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Error estimate of a stabilized second-order linear predictor–corrector scheme for the Swift–Hohenberg equation☆

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ABSTRACT

In this work, we propose a stabilized linear predictor–corrector scheme for the Swift–Hohenberg equation. More precisely, we apply a stabilized first-order scheme as the predictor and a stabilized second-order scheme as the corrector. We prove rigorously that our scheme satisfies the energy dissipation law and is second-order accurate. Numerical experiments are presented to show the accuracy and energy stability of our scheme.

1. Introduction

The Swift–Hohenberg (SH) equation was originally derived by Swift and Hohenberg [1] to describe Rayleigh–Bénard convection. Related applications can be found in complex pattern formation, complex fluids and biological tissues. The SH equation is derived from the following free energy functional

$$E(u) = \int_{\Omega} \left(\frac{1}{2}(\triangle u)^2 - |\nabla u|^2 + F(u) \right) \mathrm{d}x,$$

where Ω is a domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$, u is the density field, $F(u) = \frac{1}{4}u^4 + \frac{1-\epsilon}{2}u^2$, $0 < \epsilon < 1$ is a constant with physical significance and $\triangle$ is the Laplacian operator. The SH equation is given by

$$u_t = -\frac{\delta E}{\delta u} = -(\triangle^2 u + 2\triangle u + f(u)), \tag{1.1}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative, $f(u) = F'(u) = u^3 + (1-\epsilon)u$. The free energy is nonincreasing in time. Here we study the numerical scheme of SH equation with periodic boundary condition.

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As a nonlinear fourth-order partial differential equation, the SH equation is difficult to be solved analytically. Hence, various numerical schemes have been proposed in recent years. By applying the Crank–Nicolson scheme, a semi-implicit second-order method for the SH equation was given in [2], in which the Newton's method was used to solve the nonlinear equation at every time marching, but the convergence analysis for that scheme was not discussed. In [3], based on the operator splitting scheme, the first- and second-order Fourier spectral methods were presented for the SH equation, but the error analysis was not given. In [4], a new conservative SH equation was introduced and its first-order and second-order mass conservative operator splitting schemes were proposed, but the authors did also not discuss the convergence and error analysis. In [5], the author presented a non-iterative convex splitting scheme for the SH equation with quadratic-cubic nonlinearity, but the convergence was not given. In [6], A fast explicit high-order operator splitting scheme was presented for the SH equation with a nonlocal nonlinearity. In [7], we proposed a second-order energy stable numerical scheme for the SH equation and presented an optimal error estimate for the scheme. However, to solve the fully discrete nonlinear systems, these methods generally require the use of an iteration. Hence, the computational costs are often high and the implementations are usually complicated. There are also various linear schemes that attract the attention of many scholars, such as invariant energy quadratization (IEQ) scheme [8] and scalar auxiliary variable (SAV) scheme [9].

The main goal of this work is to give the error estimate of a linear predictor–corrector time-stepping scheme for the SH equation. To improve the stability, stabilized terms are added in the numerical scheme. Moreover, we prove rigorously that our scheme satisfies the energy dissipation law and is second-order accurate in time. Numerical results are presented to validate our theoretical analysis and show that the proposed scheme is easy to implement and is energy stable with different time step size, the energy decay is robust with respect to the stabilized constant. As a comparison, we also consider the stabilized second-order Crank–Nicolson scheme with Adam–Bashforth extrapolation for nonlinear terms. We will show that the stabilized predictor–corrector scheme is much more robust than the stabilized Crank–Nicolson scheme with Adam–Bashforth extrapolation when large time step is used.

The rest of the paper is organized as follows. In Section 2, we construct the numerical scheme and prove our scheme satisfies the energy dissipation law. In Section 3, we carry out the error estimate, which shows our scheme is second-order accurate in time. In Section 4, several numerical experiments are provided to illustrate the accuracy, robustness and energy stability of the proposed scheme. Finally, some conclusions are given in Section 5.

2. Stabilized linear predictor–corrector scheme for the SH equation and its energy stability

Let N be any positive integer, T be the final time, $\tau = T/N$ be the time step size, $t^n = n\tau$, $n = 0, 1, 2, \ldots, N$ be the time mesh points, u^n be the numerical approximation of $u(t^n)$. The stabilized linear predictor–corrector scheme is as follows.

