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On the Weak Solutions to Steady-State Mixed Navier–Stokes/Darcy Model

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Abstract In this paper, for the mixed Navier–Stokes/Darcy model with Beavers–Joseph–Saffman's interface condition, we first establish an a priori estimate for possible weak solutions by means of expanding the coupled system. Then we prove the existence of weak solutions without the small data and/or large viscosity restriction, which is required for the existence of the weak solutions in literatures. As a direct corollary, we also get the global uniqueness of the weak solution.

Keywords Porous media flow, Navier–Stokes equations, weak solution, a priori estimate, existence and global uniqueness

MR(2010) Subject Classification 76D05, 76S05, 76D03

1 Introduction

Because of the important applications in real world applications, the mixed Stokes/Darcy and Navier–Stokes/Darcy model received much attention in both theoretical and numerical aspect in last decades.

Although many numerical methods have been proposed and studied for such mixed models with Beavers–Joseph (BJ) interface condition or a simplified version, say Beavers–Joseph–Saffman (BJS) interface condition, for examples, see [1, 3, 5–7, 9, 10, 13, 16], etc., there are still some basic mathematical problems keeping unsolved. For example, for the steady-state problems, the well-posedness of the Stokes/Darcy problem with BJ interface condition under mild physical parameters, the existence of weak solutions for the Navier–Stokes/Darcy problem with BJ or even simpler BJS interface condition without the restriction of small data and/or large viscosity and the global uniqueness of the weak solution. As is pointed out by Layton

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and his co-authors in [11], even for the linear model, say the coupled Stokes/Darcy model with BJ interface condition, the well-posedness of the problem is unclear in general data case. The authors in [11] considered the steady-state coupled Stokes/Darcy model with the most accepted BJS interface condition in literatures and showed the unique solvability of the problem. Later on, Cao and his co-authors discussed the well-posedness of the coupled Stokes/Darcy model with BJ interface condition in [4] and got the unique solvability of this problem when the physical parameter appeared in the BJ condition is sufficiently small.

As far as we know, the solvability for the steady-state coupled Navier–Stokes/Darcy model with BJ or BJS interface condition is still unknown unless some very restrictive conditions are imposed on the data of the problem. For example, the solvability of the steady-state Navier–Stokes/Darcy model with BJS interface condition with mild physical parameters in weak sense and the global uniqueness of the weak solution is still an unsolved problem. When the data is small and/or the viscosity is large, the existence results have been established, for example, see [9]. Similar results can be found also in [6]. Especially, the authors in [6] slightly modified the balance of the normal force along the interface by adding some inertial force so that the interface conditions can completely compensate the nonlinear convection in the energy balance of the Navier–Stokes and derive an existence result without the small data and/or large viscosity restrictions. However, such modification is lack of physical interpolation. For mild data case, as is pointed out in [9], the difficulty for obtaining the existence of weak solutions to the coupled system comes from the interface conditions, which does not completely compensate the nonlinear convection in the energy balance in the Navier–Stokes equations and makes the nonlinear convection term unabsorbable in the dissipative energy in mild data case. Due to the same difficulty, as far as we know, the global uniqueness of the weak solution remains an unsolved open problem for lacking of a priori estimates of weak solutions.

In this paper, we try to solve the above mentioned open problems. Firstly, we get an a priori estimate of the possible weak solutions by expanding the mixed Navier–Stokes/Darcy model with BJS interface condition to a more large coupled system. By the same technique, we show the solvability of the Navier–Stokes/Darcy model with BJS interface condition without the large viscosity and/or the small data restrictions by showing the solvability of the expanded system. As a direct corollary, we establish the global uniqueness of the weak solution.

2 Mixed Navier–Stokes/Darcy Model with BJS Interface Condition

Consider the following mixed model of the Navier–Stokes equations and the Darcy equation for coupling a fluid flow and a porous media flow in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here

$$\Omega = \Omega_f \cup \Gamma \cup \Omega_p,$$

where Ω_f and Ω_p are two disjoint domains occupied by fluid flow and porous media flow and $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$ is the interface. We denote

$$\Gamma_f = \partial\Omega_f \backslash \Gamma, \quad \Gamma_p = \partial\Omega_p \backslash \Gamma,$$

and we also denote by $\boldsymbol{n}_p$ and $\boldsymbol{n}_f$ the unit outward normal vectors on $\partial\Omega_p$ and $\partial\Omega_f$, respectively. Furthermore, Γ_p consists of two disjoint parts Γ_{pd} and Γ_{pn}. We assume $|\Gamma_f|$, $|\Gamma_{pd}| > 0$. See Figure 1 for a sketch.

