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Numerical analysis of two-grid decoupling finite element scheme for Navier-Stokes/Darcy model ☆

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ABSTRACT

In this paper, for a two-grid decoupling finite element scheme for the mixed Navier-Stokes/Darcy model with Beavers-Joseph-Saffman's interface condition, we establish the optimal error estimate for the approximate solution. Our analysis shows that the fine grid decoupled problems, that is the Navier-Stokes equations and the Darcy equation, can be solved simultaneously and achieve the optimal convergence order.

1. Introduction

Because of the important applications in real world applications, the mixed Stokes/Darcy and Navier-Stokes/Darcy model received much attention in both theoretical and numerical aspect in last decades.

Many numerical methods have been studied for such mixed models, including coupled finite element methods [2,6,8,22], discontinuous Galerkin methods [12,17,21], domain decomposition methods [7,9,10,14] and Lagrange multiplier methods [13,18]. Besides these numerical methods, the authors in [20] proposed a decoupling scheme for the Stokes/Darcy model with Beavers-Joesph-Saffman's interface condition (BJS) based on two-grid finite element, which can decouple the coupled model in fine mesh level and the two decoupled problems can be numerically solved in parallel. Although the numerical experiments do suggest an optimal error order of the approximation in H^1 norm in this pioneer work, the numerical analysis only gets a half order lower error order with respect to the fluid velocity and the pressure. Later on, the authors of [5] extend such decoupling scheme to the Navier-Stokes/Darcy equations. Although their numerical experiments show the scheme can reach the optimal convergence order, the error estimation of the fluid velocity and the pressure is still half order lower than expectation. In [24], we got the optimal error order of a modified two-level decoupling scheme for the Stokes/Darcy model with BJS interface condition at the cost of changing the parallel implementation in fine mesh level to a serial implementation. In [16], by using a special auxiliary elliptic problem, the author got an optimal H^1 norm error estimation for the fluid velocity in the two-grid decoupling scheme proposed in [20] for the Stokes/Darcy model with BJS interface condition.

In this paper, we will propose a two-grid parallel decoupling scheme with coarse mesh correction for the coupled Navier-Stokes/Darcy model with BJS interface condition and try to show the optimal error order of the approximation in L^2 and H^1 norm, which has been confirmed by numerous numerical experiments.

The rest of this paper is arranged as follows. In section 2, we give a brief introduction to the Navier-Stokes/Darcy model with BJS interface condition. In section 3, we present the two-grid decoupling scheme. In section 4, we try to establish the optimal error estimations for the scheme.

2. Mixed Navier-Stokes/Darcy model with BJS interface condition

Let us consider the following mixed model of the Navier-Stokes equations and the Darcy equation for coupling a fluid flow and a porous media flow in a bounded smooth domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$. Here $\Omega = \Omega_f \cup \Gamma \cup \Omega_p$, where Ω_f and Ω_p are two disjoint, connected and smooth domains occupied by fluid flow and porous media flow and

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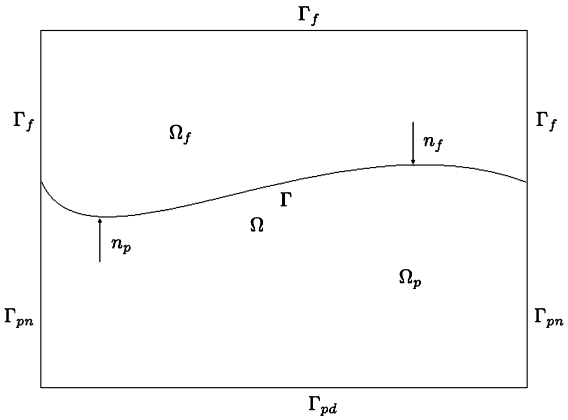

Fig. 1. A global domain Ω consisting of a fluid flow region Ω_f and a porous media flow region Ω_p separated by an interface Γ.

$\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$ is the interface. We denote $\Gamma_f = \partial\Omega_f \backslash \Gamma$, $\Gamma_p = \partial\Omega_p \backslash \Gamma$ and we also denote by $\boldsymbol{n}_p$ and $\boldsymbol{n}_f$ the unit outward normal vectors on $\partial\Omega_p$ and $\partial\Omega_f$, respectively. Furthermore, Γ_p consists of two disjoint parts Γ_{pd} and Γ_{pn}. We assume $|\Gamma_f|$, $|\Gamma_{pd}| > 0$. See Fig. 1 for a sketch.

Let us denote by $[\boldsymbol{u}_f, p_f]$ the velocity field and the pressure of the fluid flow in Ω_f and ϕ_p the piezometric head in Ω_p. The partial differential equations modeling the fluid flow and the porous media flow are

$$\begin{cases} -\nabla \cdot (\mathbb{T}_\nu(\boldsymbol{u}_f, p_f)) + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f = \boldsymbol{g}_f, & \text{in } \Omega_f, \\ \nabla \cdot \boldsymbol{u}_f = 0, & \text{in } \Omega_f, \\ -\nabla \cdot \mathbb{K}\nabla\phi_p = g_p, & \text{in } \Omega_p, \end{cases} \tag{2.1}$$

where

$$\mathbb{T}_\nu(\boldsymbol{u}_f, p_f) = -p_f \mathbb{I} + 2\nu\mathbb{D}(\boldsymbol{u}_f), \quad \mathbb{D}(\boldsymbol{u}_f) = \frac{1}{2}(\nabla \boldsymbol{u}_f + \nabla^T \boldsymbol{u}_f),$$

are the stress tensor and the deformation rate tensor, $\nu > 0$ is the kinetic viscosity and $\mathbb{K}$ is the permeability in Ω_p, which is a positive definite symmetric tensor that is allowed to vary in space. The third equation of (2.1) that describes the porous media flow motion is the Darcy's law for the piezometric head ϕ_p. For more details of these equations, we refer readers to [12] and [20]. In the rest of this paper, we always use boldface characters to denote vector valued functions or spaces of vector valued functions.

The above equations (2.1) are completed and coupled together by the following boundary conditions:

$$\boldsymbol{u}_f = 0 \quad \text{on } \Gamma_f, \quad \mathbb{K}\nabla\phi_p \cdot \boldsymbol{n}_p = 0 \quad \text{on } \Gamma_{pn}, \quad \phi_p = 0 \quad \text{on } \Gamma_{pd}, \tag{2.2}$$

and the interface conditions on Γ:

$$\begin{cases} \boldsymbol{u}_f \cdot \boldsymbol{n}_f - \mathbb{K}\nabla\phi_p \cdot \boldsymbol{n}_p = 0, \\ -[\mathbb{T}_\nu(\boldsymbol{u}_f, p_f) \cdot \boldsymbol{n}_f] \cdot \boldsymbol{n}_f = \phi_p, \\ -[\mathbb{T}_\nu(\boldsymbol{u}_f, p_f) \cdot \boldsymbol{n}_f] \cdot \boldsymbol{\tau}_i = G_i \boldsymbol{u}_f \cdot \boldsymbol{\tau}_i, \quad i = 1, \cdots, d-1. \end{cases} \tag{2.3}$$

Here $G_i > 0$, $i = 1, \cdots, d-1$, are constants depending on the nature of the porous medium and determined from experimental data, $\boldsymbol{\tau}_i$, $i = 1, \cdots, d-1$, are the orthonormal tangential unit vectors along Γ. The first condition is the mass conservation, the second one is the balance of normal force and the third one means the tangential components of the normal stress force is proportional to the tangential components of the fluid velocity, which is called the Beavers-Joseph-Saffman's (BJS) interface condition (see [3] and [23]).

