n of (2.20)-(2.21) satisfies the following uniformly boundedness*

$$\|u^n\|_{L^\infty} \le \nu, \quad n = 0, 1, 2, \cdots, K.$$

Proof. We prove this lemma by mathematical induction. Because $u^0 = u(t_0)$, $\|u^0\|_{L^\infty} \le \nu$ holds naturally. Assuming that $\|u^n\|_{L^\infty} \le \nu$ is true for $0 \le n \le M$, we derive $\|u^{M+1}\|_{L^\infty} \le \nu$ is also true by the following two steps.

(i) Taking the L^2-inner product of (2.24) with $e_u^{n+1} - e_u^n$, we get

$$\begin{aligned}
&\frac{1}{\tau}\|e_u^{n+1} - e_u^n\|^2 + \frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2 - \|(1+\Delta)e_u^n\|^2 + \|(1+\Delta)(e_u^{n+1} - e_u^n)\|^2) + \frac{\beta}{2}(\|e_u^{n+1}\|^2 - \|e_u^n\|^2 \\
&+ \|e_u^{n+1} - e_u^n\|^2) + (e_H^n W(t_{n+1}), e_u^{n+1} - e_u^n) + (H(u^n)e_W^{n+1}, e_u^{n+1} - e_u^n) = (G_u^{n+1}, e_u^{n+1} - e_u^n).
\end{aligned} \tag{2.26}$$

Taking the L^2-inner product of (2.25) with $2e_W^{n+1}$, we have

$$\begin{aligned}
&\|e_W^{n+1}\|^2 - \|e_W^n\|^2 + \|e_W^{n+1} - e_W^n\|^2 - (e_H^n(u(t_{n+1}) - u(t_n)), e_W^{n+1}) \\
&- (H(u^n)(e_u^{n+1} - e_u^n), e_W^{n+1}) = 2\tau(G_W^{n+1}, e_W^{n+1}).
\end{aligned} \tag{2.27}$$

Taking the L^2-inner product of (2.24) with $2\tau e_u^{n+1}$, we get

$$\begin{aligned}
&\|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \|e_u^{n+1} - e_u^n\|^2 + 2\tau\|(1+\Delta)e_u^{n+1}\|^2 + 2\beta\tau\|e_u^{n+1}\|^2 \\
&+ 2\tau(e_H^n W(t_{n+1}), e_u^{n+1}) + 2\tau(H(u^n)e_W^{n+1}, e_u^{n+1}) = 2\tau(G_u^{n+1}, e_u^{n+1}).
\end{aligned} \tag{2.28}$$

Combining (2.26)-(2.28), we derive

$$\begin{aligned}
&(1+\frac{\beta}{2})(\|e_u^{n+1}\|^2 - \|e_u^n\|^2) + \frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2 - \|(1+\Delta)e_u^n\|^2) + \|e_W^{n+1}\|^2 - \|e_W^n\|^2 + 2\tau\|(1+\Delta)e_u^{n+1}\|^2 \\
&+ \frac{1}{\tau}\|e_u^{n+1} - e_u^n\|^2 + (1+\frac{\beta}{2})\|e_u^{n+1} - e_u^n\|^2 + \|e_W^{n+1} - e_W^n\|^2 + \frac{1}{2}\|(1+\Delta)(e_u^{n+1} - e_u^n)\|^2 + 2\beta\tau\|e_u^{n+1}\|^2 \\
= &-(e_H^n W(t_{n+1}), e_u^{n+1} - e_u^n) + (e_H^n(u(t_{n+1}) - u(t_n)), e_W^{n+1}) - 2\tau(e_H^n W(t_{n+1}), e_u^{n+1}) \\
&- 2\tau(H(u^n)e_W^{n+1}, e_u^{n+1}) + 2\tau(G_u^{n+1}, e_u^{n+1}) + 2\tau(G_W^{n+1}, e_W^{n+1}) + (G_u^{n+1}, e_u^{n+1} - e_u^n).
\end{aligned} \tag{2.29}$$

Using Lemma 2.1, 2.2 and regularity conditions (2.22)-(2.23), we estimate each terms on the right hand side of (2.29).

$$\begin{aligned}
|(e_H^n W(t_{n+1}), e_u^{n+1} - e_u^n)| &\le \|e_H^n\|\|W(t_{n+1})\|_{L^\infty}\|e_u^{n+1} - e_u^n\| \\
&\lesssim \|e_H^n\|\|e_u^{n+1} - e_u^n\| \\
&\lesssim \frac{1}{2\tau}\|e_u^{n+1} - e_u^n\|^2 + \tau\|e_H^n\|^2 \\
&\lesssim \frac{1}{2\tau}\|e_u^{n+1} - e_u^n\|^2 + \tau\|e_u^n\|^2.
\end{aligned} \tag{2.30}$$

Since

$$\begin{aligned}
\|\nabla e_u^n\|^2 = |(\nabla e_u^n, \nabla e_u^n)| &= |(e_u^n, \Delta e_u^n)| \lesssim \|e_u^n\|^2 + \|\Delta e_u^n\|^2 \\
&= \|e_u^n\|^2 + \|(1+\Delta)e_u^n - e_u^n\|^2 \le \|e_u^n\|^2 + (\|(1+\Delta)e_u^n\| + \|e_u^n\|)^2 \\
&\lesssim \|e_u^n\|^2 + \|(1+\Delta)e_u^n\|^2,
\end{aligned}$$

we have

$$\begin{aligned}
|(e_H^n(u(t_{n+1}) - u(t_n)), e_W^{n+1})| &\le \|e_H^n\|_{L^4}\|u(t_{n+1}) - u(t_n)\|_{L^4}\|e_W^{n+1}\| \\
&\lesssim \tau\|e_H^n\|_{L^4}\|e_W^{n+1}\| \\
&\lesssim \tau\|e_W^{n+1}\|^2 + \tau\|e_H^n\|_{L^4}^2 \\
&\lesssim \tau\|e_W^{n+1}\|^2 + \tau(\|e_H^n\|^2 + \|\nabla e_H^n\|^2) \\
&\lesssim \tau\|e_W^{n+1}\|^2 + \tau(\|e_u^n\|^2 + \|\nabla e_u^n\|^2) \\
&\lesssim \tau\|e_W^{n+1}\|^2 + \tau(\|e_u^n\|^2 + \|(1+\Delta)e_u^n\|^2).
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
2\tau|(e_H^n W(t_{n+1}), e_u^{n+1})| &\le 2\tau\|e_H^n\|\|W(t_{n+1})\|_{L^\infty}\|e_u^{n+1}\| \\
&\lesssim \tau\|e_H^n\|\|e_u^{n+1}\| \\
&\lesssim \tau\|e_H^n\|^2 + \tau\|e_u^{n+1}\|^2 \\
&\lesssim \tau\|e_u^n\|^2 + \tau\|e_u^{n+1}\|^2.
\end{aligned} \tag{2.32}$$

$$\begin{aligned}2\tau|(H(u^n)e_W^{n+1},e_u^{n+1})| &\le 2\tau\|H(u^n)\|_{L^\infty}\|e_W^{n+1}\|\|e_u^{n+1}\|\\ &\lesssim \tau\|e_W^{n+1}\|\|e_u^{n+1}\|\\ &\lesssim \tau\|e_W^{n+1}\|^2+\tau\|e_u^{n+1}\|^2.\end{aligned}\tag{2.33}$$

$$\begin{aligned}2\tau|(G_u^{n+1},e_u^{n+1})+(G_W^{n+1},e_W^{n+1})| &\le 2\tau(\|G_u^{n+1}\|\|e_u^{n+1}\|+\|G_W^{n+1}\|\|e_W^{n+1}\|)\\ &\le \tau(\|G_u^{n+1}\|^2+\|G_W^{n+1}\|^2+\|e_u^{n+1}\|^2+\|e_W^{n+1}\|^2).\end{aligned}\tag{2.34}$$

$$|(G_u^{n+1},e_u^{n+1}-e_u^n)|\le \|G_u^{n+1}\|\|e_u^{n+1}-e_u^n\|\lesssim \frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|G_u^{n+1}\|^2.\tag{2.35}$$

Combining (2.30)-(2.35) with (2.29) and dropping some positive terms, we obtain

$$\begin{aligned}&(1+\frac{\beta}{2})(\|e_u^{n+1}\|^2-\|e_u^n\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2-\|(1+\Delta)e_u^n\|^2)+\|e_W^{n+1}\|^2-\|e_W^n\|^2\\ \lesssim\ &\tau(\|e_u^n\|^2+\|e_u^{n+1}\|^2+\|(1+\Delta)e_u^n\|^2+\|e_W^n\|^2+\|G_u^{n+1}\|^2+\|G_W^{n+1}\|).\end{aligned}$$

Summing up for n from 0 to m ($m\le M$) and using Lemma 2.2, we have

$$(1+\frac{\beta}{2})\|e_u^{m+1}\|^2+\frac{1}{2}\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2\le \tau\sum_{n=0}^{m+1}(\|e_u^n\|^2+\|(1+\Delta)e_u^n\|^2+\|e_W^n\|^2)+\tau^2.$$

Applying Grönwall's inequality, there exist two positive constants r_1 and r_2 such that when $\tau\le r_1$,

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2\le r_2\tau^2.\tag{2.36}$$

(ii) Since

$$\begin{aligned}\|e_u^{M+1}\|_1^2 &= \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2\\ &\lesssim r_2\tau^2,\\ \|e_u^{M+1}\|_2^2 &= \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2\\ &\lesssim r_2\tau^2,\end{aligned}$$

we have

$$\begin{aligned}\|u^{M+1}\|_{L^\infty} &\le \|e_u^{M+1}\|_{L^\infty}+\|u(t_{M+1})\|_{L^\infty}\\ &\le C_\Omega\|e_u^{M+1}\|_1^{\frac12}\|e_u^{M+1}\|_2^{\frac12}+\|u(t_{M+1})\|_{L^\infty}\\ &\le C_\Omega\sqrt{r_2}\tau+\|u(t_{M+1})\|_{L^\infty}\le \nu,\end{aligned}$$

if $\tau\le\frac{1}{C_\Omega\sqrt{r_2}}$. Thus the proof is completed by setting $r=\min\{r_1,\frac{1}{C_\Omega\sqrt{r_2}}\}$. □

Theorem 2.3. *Under the regularity conditions (2.22)-(2.23), the numerical solution u^n of (2.4)-(2.5) satisfies the following estimate:*

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2\lesssim\tau^2,\quad 0\le m\le K-1.\tag{2.37}$$

Proof. If $\tau\le r$, we have $\|u^n\|_{L^\infty}\le\nu$ for $0\le n\le K$. Hence, following the proof of Lemma 2.3, we get the result (2.37). □