Scheme (*Stabilized Linear Predictor–Corrector Scheme*)**.** Given u^n, we can calculate u^{n+1} via the following steps:

- Prediction: predict $\bar{u}^{n+1/2}$ via the stabilized linear first-order scheme

$$\frac{\bar{u}^{n+1/2} - u^n}{\tau/2} + \triangle^2 \bar{u}^{n+1/2} + 2\triangle \bar{u}^{n+1/2} + f(u^n) + S(\bar{u}^{n+1/2} - u^n) = 0, \tag{2.1}$$

- Correction: obtain u^{n+1} via the stabilized linear second-order scheme

$$\frac{u^{n+1} - u^n}{\tau} + \triangle^2 \frac{u^{n+1} + u^n}{2} + 2\triangle \frac{u^{n+1} + u^n}{2} + f(\bar{u}^{n+1/2}) + S(\frac{1}{2}(u^{n+1} + u^n) - \bar{u}^{n+1/2}) = 0, \tag{2.2}$$

where S is a given stabilization constant.

We assume that the solution u of Eq. (1.1) exists and satisfies

$$\begin{aligned}\|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|_{L^\infty(0,T;H^4(\Omega))} + \|u_{ttt}\|_{L^\infty(0,T;L^2)(\Omega)}\\ + \|u_{tt}\|_{L^\infty(0,T;H^4(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))} \le C_*.\end{aligned} \tag{2.3}$$

Let $L = 3C_*^2 + (1-\epsilon)|\Omega|$, where $|\Omega|$ is the measure of the domain Ω. Since

$$\|f'(u)\| \le \|3u^2 + 1 - \epsilon\| \le \|3u^2\| + \|1-\epsilon\| \le 3\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + (1-\epsilon)|\Omega| \le L,$$

we have

$$\|f(u) - f(v)\| \le \|f'(\xi)\|\|u - v\| \le L\|u - v\|, \tag{2.4}$$

where ξ is a number between u and v.

Theorem 2.1. *Provided $S \ge \max\{L, (L+1)/2\}$ and*

$$\tau \le \frac{2B}{A(L+S)},$$

where

$$A = \max_n\{\|u^n - \bar{u}^{n+1/2}\|^2\}, \quad B = \min_n\{\|u^{n+1} - u^n\|^2\}.$$

The stabilized predictor–corrector scheme satisfies the following energy stability property:

$$E(u^{n+1}) \le E(u^n), \quad \forall\ n \ge 1.$$

Proof. Taking the inner product of (2.1) with $\bar{u}^{n+1/2} - u^n$ and using Taylor's theorem with remainder in the integral form

$$F(\bar{u}^{n+1/2}) - F(u^n) = f(u^n)(\bar{u}^{n+1/2} - u^n) + \int_{u^n}^{\bar{u}^{n+1/2}} (\bar{u}^{n+1/2} - t)f'(t)\mathrm{d}t,$$

we have

$$\begin{aligned}&\frac{2}{\tau}\|\bar{u}^{n+1/2} - u^n\|^2 + \frac{1}{2}(\|\triangle\bar{u}^{n+1/2}\|^2 - \|\triangle u^n\|^2 + \|\triangle\bar{u}^{n+1/2} - \triangle u^n\|^2) - (\|\nabla\bar{u}^{n+1/2}\|^2 - \|\nabla u^n\|^2\\ &+ \|\nabla\bar{u}^{n+1/2} - \nabla u^n\|^2) + (F(\bar{u}^{n+1/2}) - F(u^n), 1) + S\|\bar{u}^{n+1/2} - u^n\|^2\\ =&\left(\int_{u^n}^{\bar{u}^{n+1/2}} (\bar{u}^{n+1/2} - t)f'(t)\mathrm{d}t, 1\right)\\ \le&\frac{L}{2}\|\bar{u}^{n+1/2} - u^n\|^2.\end{aligned} \tag{2.5}$$

Since

$$\begin{aligned}\|\nabla(\bar{u}^{n+1/2} - u^n)\|^2 =&(\nabla(\bar{u}^{n+1/2} - u^n), \nabla(\bar{u}^{n+1/2} - u^n))\\ \le&|(\triangle(\bar{u}^{n+1/2} - u^n), \bar{u}^{n+1/2} - u^n)|\\ \le&\frac{1}{2}\|\triangle(\bar{u}^{n+1/2} - u^n)\|^2 + \frac{1}{2}\|\bar{u}^{n+1/2} - u^n\|^2,\end{aligned}$$

we obtain from (2.5) that

$$E(\bar{u}^{n+1/2}) - E(u^n) + \frac{2}{\tau}\|\bar{u}^{n+1/2} - u^n\|^2 \le (\frac{L+1}{2} - S)\|\bar{u}^{n+1/2} - u^n\|^2.$$

If $\frac{L+1}{2} \le S$, the predictor scheme satisfies the energy dissipation law.