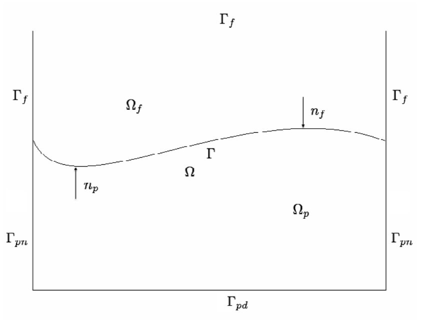

Figure 1 A global domain Ω consisting of a fluid flow region Ω_f and a porous media flow region Ω_p separated by an interface Γ

Let us denote by $[\boldsymbol{u}_f, p_f]$ the velocity field and the pressure of the fluid flow in Ω_f and ϕ_p the piezometric head in Ω_p. The partial differential equations modeling the fluid flow and the porous media flow are

$$\begin{cases} -\nabla\cdot(\mathbb{T}_\nu(\boldsymbol{u}_f, p_f)) + \boldsymbol{u}_f\cdot\nabla\boldsymbol{u}_f = \boldsymbol{g}_f, & \text{in } \Omega_f, \\ \nabla\cdot\boldsymbol{u}_f = 0, & \text{in } \Omega_f, \\ -\nabla\cdot\mathbb{K}\nabla\phi_p = g_p, & \text{in } \Omega_p, \end{cases} \tag{2.1}$$

where

$$\mathbb{T}_\nu(\boldsymbol{u}_f, p_f) = -p_f\mathbb{I} + 2\nu\mathbb{D}(\boldsymbol{u}_f), \quad \mathbb{D}(\boldsymbol{u}_f) = \frac{1}{2}(\nabla\boldsymbol{u}_f + \nabla^T\boldsymbol{u}_f),$$

are the stress tensor and the deformation rate tensor, $\nu > 0$ is the kinetic viscosity and $\mathbb{K}$ is the permeability in Ω_p, which is a positive definite symmetric tensor that is allowed to vary in space. The third equation of (2.1) that describes the porous media flow motion is the Darcy's law for the piezometric head ϕ_p. In the rest of this paper, we always use boldface characters to denote vector valued functions or spaces of vector valued functions.

The above equations (2.1) are completed and coupled together by the following boundary conditions:

$$\boldsymbol{u}_f = 0 \quad \text{on } \Gamma_f, \quad \mathbb{K}\nabla\phi_p\cdot\boldsymbol{n}_p = 0 \quad \text{on } \Gamma_{pn}, \quad \phi_p = 0 \quad \text{on } \Gamma_{pd}, \tag{2.2}$$

and the interface conditions on Γ:

$$\begin{cases} \boldsymbol{u}_f\cdot\boldsymbol{n}_f - \mathbb{K}\nabla\phi_p\cdot\boldsymbol{n}_p = 0, \\ -[\mathbb{T}_\nu(\boldsymbol{u}_f, p_f)\cdot\boldsymbol{n}_f]\cdot\boldsymbol{n}_f = \phi_p, \\ -[\mathbb{T}_\nu(\boldsymbol{u}_f, p_f)\cdot\boldsymbol{n}_f]\cdot\boldsymbol{\tau}_i = G_i\boldsymbol{u}_f\cdot\boldsymbol{\tau}_i, \quad i = 1, \ldots, d-1. \end{cases} \tag{2.3}$$

Here $G_i > 0$, $i = 1, \ldots, d-1$, are constants depending on the nature of the porous medium and determined from experimental data, $\boldsymbol{\tau}_i$, $i = 1, \ldots, d-1$, are the orthonormal tangential unit vectors along Γ. The first condition is the mass conservation, the second one is the balance of normal force and the third one means the tangential components of the normal stress force is proportional to the tangential components of the fluid velocity, which is called the Beavers–Joseph–Saffman's (BJS) interface condition (see [2] and [14]).