3. The second-order semi-discrete linear energy-stable scheme

In this section, we design the second-order linear energy-stable scheme based on the Crank-Nicolson scheme, which reads as follows

$$\frac{u^{n+1}-u^n}{\tau}+(1+\Delta)^2\frac{u^{n+1}+u^n}{2}+\beta\frac{u^{n+1}+u^n}{2}+H(\tilde u^{n+1/2})\frac{W^{n+1}+W^n}{2}=0,\tag{3.1}$$

$$W^{n+1}-W^n=\frac{1}{2}H(\tilde u^{n+1/2})(u^{n+1}-u^n),\tag{3.2}$$

where $\tilde u^{n+1/2}=\frac32 u^n-\frac12 u^{n-1}$. Thus we can write the equation (3.2) as follows,

$$W^{n+1}=\frac12 H(\tilde u^{n+1/2})u^{n+1}+c_1^n,\tag{3.3}$$

with $c_1^n=W^n-\frac12 H(\tilde u^{n+1/2})u^n$. In this way, (3.1) can be written as linear system as follows

$$\gamma^{\dagger} u^{n+1} = -G_3(u^{n+1}) + c_2^n - c_3^n, \tag{3.4}$$

with $\gamma^{\dagger} = \frac{1}{\tau}$, $c_2^n = \frac{1}{\tau}u^n$, $G_3(u^{n+1}) = \frac{1}{4}H(\tilde{u}^{n+1/2})H(\tilde{u}^{n+1/2})u^{n+1} + \frac{1}{2}(1+\Delta)^2 u^{n+1} + \frac{\beta}{2}u^{n+1}$, $c_3^n = \frac{1}{2}H(\tilde{u}^{n+1/2})c_1^n + \frac{1}{2}H(\tilde{u}^{n+1/2})W^n + \frac{1}{2}(1+\Delta)^2 u^n + \frac{\beta}{2}u^n$. Thus, u^{n+1} can be solved directly from (3.4). After we get u^{n+1}, W^{n+1} is naturally obtained in (3.3). Moreover, we note that

$$(G_3(u), v) = \frac{1}{4}(H(\tilde{u}^{n+1/2})u, H(\tilde{u}^{n+1/2})v) + \frac{1}{2}((1+\Delta)u, (1+\Delta)v) + \frac{\beta}{2}(u, v),$$

if v satisfies the identical boundary conditions as u. Hence, the linear operator $G_3(\cdot)$ is symmetric. Furthermore, for each u, we note that

$$(G_3(u), u) = \frac{1}{4}\|H(\tilde{u}^{n+1/2})u\|^2 + \frac{1}{2}\|(1+\Delta)u\|^2 + \frac{\beta}{2}\|u\|^2 \geq 0,$$

where "$=$" is valid if and only if $u \equiv 0$.

Remark 3.1. *The second-order scheme (3.1)-(3.2) involves three time levels and (u^{n+1}, W^{n+1}) can be updated after we obtain the initial values (u^0, W^0) and (u^1, W^1). Obviously, (u^0, W^0) is given by the initial conditions. To get the second-order time accuracy of the scheme, we can calculate $(\tilde{u}^1, \tilde{W}^1)$ by using the first-order scheme (3.5)-(3.6)*

$$\frac{\tilde{u}^{n+1} - u^n}{\tau} + (1+\Delta)^2 \tilde{u}^{n+1} + \beta \tilde{u}^{n+1} + H(u^n)\tilde{W}^{n+1} = 0, \tag{3.5}$$

$$\tilde{W}^{n+1} - W^n = \frac{1}{2}H(u^n)(\tilde{u}^{n+1} - u^n), \tag{3.6}$$

then apply the following corrector scheme to get (u^1, W^1),

$$\frac{u^1 - u^0}{\tau} + H(\tilde{u}^1)\frac{W^1 + W^0}{2} + (1+\Delta)^2 \frac{u^1 + u^0}{2} + \beta \frac{u^1 + u^0}{2} = 0, \tag{3.7}$$

$$W^1 - W^0 = \frac{1}{2}H(\tilde{u}^1)(u^1 - u^0), \tag{3.8}$$

with

$$H(\tilde{u}^1) = \frac{f(\tilde{u}^1)}{\sqrt{F(\tilde{u}^1) + D}}.$$

Theorem 3.1. *The linear system (3.4) can be solved uniquely, and the linear operator is a symmetric positive definite operator.*

Proof. The proof is available in the same manner as the proof of Theorem 2.1 and for brevity, we omit the details. □

Theorem 3.2. *The scheme (3.1)-(3.2) (or (3.4)) satisfies the discrete energy dissipation law as follows*

$$E_{CN}^{n+1} - E_{CN}^n = -\frac{1}{\tau}\|u^{n+1} - u^n\|^2 \leq 0, \tag{3.9}$$

where

$$E_{CN}^n = \|W^n\|^2 + \frac{1}{2}\|(1+\Delta)u^n\|^2 + \frac{\beta}{2}\|u^n\|^2 - D|\Omega|.$$

Proof. First of all, taking the L^2 inner product of $u^{n+1} - u^n$ with (3.1), we derive

$$\begin{aligned} -\frac{1}{\tau}\|u^{n+1} - u^n\|^2 = {} & \left(H(\tilde{u}^{n+1/2})\frac{W^{n+1} + W^n}{2}, u^{n+1} - u^n\right) + \frac{1}{2}\left(\|(1+\Delta)u^{n+1}\|^2 - \|(1+\Delta)u^n\|^2\right) \\ & + \frac{\beta}{2}(\|u^{n+1}\|^2 - \|u^n\|^2). \end{aligned} \tag{3.10}$$

Secondly, taking the L^2 inner product of $W^{n+1} + W^n$ with (3.2), we derive

$$\|W^{n+1}\|^2 - \|W^n\|^2 = \left(\frac{1}{2}H(\tilde{u}^{n+1/2})(u^{n+1} - u^n), W^{n+1} + W^n\right). \tag{3.11}$$

In the end, combining (3.10) and (3.11), we derive

$$\|W^{n+1}\|^2 - \|W^n\|^2 + \frac{1}{2}\left(\|(1+\Delta)u^{n+1}\|^2 - \|(1+\Delta)u^n\|^2\right) + \frac{\beta}{2}(\|u^{n+1}\|^2 - \|u^n\|^2) = -\frac{1}{\tau}\|u^{n+1} - u^n\|^2,$$

which implies that the desired result (3.9) is hold. This completes the proof. □

We now give the error analysis for the second-order scheme (3.1)-(3.2). We first formulate a truncation form for the SH system (3.1)-(3.2) as follows:

$$\frac{u(t_{n+1}) - u(t_n)}{\tau} + (1+\Delta)^2 \frac{u(t_{n+1}) + u(t_n)}{2} + \beta \frac{u(t_{n+1}) + u(t_n)}{2} + H(u(t_{n+1/2}))\frac{W(t_{n+1}) + W(t_n)}{2} = G_u^{n+1/2}, \tag{3.12}$$

$$W(t_{n+1}) - W(t_n) = \frac{1}{2}H(u(t_{n+1/2}))(u(t_{n+1}) - u(t_n)) + \tau G_W^{n+1/2}, \tag{3.13}$$

where

$$
\begin{aligned}
G_u^{n+1/2} =& \frac{u(t_{n+1})-u(t_n)}{\tau} - u_t(t_{n+1/2}) + (1+\Delta)^2\frac{u(t_{n+1})+u(t_n)}{2} - (1+\Delta)^2 u(t_{n+1/2}) \\
&+\beta\frac{u(t_{n+1})+u(t_n)}{2} - \beta u(t_{n+1/2}) + H(u(t_{n+1/2}))\frac{W(t_{n+1})+W(t_n)}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}),\\
G_W^{n+1/2} =& \frac{W(t_{n+1})-W(t_n)}{\tau} - W_t(t_{n+1/2}) + \frac{1}{2}H(u(t_{n+1/2}))u_t(t_{n+1/2})\\
&-\frac{1}{2}H(u(t_{n+1/2}))\frac{u(t_{n+1})-u(t_n)}{\tau}.
\end{aligned}
$$

We also define

$$
R_1^{n+1/2} = u(t_{n+1/2}) - \left(\frac{3}{2}u(t_n) - \frac{1}{2}u(t_{n-1})\right).
$$

To derive the error estimate, we assume that the analytic solution of the system (2.2)-(2.3) satisfies the following regularity conditions

$$
u \in L^\infty(0,T;H^4(\Omega)),\quad W \in L^\infty(0,T;L^\infty(\Omega)),\quad u_t \in L^\infty(0,T;L^\infty(\Omega)), \tag{3.14}
$$

$$
W_{tt} \in L^2(0,T;L^2(\Omega)),\quad u_{tt} \in L^2(0,T;H^4(\Omega)),\quad u_{ttt}, W_{ttt} \in L^2(0,T;L^2(\Omega)). \tag{3.15}
$$

Using Lemma 2.1, we can easily derive the following estimate for the truncation errors.

Lemma 3.1. *Under the regularity conditions (3.14)-(3.15), the truncation errors satisfy*

$$
\tau\sum_{n=0}^{K-1}(\|G_u^{n+1/2}\|^2 + \|G_W^{n+1/2}\|^2 + \|R_1^{n+1/2}\|^2 + \|\nabla R_1^{n+1/2}\|^2) \lesssim \tau^4.
$$

Proof. Since the proof is rather straight forward, we omit the details. □

We define the error function for $n = 0, 1, 2, \cdots, K-1$ as

$$
\tilde{e}_H^{n+1/2} = H(u(t_{n+1/2})) - H(\tilde{u}^{n+1/2}).
$$

Subtracting (3.1)-(3.2) from (3.12)-(3.13), respectively, we get the following error equations for $n \ge 0$,

$$
\frac{e_u^{n+1}-e_u^n}{\tau} + (1+\Delta)^2\frac{e_u^{n+1}+e_u^n}{2} + \beta\frac{e_u^{n+1}+e_u^n}{2} + \tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2} + H(\tilde{u}^{n+1/2})\frac{e_W^{n+1}+e_W^n}{2} = G_u^{n+1/2}, \tag{3.16}
$$

$$
e_W^{n+1} - e_W^n = \frac{1}{2}\tilde{e}_H^{n+1/2}(u(t_{n+1})-u(t_n)) + \frac{1}{2}H(\tilde{u}^{n+1/2})(e_u^{n+1}-e_u^n) + \tau G_W^{n+1/2} \tag{3.17}
$$

Let $\nu = \max_{0\le t\le T}\|u(t)\|_{L^\infty} + 1$, we now prove the L^∞ stability of solution u^n.

Lemma 3.2. *Under the regularity conditions (3.14)-(3.15), there exists a positive constant $\hat{r}$ (which is given in the proof), such that when $\tau \le \hat{r}$, the numerical solution u^n of (3.1)-(3.2) satisfies the following uniformly boundedness*

$$
\|u^n\|_{L^\infty} \le \nu,\quad n = 0, 1, 2, \cdots, K.
$$

Proof. We prove this lemma by mathematical induction. Because $u^0 = u(t_0)$, $\|u^0\|_{L^\infty} \le \nu$ holds naturally. Assuming that $\|u^n\|_{L^\infty} \le \nu$ is true for $0 \le n \le M$, we derive $\|u^{M+1}\|_{L^\infty} \le \nu$ is also true by the following two steps.