Similarly, using Taylor's theorem with remainder in the integral form

$$F(u^{n+1}) - F(\bar{u}^{n+1/2}) = f(\bar{u}^{n+1/2})(u^{n+1} - \bar{u}^{n+1/2}) + \int_{\bar{u}^{n+1/2}}^{u^{n+1}} (u^{n+1} - t) f'(t)\mathrm{d}t, \tag{2.6}$$

$$F(u^{n}) - F(\bar{u}^{n+1/2}) = f(\bar{u}^{n+1/2})(u^{n} - \bar{u}^{n+1/2}) + \int_{\bar{u}^{n+1/2}}^{u^{n}} (u^{n} - t) f'(t)\mathrm{d}t, \tag{2.7}$$

subtracting (2.7) from (2.6), we have

$$F(u^{n+1}) - F(u^n) = f(\bar{u}^{n+1/2})(u^{n+1} - u^n) + \int_{\bar{u}^{n+1/2}}^{u^{n+1}} (u^{n+1} - t) f'(t)\mathrm{d}t + \int_{u^n}^{\bar{u}^{n+1/2}} (u^{n} - t) f'(t)\mathrm{d}t. \tag{2.8}$$

Taking the inner product of (2.2) with $u^{n+1} - u^n$, we get

$$\begin{aligned} &\frac{1}{\tau}\|u^{n+1} - u^n\|^2 + \frac{1}{2}(\|\triangle u^{n+1}\|^2 - \|\triangle u^n\|^2) - (\|\nabla u^{n+1}\|^2 - \|\nabla u^n\|^2) + (f(\bar{u}^{n+1/2}), u^{n+1} - u^n) \\ &+ \frac{S}{2}(u^{n+1} - 2\bar{u}^{n+1/2} + u^n, u^{n+1} - u^n) = 0. \end{aligned} \tag{2.9}$$

Since

$$\begin{aligned} &\frac{S}{2}(u^{n+1} - 2\bar{u}^{n+1/2} + u^n, u^{n+1} - u^n) \\ &\quad = \frac{S}{2}((u^{n+1} - \bar{u}^{n+1/2}) - (\bar{u}^{n+1/2} - u^n), (u^{n+1} - \bar{u}^{n+1/2}) + (\bar{u}^{n+1/2} - u^n)) \\ &\quad = \frac{S}{2}(\|u^{n+1} - \bar{u}^{n+1/2}\|^2 - \|\bar{u}^{n+1/2} - u^n\|^2), \end{aligned}$$

we obtain from (2.8) and (2.9) that

$$\begin{aligned} &E(u^{n+1}) - E(u^n) + \frac{1}{\tau}\|u^{n+1} - u^n\|^2 + \frac{S}{2}\|u^{n+1} - \bar{u}^{n+1/2}\|^2 - \frac{S}{2}\|\bar{u}^{n+1/2} - u^n\|^2 \\ &\le \int_{\bar{u}^{n+1/2}}^{u^{n+1}} (u^{n+1} - t) f'(t)\mathrm{d}t + \int_{u^n}^{\bar{u}^{n+1/2}} (u^{n} - t) f'(t)\mathrm{d}t \\ &\le \frac{L}{2}\|u^{n+1} - \bar{u}^{n+1/2}\|^2 + \frac{L}{2}\|\bar{u}^{n+1/2} - u^n\|^2, \end{aligned}$$

That is

$$E(u^{n+1}) - E(u^n) \le \frac{L-S}{2}\|u^{n+1} - \bar{u}^{n+1/2}\|^2 + \frac{L+S}{2}\|\bar{u}^{n+1/2} - u^n\|^2 - \frac{1}{\tau}\|u^{n+1} - u^n\|^2.$$

Let

$$A = \max_n\{\|u^n - \bar{u}^{n+1/2}\|^2\}, \quad B = \min_n\{\|u^{n+1} - u^n\|^2\}.$$

Generally, $A \neq 0$. If $L \le S$ and $\frac{L+S}{2}A - \frac{B}{\tau} \le 0$, i.e. $\tau \le \frac{2B}{A(L+S)}$, the corrector scheme satisfies the energy dissipation law. □

3. Error estimate

Denote $e^n = u^n - u(t^n)$ and $\bar{e}^{n+1/2} = \bar{u}^{n+1/2} - u(t^{n+1/2})$, where $t^{n+1/2} = (t^n + t^{n+1})/2$. We now derive the error analysis of the proposed scheme, which shows the second-order convergence in time.