3 Preliminaries

Later on we need the following Hilbert spaces

$$\boldsymbol{X}_f = \{\boldsymbol{v}_f \in \boldsymbol{H}^1(\Omega_f) : \boldsymbol{v}_f|_{\Gamma_f} = 0\}, \quad Q_f = L^2(\Omega_f),$$
$$X_p = \{\psi_p \in H^1(\Omega_p) : \psi_p|_{\Gamma_{pd}} = 0\},$$

where $[\boldsymbol{X}_f, Q_f]$ is the space pair for the velocity and the pressure in the fluid flow region Ω_f and X_p is the space for the piezometric head in the porous medium region Ω_p. Let us denote by $\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)$ the interpolation space [12]

$$\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma) = [\boldsymbol{L}^2(\Gamma), \boldsymbol{H}_0^1(\Gamma)]_{\frac{1}{2}}.$$

In addition we also introduce a vector valued Hilbert space over Ω_p,

$$\boldsymbol{X}_p = \{\boldsymbol{v}_p \in \boldsymbol{H}^1(\Omega_p) : \boldsymbol{v}_p|_{\Gamma_p} = 0\},$$

and a lifting operator γ^{-1} from a subspace of $\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)$ into $\boldsymbol{X}_p$: for any $\boldsymbol{\zeta} \in \boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)$ with $\int_\Gamma \boldsymbol{\zeta} \cdot \boldsymbol{n}_p ds = 0$,

$$\gamma^{-1}\boldsymbol{\zeta} \in \boldsymbol{X}_p, \quad (\gamma^{-1}\boldsymbol{\zeta})|_\Gamma = \boldsymbol{\zeta}, \quad \nabla \cdot (\gamma^{-1}\boldsymbol{\zeta}) = 0.$$

We assume

$$\boldsymbol{g}_f \in \boldsymbol{X}_f', \quad g_p \in X_p', \quad \mathbb{K} \in L^\infty(\Omega_p)^{d\times d}, \tag{3.1}$$

and there exist $\lambda_{\max} > 0$ and $\lambda_{\min} > 0$ such that

$$\text{a.e. } \mathbf{x} \in \Omega_p, \quad \lambda_{\min}|\mathbf{x}|^2 \le \mathbb{K}\mathbf{x}\cdot\mathbf{x} \le \lambda_{\max}|\mathbf{x}|^2. \tag{3.2}$$

Here $\boldsymbol{X}_f'$ and X_p' are the dual spaces of $\boldsymbol{X}_f$ and X_p, respectively.

For simplicity, we always use $(\cdot,\cdot)_D$ and $\|\cdot\|_D$ to denote the L^2 inner product and the corresponding norm on any given domain D. Since $|\Gamma_f|$, $|\Gamma_{pd}| > 0$, we know that $\|\mathbb{D}(\cdot)\|_{\Omega_f}$ and $\|\mathbb{K}^{\frac{1}{2}}\nabla\cdot\|_{\Omega_p}$ are equivalent norms of the usual Sobolev norms in $\boldsymbol{X}_f$ and X_p due to the Korn's inequality, the Poincaré inequality and (3.2).

For any $[\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$, let us introduce:

$$\begin{aligned} &B([\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p]) \\ &\quad = a([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) + b_f(\boldsymbol{u}_f, \boldsymbol{u}_f, \boldsymbol{v}_f) + d_f(p_f, \boldsymbol{v}_f) - d_f(q_f, \boldsymbol{u}_f) + a_\Gamma([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]), \end{aligned}$$

where

$$\begin{aligned} &a([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + a_p(\phi_p, \psi_p), \\ &\quad = \left[2\nu(\mathbb{D}(\boldsymbol{u}_f), \mathbb{D}(\boldsymbol{v}_f))_{\Omega_f} + \sum_{i=1}^{d-1} G_i(\boldsymbol{u}_f\cdot\boldsymbol{\tau}_i, \boldsymbol{v}_f\cdot\boldsymbol{\tau}_i)_\Gamma\right] + (\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p}, \\ &b_f(\boldsymbol{u}_f, \boldsymbol{w}_f, \boldsymbol{v}_f) = b_{1f}(\boldsymbol{u}_f, \boldsymbol{w}_f, \boldsymbol{v}_f) + b_{2f}(\boldsymbol{u}_f, \boldsymbol{w}_f, \boldsymbol{v}_f), \\ &\quad = ((\boldsymbol{u}_f\cdot\nabla)\boldsymbol{w}_f, \boldsymbol{v}_f)_{\Omega_f} + \frac{1}{2}((\nabla\cdot\boldsymbol{u}_f)\boldsymbol{w}_f, \boldsymbol{v}_f)_{\Omega_f}, \\ &d_f(p_f, \boldsymbol{v}_f) = (p_f, \nabla\cdot\boldsymbol{v}_f)_{\Omega_f}, \quad \tilde{a}_\Gamma(\phi_p, \boldsymbol{v}_f) = (\phi_p, \boldsymbol{v}_f\cdot\boldsymbol{n}_f)_\Gamma, \\ &a_\Gamma([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = \tilde{a}_\Gamma(\phi_p, \boldsymbol{v}_f) - \tilde{a}_\Gamma(\psi_p, \boldsymbol{u}_f). \end{aligned}$$