(i) Taking the L^2-inner product of (3.16) with $e_u^{n+1} - e_u^n$, we get

$$
\begin{aligned}
&\frac{1}{\tau}\|e_u^{n+1}-e_u^n\|^2 + \frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2 - \|(1+\Delta)e_u^n\|^2) + \left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2}, e_u^{n+1}-e_u^n\right)\\
&+\left(H(\tilde{u}^{n+1/2})\frac{e_W^{n+1}+e_W^n}{2}, e_u^{n+1}-e_u^n\right) + \frac{\beta}{2}(\|e_u^{n+1}\|^2 - \|e_u^n\|^2) = (G_u^{n+1/2}, e_u^{n+1}-e_u^n).
\end{aligned} \tag{3.18}
$$

Taking the L^2-inner product of (3.17) with $e_W^{n+1} + e_W^n$, we have

$$
\begin{aligned}
&\|e_W^{n+1}\|^2 - \|e_W^n\|^2 - \frac{1}{2}(\tilde{e}_H^{n+1/2}(u(t_{n+1})-u(t_n)), e_W^{n+1}+e_W^n)\\
&-\frac{1}{2}(H(\tilde{u}^{n+1/2})(e_u^{n+1}-e_u^n), e_W^{n+1}+e_W^n) = \tau(G_W^{n+1/2}, e_W^{n+1}+e_W^n).
\end{aligned} \tag{3.19}
$$

Taking the L^2-inner product of (3.16) with $\tau(e_u^{n+1}+e_u^n)$, we get

$$
\begin{aligned}
&\|e_u^{n+1}\|^2 - \|e_u^n\|^2 + \frac{1}{2}\tau\|(1+\Delta)(e_u^{n+1}+e_u^n)\|^2 + \tau\left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2}, e_u^{n+1}+e_u^n\right)\\
&+\tau\left(H(\tilde{u}^{n+1/2})\frac{e_W^{n+1}+e_W^n}{2}, e_u^{n+1}+e_u^n\right) + \frac{1}{2}\tau\beta\|e_u^{n+1}+e_u^n\|^2 = \tau(G_u^{n+1/2}, e_u^{n+1}+e_u^n).
\end{aligned} \tag{3.20}
$$

Combining (3.18)-(3.20) and dropping some positive terms, we derive

$$\begin{aligned}
&(1+\frac{\beta}{2})(\|e_u^{n+1}\|^2-\|e_u^n\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2-\|(1+\Delta)e_u^n\|^2)\\
&+\|e_W^{n+1}\|^2-\|e_W^n\|^2+\frac{1}{\tau}\|e_u^{n+1}-e_u^n\|^2+\frac{1}{2}\tau\beta\|e_u^{n+1}+e_u^n\|^2\\
=&-\left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2},e_u^{n+1}-e_u^n\right)+(G_u^{n+1/2},e_u^{n+1}-e_u^n)+\frac{1}{2}(\tilde{e}_H^{n+1/2}(u(t_{n+1})-u(t_n)),e_W^{n+1}-e_W^n)\\
&+\tau(G_W^{n+1/2},e_W^{n+1}-e_W^n)-\tau\left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2},e_u^{n+1}-e_u^n\right)-\tau\left(H(\tilde{u}^{n+1/2})\frac{e_W^{n+1}+e_W^n}{2},e_u^{n+1}-e_u^n\right)\\
&+\tau(G_u^{n+1/2},e_u^{n+1}-e_u^n).
\end{aligned}\tag{3.21}$$

Using Lemma 2.1 and regularity conditions (3.14)-(3.15), we estimate each terms on the right hand side of (3.21).

$$\begin{aligned}
&\left|\left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2},e_u^{n+1}-e_u^n\right)\right|\\
\le&\|\tilde{e}_H^{n+1/2}\|(\|W(t_{n+1})\|_{L^\infty}+\|W(t_n)\|_{L^\infty})\|e_u^{n+1}-e_u^n\|\\
\lesssim&\|\tilde{e}_H^{n+1/2}\|\|e_u^{n+1}-e_u^n\|\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|\tilde{e}_H^{n+1/2}\|^2\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|u(t_{n+1/2})-\tilde{u}^{n+1/2}\|^2\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|u(t_{n+1/2})-(\frac{3}{2}u(t_n)-\frac{1}{2}u(t_{n-1}))\\
&+(\frac{3}{2}u(t_n)-\frac{1}{2}u(t_{n-1}))-(\frac{3}{2}u^n-\frac{1}{2}u^{n-1})\|^2\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|R_1^{n+1/2}+\frac{3}{2}e_u^n-\frac{1}{2}e_u^{n-1}\|^2\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+2\tau\|R_1^{n+1/2}\|^2+2\tau\|\frac{3}{2}e_u^n-\frac{1}{2}e_u^{n-1}\|^2\\
\lesssim&\frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+2\tau\|R_1^{n+1/2}\|^2+9\tau\|e_u^n\|^2+\tau\|e_u^{n-1}\|^2.
\end{aligned}\tag{3.22}$$

$$\begin{aligned}
&\frac{1}{2}|(\tilde{e}_H^{n+1/2}(u(t_{n+1})-u(t_n)),e_W^{n+1}+e_W^n)|\\
\le&\|\tilde{e}_H^{n+1/2}\|_{L^4}\|u(t_{n+1})-u(t_n)\|_{L^4}\|e_W^{n+1}+e_W^n\|\\
\lesssim&\tau\|\tilde{e}_H^{n+1/2}\|_{L^4}\|e_W^{n+1}+e_W^n\|\\
\lesssim&\tau\|e_W^{n+1}+e_W^n\|^2+\tau\|\tilde{e}_H^{n+1/2}\|_{L^4}^2\\
\lesssim&\tau\|e_W^{n+1}\|^2+\tau\|e_W^n\|^2+\tau(\|\tilde{e}_H^{n+1/2}\|^2+\|\nabla\tilde{e}_H^{n+1/2}\|^2)\\
\lesssim&\tau(\|e_W^{n+1}\|^2+\|e_W^n\|^2+\|R_1^{n+1/2}\|^2+\|\nabla R_1^{n+1/2}\|^2+\|e_u^n\|^2\\
&+\|e_u^{n-1}\|^2+\|\nabla e_u^n\|^2+\|\nabla e_u^{n-1}\|^2)\\
\lesssim&\tau(\|e_W^{n+1}\|^2+\|e_W^n\|^2+\|R_1^{n+1/2}\|^2+\|\nabla R_1^{n+1/2}\|^2+\|e_u^n\|^2\\
&+\|e_u^{n-1}\|^2+\|(1+\Delta)e_u^n\|^2+\|(1+\Delta)e_u^{n-1}\|^2).
\end{aligned}\tag{3.23}$$

$$\begin{aligned}
&\tau\left|\left(\tilde{e}_H^{n+1/2}\frac{W(t_{n+1})+W(t_n)}{2},e_u^{n+1}+e_u^n\right)\right|\\
\le&\tau\|\tilde{e}_H^{n+1/2}\|(\|W(t_{n+1})\|_{L^\infty}+\|W(t_n)\|_{L^\infty})\|e_u^{n+1}+e_u^n\|\\
\lesssim&\tau\|\tilde{e}_H^{n+1/2}\|\|e_u^{n+1}+e_u^n\|\\
\lesssim&\tau\|\tilde{e}_H^{n+1/2}\|^2+\tau\|e_u^{n+1}+e_u^n\|^2\\
\lesssim&\tau\|R_1^{n+1/2}\|^2+\tau(\|e_u^{n+1}\|^2+\|e_u^n\|^2+\|e_u^{n-1}\|^2).
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
&\tau\left|\left(H(\tilde{u}^{n+1/2})\frac{e_W^{n+1}+e_W^n)}{2},e_u^{n+1}+e_u^n\right)\right|\\
\lesssim&\tau\|H(\tilde{u}^{n+1/2})\|_{L^\infty}\|e_W^{n+1}+e_W^n\|\|e_u^{n+1}+e_u^n\|\\
\lesssim&\tau\|e_W^{n+1}+e_W^n\|^2+\tau\|e_u^{n+1}+e_u^n\|^2\\
\lesssim&\tau(\|e_W^{n+1}\|^2+\|e_W^n\|^2+\|e_u^{n+1}\|^2+\|e_u^n\|^2).
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
&\tau|(G_W^{n+1/2}, e_W^{n+1}+e_W^n)+(G_u^{n+1/2}, e_u^{n+1}+e_u^n)| \\
\lesssim\ & \tau(\|G_W^{n+1/2}\|\|e_W^{n+1}+e_W^n\|+\|G_u^{n+1/2}\|\|e_u^{n+1}+e_u^n\|) \\
\lesssim\ & \tau(\|G_W^{n+1/2}\|^2+\|G_u^{n+1/2}\|^2+\|e_W^{n+1}+e_W^n\|^2+\|e_u^{n+1}+e_u^n\|^2) \\
\lesssim\ & \tau(\|G_W^{n+1/2}\|^2+\|G_u^{n+1/2}\|^2+\|e_W^{n+1}\|^2+\|e_W^n\|^2+\|e_u^{n+1}\|^2+\|e_u^n\|^2).
\end{aligned} \tag{3.26}$$

$$|(G_u^{n+1/2}, e_u^{n+1}-e_u^n)| \le \|G_u^{n+1/2}\|\|e_u^{n+1}-e_u^n\| \lesssim \frac{1}{2\tau}\|e_u^{n+1}-e_u^n\|^2+\tau\|G_u^{n+1/2}\|^2. \tag{3.27}$$

Combining (3.22)-(3.27) with (3.21), we obtain

$$\begin{aligned}
&(1+\frac{\beta}{2})(\|e_u^{n+1}\|^2-\|e_u^n\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2-\|(1+\Delta)e_u^n\|^2)+\|e_W^{n+1}\|^2-\|e_W^n\|^2 \\
\lesssim\ & \tau(\|R_1^{n+1/2}\|^2+\|\nabla R_1^{n+1/2}\|^2+\|G_W^{n+1/2}\|^2+\|G_u^{n+1/2}\|^2+\|e_u^{n+1}\|^2+\|e_u^n\|^2+\|e_u^{n-1}\|^2 \\
&+\|e_W^{n+1}\|^2+\|e_W^n\|^2+\|(1+\Delta)e_u^n\|^2+\|(1+\Delta)e_u^{n-1}\|^2).
\end{aligned}$$

Summing up n from 0 to m ($m \le M$) and using Lemma 3.1, we have

$$(1+\frac{\beta}{2})\|e_u^{m+1}\|^2+\frac{1}{2}\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2 \lesssim \tau\sum_{n=0}^{m+1}(\|e_u^n\|^2+\|(1+\Delta)e_u^n\|^2+\|e_W^n\|^2)+\tau^4.$$

Applying Grönwall's inequality, there exist two positive constants $\hat{r}_1$ and $\hat{r}_2$ such that when $\tau \le \hat{r}_1$,

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2 \lesssim \hat{r}_2\tau^4.$$

(ii) Since

$$\begin{aligned}
\|e_u^{M+1}\|_1^2 =& \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2 \\
\lesssim& \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2 \\
\lesssim& \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2 \\
\lesssim& \hat{r}_2\tau^4, \\
\|e_u^{M+1}\|_2^2 =& \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2 \\
\lesssim& \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2 \\
\lesssim& \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2 \\
\lesssim& \hat{r}_2\tau^4,
\end{aligned}$$

we have

$$\begin{aligned}
\|u^{M+1}\|_{L^\infty} \le& \|e_u^{M+1}\|_{L^\infty}+\|u(t_{M+1})\|_{L^\infty} \\
\le& C_\Omega\|e_u^{M+1}\|_1^{\frac{1}{2}}\|e_u^{M+1}\|_2^{\frac{1}{2}}+\|u(t_{M+1})\|_{L^\infty} \\
\le& C_\Omega\sqrt{\hat{r}_2}\tau^2+\|u(t_{M+1})\|_{L^\infty} \le \nu,
\end{aligned}$$

if $\tau \le \frac{1}{\sqrt{C_\Omega\sqrt{\hat{r}_2}}}$. Thus the proof is completed by setting $r=\min\{\hat{r}_1, \frac{1}{\sqrt{C_\Omega\sqrt{\hat{r}_2}}}\}$. □