Let us introduce the following Hilbert spaces

$$\boldsymbol{X}_f = \{\boldsymbol{v}_f \in \boldsymbol{H}^1(\Omega_f) : \boldsymbol{v}_f|_{\Gamma_f} = 0\}, \quad Q_f = L^2(\Omega_f),$$
$$X_p = \{\psi_p \in H^1(\Omega_p) : \psi_p|_{\Gamma_{pd}} = 0\},$$

where $[\boldsymbol{X}_f, Q_f]$ is the space pair for the velocity and the pressure in the fluid flow region Ω_f and X_p is the space for the piezometric head in the porous medium region Ω_p. Furthermore, we assume

$$\boldsymbol{g}_f \in \boldsymbol{X}'_f, \quad g_p \in X'_p. \tag{2.4}$$

Here $\boldsymbol{X}'_f$ and X'_p are the dual spaces of $\boldsymbol{X}_f$ and X_p, respectively.

For simplicity, we always use $(\cdot,\cdot)_D$ and $\|\cdot\|_D$ to denote the L^2 inner product and the corresponding norm on any given domain D. Since $|\Gamma_f|$, $|\Gamma_{pd}| > 0$, we know that $\|\mathbb{D}(\cdot)\|_{\Omega_f}$ and $\|\mathbb{K}^{\frac{1}{2}}\nabla\cdot\|_{\Omega_p}$ are equivalent norms of the usual Sobolev norms in $\boldsymbol{X}_f$ and X_p due to the Korn's and the Poincaré inequalities.

For any $[\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$, let us introduce:

$$B([\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p]) = a([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) + b_f(\boldsymbol{u}_f, \boldsymbol{u}_f, \boldsymbol{v}_f)$$
$$+ d_f(p_f, \boldsymbol{v}_f) - d_f(q_f, \boldsymbol{u}_f) + a_\Gamma([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]),$$

where

$$a([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_p(\phi_p, \psi_p),$$
$$= [2\nu(\mathbb{D}(\boldsymbol{u}_f), \mathbb{D}(\boldsymbol{v}_f))_{\Omega_f} + \sum_{i=1}^{d-1} G_i(\boldsymbol{u}_f \cdot \boldsymbol{\tau}_i, \boldsymbol{v}_f \cdot \boldsymbol{\tau}_i)_\Gamma] + (\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p},$$
$$b_f(\boldsymbol{u}_f, \boldsymbol{w}_f, \boldsymbol{v}_f) = ((\boldsymbol{u}_f \cdot \nabla)\boldsymbol{w}_f, \boldsymbol{v}_f)_{\Omega_f},$$
$$d_f(p_f, \boldsymbol{v}_f) = (p_f, \nabla \cdot \boldsymbol{v}_f)_{\Omega_f}, \quad \tilde{a}_\Gamma(\phi_p, \boldsymbol{v}_f) = (\phi_p, \boldsymbol{v}_f \cdot \boldsymbol{n}_f)_\Gamma,$$
$$a_\Gamma([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = \tilde{a}_\Gamma(\phi_p, \boldsymbol{v}_f) - \tilde{a}_\Gamma(\psi_p, \boldsymbol{u}_f).$$

Now the weak formulation of the mixed Navier-Stokes/Darcy model with BJS interface condition reads as follows (see [5], [12], [18] and [20] for details): for $\boldsymbol{g}_f \in \boldsymbol{X}'_f$, $g_p \in X'_p$, find $[\boldsymbol{u}_f, p_f, \phi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$ such that $\forall [\boldsymbol{v}_f, q_f, \psi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$

$$(Q) \qquad B([\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p]) = (\boldsymbol{g}_f, \boldsymbol{v}_f)_{\Omega_f} + (g_p, \psi_p)_{\Omega_p}.$$

Thanks to [12], we know that there exists a positive constant $\beta > 0$ such that the following Ladyzhenskaya-Babuška-Brezzi (LBB) condition holds:

$$\inf_{q_f \in Q_f} \sup_{\boldsymbol{v}_f \in \boldsymbol{X}_f} \frac{d_f(q_f, \boldsymbol{v}_f)}{\|q_f\|_{\Omega_f} \|\mathbb{D}(\boldsymbol{v}_f)\|_{\Omega_f}} \geq \beta. \tag{2.5}$$

For the purpose of later analysis, for any bounded domain $D \in \mathbf{R}^d$, we recall some inequalities and identity:

$$\|v\|_{L^2(\partial D)} \leq c\|v\|^{\frac{1}{2}}_{L^2(D)} \|v\|^{\frac{1}{2}}_{H^1(D)} \leq c\|v\|_{H^1(D)} \quad \forall v \in H^1(D), \tag{2.6}$$

$$\|v\|_{L^4(\partial D)} \leq c\|v\|_{H^1(D)} \quad \forall v \in H^1(D), \tag{2.7}$$

$$|((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v})_D| \leq c\|\boldsymbol{w}\|^{\frac{1}{2}}_{L^2(D)} \|\boldsymbol{w}\|^{\frac{1}{2}}_{H^1(D)} \|\boldsymbol{u}\|_{H^1(D)} \|\boldsymbol{v}\|_{H^1(D)}, \quad \forall \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(D). \tag{2.8}$$

Here c is a positive constant only depending on D. Indeed, for the inequalities (2.6) and (2.7), we refer readers to Theorem 1.6.6 in [4] and Theorem 5.22 in [1], respectively.

3. A two-grid decoupling scheme

In the rest, we assume Ω_f and Ω_p are smooth domains and we denote

$$Y = \boldsymbol{X}_f \times Q_f \times X_p, \quad W = \boldsymbol{X}_f \times Q_f.$$

And for any given positive constant $\mu > 0$, we denote by

$$Y_\mu = \boldsymbol{X}_{f\mu} \times Q_{f\mu} \times X_{p\mu}, \quad W_\mu = \boldsymbol{X}_{f\mu} \times Q_{f\mu},$$

the corresponding finite element spaces. In what follows, we assume that MINI element and piecewise linear continuous element are applied in the fluid and the porous media regions, respectively. We know that such finite element space pair $[\boldsymbol{X}_{f\mu}, Q_{f\mu}]$ satisfies the following discrete LBB condition: there exists a positive constant independent of μ, which we still denote as $\beta > 0$, such that

$$\inf_{q_{f\mu}\in Q_{f\mu}} \sup_{\boldsymbol{v}_{f\mu}\in \boldsymbol{X}_{f\mu}} \frac{d_f(q_{f\mu}, \boldsymbol{v}_{f\mu})}{\|q_{f\mu}\|_Q \|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f}} \geq \beta. \tag{3.1}$$

In addition, we assume the following local regularity

$$u_f \in \boldsymbol{H}^2(\Omega_f), \quad \phi_p \in H^2(\Omega_p), \quad p_f \in H^1(\Omega_f). \tag{3.2}$$

In the rest, we use $c > 0$ to denote a mesh sizes independent constant, which may depend on certain combination of the norms of $\boldsymbol{u}_f$, p_f and ϕ_p.