Now the weak formulation of the mixed Navier–Stokes/Darcy model with BJS interface condition reads as follows: for $\boldsymbol{g}_f \in \boldsymbol{X}_f'$, $g_p \in X_p'$, find $[\boldsymbol{u}_f, p_f, \phi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$ such that $\forall [\boldsymbol{v}_f, q_f, \psi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$

$$\text{(Q)} \qquad B([\boldsymbol{u}_f, p_f, \phi_p], [\boldsymbol{v}_f, q_f, \psi_p]) = (\boldsymbol{g}_f, \boldsymbol{v}_f)_{\Omega_f} + (g_p, \psi_p)_{\Omega_p}.$$

Since $\nabla \cdot \boldsymbol{u}_f = 0$, we know that $b_{2f}(\boldsymbol{u}_f, \boldsymbol{u}_f, \boldsymbol{v}_f) = \frac{1}{2}((\nabla \cdot \boldsymbol{u}_f)\boldsymbol{u}_f, \boldsymbol{v}_f)_{\Omega_f} = 0$ in the problem (Q).

Thanks to [9], we know that there exists a positive constant $\beta > 0$ such that the following Ladyzhenskaya–Babuška–Brezzi (LBB) condition holds:

$$\inf_{q_f \in Q_f} \sup_{\boldsymbol{v}_f \in \boldsymbol{X}_f} \frac{d_f(q_f, \boldsymbol{v}_f)}{\|q_f\|_{\Omega_f} \|\mathbb{D}(\boldsymbol{v}_f)\|_{\Omega_f}} \geq \beta. \tag{3.3}$$

If we introduce the following divergence-free space

$$\boldsymbol{V}_f = \{\boldsymbol{v}_f \in \boldsymbol{X}_f : \nabla \cdot \boldsymbol{v}_f = 0\},$$

the restriction of the test function $\boldsymbol{v}_f$ to $\boldsymbol{V}_f$ in (Q) leads to the following reduced weak form: find $[\boldsymbol{u}_f, \phi_p] \in \boldsymbol{V}_f \times X_p$ such that $\forall [\boldsymbol{v}_f, \psi_p] \in \boldsymbol{V}_f \times X_p$,

$$\text{(P)} \qquad B^{\boldsymbol{V}}([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = (\boldsymbol{g}_f, \boldsymbol{v}_f)_{\Omega_f} + (g_p, \psi_p)_{\Omega_p},$$

where $B^{\boldsymbol{V}}([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = B([\boldsymbol{u}_f, 0, \phi_p], [\boldsymbol{v}_f, 0, \psi_p])$. By the same argument in [8], we know that the problem (Q) and (P) are equivalent.

In the rest of this paper, we assume Ω_f and Ω_p are polygons or polyhedrons for simplicity. For a given small positive parameter $\mu > 0$, let us denote by T_f^μ, T_p^μ the regular triangulations of Ω_f, Ω_p and we assume that the two meshes coincide on Γ. Let us denote by $\boldsymbol{X}_{f\mu} \subset \boldsymbol{X}_f$, $Q_{f\mu} \subset Q_f$, $X_{p\mu} \subset X_p$ and $\boldsymbol{X}_{p\mu} \subset \boldsymbol{X}_p$ the finite element spaces defined on Ω_f and Ω_p based on the above triangulations. And we assume that $[\boldsymbol{X}_{f\mu}, Q_{f\mu}]$ is a stable finite element pair. In addition, let us denote by Π_f^μ the Scott–Zhang finite element interpolator[15] from $\boldsymbol{X}_f$ onto $\boldsymbol{X}_{f\mu}$ with the following property

$$\|\boldsymbol{v}_f - \Pi_f^\mu \boldsymbol{v}_f\|_{\Omega_f} \leq c\mu \|\mathbb{D}(\boldsymbol{v}_f)\|_{\Omega_f}, \quad \forall \boldsymbol{v}_f \in \boldsymbol{X}_f. \tag{3.4}$$

Here and after, we always use c to denote a generic positive constant which may take different values in different occasions.