Theorem 3.3. *Under the regularity conditions (3.14)-(3.15), the numerical solution u^n of (3.1)-(3.2) satisfies the following estimate:*

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2 \lesssim \tau^2, \quad 0 \le m \le K-1. \tag{3.28}$$

Proof. If $\tau \le \hat{r}$, we have $\|u^n\|_{L^\infty} \le \nu$ for $0 \le n \le K$. Hence, following the proof of Lemma 3.2, we get the result (3.28). □

4. The fully discrete scheme and its error analysis

In this section, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete scheme. Since the proof for the first-order scheme is essentially the same as for the second-order scheme, for the sake of brevity, we shall provide the details only for the second-order Crank-Nicolson scheme. Let $\xi=(1+\Delta)u$, the system (2.2)-(2.3) can be rewritten as

$$u_t+(1+\Delta)\xi+\beta u+H(u)W=0, \tag{4.1}$$

$$\xi=(1+\Delta)u, \tag{4.2}$$

$$W_t=\frac{1}{2}H(u)u_t, \tag{4.3}$$

with $u(t_0)=u^0$, $\xi(t_0)=\xi^0:=(1+\Delta)u^0$ and $W(t_0)=W^0:=\sqrt{F(u^0)+D}$. The weak form of the above system (4.1)-(4.3) is

$$(u_t, \psi) + (\xi, \psi) - (\nabla \xi, \nabla \psi) + \beta(u, \psi) + (H(u)W, \psi) = 0,\ \forall\ \psi \in H^1(\Omega) \tag{4.4}$$

$$(\xi, v) = (u, v) - (\nabla u, \nabla v),\ \forall\ v \in H^1(\Omega) \tag{4.5}$$

$$(W_t, \zeta) = \frac{1}{2}(H(u)u_t, \zeta),\ \forall\ \zeta \in L^2(\Omega). \tag{4.6}$$

We denote by V_N the space of polynomials of degree $\le N$ in each direction, and for any $\varphi \in H^k(\Omega)$, we define a projection $\Pi_N : H^k(\Omega) \to V_N$ by

$$(\Pi_N \varphi - \varphi, 1) = 0,\ (\nabla(\Pi_N \varphi - \varphi), \nabla \psi) = 0,\ \forall\ \psi \in V_N. \tag{4.7}$$

It is well known that the following estimate holds [1]:

$$\|\varphi - \Pi_N \varphi\|_s \lesssim N^{s-k}\|\varphi\|_k,\ s = 0, 1,\ \forall\ \varphi \in H^k(\Omega),\ k \ge 1. \tag{4.8}$$

Let $L_0^2(\Omega) = \{v \in L^2(\Omega) : (v, 1) = 0\}$. The discrete Laplacian $\Delta_N : V_N \cap L_0^2 \to V_N \cap L_0^2$ is defined as follows: for any $\psi_N \in V_N \cap L_0^2$, let $\Delta_N \psi_N$ be the unique solution to

$$(\Delta_N \psi_N, \chi) = -(\nabla \psi_N, \nabla \chi),\ \forall\ \chi \in V_N. \tag{4.9}$$

The fully discrete form of (4.1)-(4.3) is

$$\frac{u_N^{n+1} - u_N^n}{\tau} + (1 + \Delta_N)\frac{\xi_N^{n+1} + \xi_N^n}{2} + \beta\frac{u_N^{n+1} + u_N^n}{2} + H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1} + W_N^n}{2} = 0, \tag{4.10}$$

$$\xi_N^n = (1 + \Delta_N)u_N^n, \tag{4.11}$$

$$\frac{W_N^{n+1} - W_N^n}{\tau} = \frac{1}{2}H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1} - u_N^n}{\tau}. \tag{4.12}$$

where $\tilde{u}_N^{n+1/2} = \frac{3}{2}u_N^n - \frac{1}{2}u_N^{n-1}$. The spectral-Galerkin method for the scheme (4.10)-(4.12) reads: given $u_N^0 = \Pi_N u^0$, $\xi_N^0 = \Pi_N \xi^0$ and $W_N^0 = \Pi_N W^0$, find $u_N^{n+1} \in V_N$ such that

$$\begin{aligned}&\left(\frac{u_N^{n+1} - u_N^n}{\tau}, \psi\right) + \left(\frac{\xi_N^{n+1} + \xi_N^n}{2}, \psi\right) - \left(\frac{\nabla\xi_N^{n+1} + \nabla\xi_N^n}{2}, \nabla\psi\right) + \beta\left(\frac{u_N^{n+1} + u_N^n}{2}, \psi\right)\\ &+ \left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1} + W_N^n}{2}, \psi\right) = 0,\ \forall\ \psi \in V_N\end{aligned} \tag{4.13}$$

$$(\xi_N^n, v) = (u_N^n, v) - (\nabla u_N^n, \nabla v),\ \forall\ v \in V_N, \tag{4.14}$$

$$\left(\frac{W_N^{n+1} - W_N^n}{\tau}, \zeta\right) = \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1} - u_N^n}{\tau}, \zeta\right),\ \forall\ \zeta \in V_N. \tag{4.15}$$

Remark 4.1. *Since the above scheme involves three time levels, we need to apply the spectral-Galerkin method to the initialization step (3.5)-(3.8) to calculate (u_N^1, ξ_N^1, W_N^1) and then start the above scheme.*

We now give the following energy decreasing property of the fully discrete scheme.

Theorem 4.1. *The fully scheme (4.10)-(4.12) satisfies the discrete energy dissipation law as follows*

$$\mathcal{E}_{CN}^{n+1} - \mathcal{E}_{CN}^n = -\frac{1}{\tau}\|u_N^{n+1} - u_N^n\|^2 \le 0, \tag{4.16}$$

where

$$\mathcal{E}_{CN}^n = \|W_N^n\|^2 + \frac{1}{2}\|\xi_N^n\|^2 + \frac{\beta}{2}\|u_N^n\|^2 - D|\Omega|.$$

Proof. Let $\psi = u_N^{n+1} - u_N^n$ in (4.13), we have

$$\begin{aligned}&\frac{1}{\tau}\|u_N^{n+1} - u_N^n\|^2 + \frac{1}{2}(\xi_N^{n+1} + \xi_N^n, u_N^{n+1} - u_N^n) - \frac{1}{2}(\nabla\xi_N^{n+1} + \nabla\xi_N^n, \nabla u_N^{n+1} - \nabla u_N^n)\\ &+ \frac{\beta}{2}(\|u_N^{n+1}\|^2 - \|u_N^n\|^2) + \frac{1}{2}(H(\tilde{u}_N^{n+1/2})(W_N^{n+1} + W_N^n), u_N^{n+1} - u_N^n) = 0.\end{aligned} \tag{4.17}$$

From (4.14), we naturally have

$$(\xi_N^{n+1} - \xi_N^n, v) = (u_N^{n+1} - u_N^n, v) - (\nabla u_N^{n+1} - \nabla u_N^n, \nabla v). \tag{4.18}$$

Taking $v = (\xi_N^{n+1} + \xi_N^n)/2$ in (4.18), we obtain

$$\frac{1}{2}(\|\xi_N^{n+1}\|^2 - \|\xi_N^n\|^2) = \frac{1}{2}(u_N^{n+1} - u_N^n, \xi_N^{n+1} + \xi_N^n) - \frac{1}{2}(\nabla u_N^{n+1} - \nabla u_N^n, \nabla\xi_N^{n+1} + \nabla\xi_N^n). \tag{4.19}$$

Taking $\zeta = \tau(W_N^{n+1} + W_N^n)$ in (4.15), we have

$$\|W_N^{n+1}\|^2 - \|W_N^n\|^2 = \frac{1}{2}(H(\tilde{u}_N^{n+1/2})(u_N^{n+1} - u_N^n), W_N^{n+1} + W_N^n). \tag{4.20}$$

Combining (4.17), (4.19) and (4.20), we get

$$\frac{1}{\tau}\|u_N^{n+1} - u_N^n\|^2 + \frac{1}{2}(\|\xi_N^{n+1}\|^2 - \|\xi_N^n\|^2) + \frac{\beta}{2}(\|u_N^{n+1}\|^2 - \|u_N^n\|^2) + (\|W_N^{n+1}\|^2 - \|W_N^n\|^2) = 0,$$

which implies the desired result. $\square$

In this work, we assume that the initial data satisfies the following stability:

$$\mathcal{E}_{CN}^0 = \|W_N^0\|^2 + \frac{1}{2}\|\xi_N^0\|^2 + \frac{\beta}{2}\|u_N^0\|^2 - D|\Omega| \le C. \tag{4.21}$$

We define the discrete H^2-norm as

$$\|\phi_N^n\|_{H^2} = \|\phi_N^n\| + \|\nabla\phi_N^n\| + \|\Delta_N\phi_N^n\|, \ \forall\ \phi_N^n \in V_N.$$

To derive the error estimate, we first give the H^2-boundedness for the numerical solution.

Lemma 4.1. *Assuming that u_N^n is the solution of the scheme (4.10)-(4.12), there is a constant $C > 0$ such that*

$$\|u_N^n\|_{H^2} \le C. \tag{4.22}$$

Proof. From Theorem 4.1 and (4.21), we know that there is constant $C \ge 0$ such that

$$\frac{1}{2}\|W_N^n\|^2 + \frac{1}{4}\|\xi_N^n\|^2 + \frac{\beta}{4}\|u_N^n\|^2 \le C,\ 0 \le n \le K,$$

hence, we have

$$\|u_N^n\| \le C, \quad \|\xi_N^n\| \le C. \tag{4.23}$$

By applying (4.14), we have

$$(\nabla u_N^n, \nabla u_N^n) = (u_N^n, u_N^n) - (\xi_N^n, u_N^n),$$
$$(\nabla u_N^n, \nabla\Delta_N u_N^n) = (u_N^n, \Delta_N u_N^n) - (\xi_N^n, \Delta_N u_N^n).$$

Hence, it holds that

$$\|\nabla u_N^n\|^2 \le \|u_N^n\|^2 + \|u_N^n\|\|\xi_N^n\| \le C, \tag{4.24}$$
$$\|\Delta_N u_N^n\|^2 \le \|u_N^n\|\|\Delta_N u_N^n\| + \|\xi_N^n\|\|\Delta_N u_N^n\|,$$
$$\|\Delta_N u_N^n\| \le \|u_N^n\| + \|\xi_N^n\| \le C. \tag{4.25}$$