First of all, we present the classical finite element discretization of the coupled model (Q): find $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$ such that $\forall [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}] \in Y_\mu$

$$(Q_\mu) \qquad B([\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]) = (\boldsymbol{g}_f, \boldsymbol{v}_{f\mu})_{\Omega_f} + (g_p, \psi_{p\mu})_{\Omega_p}.$$

See [11], for a nonsingular solution $[\boldsymbol{u}_f, p_f, \phi_p] \in Y$ of the problem (Q), there is a nonsingular solution $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$ of (Q_μ) near $[\boldsymbol{u}_f, p_f, \phi_p]$ when μ is smaller than some prescribed $\mu_0 > 0$. Based on this observation, the authors of [5] get the following error estimates of the coupled scheme (Q_μ):

$$\begin{cases} \|\mathbb{D}(\boldsymbol{u}_f - \boldsymbol{u}_{f\mu})\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p - \phi_{p\mu})\|_{\Omega_p} + \|p_f - p_{f\mu}\|_{\Omega_f} \leq c\mu, \\ \|\boldsymbol{u}_f - \boldsymbol{u}_{f\mu}\|_{\Omega_f} + \|\phi_p - \phi_{p\mu}\|_{\Omega_p} \leq c\mu^2. \end{cases} \tag{3.3}$$

For the above mentioned $\mu_0 > 0$, $\mu < \mu_0$ and the nonsingular solution $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$, we denote for any $[\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}] \in Y_\mu$

$$\begin{aligned} \mathcal{L}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]) &= a_f(\boldsymbol{w}_{f\mu}, \boldsymbol{v}_{f\mu}) + b_f(\boldsymbol{u}_{f\mu}, \boldsymbol{w}_{f\mu}, \boldsymbol{v}_{f\mu}) \\ &+ b_f(\boldsymbol{w}_{f\mu}, \boldsymbol{u}_{f\mu}, \boldsymbol{v}_{f\mu}) + d_f(r_{f\mu}, \boldsymbol{v}_{f\mu}) - d_f(q_{f\mu}, \boldsymbol{w}_{f\mu}) + a_p(\sigma_{p\mu}, \psi_{p\mu}) \\ &+ a_\Gamma([\boldsymbol{w}_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, \psi_{p\mu}]), \end{aligned}$$

the bilinear form associated with the Frechet derivative of the Navier-Stokes/Darcy operator corresponding to (Q_μ) at the nonsingular point $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$ in Y_μ. We know from [11], the Fredholm alternative and the similar procedure in [19] that there exists a μ independent constant $\gamma > 0$ such that

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]\in Y_\mu} \frac{\mathcal{L}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{p\mu}\|_{\Omega_p}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\sigma_{p\mu}\|_{\Omega_p}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]\in Y_\mu} \frac{\mathcal{L}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}], [\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{p\mu}\|_{\Omega_p}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\sigma_{p\mu}\|_{\Omega_p}). \end{aligned} \tag{3.5}$$

Furthermore, for any $\tilde{\boldsymbol{u}}_f \in \boldsymbol{X}_f$, we have

$$\begin{aligned} &\mathcal{L}_{\tilde{\boldsymbol{u}}_f}([\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]) \\ &\quad = \mathcal{L}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]) \\ &\qquad + b_f(\tilde{\boldsymbol{u}}_f - \boldsymbol{u}_{f\mu}, \boldsymbol{w}_{f\mu}, \boldsymbol{v}_{f\mu}) + b_f(\boldsymbol{w}_{f\mu}, \tilde{\boldsymbol{u}}_f - \boldsymbol{u}_{f\mu}, \boldsymbol{v}_{f\mu}). \end{aligned} \tag{3.6}$$

If $\tilde{\boldsymbol{u}}_f \in \boldsymbol{X}_f$ is closed to $\boldsymbol{u}_{f\mu}$ such that $\|\mathbb{D}(\boldsymbol{u}_{f\mu} - \tilde{\boldsymbol{u}}_f)\|_{\Omega_f}$ is small enough, we can get that there exists a positive constant independent of μ, which we still denote as $\gamma > 0$, such that

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]\in Y_\mu} \frac{\mathcal{L}_{\tilde{\boldsymbol{u}}_f}([\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{p\mu}\|_{\Omega_p}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\sigma_{p\mu}\|_{\Omega_p}), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}]\in Y_\mu} \frac{\mathcal{L}_{\tilde{\boldsymbol{u}}_f}([\boldsymbol{v}_{f\mu}, q_{f\mu}, \psi_{p\mu}], [\boldsymbol{w}_{f\mu}, r_{f\mu}, \sigma_{p\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{p\mu}\|_{\Omega_p}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\sigma_{p\mu}\|_{\Omega_p}). \end{aligned} \tag{3.8}$$

We refer readers to Lemma 3.3 and Theorem 3.1 in Chapter IV of [11] for detail proof of (3.7) and (3.8) as long as $\|\mathbb{D}(\boldsymbol{u}_{f\mu} - \tilde{\boldsymbol{u}}_f)\|_{\Omega_f}$ is small.

From now on, we always assume $[\boldsymbol{u}_f, p_f, \phi_p] \in Y$ is a nonsingular solution to the problem (Q) and $\mu < \mu_0$ such that $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$ is a nonsingular solution to (Q_μ) near $[\boldsymbol{u}_f, p_f, \phi_p]$ satisfying the estimation (3.3).

In the rest, we also denote

$$\mathcal{L}^{NS}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}]) = \mathcal{L}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}, 0], [\boldsymbol{v}_{f\mu}, q_{f\mu}, 0]).$$

Since $[\boldsymbol{u}_{f\mu}, p_{f\mu}, \phi_{p\mu}] \in Y_\mu$ is a nonsingular solution to (Q_μ), $[\boldsymbol{u}_{f\mu}, p_{f\mu}] \in \boldsymbol{X}_{f\mu} \times Q_{f\mu}$ is a nonsingular solution to the Navier-Stokes equations in (Q_μ) with the given piezometric head $\phi_{p\mu} \in X_{p\mu}$. Thus the bilinear form $\mathcal{L}^{NS}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}])$ also satisfies the following *inf-sup* conditions like (3.4)-(3.5):

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}]\in W_\mu} \frac{\mathcal{L}^{NS}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{w}_{f\mu}, r_{f\mu}], [\boldsymbol{v}_{f\mu}, q_{f\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f}), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{f\mu}, q_{f\mu}]\in W_\mu} \frac{\mathcal{L}^{NS}_{\boldsymbol{u}_{f\mu}}([\boldsymbol{v}_{f\mu}, q_{f\mu}], [\boldsymbol{w}_{f\mu}, r_{f\mu}])}{\|\mathbb{D}(\boldsymbol{v}_{f\mu})\|_{\Omega_f} + \|q_{f\mu}\|_{\Omega_f}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{f\mu})\|_{\Omega_f} + \|r_{f\mu}\|_{\Omega_f}). \end{aligned} \tag{3.10}$$

Now we state the following two-grid decoupling algorithm.