For the purpose of later analysis, for any bounded domain $D \in \mathbb{R}^d$, we recall some inequalities and identity:

$$\|v\|_{L^2(\partial D)} \leq c\|v\|_{L^2(D)}^{\frac{1}{2}} \|v\|_{H^1(D)}^{\frac{1}{2}} \leq c\|v\|_{H^1(D)}, \quad \forall v \in H^1(D), \tag{3.5}$$

$$\|v\|_{L^4(\partial D)} \leq c\|v\|_{H^1(D)}, \quad \forall v \in H^1(D), \tag{3.6}$$

$$\begin{aligned} &|((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v})_D|, \ |((\nabla \cdot \boldsymbol{u})\boldsymbol{w}, \boldsymbol{v})_D| \\ &\quad \leq c\|\boldsymbol{w}\|_{L^2(D)}^{\frac{1}{2}} \|\boldsymbol{w}\|_{H^1(D)}^{\frac{1}{2}} \|\boldsymbol{u}\|_{H^1(D)} \|\boldsymbol{v}\|_{H^1(D)}, \quad \forall \boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}^1(D), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\int_D (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} + \int_D (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} \cdot \boldsymbol{v} \\ &\quad = \int_{\partial D} (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u} \cdot \boldsymbol{n} - \int_D (\nabla \cdot \boldsymbol{u})\boldsymbol{v} \cdot \boldsymbol{w}, \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(D). \end{aligned} \tag{3.8}$$

4 An a Priori Estimate of Weak Solutions

We know from [9] that the difficulty for obtaining an a priori estimate of weak solutions to the Navier–Stokes/Darcy model comes from the unbalance of the energy caused by the nonlinear convection in the Navier–Stokes equations. In one sentence, there is energy exchange due to convection along the interface Γ. Generally, we need some small data and/or large viscosity restrictions to make such energy exchange to be absorbed by the dissipative energy in the energy balance of the Navier–Stokes equations. On the other hand, such energy exchange along the interface Γ must come from the outside region of the Navier–Stokes equations. If we can mimic an outside world of the Navier–Stokes equations, for example an outside world in Ω_p, such that the convection in the outside system can completely or nearly compensate the above mentioned energy exchange, it is possible for us to make the energy exchange along Γ controllable.

To do this, for certain fixed $\mu > 0$ and some constant $\sigma > 0$, we introduce the following auxiliary linear equations in Ω_p: for any given $\boldsymbol{u}_f \in \boldsymbol{V}_f$,

$$\begin{cases} -2\sigma\nabla\cdot\mathbb{D}(\boldsymbol{u}_p) + (\boldsymbol{u}_p^0\cdot\nabla)\boldsymbol{u}_p = 0 \quad \text{in } \Omega_p, \\ \boldsymbol{u}_p|_{\Gamma_p} = 0, \quad \boldsymbol{u}_p|_{\Gamma} = (\Pi_f^\mu \boldsymbol{u}_f)|_\Gamma, \end{cases} \tag{4.1}$$

where $\boldsymbol{u}_p^0 = \gamma^{-1}(\boldsymbol{u}_f|_\Gamma) \in \boldsymbol{X}_p$ with $\nabla\cdot\boldsymbol{u}_p^0 = 0$. We can easily show that the auxiliary system is well-posed for any given $\mu > 0$, $\sigma > 0$ and $\boldsymbol{u}_f \in \boldsymbol{V}_f$.

Now we consider the Galerkin approximation of (4.1) in the finite element space $\boldsymbol{X}_{p\mu}$: find $\boldsymbol{u}_{p\mu} \in \boldsymbol{X}_{p\mu}$ such that

$$(\mathrm{A}_\mu) \quad \begin{cases} 2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\boldsymbol{u}_p^0\cdot\nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\boldsymbol{v}_{p\mu} = 0, \quad \forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu}, \\ \boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_f)|_\Gamma. \end{cases}$$

It is clear that (P) and (A_μ) form a weakly coupled system

$$(\mathrm{C}) \quad \begin{cases} B^{\boldsymbol{V}}([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]) = (\boldsymbol{g}_f, \boldsymbol{v}_f)_{\Omega_f} + (g_p, \psi_p)_{\Omega_p}, \quad \forall [\boldsymbol{v}_f, \psi_p] \in \boldsymbol{V}_f \times X_p, \\ 2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\boldsymbol{u}_p^0\cdot\nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma\displaystyle\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\boldsymbol{v}_{p\mu} = 0, \quad \forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu}, \\ \boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_f)|_\Gamma. \end{cases}$$

We call (C) a weakly coupled system since (A_μ) is subjected to (P) while (P) is independent of (A_μ).

Now it is ready for us to derive the a priori estimate of the possible solutions to (P).