From (4.23), (4.24) and (4.25), we can deduce (4.22). $\square$

Let us denote

$$\sigma_u^n := u_N^n - \Pi_N u(t_n), \quad \rho_u^n := \Pi_N u(t_n) - u(t_n),$$
$$\sigma_\xi^n := \xi_N^n - \Pi_N \xi(t_n), \quad \rho_\xi^n := \Pi_N \xi(t_n) - \xi(t_n),$$
$$\sigma_W^n := W_N^n - \Pi_N W(t_n), \quad \rho_W^n := \Pi_N W(t_n) - W(t_n),$$

thus,

$$\varepsilon_u^n := u_N^n - u(t_n) = u_N^n - \Pi_N u(t_n) + \Pi_N u(t_n) - u(t_n) = \sigma_u^n + \rho_u^n,$$
$$\varepsilon_\xi^n := \xi_N^n - \xi(t_n) = \xi_N^n - \Pi_N \xi(t_n) + \Pi_N \xi(t_n) - \xi(t_n) = \sigma_\xi^n + \rho_\xi^n,$$
$$\varepsilon_W^n := W_N^n - W(t_n) = W_N^n - \Pi_N W(t_n) + \Pi_N W(t_n) - W(t_n) = \sigma_W^n + \rho_W^n.$$

By the definition of the projection Π_N, we have

$$(\nabla\rho_u^n, \nabla\psi) = (\nabla\rho_\xi^n, \nabla\psi) = 0, \quad \forall\ \psi \in V_N.$$

We also denote

$$T_1^{n+1/2} = \frac{u(t_{n+1}) - u(t_n)}{\tau} - u_t(t_{n+1/2}), \quad T_2^{n+1/2} = \frac{\xi(t_{n+1}) + \xi(t_n)}{2} - \xi(t_{n+1/2}),$$
$$T_3^{n+1/2} = \frac{u(t_{n+1}) + u(t_n)}{2} - u(t_{n+1/2}), \quad T_4^{n+1/2} = \frac{W(t_{n+1}) - W(t_n)}{\tau} - W_t(t_{n+1/2}),$$
$$T_5^{n+1/2} = \frac{W(t_{n+1}) + W(t_n)}{2} - W(t_{n+1/2}), \quad T_6^{n+1/2} = \frac{3}{2}u(t_n) - \frac{1}{2}u(t_{n-1}) - u(t_{n+1/2}).$$

By using the Taylor expansion, we can easily derive the following estimates:

$$\|T_1^{n+1/2}\| \le \frac{1}{24}\|u\|_{W^{3,\infty}(0,T;L^2(\Omega))}\tau^2, \quad \|T_2^{n+1/2}\| \le \frac{1}{8}\|\xi\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2, \tag{4.26}$$

$$\|T_3^{n+1/2}\| \le \frac{1}{8}\|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2, \quad \|T_4^{n+1/2}\| \le \frac{1}{24}\|W\|_{W^{3,\infty}(0,T;L^2(\Omega))}\tau^2, \tag{4.27}$$

$$\|T_5^{n+1/2}\| \le \frac{1}{8}\|W\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2, \quad \|T_6^{n+1/2}\| \le \frac{9}{8}\|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2. \tag{4.28}$$

Theorem 4.2. *Assuming that $u \in W^{3,\infty}(0,T;H^k(\Omega))$, $\xi \in W^{2,\infty}(0,T;H^{k+2}(\Omega))$ and $W \in W^{3,\infty}(0,T;H^k(\Omega))$, then we have the following error estimate*

$$\|u(t_n) - u_N^n\| \lesssim N^{-k} + (\|u\|_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^2, \quad 0 \le n \le K.$$

Proof. Subtracting (4.4)-(4.6) from (4.13)-(4.15) at $t_{n+1/2}$, we get

$$\begin{aligned}
&\left(\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau},\psi\right) + \left(\frac{\rho_u^{n+1}-\rho_u^n}{\tau},\psi\right) + (T_1^{n+1/2},\psi) + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2},\psi\right) + \left(\frac{\rho_\xi^{n+1}+\rho_\xi^n}{2},\psi\right)\\
&+ (T_2^{n+1/2},\psi) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^n}{2},\nabla\psi\right) - (\nabla T_2^{n+1/2},\nabla\psi) + \beta\left(\frac{\sigma_u^{n+1}+\sigma_u^n}{2},\psi\right)\\
&+ \beta\left(\frac{\rho_u^{n+1}+\rho_u^n}{2},\psi\right) + \beta(T_3^{n+1/2},\psi) + \left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}),\psi\right) = 0,
\end{aligned} \tag{4.29}$$

$$(\sigma_\xi^n, v) - (\sigma_u^n, v) + (\nabla\sigma_u^n, \nabla v) = (\rho_u^n, v) - (\rho_\xi^n, v), \tag{4.30}$$

$$\left(\frac{\varepsilon_W^{n+1}-\varepsilon_W^n}{\tau},\zeta\right) + (T_4^{n+1/2},\zeta) = \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1}-u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}),\zeta\right). \tag{4.31}$$

Arranging (4.29), we have

$$\begin{aligned}
&\left(\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau},\psi\right) + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2},\psi\right) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^n}{2},\nabla\psi\right) + \beta\left(\frac{\sigma_u^{n+1}+\sigma_u^n}{2},\psi\right)\\
&= -\left(\frac{\rho_u^{n+1}-\rho_u^n}{\tau},\psi\right) - (T_1^{n+1/2},\psi) - \left(\frac{\rho_\xi^{n+1}+\rho_\xi^n}{2},\psi\right) - (T_2^{n+1/2},\psi) + (\nabla T_2^{n+1/2},\nabla\psi)\\
&\quad - \beta\left(\frac{\rho_u^{n+1}+\rho_u^n}{2},\psi\right) - \beta(T_3^{n+1/2},\psi) - \left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}),\psi\right).
\end{aligned} \tag{4.32}$$

From (4.30), we have

$$\begin{aligned}
&\left(\frac{\sigma_\xi^{n+1}-\sigma_\xi^n}{\tau},v\right) - \left(\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau},v\right) + \left(\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^n}{\tau},\nabla v\right)\\
&= \left(\frac{\rho_u^{n+1}-\rho_u^n}{\tau},v\right) - \left(\frac{\rho_\xi^{n+1}-\rho_\xi^n}{\tau},v\right).
\end{aligned} \tag{4.33}$$

Taking $\psi = (\sigma_u^{n+1} - \sigma_u^n)/\tau$ in (4.32), we get

$$\begin{aligned}
&\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^n}{2},\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^n}{\tau}\right)\\
&+ \frac{\beta}{2\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^n\|^2)\\
&= -\left(\frac{\rho_u^{n+1}-\rho_u^n}{\tau},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) - \left(T_1^{n+1/2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) - \left(\frac{\rho_\xi^{n+1}+\rho_\xi^n}{2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right)\\
&\quad - \left(T_2^{n+1/2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) + \left(\nabla T_2^{n+1/2},\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^n}{\tau}\right) - \beta\left(\frac{\rho_u^{n+1}+\rho_u^n}{2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right)\\
&\quad - \beta\left(T_3^{n+1/2},\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) - \left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}),\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right).
\end{aligned} \tag{4.34}$$

Taking $v = (\sigma_\xi^{n+1} + \sigma_\xi^n)/2$ in (4.33), we have

$$\begin{aligned}
&\left(\frac{\sigma_\xi^{n+1}-\sigma_\xi^n}{\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right) - \left(\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right) + \left(\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^n}{\tau},\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^n}{2}\right)\\
&= \left(\frac{\rho_u^{n+1}-\rho_u^n}{\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right) - \left(\frac{\rho_\xi^{n+1}-\rho_\xi^n}{\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right).
\end{aligned} \tag{4.35}$$

Taking $\zeta = (\varepsilon_W^{n+1} + \varepsilon_W^n)/2$ in (4.31), we obtain

$$\frac{1}{2\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^n\|^2) = -\left(T_4^{n+1/2}, \frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right) + \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1} - u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}), \frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right). \tag{4.36}$$

Combining (4.34), (4.35) and (4.36), we have

$$\begin{aligned}
&\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 + \frac{\beta}{2\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^n\|^2) + \frac{1}{2\tau}(\|\sigma_\xi^{n+1}\|^2 - \|\sigma_\xi^n\|^2) + \frac{1}{2\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^n\|^2) \\
&= -\left(\frac{\rho_u^{n+1} - \rho_u^n}{\tau}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) - \left(T_1^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) - \left(\frac{\rho_\xi^{n+1} + \rho_\xi^n}{2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \\
&\quad - \left(T_2^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) + \left(\nabla T_2^{n+1/2}, \frac{\nabla\sigma_u^{n+1} - \nabla\sigma_u^n}{\tau}\right) - \beta\left(\frac{\rho_u^{n+1} + \rho_u^n}{2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \\
&\quad - \beta\left(T_3^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) - \left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1} + W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}), \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \\
&\quad + \left(\frac{\rho_u^{n+1} - \rho_u^n}{\tau}, \frac{\sigma_\xi^{n+1} + \sigma_\xi^n}{2}\right) - \left(\frac{\rho_\xi^{n+1} - \rho_\xi^n}{\tau}, \frac{\sigma_\xi^{n+1} + \sigma_\xi^n}{2}\right) - \left(T_4^{n+1/2}, \frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right) \\
&\quad + \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1} - u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}), \frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right).
\end{aligned} \tag{4.37}$$

Now, we estimate each term at the right hand side of (4.37).

$$-\left(\frac{\rho_u^{n+1} - \rho_u^n}{\tau}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2, \tag{4.38}$$

$$\begin{aligned}
-\left(T_1^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) &\le 4\|T_1^{n+1/2}\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 \\
&\le \frac{1}{144}\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2,
\end{aligned} \tag{4.39}$$

$$-\left(\frac{\rho_\xi^{n+1} + \rho_\xi^n}{2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2, \tag{4.40}$$

$$\begin{aligned}
-\left(T_2^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) &\le 4\|T_2^{n+1/2}\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 \\
&\le \frac{1}{16}\|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2,
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\left(\nabla T_2^{n+1/2}, \frac{\nabla\sigma_u^{n+1} - \nabla\sigma_u^n}{\tau}\right) &\le \left|\left(\Delta T_2^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right)\right| \\
&\le 4\|\Delta T_2^{n+1/2}\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 \\
&\le \frac{1}{16}\|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2,
\end{aligned} \tag{4.42}$$

$$-\beta\left(\frac{\rho_u^{n+1} + \rho_u^n}{2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2, \tag{4.43}$$

$$\begin{aligned}
-\beta\left(T_3^{n+1/2}, \frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right) &\le 4\beta^2\|T_3^{n+1/2}\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 \\
&\le \frac{\beta^2}{16}\|u\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2,
\end{aligned} \tag{4.44}$$

$$\begin{aligned}
\left(\frac{\rho_u^{n+1} - \rho_u^n}{\tau}, \frac{\sigma_\xi^{n+1} + \sigma_\xi^n}{2}\right) &\le \frac{1}{8}\left\|\frac{\rho_u^{n+1} - \rho_u^n}{\tau}\right\|^2 + 2\left\|\frac{\sigma_\xi^{n+1} + \sigma_\xi^n}{2}\right\|^2 \\
&\le CN^{-2k} + \|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^n\|^2,
\end{aligned} \tag{4.45}$$

$$\left(\frac{\rho_\xi^{n+1}-\rho_\xi^n}{\tau}, \frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right) \le \frac{1}{8}\left\|\frac{\rho_\xi^{n+1}-\rho_\xi^n}{\tau}\right\|^2 + 2\left\|\frac{\sigma_\xi^{n+1}+\sigma_\xi^n}{2}\right\|^2$$
$$\le CN^{-2k} + \|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^n\|^2, \tag{4.46}$$

$$-\left(T_4^{n+1/2}, \frac{\epsilon_W^{n+1}+\epsilon_W^n}{2}\right) \le \frac{1}{2}\|T_4^{n+1/2}\|^2 + \frac{1}{2}\left\|\frac{\epsilon_W^{n+1}+\epsilon_W^n}{2}\right\|^2 \tag{4.47}$$
$$\le \frac{1}{1152}\|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{4}(\|\epsilon_W^{n+1}\|^2 + \|\epsilon_W^n\|^2). \tag{4.48}$$