Two-grid decoupling algorithm

1. Solve (Q_H) with a coarse mesh size $H < \mu_0$ to get $[\boldsymbol{u}_{fH}, p_{fH}, \phi_{pH}] \in Y_H$.
2. For $h < H < \mu_0$, find $[\boldsymbol{u}^{fh}, p^{fh}, \phi^{ph}] \in Y_h$ such that $\forall [\boldsymbol{v}_{fh}, q_{fh}, \psi_{ph}] \in Y_h$

$$\begin{aligned} &\mathcal{L}^{NS}_{\boldsymbol{u}_{fH}}([\boldsymbol{u}^{fh}, p^{fh}], [\boldsymbol{v}_{fh}, q_{fh}]) + a_p(\phi^{ph}, \psi_{ph}) \\ &\quad = b_f(\boldsymbol{u}_{fH}, \boldsymbol{u}_{fH}, \boldsymbol{v}_{fh}) - a_\Gamma([\boldsymbol{u}_{fH}, \phi_{pH}], [\boldsymbol{v}_{fh}, \psi_{ph}]) \\ &\quad + (\boldsymbol{g}_f, \boldsymbol{v}_{fh})_{\Omega_f} + (g_p, \psi_{ph})_{\Omega_p}. \end{aligned} \tag{3.11}$$

3. Find $[\hat{\boldsymbol{u}}^{fH}, \hat{p}^{fH}, \hat{\phi}^{pH}] \in Y_H$ such that $\forall [\boldsymbol{v}_{fH}, q_{fH}, \psi_{pH}] \in Y_H$

$$\begin{aligned} &\mathcal{L}_{\boldsymbol{u}^{fh}}([\hat{\boldsymbol{u}}^{fH}, \hat{p}^{fH}, \hat{\phi}^{pH}], [\boldsymbol{v}_{fH}, q_{fH}, \psi_{pH}] \\ &\quad = (\boldsymbol{g}_f, \boldsymbol{v}_{fH})_{\Omega_f} + (g_p, \psi_{pH})_{\Omega_p} - a_f(\boldsymbol{u}^{fh}, \boldsymbol{v}_{fH}) \\ &\quad - b_f(\boldsymbol{u}^{fh}, \boldsymbol{u}^{fh}, \boldsymbol{v}_{fH}) - d_f(p^{fh}, \boldsymbol{v}_{fH}) + d_f(q_{fH}, \boldsymbol{u}^{fh}) \\ &\quad - a_p(\phi^{fh}, \psi_{pH}) - a_\Gamma([\boldsymbol{u}^{fh}, \phi^{ph}], [\boldsymbol{v}_{fH}, \psi_{pH}]). \end{aligned} \tag{3.12}$$

   Update $[\boldsymbol{u}^{fh}, p^{fh}, \phi^{ph}]$ as

$$[\boldsymbol{u}^{fh}_H, p^{fh}_H, \phi^{ph}_H] = [\boldsymbol{u}^{fh}, p^{fh}, \phi^{ph}] + [\hat{\boldsymbol{u}}^{fH}, \hat{p}^{fH}, \hat{\phi}^{pH}].$$

Remark 1. Thanks to (3.9)-(3.10) and the identity (3.6), we can show that $\forall [\boldsymbol{w}_{fh}, r_{fh}] \in W_h$

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{fh}, q_{fh}]\in W_h} \frac{\mathcal{L}^{NS}_{\boldsymbol{u}_{fH}}([\boldsymbol{w}_{fh}, r_{fh}], [\boldsymbol{v}_{fh}, q_{fh}])}{\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f} + \|q_{fh}\|_{\Omega_f}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{fh})\|_{\Omega_f} + \|r_{fh}\|_{\Omega_f}), \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\sup_{[\boldsymbol{v}_{fh}, q_{fh}]\in W_h} \frac{\mathcal{L}^{NS}_{\boldsymbol{u}_{fH}}([\boldsymbol{v}_{fh}, q_{fh}], [\boldsymbol{w}_{fh}, r_{fh}])}{\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f} + \|q_{fh}\|_{\Omega_f}} \\ &\qquad \geq \gamma(\|\mathbb{D}(\boldsymbol{w}_{fh})\|_{\Omega_f} + \|r_{fh}\|_{\Omega_f}), \end{aligned} \tag{3.14}$$

by the same procedure in [19]. Now, in the second step, once $[\boldsymbol{u}_{fH}, p_{fH}, \phi_{pH}] \in Y_H$ is obtained in the first step, (3.11) is actually equivalent to the following two decoupled equations: $\forall [\boldsymbol{v}_{fh}, q_{fh}] \in W_h, \psi_{ph} \in X_{ph}$

$$\mathcal{L}^{NS}_{\boldsymbol{u}_{fH}}([\boldsymbol{u}^{fh},p^{fh}],[\boldsymbol{v}_{fh},q_{fh}])$$
$$=(\boldsymbol{g}_f,\boldsymbol{v}_{fh})_{\Omega_f}+b_f(\boldsymbol{u}_{fH},\boldsymbol{u}_{fH},\boldsymbol{v}_{fh})-\tilde{a}_\Gamma(\phi_{pH},\boldsymbol{v}_{fh}),$$

and

$$a_p(\phi^{ph},\psi_{ph})=(g_p,\psi_{pH})_{\Omega_p}+\tilde{a}_\Gamma(\psi_{ph},\boldsymbol{u}_{fH}).$$

These two equations are uniquely solvable because of the *inf-sup* conditions stated above and $X_{ph}-$ coercive of $a_p(\cdot,\cdot)$. In addition, if $\|\mathbb{D}(\boldsymbol{u}^{fh}-\boldsymbol{u}_{fH})\|_{\Omega_f}$ is small enough, the (3.7) is valid if we replace $\tilde{\boldsymbol{u}}_f$ with $\boldsymbol{u}^{fh}$. Then (3.12) is uniquely solvable. Therefore, for sufficiently small $h<H<\mu_0$, the above two-grid algorithm admits a unique solution $[\boldsymbol{u}^{fh}_H,p^{fh}_H,\phi^{ph}_H]\in Y_h$.

Remark 2. By the same procedure in [5], if we denote $[\boldsymbol{u}_{fh},p_{fh},\phi_{ph}]\in Y_h$ as the nonsingular solution of (Q_h) near the nonsingular solution $[\boldsymbol{u}_f,p_f,\phi_p]\in Y$, one can obtained the following error estimate of ϕ^{ph}

$$\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_{ph}-\phi^{ph})\|_{\Omega_p}\le cH^2. \tag{3.15}$$

Later on, we call it an optimal error estimation if we configure H and h as $h\sim H^2$ in the choice of the coarse mesh and fine mesh finite element subspaces in this paper.