Theorem 4.1 *There holds the following a priori estimate for the possible solutions* $[\boldsymbol{u}_f, \phi_p] \in \boldsymbol{V}_f \times X_p$ *to* (P)

$$\nu\|\mathbb{D}(\boldsymbol{u}_f)\|_{\Omega_f}^2 + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 \le \mathcal{C}^2,$$

where

$$\mathcal{C}^2 = c\nu^{-1}\|\boldsymbol{g}_f\|_{\boldsymbol{X}_f'}^2 + c\lambda_{\min}^{-1}\|g_p\|_{X_p'}^2.$$

Proof In the proof, we assume that $[\boldsymbol{u}_f, \phi_p] \in \boldsymbol{V}_f \times X_p$ is a solution to the problem (P) and $\boldsymbol{u}_{p\mu} \in \boldsymbol{X}_{p\mu}$ is the corresponding unique solution of (A_μ). For this solution $\boldsymbol{u}_f \in \mathbf{V}_f$, we denote

$$M_{\boldsymbol{u}_f} = \|\mathbb{D}(\boldsymbol{u}_f)\|_{\Omega_f}. \tag{4.2}$$

Taking $[\boldsymbol{v}_f, \psi_p] = [\boldsymbol{u}_f, \phi_p]$ in (P) and omitting the non-negative interface term $\sum_{i=1}^{d-1} G_i(\boldsymbol{u}_f \cdot \boldsymbol{\tau}_i, \boldsymbol{u}_f \cdot \boldsymbol{\tau}_i)_\Gamma$, we get

$$\begin{aligned} &2\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} + ((\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f, \boldsymbol{u}_f)_{\Omega_f} \\ &\quad \le (\boldsymbol{g}_f, \boldsymbol{u}_f)_{\Omega_f} + (g_p, \phi_p)_{\Omega_p}. \end{aligned} \tag{4.3}$$

Taking $\boldsymbol{v}_{p\mu} = \boldsymbol{u}_{p\mu}$ in (A_μ) and using the boundary condition $\boldsymbol{u}_{p\mu}|_\Gamma = (\Pi^\mu_f \boldsymbol{u}_f)|_\Gamma$ lead to

$$2\sigma\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|^2_{\Omega_p} + ((\boldsymbol{u}^0_p\cdot\nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{u}_{p\mu})_{\Omega_p} - \sigma\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\Pi^\mu_f\boldsymbol{u}_f = 0. \tag{4.4}$$

Being aware of $\boldsymbol{n}_f = -\boldsymbol{n}_p$ on Γ, $\nabla\cdot\boldsymbol{u}_f = \nabla\cdot\boldsymbol{u}^0_p = 0$ and the identity (3.8), it is easy to verify that

$$\begin{aligned} ((\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f, \boldsymbol{u}_f)_{\Omega_f} &= \frac{1}{2}\int_\Gamma |\boldsymbol{u}_f|^2\boldsymbol{u}_f\cdot\boldsymbol{n}_f, \\ ((\boldsymbol{u}^0_p\cdot\nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{u}_{p\mu})_{\Omega_p} &= \frac{1}{2}\int_\Gamma |\boldsymbol{u}_{p\mu}|^2\boldsymbol{u}^0_p\cdot\boldsymbol{n}_p = -\frac{1}{2}\int_\Gamma |\Pi^\mu_f\boldsymbol{u}_f|^2\boldsymbol{u}_f\cdot\boldsymbol{n}_f. \end{aligned}$$

By using (3.5), (3.6), (3.4), the assumption (4.2), the Korn's and the Poincaré inequality, summation of the above two identities leads to

$$\begin{aligned} &((\boldsymbol{u}_f\cdot\nabla)\boldsymbol{u}_f, \boldsymbol{u}_f)_{\Omega_f} + ((\boldsymbol{u}^0_p\cdot\nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{u}_{p\mu})_{\Omega_p} \\ &\quad = \frac{1}{2}\int_\Gamma [|\boldsymbol{u}_f|^2\boldsymbol{u}_f\cdot\boldsymbol{n}_f - |\Pi^\mu_f\boldsymbol{u}_f|^2\boldsymbol{u}_f\cdot\boldsymbol{n}_f] \\ &\quad = \frac{1}{2}\int_\Gamma [(\boldsymbol{u}_f - \Pi^\mu_f\boldsymbol{u}_f)\cdot(\boldsymbol{u}_f + \Pi^\mu_f\boldsymbol{u}_f)\boldsymbol{u}_f\cdot\boldsymbol{n}_f] \\ &\quad \le c\|\boldsymbol{u}_f - \Pi^\mu_f\boldsymbol{u}_f\|_{L^2(\Gamma)}\|\boldsymbol{u}_f + \Pi^\mu_f\boldsymbol{u}_f\|_{L^4(\Gamma)}\|\boldsymbol{u}_f\|_{L^4(\Gamma)} \\ &\quad \le cM_{\boldsymbol{u}_f}\mu^{\frac{1}{2}}\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}. \end{aligned} \tag{4.5}$$

This means the auxiliary system can almost compensate the nonlinear convection of the Navier–Stokes equations in the energy balance.