Since $F(u) \ge -\frac{(\epsilon+\beta)^2}{4}$, we can choose $D > \frac{5}{4}(\epsilon+\beta)^2$ such that $\sqrt{F(u)+D} > \epsilon+\beta$. Thus, we have

$$\|H(u)\| = \left\|\frac{f(u)}{\sqrt{F(u)+D}}\right\| \le \frac{1}{\epsilon+\beta}\|u^3-(\epsilon+\beta)u\| \tag{4.49}$$
$$\le \frac{1}{\epsilon+\beta}(\|u\|^3_{L^\infty(0,T;L^2(\Omega))} + (\epsilon+\beta)\|u\|_{L^\infty(0,T;L^2(\Omega))}) \le C. \tag{4.50}$$

Using (4.22), we have

$$\|\tilde{u}_N^{n+1/2}\| = \left\|\frac{3}{2}u_N^n - \frac{1}{2}u_N^{n-1}\right\| \le \frac{3}{2}\|u_N^n\| + \frac{1}{2}\|u_N^{n-1}\| \le C,$$

we also have

$$\|H(\tilde{u}_N^{n+1/2})\| \le \frac{1}{\epsilon+\beta}\|(\tilde{u}_N^{n+1/2})^3 - (\epsilon+\beta)\tilde{u}_N^{n+1/2}\| \le \frac{1}{\epsilon+\beta}(\|\tilde{u}_N^{n+1/2}\|^3 + (\epsilon+\beta)\|\tilde{u}_N^{n+1/2}\|) \le C. \tag{4.51}$$

Applying Lemma 2.1, we have

$$\begin{aligned}
&\|H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2}))\| \\
\le\ & \hat{C}_0\|\tilde{u}_N^{n+1/2} - u(t_{n+1/2})\| \\
=\ & \hat{C}_0\left\|\frac{3}{2}u_N^n - \frac{1}{2}u_N^{n-1} - \left(\frac{3}{2}u(t_n) - \frac{1}{2}u(t_{n-1})\right) + \left(\frac{3}{2}u(t_n) - \frac{1}{2}u(t_{n-1})\right) - u(t_{n+1/2})\right\| \\
=\ & \hat{C}_0\left\|\frac{3}{2}(u_N^n - u(t_n)) - \frac{1}{2}(u_N^{n-1} - u(t_{n-1})) + T_6^{n+1/2}\right\| \\
\le\ & \frac{3}{2}\hat{C}_0\|u_N^n - u(t_n)\| + \frac{1}{2}\hat{C}_0\|u_N^{n-1} - u(t_{n-1})\| + \hat{C}_0\|T_6^{n+1/2}\| \\
\le\ & C(\|\sigma_u^n\| + \|\rho_u^n\| + \|\sigma_u^{n-1}\| + \|\rho_u^{n-1}\|) + C\|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2.
\end{aligned} \tag{4.52}$$

Since

$$\begin{aligned}
&H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}) \\
=\ & H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(\tilde{u}_N^{n+1/2})\frac{W(t_{n+1})+W(t_n)}{2} \\
&+ H(\tilde{u}_N^{n+1/2})\frac{W(t_{n+1})+W(t_n)}{2} - H(u(t_{n+1/2}))\frac{W(t_{n+1})+W(t_n)}{2} \\
&+ H(u(t_{n+1/2}))\frac{W(t_{n+1})+W(t_n)}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}) \\
=\ & H(\tilde{u}_N^{n+1/2})\frac{\epsilon_W^{n+1}+\epsilon_W^n}{2} + (H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2})))\frac{W(t_{n+1})+W(t_n)}{2} + H(u(t_{n+1/2}))T_5^{n+1/2},
\end{aligned}$$

we have

$$\begin{aligned}
&\left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}), \frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) \\
=\ & \left(H(\tilde{u}_N^{n+1/2})\frac{\epsilon_W^{n+1}+\epsilon_W^n}{2}, \frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) \\
&+ \left((H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2})))\frac{W(t_{n+1})+W(t_n)}{2}, \frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) + \left(H(u(t_{n+1/2}))T_5^{n+1/2}, \frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) \\
=:\ & I_1 + I_2 + I_3.
\end{aligned} \tag{4.53}$$

Using (4.49)-(4.52), we have the following estimates:

$$
\begin{aligned}
I_1 &\le 8\|H(\tilde{u}_N^{n+1/2})\|^2 \left\|\frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \\
&\le C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2) + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2, \qquad (4.54)\\
I_2 &\le 8\|H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2}))\|^2 \left\|\frac{W(t_{n+1})+W(t_n)}{2}\right\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \\
&\le C(\|\sigma_u^n\| + \|\rho_u^n\| + \|\sigma_u^{n-1}\| + \|\rho_u^{n-1}\| + \|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2)^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \\
&\le C\|\sigma_u^n\|^2 + C\|\sigma_u^{n-1}\|^2 + CN^{-2k} + C\|u\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \qquad (4.55)\\
I_3 &\le 8\|H(u(t_{n+1/2}))\|^2\|T_5^{n+1/2}\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \\
&\le C\|W\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2. \qquad (4.56)
\end{aligned}
$$

Combining (4.53)-(4.56), we get

$$
\begin{aligned}
&\left(H(\tilde{u}_N^{n+1/2})\frac{W_N^{n+1}+W_N^n}{2} - H(u(t_{n+1/2}))W(t_{n+1/2}), \frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right) \\
\le\ & \frac{1}{4}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2 + \|\sigma_u^n\|^2 + \|\sigma_u^{n-1}\|^2) \\
&+ C\|u\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^4 + C\|W\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + CN^{-2k}. \qquad (4.57)
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
&H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1}-u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}) \\
=\ & H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1}-u_N^n}{\tau} - H(\tilde{u}_N^{n+1/2})\frac{\Pi_N u(t_{n+1}) - \Pi_N u(t_n)}{\tau} + H(\tilde{u}_N^{n+1/2})\frac{\Pi_N u(t_{n+1}) - \Pi_N u(t_n)}{\tau} \\
&- H(\tilde{u}_N^{n+1/2})\frac{u(t_{n+1})-u(t_n)}{\tau} + H(\tilde{u}_N^{n+1/2})\frac{u(t_{n+1})-u(t_n)}{\tau} - H(u(t_{n+1/2}))\frac{u(t_{n+1})-u(t_n)}{\tau} \\
&+ H(u(t_{n+1/2}))\frac{u(t_{n+1})-u(t_n)}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}) \\
=\ & H(\tilde{u}_N^{n+1/2})\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau} + H(\tilde{u}_N^{n+1/2})\frac{\rho_u^{n+1}-\rho_u^n}{\tau} + (H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2})))\frac{u(t_{n+1})-u(t_n)}{\tau} \\
&+ H(u(t_{n+1/2}))T_1^{n+1/2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1}-u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}), \frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right) \\
=\ & \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right) + \frac{1}{2}\left(H(\tilde{u}_N^{n+1/2})\frac{\rho_u^{n+1}-\rho_u^n}{\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right) \\
&+ \frac{1}{2}\left((H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2})))\frac{u(t_{n+1})-u(t_n)}{\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right) \\
&+ \frac{1}{2}\left(H(u(t_{n+1/2}))T_1^{n+1/2}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right) \\
=:\ & J_1 + J_2 + J_3 + J_4. \qquad (4.58)
\end{aligned}
$$

Using (4.49)-(4.52), we have the following estimates:

$$
\begin{aligned}
J_1 &\le \|H(\tilde{u}_N^{n+1/2})\|^2 \left\|\frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2 \\
&\le C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2) + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^n}{\tau}\right\|^2, \qquad (4.59)\\
J_2 &\le \frac{1}{4}\|H(\tilde{u}_N^{n+1/2})\|^2 \left\|\frac{\rho_u^{n+1}-\rho_u^n}{\tau}\right\|^2 + \frac{1}{4}\left\|\frac{\varepsilon_W^{n+1}+\varepsilon_W^n}{2}\right\|^2
\end{aligned}
$$

$$\leq CN^{-2k} + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2), \tag{4.60}$$

$$J_3 \leq \frac{1}{4}\|H(\tilde{u}_N^{n+1/2}) - H(u(t_{n+1/2}))\|^2 \left\|\frac{u(t_{n+1}) - u(t_n)}{\tau}\right\|^2 + \frac{1}{4}\left\|\frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right\|^2$$

$$\leq C(\|\sigma_u^n\| + \|\rho_u^n\| + \|\sigma_u^{n-1}\| + \|\rho_u^{n-1}\| + \|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2)^2 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2)$$

$$\leq C(\|\sigma_u^n\|^2 + \|\sigma_u^{n-1}\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2) + CN^{-2k} + \|u\|^2_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^4, \tag{4.61}$$

$$J_4 \leq \frac{1}{4}\|H(u(t_{n+1/2}))\|^2\|T_1^{n+1/2}\|^2 + \frac{1}{4}\left\|\frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right\|^2$$

$$\leq C\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2). \tag{4.62}$$

Combining (4.58)-(4.62), we get

$$\left(H(\tilde{u}_N^{n+1/2})\frac{u_N^{n+1} - u_N^n}{\tau} - H(u(t_{n+1/2}))u_t(t_{n+1/2}), \frac{\varepsilon_W^{n+1} + \varepsilon_W^n}{2}\right)$$

$$\leq \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2 + \|\sigma_u^n\|^2 + \|\sigma_u^{n-1}\|^2)$$

$$+ C\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + CN^{-2k}. \tag{4.63}$$

Substituting (4.38)-(4.47), (4.57) and (4.63) into (4.37), we get

$$\frac{1}{4}\left\|\frac{\sigma_u^{n+1} - \sigma_u^n}{\tau}\right\|^2 + \frac{\beta}{2\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^n\|^2) + \frac{1}{2\tau}(\|\sigma_\xi^{n+1}\|^2 - \|\sigma_\xi^n\|^2) + \frac{1}{2\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^n\|^2)$$

$$\leq CN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4$$

$$+ C(\|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^n\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2 + \|\sigma_u^n\|^2 + \|\sigma_u^{n-1}\|^2). \tag{4.64}$$

Multiplying (4.64) by τ, dropping some nonnegative terms, summing n from 0 to l $(0 < l \leq K-1)$, we get

$$\frac{\beta}{2}\|\sigma_u^{l+1}\|^2 + \frac{1}{4}\|\sigma_\xi^{l+1}\|^2 + \frac{1}{4}\|\varepsilon_W^{l+1}\|^2 \leq C\tau\sum_{n=1}^{l}(\|\sigma_u^n\|^2 + \|\sigma_u^{n-1}\|^2 + \|\sigma_\xi^n\|^2 + \|\sigma_\xi^{n+1}\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2)$$

$$+ CTN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4$$

$$+ \|\sigma_u^1\|^2 + \|\sigma_\xi^1\|^2 + \|\varepsilon_W^1\|^2. \tag{4.65}$$

Applying the same approach to the initialization step (3.5)-(3.8) and noting that $\sigma_u^0 = \sigma_\xi^0 = \sigma_W^0 = 0$, we have

$$\|\sigma_u\|^2 + \|\sigma_\xi\|^2 + \|\varepsilon_W^1\|^2 \leq CTN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4. \tag{4.66}$$

Combining (4.65) and (4.66) and using the discrete Grönwall's inequality, we obtain

$$\|\sigma_u^{l+1}\|^2 + \|\sigma_\xi^{l+1}\|^2 + \|\varepsilon_W^{l+1}\|^2 \lesssim N^{-2k} + (\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4.$$

In addition, because $\|u_N^l - u(t_l)\| \leq \|\sigma_u^l\| + \|\rho_u^l\|$ and (4.8), we get the desired result. □

5. Numerical experiments

In this section, we give several numerical experiments for the SH equation to verify the accuracy and energy stability of the proposed schemes.