4. Error estimations

Although numerical results in literatures do suggest an optimal error order for $[\boldsymbol{u}^{fh},p^{fh}]\in \boldsymbol{X}_{fh}\times Q_{fh}$, by the same method in [5], we can only get the error estimation of it that is half order lower than the optimal order like following

$$\|\mathbb{D}(\boldsymbol{u}_{fh}-\boldsymbol{u}^{fh})\|_{\Omega_f}+\|p_{fh}-p^{fh}\|_{\Omega_f}\le cH^{\frac{3}{2}}. \tag{4.1}$$

To get the error estimations of the final approximation $[\boldsymbol{u}^{fh}_H,p^{fh}_H,\phi^{ph}_H]$, we first re-consider the error estimations of $[\boldsymbol{u}^{fh},p^{fh},\phi^{ph}]$, especially the error estimation of $[\boldsymbol{u}^{fh},p^{fh}]$. In the follows, we will first give some more rigorous estimates on the fine grid approximation $[\boldsymbol{u}^{fh},p^{fh},\phi^{ph}]\in Y_h$, actually a rigorous estimation of $\|\mathbb{D}(u_{fh}-u^{fh})\|_{\Omega_f}$ and $\|p_{fh}-p^{fh}\|_{\Omega_f}$.

Theorem 4.1. *Assume Ω_f and Ω_p are smooth domains such that the mixed Navier-Stokes/Darcy problem satisfies the local regularity assumption (3.2) and $h<H<\mu_0$ is small enough such that (3.4) holds. Let $[\boldsymbol{u}_{fh},p_{fh},\phi_{ph}]\in Y_h$ and $[\boldsymbol{u}^{fh},p^{fh},\phi^{ph}]\in Y_h$ be defined by the coupled Galerkin scheme (Q_h) and the two-grid decoupling algorithm. The following error estimate holds:*

$$\|\mathbb{D}(\boldsymbol{u}_{fh}-\boldsymbol{u}^{fh})\|_{\Omega_f}+\|p_{fh}-p^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_{ph}-\phi^{ph})\|_{\Omega_p}\le cH^2. \tag{4.2}$$

Proof. In what follows, we will only try to establish the error estimate of the velocity and the pressure in (4.2).

First of all, let us denote

$$\boldsymbol{e}_{\boldsymbol{u}_f}=\boldsymbol{u}_{fh}-\boldsymbol{u}^{fh},\quad e_{p_f}=p_{fh}-p^{fh},\quad e_{\phi_p}=\phi_{ph}-\phi^{ph}.$$

The subtraction of (3.11) from (Q_h) with $\psi_{ph}=0$ admits

$$\begin{aligned}\mathcal{L}^{NS}_{\boldsymbol{u}_{fH}}([\boldsymbol{e}_{\boldsymbol{u}_f},e_{p_f}],[\boldsymbol{v}_{fh},q_{fh}])&=(\phi_{ph}-\phi_{pH},\boldsymbol{v}_{fh}\cdot\boldsymbol{n}_f)_\Gamma\\&+b_f(\boldsymbol{u}_{fh}-\boldsymbol{u}_{fH},\boldsymbol{u}_{fh}-\boldsymbol{u}_{fH},\boldsymbol{v}_{fh}).\end{aligned} \tag{4.3}$$

Thanks to (2.8) and (4.1), we have

$$\begin{aligned}&b_f(\boldsymbol{u}_{fh}-\boldsymbol{u}_{fH},\boldsymbol{u}_{fh}-\boldsymbol{u}_{fH},\boldsymbol{v}_{fh})\\&\le c\|\mathbb{D}(\boldsymbol{u}_{fh}-\boldsymbol{u}_{fH})\|^2_{\Omega_f}\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}\le cH^2\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}.\end{aligned} \tag{4.4}$$

To estimate the first term on the right hand side of (4.3), we introduce an elliptic extension of $\boldsymbol{v}_{fh}\cdot\boldsymbol{n}_f$ into the porous media region. That is to find $\Phi_p\in H^1(\Omega_p)$ satisfying

$$(\textbf{Auxiliary Problem})\qquad \begin{cases}-\nabla\cdot(\mathbb{K}\nabla\Phi_p)=0 & \text{in }\ \Omega_p,\\ \mathbb{K}\nabla\Phi_p\cdot\boldsymbol{n}_p=\boldsymbol{v}_{fh}\cdot\boldsymbol{n}_p & \text{on }\ \Gamma,\\ \mathbb{K}\nabla\Phi_p\cdot\boldsymbol{n}_p=0 & \text{on }\ \Gamma_{pn},\\ \Phi_p=0 & \text{on }\ \Gamma_{pd}.\end{cases}$$

It is classical that

$$\|\mathbb{K}^{\frac{1}{2}}\nabla\Phi_p\|_{\Omega_p}\le c\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}. \tag{4.5}$$

Furthermore, if the interface Γ is smooth, for example Γ is smooth such that $\boldsymbol{n}_p$ is continuous along it, we have for $\boldsymbol{v}_{fh}\in\boldsymbol{X}_f$,

$$\|\mathbb{K}^{\frac{1}{2}}\nabla\Phi_p\|_{H^1(\Omega_p)}\le c\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}. \tag{4.6}$$

Now for the first term on the right hand side of (4.3), we have

$$\begin{aligned}|(\phi_{ph}-\phi_{pH},\boldsymbol{v}_{fh}\cdot\boldsymbol{n}_f)_\Gamma|&=|-\int_\Gamma(\phi_{ph}-\phi_{pH})\mathbb{K}\nabla\Phi_p\cdot\boldsymbol{n}_p|\\&=|-\int_{\partial\Omega_p}(\phi_{ph}-\phi_{pH})\mathbb{K}\nabla\Phi_p\cdot\boldsymbol{n}_p|=|-\int_{\Omega_p}\nabla\cdot((\phi_{ph}-\phi_{pH})\mathbb{K}\nabla\Phi_p)|\\&=|-\int_{\Omega_p}(\phi_{ph}-\phi_{pH})\nabla\cdot(\mathbb{K}\nabla\Phi_p)-\int_{\Omega_p}\mathbb{K}\nabla(\phi_{ph}-\phi_{pH})\cdot\nabla\Phi_p|\\&=|-\int_{\Omega_p}\mathbb{K}\nabla(\phi^{ph}-\phi_{pH})\cdot\nabla\Phi_p+\int_{\Omega_p}\mathbb{K}\nabla e_{\phi_p}\cdot\nabla\Phi_p|\\&\le|\int_{\Omega_p}\mathbb{K}\nabla(\phi^{ph}-\phi_{pH})\cdot\nabla\Phi_p|+\|\mathbb{K}^{\frac{1}{2}}\nabla e_{\phi_p}\|_{\Omega_p}\|\mathbb{K}^{\frac{1}{2}}\nabla\Phi_p\|_{\Omega_p}.\end{aligned}$$

From the definition of ϕ_{pH} in (Q_H) and ϕ^{ph} in (3.11), we can easily verify that

$$\int_{\Omega_p}\mathbb{K}\nabla(\phi^{ph}-\phi_{pH})\cdot\nabla\Phi_{pH}=0\quad\forall\Phi_{pH}\in X_{pH},$$

which means the projection associated with $a_p(\cdot,\cdot)$ of ϕ^{ph} onto X_{pH} is the coarse mesh approximation ϕ_{pH}.