Theorem 3.1. *Assuming the analytical solution of (1.1) satisfies regularity condition (2.3). For $S > 5$, we have the following error estimate*

$$\|e^N\|^2 \le 4\exp(C_1 T) C_2 T C_*^2 \tau^4, \tag{3.1}$$

where

$$C_1 = \frac{5(L^2+S^2)}{2S}\left(\frac{4}{3} + \frac{L^2+S^2}{S}\tau\right), \quad C_2 = \max\left\{\frac{5(L^2+S^2)}{3S}, \frac{5}{2S}\right\}.$$

Proof. Firstly, we derive the estimate of $\bar{e}^{n+1/2}$. At time level t^n, Eq. (1.1) becomes

$$u_t(t^n) + \triangle^2 u(t^n) + 2\triangle u(t^n) + f(u(t^n)) = 0. \tag{3.2}$$

Subtracting (3.2) from (2.1), we obtain

$$\frac{\bar{e}^{n+1/2} - e^n}{\tau/2} + \triangle^2 \bar{e}^{n+1/2} + 2\triangle \bar{e}^{n+1/2} + f(u^n) - f(u(t^n)) + S(\bar{e}^{n+1/2} - e^n) + \bar{G}^n = 0, \tag{3.3}$$

where the truncation error $\bar{G}^n = \bar{G}_1^n + \bar{G}_2^n$ with

$$\begin{aligned} \bar{G}_1^n &= \frac{u(t^{n+1/2}) - u(t^n)}{\tau/2} - u_t(t^n), \\ \bar{G}_2^n &= \triangle^2(u(t^{n+1/2}) - u(t^n)) + 2\triangle(u(t^{n+1/2}) - u(t^n)) + S(u(t^{n+1/2}) - u(t^n)) \\ &= (\triangle^2 + 2\triangle + S)(u(t^{n+1/2}) - u(t^n)). \end{aligned}$$

By Taylor expansion, we have

$$\|\bar{G}_1^n\|^2 \le \frac{1}{16}\|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}\tau^2, \quad \|\bar{G}_2^n\|^2 \le \frac{1}{4}\|u_t\|^2_{L^\infty(0,T;H^4(\Omega))}\tau^2.$$

Taking the inner product of (3.3) with $\bar{e}^{n+1/2}$, we derive

$$\begin{aligned} &\frac{1}{\tau}(\|\bar{e}^{n+1/2}\|^2 - \|e^n\|^2 + \|\bar{e}^{n+1/2} - e^n\|^2) + \|\triangle \bar{e}^{n+1/2}\|^2 + S\|\bar{e}^{n+1/2}\|^2 \\ =& -2(\triangle \bar{e}^{n+1/2}, \bar{e}^{n+1/2}) - (f(u^n) - f(u(t^n)), \bar{e}^{n+1/2}) + S(e^n, \bar{e}^{n+1/2}) - (\bar{G}^n, \bar{e}^{n+1/2}) \\ \le& S\|\bar{e}^{n+1/2}\|^2 + \frac{3}{S}\|\triangle \bar{e}^{n+1/2}\|^2 + \frac{3(L^2+S^2)}{4S}\|e^n\|^2 + \frac{1}{4\tau}\|\bar{e}^{n+1/2}\|^2 + \tau\|\bar{G}^n\|^2, \end{aligned}$$

that is

$$\frac{3}{4\tau}\|\bar{e}^{n+1/2}\|^2 + (1 - \frac{3}{S})\|\triangle \bar{e}^{n+1/2}\|^2 \le \left(\frac{1}{\tau} + \frac{3(L^2+S^2)}{4S}\right)\|e^n\|^2 + \tau\|\bar{G}^n\|^2.$$