Taking the above estimation into account, the summation of (4.3) and (4.4) yields

$$\begin{aligned} &2\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} + 2\sigma\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|^2_{\Omega_p} \\ &\quad \le |(\boldsymbol{g}_f, \boldsymbol{u}_f)_{\Omega_f}| + |(g_p, \phi_p)_{\Omega_p}| + \sigma\left|\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\Pi^\mu_f\boldsymbol{u}_f\right| + cM_{\boldsymbol{u}_f}\mu^{\frac{1}{2}}\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}. \end{aligned} \tag{4.6}$$

For the first two terms on the right hand side of the above inequality, by using the Korn's inequality, the Poincaré inequality and (3.2) we have

$$\begin{aligned} &|(\boldsymbol{g}_f, \boldsymbol{u}_f)_{\Omega_f}| + |(g_p, \phi_p)_{\Omega_p}| \\ &\quad \le c\|\boldsymbol{g}_f\|_{\boldsymbol{X}'_f}\|\mathbb{D}(\boldsymbol{u}_f)\|_{\Omega_f} + c\lambda^{-\frac{1}{2}}_{\min}\|g_p\|_{X'_p}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p} \\ &\quad \le \frac{\nu}{2}\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f} + \frac{1}{2}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} + c\nu^{-1}\|\boldsymbol{g}_f\|^2_{\boldsymbol{X}'_f} + c\lambda^{-1}_{\min}\|g_p\|^2_{X'_p}. \end{aligned} \tag{4.7}$$

For the third term on the right hand side of (4.6), by using (3.5), the Korn's inequality, the Poincaré inequality and the following inequality

$$\|\boldsymbol{v}_{p\mu}\|_{L^2(\partial K)} \le c\mu^{-\frac{1}{2}}\|\boldsymbol{v}_{p\mu}\|_{L^2(K)}, \quad \forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu},$$

we have

$$
\begin{aligned}
\sigma\left|\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\Pi_f^\mu \boldsymbol{u}_f\right| &\le \sigma\left\|\frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p}\right\|_{L^2(\Gamma)}\|\Pi_f^\mu \boldsymbol{u}_f\|_{L^2(\Gamma)}\\
&\le c\sigma\|\nabla \boldsymbol{u}_{p\mu}\cdot \boldsymbol{n}_p\|_{L^2(\partial\Omega_p)}\|\Pi_f^\mu \boldsymbol{u}_f\|_{L^2(\partial\Omega_f)}\\
&\le c\sigma\bigg(\sum_{K\in T_p^\mu}\|\nabla \boldsymbol{u}_{p\mu}\|^2_{L^2(\partial K)}\bigg)^{\frac{1}{2}}\|\Pi_f^\mu \boldsymbol{u}_f\|_{H^1(\Omega_f)}\\
&\le c\sigma\mu^{-\frac{1}{2}}\|\nabla \boldsymbol{u}_{p\mu}\|_{\Omega_p}\|\nabla \boldsymbol{u}_f\|_{\Omega_f}\le c\sigma\mu^{-\frac{1}{2}}\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|_{\Omega_p}\|\mathbb{D}(\boldsymbol{u}_f)\|_{\Omega_f}\\
&\le c\sigma^2\mu^{-1}\nu^{-1}\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|^2_{\Omega_p}+\frac{\nu}{2}\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}. \qquad (4.8)
\end{aligned}
$$

If we choose μ small enough and σ small enough such that

$$cM_{\boldsymbol{u}_f}\mu^{\frac{1}{2}} < \frac{\nu}{2}, \quad 0<\sigma\le c\nu\mu,$$

combination of (4.3), (4.4), (4.5), (4.7) and (4.8) admits

$$
\begin{aligned}
&\sigma\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|^2_{\Omega_p}+\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} \qquad (4.9)\\
&\le c\nu^{-1}\|\boldsymbol{g}_f\|^2_{\boldsymbol{X}_f'}+c\lambda_{\min}^{-1}\|g_p\|^2_{X_p'}\triangleq \mathcal{C}^2.
\end{aligned}
$$