Noticing (4.5), (4.6), (3.15) and (3.3), we have $\forall\Phi_{pH}\in X_{pH}$

$$\begin{aligned}&|\int_{\Omega_p}\mathbb{K}\nabla(\phi^{ph}-\phi_{pH})\cdot\nabla\Phi_p|\\&\le\inf_{\Phi_{pH}\in X_{pH}}|\int_{\Omega_p}\mathbb{K}\nabla(\phi^{ph}-\phi_{pH})\cdot\nabla(\Phi_p-\Phi_{pH})|\\&\le\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi^{ph}-\phi_{pH})\|_{\Omega_p}\inf_{\Phi_{pH}\in X_{pH}}\|\mathbb{K}^{\frac{1}{2}}\nabla(\Phi_p-\Phi_{pH})\|_{\Omega_p}\\&\le cH(\|\mathbb{K}^{\frac{1}{2}}\nabla e_{\phi_p}\|_{\Omega_p}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_{ph}-\phi_{pH})\|_{\Omega_p})\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}\\&\le cH^2\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}.\end{aligned}$$

By using (4.5) and (3.3) and the above estimates, we get

$$|(\phi_{ph}-\phi_{pH},\boldsymbol{v}_{fh}\cdot\boldsymbol{n}_f)_\Gamma|\le cH^2\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}. \tag{4.7}$$

Now, combination of (4.4) and (4.7) with (4.3) and the usage of (3.13) with $[\boldsymbol{w}_{fh},r_{fh}]=[\boldsymbol{e}_{\boldsymbol{u}_f},e_{p_f}]$ lead to

$$\|\mathbb{D}(\boldsymbol{u}_{fh}-\boldsymbol{u}^{fh})\|_{\Omega_f}+\|p_{fh}-p^{fh}\|_{\Omega_f}\le cH^2.$$

This concludes the proof of this theorem. □

Now let us turn to the error estimate of the final approximation $[\boldsymbol{u}^{fh}_H,p^{fh}_H,\phi^{ph}_H]$.

Thanks to (3.3) and the results in Theorem 4.1, we can check that

$$\|\mathbb{D}(\boldsymbol{u}^{fh}-\boldsymbol{u}_{fH})\|_{\Omega_f}\le c(H+h).$$

If $H>0$ is small enough, from (3.6), we have

$$\sup_{[\boldsymbol{v}_{fH},q_{fH},\psi_{pH}]\in Y_H}\frac{\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{v}_{fH},q_{fH},\psi_{pH}],[\boldsymbol{w}_{fH},r_{fH},\sigma_{pH}])}{\|\mathbb{D}(\boldsymbol{v}_{fH})\|_{\Omega_f}+\|q_{fH}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{pH}\|_{\Omega_p}}$$
$$\geq\gamma(\|\mathbb{D}(\boldsymbol{w}_{fH})\|_{\Omega_f}+\|r_{fH}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla\sigma_{pH}\|_{\Omega_p}).$$

Due to the above inequality, we can introduce the following projection from Y_h onto Y_H: for given $\chi_h=[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}]\in Y_h$, find

$$P_H\chi_h=[P_H^{\boldsymbol{u}}(\chi_h)\boldsymbol{v}_{fh},P_H^p(\chi_h)q_{fh},P_H^{\phi}(\chi_h)\psi_{ph}]\in Y_H,$$

which we simply denote as $P_H\chi_h=[P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},P_H^pq_{fh},P_H^{\phi}\psi_{ph}]\in Y_H$ and $Q_H\chi_h=(I_h-P_H)\chi_h=[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{fh}]$, such that $\forall[\boldsymbol{w}_{fH},r_{fH},\sigma_{pH}]\in Y_H$

$$\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{w}_{fH},r_{fH},\sigma_{pH}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}])=0. \tag{4.8}$$

Then we can rewrite (3.12) as

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\hat{\boldsymbol{u}}^{fH},\hat{p}^{fH},\hat{\phi}^{pH}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=(\boldsymbol{g}_f,P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,P_H^{\phi}\psi_{ph})_{\Omega_p}-a_f(\boldsymbol{u}^{fh},P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})\\&\quad-b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})-d_f(p^{fh},P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})+d_f(P_H^pq_{fh},\boldsymbol{u}^{fh})\\&\quad-a_p(\phi^{ph},P_H^{\phi}\psi_{ph})-a_\Gamma([\boldsymbol{u}^{fh},\phi^{ph}],[P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},P_H^{\phi}\psi_{ph}]).\end{aligned}$$

Notice that

$$\begin{aligned}&a_f(\boldsymbol{u}^{fh},\boldsymbol{v}_{fh})+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},\boldsymbol{v}_{fh})+d_f(p^{fh},\boldsymbol{v}_{fh})-d_f(q_{fh},\boldsymbol{u}^{fh})\\&\quad+a_p(\phi^{ph},\psi_{ph})+a_\Gamma([\boldsymbol{u}^{fh},\phi^{ph}],[\boldsymbol{v}_{fh},\psi_{ph}])\\&=\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}^{fh},p^{ph},\phi^{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])-b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},\boldsymbol{v}_{fh}),\end{aligned}$$

we have the final approximation $[\boldsymbol{u}_H^{fh},p_H^{fh},\phi_H^{ph}]\in Y_h$ satisfies

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}_H^{fh},p_H^{fh},\phi_H^{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=(\boldsymbol{g}_f,P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,P_H^{\phi}\psi_{ph})_{\Omega_p}+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},P_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})\\&\quad+\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}^{fh},p^{fh},\phi^{ph}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}]).\end{aligned}\tag{4.9}$$

Now let us consider the following linearized approximation of the problem (Q_h): find $[\bar{u}_{fh},\bar{p}_{fh},\bar{\phi}_{ph}]\in Y_h$ such that $\forall[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}]\in Y_h$

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh},\bar{p}_{fh},\bar{\phi}_{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=(\boldsymbol{g}_f,\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,\psi_{ph})_{\Omega_p}+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},\boldsymbol{v}_{fh}).\end{aligned}\tag{4.10}$$

With the same definition of $\boldsymbol{e}_{\boldsymbol{u}_f}$ as defined in the proof of Theorem 4.1, the problem (Q_h) can be rewritten as

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}_{fh},p_{fh},\phi_{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=(\boldsymbol{g}_f,\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,\psi_{ph})_{\Omega_p}+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},\boldsymbol{v}_{fh})-b_f(\boldsymbol{e}_{\boldsymbol{u}_f},\boldsymbol{e}_{\boldsymbol{u}_f},\boldsymbol{v}_{fh}).\end{aligned}\tag{4.11}$$

By using (2.8) and the error estimate of $\boldsymbol{e}_{\boldsymbol{u}_f}$ in Theorem 4.1, we have

$$|b_f(\boldsymbol{e}_{\boldsymbol{u}_f},\boldsymbol{e}_{\boldsymbol{u}_f},\boldsymbol{v}_{fh})|\leq c\|\mathbb{D}(\boldsymbol{e}_{\boldsymbol{u}_f})\|_{\Omega_f}^2\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}\leq cH^4\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f}.$$

Comparing the above two equations (4.11) and (4.10) and using (3.7), we can easily get

$$\|\mathbb{D}(\boldsymbol{u}_{fh}-\bar{\boldsymbol{u}}_{fh})\|_{\Omega_f}+\|p_{fh}-\bar{p}_{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_{ph}-\bar{\phi}_{ph}\|_{\Omega_p}\leq cH^4. \tag{4.12}$$

Based on this estimation, we try to compare the final approximation $[\boldsymbol{u}_H^{fh},p_H^{fh},\phi_H^{ph}]$ with $[\bar{\boldsymbol{u}}_{fh},\bar{p}_{fh},\bar{\phi}_{ph}]$ rather than $[\boldsymbol{u}_{fh},p_{fh},\phi_{ph}]$.