If $S > 3$, dropping the nonnegative terms, we have

$$\begin{aligned} \|\bar{e}^{n+1/2}\|^2 &\le \left(\frac{4}{3} + \frac{L^2+S^2}{S}\tau\right)\|e^n\|^2 + \frac{8}{3}\tau^2(\|\bar{G}_1^n\|^2 + \|\bar{G}_2^n\|^2) \\ &\le \left(\frac{4}{3} + \frac{L^2+S^2}{S}\tau\right)\|e^n\|^2 + \frac{2}{3}(\|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_t\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^4. \end{aligned} \tag{3.4}$$

We now derive the estimate of e^{n+1}. At time level $t^{n+1/2}$, Eq. (1.1) becomes

$$u_t(t^{n+1/2}) + \triangle^2 u(t^{n+1/2}) + 2\triangle u(t^{n+1/2}) + f(u(t^{n+1/2})) = 0. \tag{3.5}$$

Subtracting (3.5) from (2.2), we have

$$\begin{aligned} &\frac{e^{n+1} - e^n}{\tau} + \triangle^2 \frac{e^{n+1} + e^n}{2} + 2\triangle \frac{e^{n+1} + e^n}{2} + f(\bar{u}^{n+1/2}) - f(u(t^{n+1/2})) \\ &\qquad + S(\frac{1}{2}(e^{n+1} + e^n) - \bar{e}^{n+1/2}) + G_1^n + G_2^n = 0, \end{aligned} \tag{3.6}$$

where the truncation errors are

$$\begin{aligned} G_1^n &= \frac{u(t^{n+1}) - u(t^n)}{\tau} - u_t(t^{n+1/2}), \\ G_2^n &= \triangle^2\left(\frac{u(t^{n+1}) + u(t^n)}{2} - u(t^{n+1/2})\right) + 2\triangle\left(\frac{u(t^{n+1}) + u(t^n)}{2} - u(t^{n+1/2})\right) \end{aligned}$$

$$
\begin{aligned}
&+ S\left(\frac{u(t^{n+1})+u(t^n)}{2} - u(t^{n+1/2})\right)\\
=&(\triangle^2+2\triangle+S)\left(\frac{u(t^{n+1})+u(t^n)}{2} - u(t^{n+1/2})\right).
\end{aligned}
$$

By Taylor expansion, we have

$$
\|G_1^n\|^2 \le \frac{1}{576}\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}\tau^4, \quad \|G_2^n\|^2 \le \frac{1}{128}\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))}\tau^4.
$$

Taking the inner product of (3.6) with $e^{n+1}+e^n$, we obtain

$$
\begin{aligned}
&\frac{1}{\tau}(\|e^{n+1}\|^2-\|e^n\|^2)+\frac{1}{2}\|\triangle(e^{n+1}+e^n)\|^2+\frac{S}{2}\|e^{n+1}+e^n\|^2\\
=&-(\triangle(e^{n+1}+e^n),e^{n+1}+e^n)-(f(\bar{u}^{n+1/2})-f(u(t^{n+1/2})),e^{n+1}+e^n)+S(\bar{e}^{n+1/2},e^{n+1}+e^n)\\
&-(G_1^n,e^{n+1}+e^n)-(G_2^n,e^{n+1}+e^n)\\
=&\frac{S}{2}\|e^{n+1}+e^n\|^2+\frac{5}{2S}\|\triangle(e^{n+1}+e^n)\|^2+\frac{5(L^2+S^2)}{2S}\|\bar{e}^{n+1/2}\|^2+\frac{5}{2S}(\|G_1^n\|^2+\|G_2^n\|^2),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{1}{\tau}\|e^{n+1}\|^2-\frac{1}{\tau}\|e^n\|^2+\left(\frac{1}{2}-\frac{5}{2S}\right)\|\triangle(e^{n+1}+e^n)\|^2\\
\le&\frac{5(L^2+S^2)}{2S}\|\bar{e}^{n+1/2}\|^2+\frac{5}{2S}(\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^4.
\end{aligned}
$$

If $S>5$, dropping the nonnegative terms and using (3.4), we have

$$
\begin{aligned}
\|e^{n+1}\|^2-\|e^n\|^2\le&\frac{5(L^2+S^2)}{2S}\tau\|\bar{e}^{n+1/2}\|^2+\frac{5}{2S}(\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^5\\
\le&\frac{5(L^2+S^2)}{2S}\left(\frac{4}{3}+\frac{L^2+S^2}{S}\tau\right)\tau\|e^n\|^2\\
&+\frac{5(L^2+S^2)}{3S}(\|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^5\\
&+\frac{5}{2S}(\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^5.
\end{aligned}
$$