5.1. Temporal accuracy test

We first test the convergence rates of the two proposed schemes. The parameter is $\epsilon = 0.025$, $\beta = 1$, $D = 50$. Because it is difficult to obtain the analytical solution for SH equation, we add a suitable source term such that the exact solution is

$$u(x, y, t) = \cos(t)\sin(\frac{2\pi}{64}x)\cos(\frac{2\pi}{64}y).$$

Set the computational domain to be $\Omega = [0, 128] \times [0, 128]$. We set $N = 256$ so that the spatial discretization errors are negligible compared with the time discretization errors. In Table 1, we show the L^2 errors of the phase variable between the analytical solution and numerical solution with different time step sizes at $T = 10$. From Table 1, we can observe that the two schemes give desired orders of accuracy in time. The spatial L^2 errors are plotted in Fig. 1.

Table 1
The errors and order of convergence at $T = 10$ for the phase variable u that are computed by the first-order scheme and Crank-Nicolson scheme, using different time step sizes and $N = 256$. The physical parameter is $\epsilon = 0.025$.

	τ	L^2 error	order		τ	L^2 error	order
first-order scheme	1/2	9.9620e-01	-	Crank-Nicolson scheme	1/2	1.8092e-01	-
	1/4	8.4500e-01	0.24		1/4	2.6606e-02	2.77
	1/8	3.6103e-01	1.23		1/8	4.3496e-03	2.61
	1/16	1.6450e-01	1.13		1/16	7.5820e-04	2.52
	1/32	7.8230e-02	1.07		1/32	1.5349e-04	2.30
	1/64	3.8117e-02	1.04		1/64	3.4175e-05	2.17
	1/128	1.8810e-02	1.02		1/128	8.0541e-06	2.09
	1/256	9.3434e-03	1.01		1/256	1.9550e-06	2.04
	1/512	4.6563e-03	1.00		1/512	4.8163e-07	2.02
	1/1024	2.3243e-03	1.00		1/1024	1.1953e-07	2.01

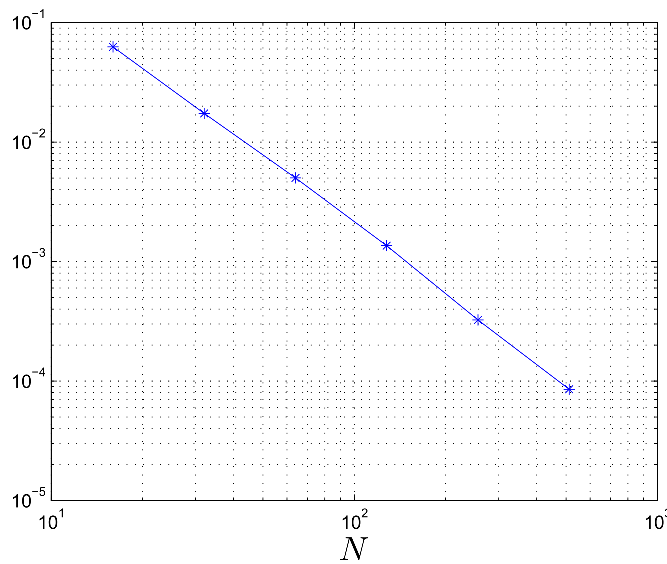


Fig. 1. The spatial L^2 error.

5.2. Energy stability test

In this subsection, we consider the smooth initial condition (5.1) to verify the energy stability of our schemes.

$$\begin{aligned} u(x, y, 0) =& 0.07 - 0.02\cos(\frac{\pi(x-12)}{16})\sin(\frac{\pi(y-1)}{16}) - 0.01\sin^2(\frac{\pi x}{8})\sin^2(\frac{\pi(y-6)}{8}) \\ &+ 0.02\cos^2(\frac{\pi(x+10)}{32})\cos^2(\frac{\pi(y+3)}{32}). \end{aligned} \tag{5.1}$$

The parameters are $\epsilon = 0.025$, $\beta = 1$, $D = 50$, $T = 100$, $N = 64$ and $\Omega = [0, 32]^2$. In Fig. 2, we present the evolution of the discrete energy with different time step sizes of $\tau = 0.01, 0.1, 1, 2, 5, 10, 20, 25$ using the first-order and Crank-Nicolson schemes, respectively. We see that the energy is nonincreasing, which validates that our schemes satisfy the unconditional energy stability. We can also find that the first-order scheme is more stable than the Crank-Nicolson scheme since its energy is much lower than the Crank-Nicolson scheme.

5.3. Phase transition behaviors

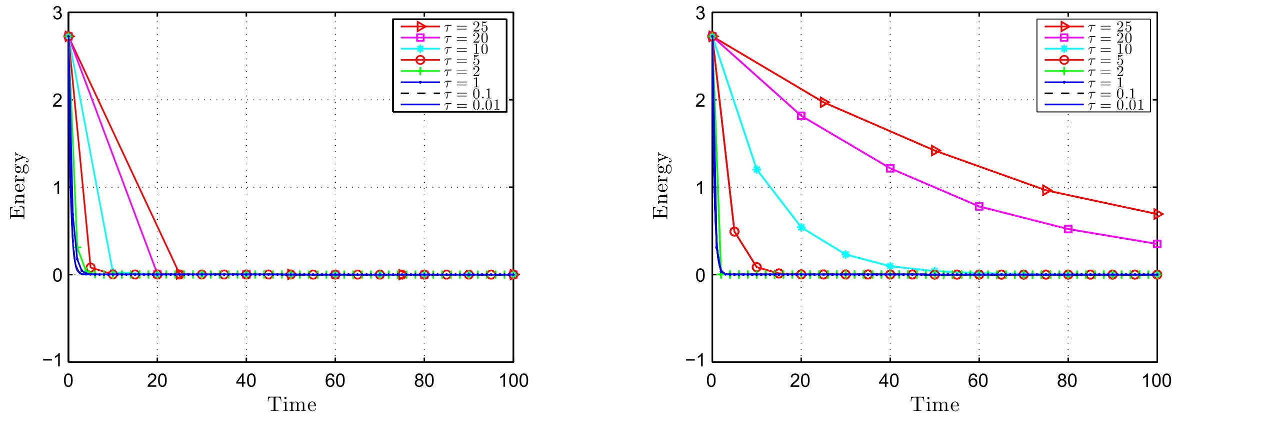
In this subsection, we apply the Crank-Nicolson scheme to check the evolution from a non-equilibrium state to a steady state. Since the first-order scheme provides similar numerical result, for simplicity, we only consider the Crank-Nicolson scheme in the following simulations.

5.3.1. 2D case with random initial condition

With the initial condition $u^0 = \overline{u}$+rand, where $\overline{u} = -0.2$ and rand is a randomly chosen number between -0.4 and 0.4 at the grid points, we set $N = 128$ and the 2D computational domain is $[-30, 30] \times [-30, 30]$. Let the time step be $\tau = 1$ and the parameter be $\epsilon = 0.025$, $\beta = 1$, $D = 50$. Fig. 3 shows the time evolution of the phase transition behavior, which validates that our scheme does lead to the expected states. Fig. 4 displays the results at $t = 100$ with respect to four different values of ϵ, i.e., $\epsilon = 0.05, 0.1, 0.25$ and 0.5. We can find that a large value of ϵ accelerates the formation of regular laminar pattern.

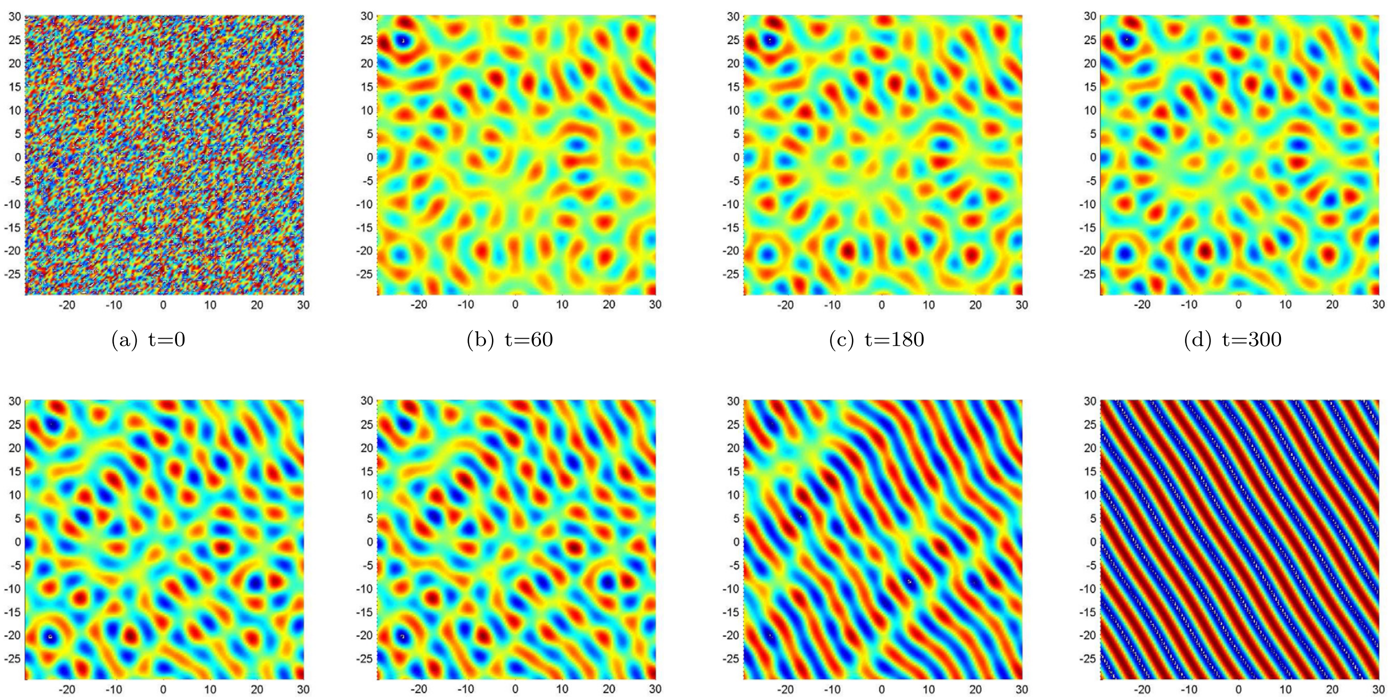
5.3.2. 3D case with random initial condition

With the initial condition $u^0 = \overline{u}$+rand, where $\overline{u} = -0.5$ and rand is a randomly chosen number between -1 and 1 at the grid points, $\Omega = [-10, 10]^3$, $N = 40$. Let the time step be $\tau = 1$ and the parameter be $\epsilon = 0.35$, $\beta = 1$, $D = 50$. Fig. 5 shows the time evolution of the phase transition behavior, which validates that our scheme does lead to the expected states. Fig. 7 (a) shows the energy evolution with the random initial condition in 3D.