Let us denote

$$\bar{\boldsymbol{e}}_{\boldsymbol{u}_f}=\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}^{fh},\quad \bar{e}_{p_f}=\bar{p}_{fh}-p^{fh},\quad \bar{e}_{\phi_p}=\bar{\phi}_{ph}-\phi^{ph}.$$

By using (4.2), (4.12) and the triangle inequality, we have

$$\|\mathbb{D}(\bar{\boldsymbol{e}}_{\boldsymbol{u}_f})\|_{\Omega_f}+\|\bar{e}_{p_f}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla\bar{e}_{\phi_p}\|_{\Omega_p}\leq cH^2. \tag{4.13}$$

Due to (4.10), we have

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}^{fh},p^{fh},\phi^{ph}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}])\\&=\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh},\bar{p}_{fh},\bar{\phi}_{ph}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}])\\&\quad-\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{e}}_{\boldsymbol{u}_f},\bar{e}_{p_f},\bar{e}_{\phi_p}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}])\\&=(\boldsymbol{g}_f,Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,Q_H^{\phi}\psi_{ph})_{\Omega_p}+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh})\\&\quad-\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{e}}_{\boldsymbol{u}_f},\bar{e}_{p_f},\bar{e}_{\phi_p}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}]).\end{aligned}$$

Therefore we can rewrite (4.9) as

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{u}_H^{fh},p_H^{fh},\phi_H^{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=(\boldsymbol{g}_f,\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,\psi_{ph})_{\Omega_p}+b_f(\boldsymbol{u}^{fh},\boldsymbol{u}^{fh},\boldsymbol{v}_{fh})\\&\quad-\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{e}}_{\boldsymbol{u}_f},\bar{e}_{p_f},\bar{e}_{\phi_p}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}]).\end{aligned}\tag{4.14}$$

Then we have

$$\begin{aligned}&\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh},\bar{p}_{fh}-p_H^{fh},\bar{\phi}_{ph}-\phi_H^{ph}],[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}])\\&=\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{e}}_{\boldsymbol{u}_f},\bar{e}_{p_f},\bar{e}_{\phi_p}],[Q_H^{\boldsymbol{u}}\boldsymbol{v}_{fh},Q_H^pq_{fh},Q_H^{\phi}\psi_{ph}]),\end{aligned}\tag{4.15}$$

and it is obvious that $\forall[\boldsymbol{v}_{fH},q_{fH},\psi_{pH}]\in Y_H$

$$\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh},\bar{p}_{fh}-p_H^{fh},\bar{\phi}_{ph}-\phi_H^{ph}],[\boldsymbol{v}_{fH},q_{fH},\psi_{pH}])=0. \tag{4.16}$$

Thanks to (3.7) and (4.13), we can easily get

$$\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|\bar{p}_{fh}-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p}\leq cH^2. \tag{4.17}$$

To give the L^2 error estimations of $\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh}$ and $\bar{\phi}_{ph}-\phi_H^{ph}$, we will use the Aubin-Nitche technique. We first make the following usual assumption:

(A) for any given $\boldsymbol{f}_f\in\boldsymbol{L}^2(\Omega_f)$ and $f_p\in L^2(\Omega_p)$, the solution of the following problem

$$\mathcal{L}_{\boldsymbol{u}^{fh}}([\boldsymbol{v}_{fh},q_{fh},\psi_{ph}],[\boldsymbol{w}_{fh},r_{fh},\sigma_{ph}])=(\boldsymbol{f}_p,\boldsymbol{v}_{fh})_{\Omega_f}+(f_p,\psi_{ph})_{\Omega_p},\quad\forall[\boldsymbol{v}_{fh},q_{fh},\psi_{ph}]\in Y_h,$$

satisfies

$$\|\boldsymbol{w}_{fh}\|_{H^2(\Omega_f)}+\|r_{fh}\|_{H^1(\Omega_f)}+\|\sigma_{fh}\|_{H^2(\Omega_p)}\leq c(\|\boldsymbol{f}_f\|_{\Omega_f}+\|f_p\|_{\Omega_p}).$$

Here the H^2 norm is in the piecewise sense.

Now by taking $\boldsymbol{f}_p=\boldsymbol{v}_{fh}=\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh}$, $f_p=\psi_{ph}=\bar{\phi}_{ph}-\phi_H^{ph}$ and $q_{fh}=\bar{p}_{fh}-p_H^{fh}$ and being aware of (4.16), we get $\forall[\boldsymbol{w}_{fH},r_{fH},\sigma_{pH}]\in Y_H$

$$\begin{aligned}&\|\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh}\|_{\Omega_f}^2+\|\bar{\phi}_{ph}-\phi_H^{ph}\|_{\Omega_p}^2\\&=\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh},\bar{p}_{fh}-p_H^{fh},\bar{\phi}_{ph}-\phi_H^{ph}],[\boldsymbol{w}_{fh},r_{fh},\sigma_{ph}])\\&=\mathcal{L}_{\boldsymbol{u}^{fh}}([\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh},\bar{p}_{fh}-p_H^{fh},\bar{\phi}_{ph}-\phi_H^{ph}],\\&\qquad[\boldsymbol{w}_{fh}-\boldsymbol{w}_{fH},r_{fh}-r_{fH},\sigma_{ph}-\sigma_{pH}])\\&\leq c\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}\|\mathbb{D}(\boldsymbol{w}_{fh}-\boldsymbol{w}_{fH})\|_{\Omega_f}\\&\quad+c\|\bar{p}_{fh}-p_H^{fh}\|_{\Omega_f}\|\mathbb{D}(\boldsymbol{w}_{fh}-\boldsymbol{w}_{fH})\|_{\Omega_f}\\&\quad+c\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p}\|\mathbb{K}^{\frac{1}{2}}\nabla(\sigma_{ph}-\sigma_{pH})\|_{\Omega_p}\\&\quad+c\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}\|\mathbb{K}^{\frac{1}{2}}\nabla(\sigma_{ph}-\sigma_{pH})\|_{\Omega_p}\\&\quad+c\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p}\|\mathbb{D}(\boldsymbol{w}_{fh}-\boldsymbol{w}_{fH})\|_{\Omega_f}\\&\leq c(\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|\bar{p}_{fh}-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p})\\&\quad\times(\inf_{\boldsymbol{w}_{fH}\in\boldsymbol{X}_{fH}}\|\mathbb{D}(\boldsymbol{w}_{fh}-\boldsymbol{w}_{fH})\|_{\Omega_f}+\inf_{\sigma_{pH}\in X_{pH}}\|\mathbb{K}^{\frac{1}{2}}\nabla(\sigma_{ph}-\sigma_{pH})\|_{\Omega_p})\end{aligned}$$

Table 1
Numerical results of the two-grid decoupling scheme with $h = H^2$.