Summing up for n from 0 to $N-1$, and noting that $e^0=0$, we have

$$
\begin{aligned}
\|e^N\|^2\le&\tau\sum_{n=1}^{N-1}C_1\|e^n\|^2+C_2T(\|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^2_{L^\infty(0,T;H^4(\Omega))}+\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}\\
&+\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^4,
\end{aligned}
$$

where

$$
C_1=\frac{5(L^2+S^2)}{2S}\left(\frac{4}{3}+\frac{L^2+S^2}{S}\tau\right),\quad C_2=\max\left\{\frac{5(L^2+S^2)}{3S},\frac{5}{2S}\right\}.
$$

Applying the discrete Gronwall's inequality and the regularity assumption (2.3), we have

$$
\begin{aligned}
\|e^N\|^2\le&\exp(C_1T)C_2T(\|u_{tt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_t\|^2_{L^\infty(0,T;H^4(\Omega))}\\
&+\|u_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_{tt}\|^2_{L^\infty(0,T;H^4(\Omega))})\tau^4\\
\le&4\exp(C_1T)C_2TC_*^2\tau^4,
\end{aligned}
$$

which is the desired result. $\square$

Table 1
The errors and rates of convergence at $T = 1$ for the phase variable u with different time step size. The physical parameter is $\epsilon = 0.5$ and the stabilized constant is $S = 6$.

τ	L^2 error	Rate
1/8	4.8788e−02	–
1/16	1.5893e−02	1.62
1/32	4.6234e−03	1.78
1/64	1.2545e−03	1.88
1/128	3.2731e−04	1.94
1/256	8.3618e−05	1.97
1/512	2.1121e−05	1.99
1/1024	5.2940e−06	2.00

4. Numerical experiments

In this section, we give several numerical experiments for the SH equation to verify the accuracy and energy stability of the proposed scheme. We apply the Fourier pseudo spectral method for spatial discretization and fast Fourier transform (FFT) is applied for all numerical computations to solve the SH equation with the periodic boundary condition.

4.1. Accuracy and energy stability

We first test the temporal convergence rate of our scheme with the initial value condition

$$u(x,y) = \sin(\frac{\pi x}{16})\cos(\frac{\pi y}{16})$$

on the domain $\Omega = [0,32]\times[0,32]$. We apply 64^2 Fourier modes so that the spatial discretization errors are negligible compared with the temporal discretization errors. We take $\epsilon = 0.5$, $S = 6$. The errors are calculated by comparison with the reference solution with $\tau = 2^{-14}$. In Table 1, we show the L^2 errors of the phase variable with different time step size at $T = 1$ and we can observe that our scheme gives desired rate of accuracy in time. Fig. 1(a) shows that the energy decay is robust with respect to the stabilized constant S. Fig. 1(b) shows the energy evolution with different time step size τ. Fig. 2 shows the energy evolution with $S = 0$ and $S = 6$. We observe that the scheme quickly blows up if $S = 0$, which implies the stabilization terms are necessary. As a comparison, we also consider the following stabilized second-order Crank–Nicolson scheme with Adam–Bashforth extrapolation for nonlinear terms (CN/AB),

$$\frac{u^{n+1}-u^n}{\tau} + \triangle^2\frac{u^{n+1}+u^n}{2} + 2\triangle\frac{u^{n+1}+u^n}{2} + f(\tilde{u}^{n+1/2}) + S(u^{n+1}-2u^n+u^{n-1}) = 0,$$