(a) first-order scheme (b) Crank-Nicolson scheme

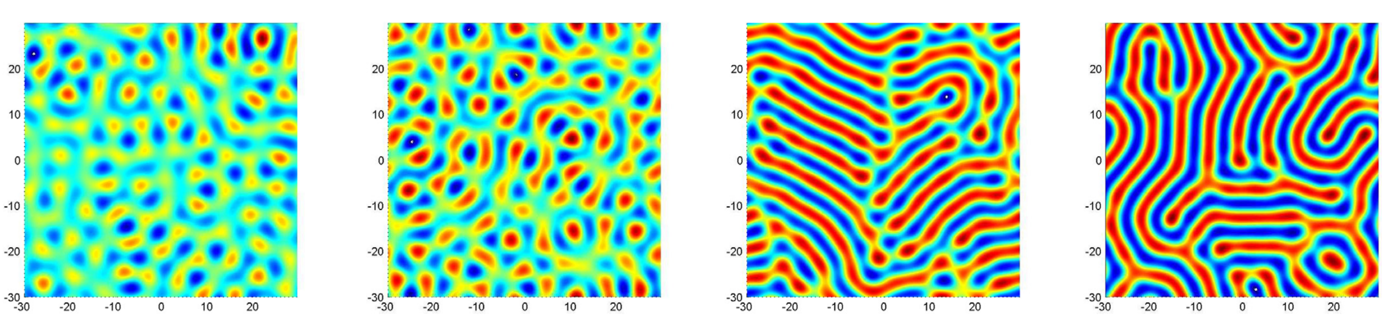
Fig. 2. Evolution of the energy with different time step size of $\tau = 0.01$, 0.1, 1, 2, 5, 10, 20, 25 using the first-order and Crank-Nicolson schemes, respectively, where $\epsilon = 0.025$.



(a) t=0 (b) t=60 (c) t=180 (d) t=300

(e) t=540 (f) t=720 (g) t=1200 (h) t=3000

Fig. 3. The evolution of the phase transition behavior in 2D with $\overline{u} = -0.2$. Snapshots of the numerical approximation of the density field u are taken at $t = 0$, 60, 180, 300, 540, 720, 1200, 3000. The computational domain is $[-30, 30] \times [-30, 30]$. The parameters are $\epsilon = 0.025$, $T = 3000$, $N = 128$. The time step is $\tau = 1$.



(a) $\epsilon = 0.05$ (b) $\epsilon = 0.1$ (c) $\epsilon = 0.25$ (d) $\epsilon = 0.5$

Fig. 4. Snapshots at $t = 100$ with respect to different values of ϵ.

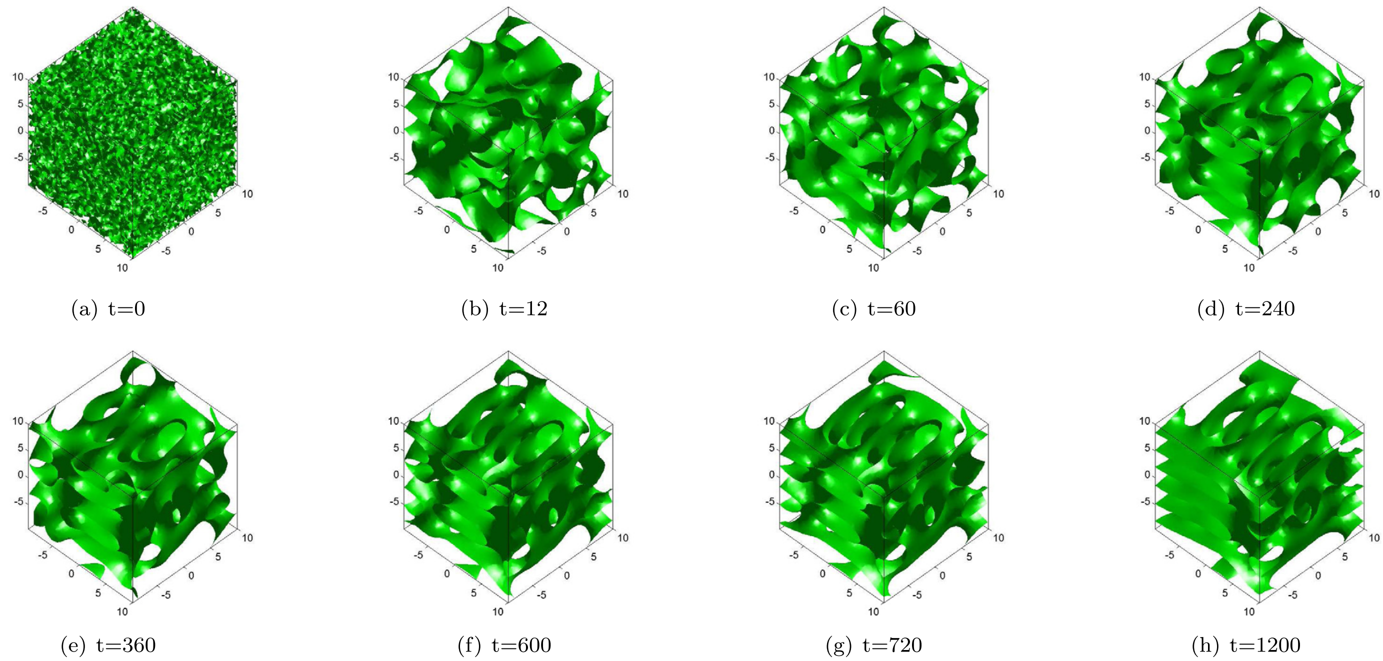


Fig. 5. The evolution of the phase transition behavior in $3D$ with $\bar{u} = -0.5$. Snapshots of the numerical approximation of the density field u are taken at $t = 0$, 12, 60, 240, 360, 600, 720, 1200. The computational domain is $[-10, 10]^3$. The parameters are $\epsilon = 0.35$, $T = 1200$, $N = 40$. The time step is $\tau = 1$.

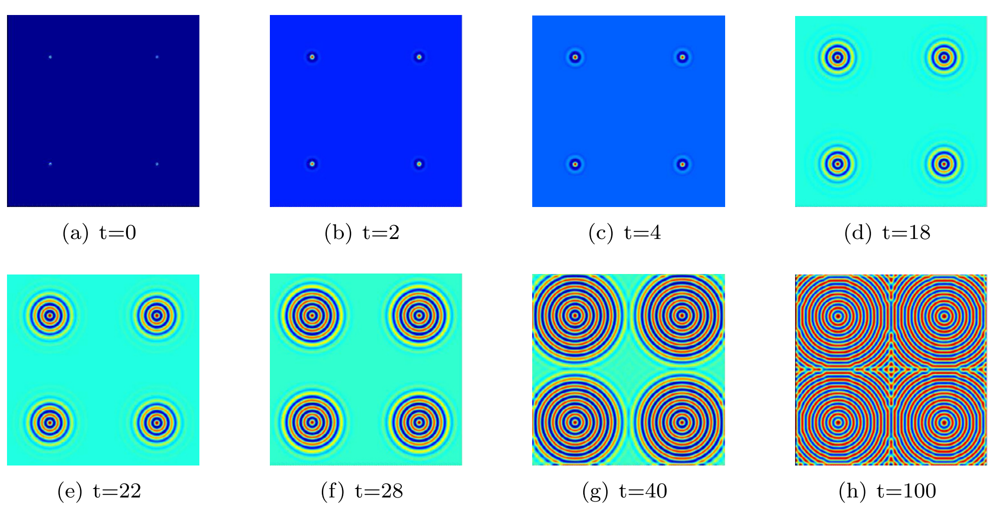


Fig. 6. Temporal evolution of 2D polycrystal growth in a supercooled liquid.

5.3.3. *2D polycrystal growth in a supercooled liquid*

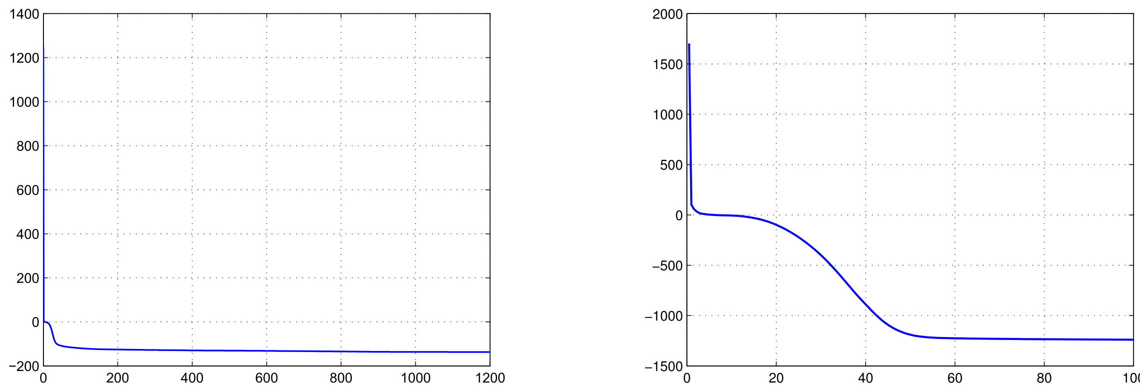
The polycrystal growth in a supercooled liquid was considered as an important benchmark test in two-dimensional space. Here, we consider the growth of four crystal nucleuses with the following initial condition:

$$u(x, y, 0) = 0.14 + \alpha \mathrm{rand}(x, y),$$

where α takes the values of $\alpha = 1$ for four crystal nucleuses locating at $(40, 40)$, $(140, 40)$, $(140, 40)$, $(140, 140)$, respectively. The diameter of each nucleus is 4. The computational domain is $\Omega = [0, 180]^2$. $N = 512$, $\tau = 1/2$, $\epsilon = 0.5$, $\beta = 1$, $D = 50$ and $T = 100$. Fig. 6 shows the growth of the four nuclei in time and formation of the obvious grain boundaries. Fig. 7 (b) shows the energy evolution of the 2D polycrystal growth in a supercooled liquid.

6. Conclusions

In this paper, we propose and analyze first- and second-order linear energy-stable schemes for the SH equation. We prove rigorously that the schemes satisfy the energy dissipation property and derive the error estimate. Numerical tests are given to show the accuracy and energy stability of the proposed schemes.



(a) 3D case with random initial condition

(b) polycrystal growth in a supercooled liquid in 2D

Fig. 7. Evolution of the energy with the random initial condition in 3D and polycrystal growth in a supercooled liquid in 2D.

Data availability

No data was used for the research described in the article.

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