H	h	$\|\mathbb{D}e_{\boldsymbol{u}}\|_{\Omega_f}$	$R_{\boldsymbol{u}}^1$	$\|e_p\|_{\Omega_f}$	R_p^0	$\|\mathbb{K}^{\frac{1}{2}}\nabla e_\phi\|_{\Omega_p}$	R_ϕ^1
$\frac{1}{4}$	$\frac{1}{16}$	8.4856e-2	\	2.4733e-2	\	3.8090e-2	\
$\frac{1}{8}$	$\frac{1}{64}$	2.0649e-2	1.0195	2.8405e-3	1.5611	9.5124e-3	1.0008
$\frac{1}{16}$	$\frac{1}{256}$	5.1691e-3	0.9990	3.5333e-4	1.5035	2.3745e-3	1.0011

$$\le cH(\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|\bar{p}_{fh}-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p})$$
$$\times(\|\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh}\|_{\Omega_f}+\|\bar{\phi}_{ph}-\phi_H^{ph}\|_{\Omega_p}).$$

Then by applying (4.17), we get

$$\begin{aligned}&\|\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh}\|_{\Omega_f}+\|\bar{\phi}_{ph}-\phi_H^{ph}\|_{\Omega_p}\\&\quad\le cH(\|\mathbb{D}(\bar{\boldsymbol{u}}_{fh}-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|\bar{p}_{fh}-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\bar{\phi}_{ph}-\phi_H^{ph})\|_{\Omega_p})\\&\quad\le cH^3.\end{aligned}$$

Finally, combination of the above estimate with (4.12) yields

$$\|\boldsymbol{u}_{fh}-\boldsymbol{u}_H^{fh}\|_{\Omega_f}+\|\phi_{ph}-\phi_H^{ph}\|_{\Omega_p}\le cH^3. \tag{4.18}$$

Then we obtain the following theorem by means of triangle inequality and the classical results in (3.3).

Theorem 4.2. *Under the assumptions in Theorem 4.1, there hold the following error estimates:*

$$\begin{aligned}&\|\mathbb{D}(\boldsymbol{u}_f-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|p_f-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p-\phi_H^{ph})\|_{\Omega_p}\\&\quad\le c(h+H^2),\end{aligned} \tag{4.19}$$

$$\|\boldsymbol{u}_f-\boldsymbol{u}_H^{fh}\|_{\Omega_f}+\|\phi_p-\phi_H^{ph}\|_{\Omega_p}\le c(h^2+H^3). \tag{4.20}$$

Remark 3. To balance the two error terms in (4.19), we need to choose

$$h\sim H^2,$$

and we actually get

$$\|\mathbb{D}(\boldsymbol{u}_f-\boldsymbol{u}_H^{fh})\|_{\Omega_f}+\|p_f-p_H^{fh}\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p-\phi_H^{ph})\|_{\Omega_p}\le ch.$$

On the other hand, to balance the two error terms in (4.20), we take

$$h\sim H^{\frac{3}{2}}.$$

Thus we have

$$\|\boldsymbol{u}_f-\boldsymbol{u}_H^{fh}\|_{\Omega_f}+\|\phi_p-\phi_H^{ph}\|_{\Omega_p}\le ch^2.$$

In such sense, we say that we get the optimal error estimations for the two-grid decoupling scheme.

5. Numerical experiments

In this section, we present some numerical experiments to verify the theoretical results of the proposed two-grid decoupling scheme. Let the computational domain Ω be composed of $\Omega_f=(0,1)\times(1,2)$ and $\Omega_p=(0,1)\times(0,1)$ with the interface $\Gamma=(0,1)\times\{1\}$. For simplicity, all the physical parameters are set to 1 and $\mathbb{K}=\mathbf{I}$. We choose the following exact solution to the mixed Navier-Stokes/Darcy model, which satisfies the three interface conditions (see [5]).

$$\begin{cases}\boldsymbol{u}_f=\left(\cos(\frac{\pi}{2}y)^2\sin(\frac{\pi}{2}x),-\cos(\frac{\pi}{2}x)(\frac{1}{4}\sin(\pi y)+\frac{\pi}{4}y)\right), & (x,y)\in\Omega_f,\\ p_f=\frac{\pi}{4}\cos(\frac{\pi}{2}x)(y-1-\cos(\pi y)), & (x,y)\in\Omega_f,\\ \phi_p=\frac{\pi}{4}\cos(\frac{\pi}{2}x), & (x,y)\in\Omega_p.\end{cases}$$

Table 2
Numerical results of the two-grid decoupling scheme with $h = H^{\frac{3}{2}}$.

H	h	$\|e_{\boldsymbol{u}}\|_{\Omega_f}$	$R_{\boldsymbol{u}}^0$	$\|e_\phi\|_{\Omega_p}$	R_ϕ^0
$\frac{1}{16}$	$\frac{1}{64}$	7.6971e-5	\	4.3344e-5	\
$\frac{1}{25}$	$\frac{1}{125}$	1.9208e-5	2.0736	1.0405e-5	2.1315
$\frac{1}{49}$	$\frac{1}{343}$	2.5403e-6	2.0041	1.4637e-6	1.9430

Then we can get the external force term $\boldsymbol{g}_f$ in Ω_f and the source term g_p in Ω_p. And we impose the boundary conditions accordingly.

The finite element spaces we used are the MINI elements, say $P1b-P1$ finite element pair, for the fluid region and the linear continuous Lagrangian element, that is $P1$ element, for the porous media region. In the following two tables, we denote

$$e_{\boldsymbol{u}}=\boldsymbol{u}_f-\boldsymbol{u}_H^{fh},\quad e_p=p_f-p_H^{fp},\quad e_\phi=\phi_p-\phi_H^{ph},$$

and

$$R_{\boldsymbol{u}}^0,\quad R_\phi^0,\quad R_{\boldsymbol{u}}^1,\quad R_\phi^1,\quad R_p^0,$$

the convergence orders of $\boldsymbol{u}_H^{fh}$ and ϕ_H^{ph} in L^2 and H^1 norm, the convergence order in L^2 norm of p_H^{fh}, respectively.

We compute the errors between the exact solution and its numerical approximation by the proposed two-grid decoupling scheme. In Table 1, we show the errors and convergence orders with respect to the fine mesh size h in H^1 norm for $\boldsymbol{u}_f$, ϕ_p and L^2 norm for p_f with the mesh size configuration $h=H^2$. Table 2 presents the errors in L^2 norm of $\boldsymbol{u}_f$ and ϕ_p with $h=H^{\frac{3}{2}}$. By analyzing the data in the two tables, we can conclude that the convergence orders for the two-grid decoupling scheme are optimal both in H^1 and L^2 norm. These are consistent with the theoretical results obtained in Theorem 4.2.

All the computations are completed by using the open source software package FreeFem++ [15]. We also thank Dr. Yuhong Zhang from Hunan Normal University, China, for his help in programing with FreeFem++.

Link to the Reproducible Capsule

https://codeocean.com/capsule/6782045/tree/v1

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