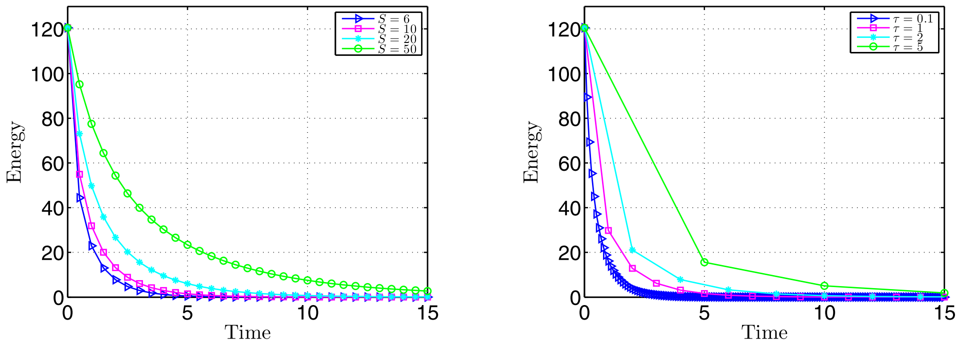
where

$$\tilde{u}^{n+1/2} = \frac{3}{2}u^n - \frac{1}{2}u^{n-1}.$$

The initialization step is

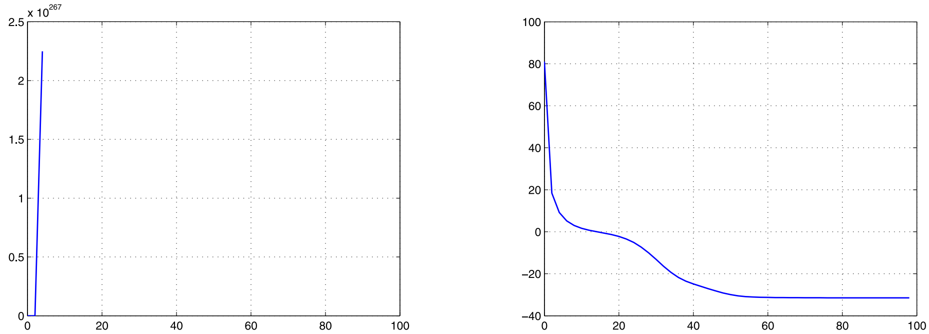
$$\frac{u^1-u^0}{\tau} + \triangle^2 u^1 + 2\triangle u^1 + f(u^0) + S(u^1-u^0) = 0.$$

Fig. 3 shows the energy evolution of the stabilized CN/AB scheme and our stabilized predictor–corrector scheme, which implies the stabilized predictor–corrector scheme is much more robust when large time step is used.



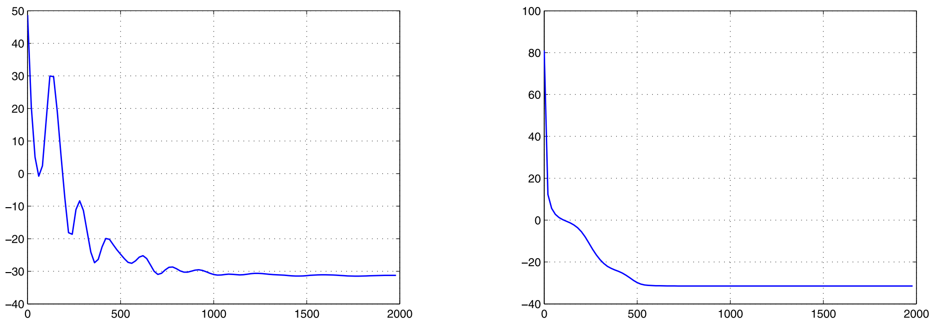
(a) (b)

Fig. 1. Evolution of the energy with different stabilized constant S and different time step size τ.



(a) S=0 (b) S=6

Fig. 2. Evolution of the energy with $S = 0$ and $S = 6$, the time step size is $\tau = 2$.

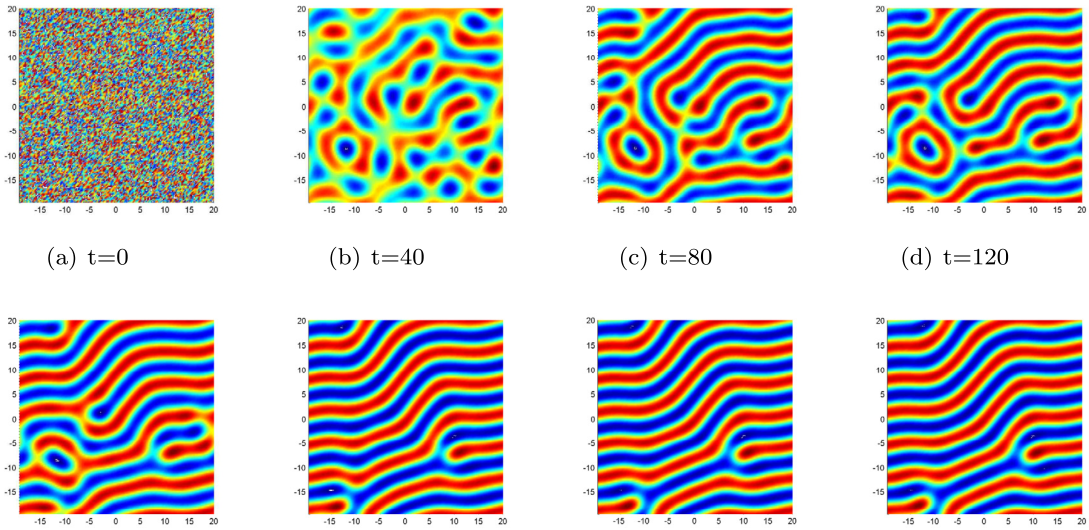


(a) (b)

Fig. 3. Evolution of the energy of stabilized CN/AB scheme (a) and the stabilized predictor–corrector (b), the time step size is $\tau = 20$.

4.2. Phase transition behaviors

We apply our scheme to check the evolution from a randomly perturbed nonequilibrium state to a steady state. With the initial condition $u(x, y) = 0.2 + \text{rand}$, where rand is random number between -0.02 and 0.02 at the grid points. We set $\epsilon = 0.5$, $S = 6$, $\tau = 1$ and $T = 400$. We use 128^2 Fourier modes to discrete the



(a) t=0 (b) t=40 (c) t=80 (d) t=120

(e) t=160 (f) t=240 (g) t=320 (h) t=400

Fig. 4. The evolution of the phase transition behavior. Snapshots of the numerical approximation of the phase variable u are taken at $t = 0, 40, 80, 120, 160, 240, 320, 400$. The computational domain is $[-20, 20] \times [-20, 20]$. The parameters are $\epsilon = 0.5$, $S = 6$, $\tau = 1$, $T = 400$. 128^2 Fourier modes are used to discrete the space.

domain $\Omega = [-20, 20] \times [-20, 20]$. Fig. 4 shows the time evolution of the phase transition behavior, which validates that our scheme does lead to the expected states.

5. Conclusions

In the work, we design a stabilized linear predictor–corrector scheme for the SH equation. We prove the scheme satisfies energy dissipation law. Rigorous results about convergence and error estimate are derived, which shows the second-order convergence in time of our proposed scheme. Numerical tests show our scheme is energy stable with large enough time step size and the energy decay is robust with the stabilized constant. Due to the generality of the theoretical and numerical approach, the results in this work can be easily applied to construct corresponding second-order schemes for other phase-field models, such as Allen–Cahn model, Cahn–Hilliard model and phase field crystal model.

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References

[1] J.B. Swift, P.C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A 15 (1) (1977) 319–328.
[2] Hector Gomez, Xesus Nogueira, A new space–time discretization for the Swift-Hohenberg equation that strictly respects the Lyapunov functional, Commun. Nonlinear Sci. Numer. Simul. 17 (12) (2012) 4930–4946.
[3] Hyun Geun Lee, A semi-analytical Fourier spectral method for the Swift-Hohenberg equation, Comput. Math. Appl. 74 (8) (2017) 1885–1896.
[4] Hyun Geun Lee, A new conservative Swift-Hohenberg equation and its mass conservative method, J. Comput. Appl. Math. 375 (2020) 112815, http://dx.doi.org/10.1016/j.cam.2020.112815.
[5] Hyun Geun Lee, A non-iterative and unconditionally energy stable method for the Swift-Hohenberg equation with quadratic-cubic nonlinearity, Appl. Math. Lett. 123 (2022) 107579, http://dx.doi.org/10.1016/j.aml.2021.107579.

[6] Zhifeng Weng, Yangfang Deng, Qingqu Zhuang, Shuying Zhai, A fast and efficient numerical algorithm for Swift-Hohenberg equation with a nonlocal nonlinearity, Appl. Math. Lett. 118 (2021) 107170, http://dx.doi.org/10.1016/j.aml.2021.107170.

[7] Longzhao Qi, Yanren Hou, A second order energy stable BDF numerical scheme for the Swift-Hohenberg equation, J. Sci. Comput. 88 (3) (2021) 1–25.

[8] Xiaofeng Yang, Linear, first and second-order, unconditionally energy stable numerical schemes for the phase field model of homopolymer blends, J. Comput. Phys. 327 (2016) 294–316, http://dx.doi.org/10.1016/j.jcp.2016.09.029.

[9] Jie Shen, Jie Xu, Jiang Yang, The scalar auxiliary variable (SAV) approach for gradient flows, J. Comput. Phys. 353 (2018) 407–416, http://dx.doi.org/10.1016/j.jcp.2017.10.021.