Since the solutions of (P) is independent of the system (A_μ), the above a priori estimate actually gives a μ and σ independent a priori estimate of possible solutions to (P)

$$\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p}\le \mathcal{C}^2. \qquad \square$$

Remark 4.2 The result of Theorem 4.1 means that there exists a positive constant $\mathcal{C}$ only depending on the data of the problem (P) such that all possible solutions $[\boldsymbol{u}_f,\phi_p]\in \boldsymbol{V}_f\times X_p$ to (P) is bounded by this constant, especially

$$\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f}\le \mathcal{C}^2.$$

Therefore

$$M_{\boldsymbol{u}_f}\le \frac{\mathcal{C}}{\nu^{\frac{1}{2}}}.$$

In the next section, we fix $\mu>0$ and $\sigma>0$ such that

$$c\mathcal{C}\mu^{\frac{1}{2}}<\frac{\nu^{\frac{3}{2}}}{2} \quad \text{and} \quad \sigma\le c\nu\mu. \qquad (4.10)$$

5 Existence and Global Uniqueness of the Weak Solution

In this section, we will use the Galerkin method to show the solvability of (P) (and (Q)), and then give the global uniqueness of the solution.

5.1 Solvability of the Coupled FEM Scheme

For any given $h>0$, we denote by $\boldsymbol{X}_{fh}$, Q_{fh} and X_{ph} the corresponding finite element spaces of $\boldsymbol{X}_f$, Q_f and X_p, which converge to $\boldsymbol{X}_f$, Q_f and X_p when $h\to 0$. Furthermore, we introduce

$$\boldsymbol{V}_{fh}=\{\boldsymbol{v}_{fh}\in \boldsymbol{X}_{fh}:\ d_f(q_{fh},\boldsymbol{v}_{fh})=0,\ \forall q_{fh}\in Q_{fh}\}.$$

We consider the Galerkin approximation of (P) in $\boldsymbol{V}_{fh}\times X_{ph}$: find $[\boldsymbol{u}_{fh},\phi_{ph}]\in \boldsymbol{V}_{fh}\times X_{ph}$ such that $\forall[\boldsymbol{v}_{fh},\psi_{ph}]\in \boldsymbol{V}_{fh}\times X_{ph}$,

$$(\mathrm{P}_h) \qquad B^{\boldsymbol{V}}([\boldsymbol{u}_{fh},\phi_{ph}],[\boldsymbol{v}_{fh},\psi_{ph}])=(\boldsymbol{g}_f,\boldsymbol{v}_{fh})_{\Omega_f}+(g_p,\psi_{ph})_{\Omega_p}.$$

First of all, we need to show the solvability of (P_h). Let us denote

$$\boldsymbol{U}_{h\mu} = \boldsymbol{V}_{fh} \times \boldsymbol{X}_{p\mu} \times X_{ph}.$$

We consider the following more large coupled system: find $[\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}] \in \boldsymbol{U}_{h\mu}$ such that $\forall [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}] \in \boldsymbol{U}_{h\mu}$,

$$(\mathrm{C}_h) \qquad \begin{cases} B^{\boldsymbol{V}}([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]) = (\boldsymbol{g}_f, \boldsymbol{v}_{fh})_{\Omega_f} + (g_p, \psi_{ph})_{\Omega_p}, \\ 2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\boldsymbol{u}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma \displaystyle\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p} \boldsymbol{v}_{p\mu} = 0, \\ \boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma, \end{cases}$$

where $\boldsymbol{u}_p^0 = \gamma^{-1}(\boldsymbol{u}_{fh}|_\Gamma)$ with $\nabla \cdot \boldsymbol{u}_p^0 = 0$. It is obvious that if $[\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}] \in \boldsymbol{U}_{h\mu}$ is a solution to (C_h), $[\boldsymbol{u}_{fh}, \phi_{ph}]$ solves (P_h). On the other hand, if $[\boldsymbol{u}_{fh}, \phi_{ph}] \in \boldsymbol{V}_{fh} \times X_{ph}$ is a solution to (P_h), one can get a unique $\boldsymbol{u}_{p\mu} \in \boldsymbol{X}_{p\mu}$ such that $[\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}] \in \boldsymbol{U}_{h\mu}$ solves (C_h). Therefore, to show the solvability of (P_h), we turn to show the solvability of (C_h).

For this purpose, for any given $\tilde{\boldsymbol{u}}_{fh} \in \boldsymbol{V}_{fh}$ satisfying $\nu\|\mathbb{D}(\tilde{\boldsymbol{u}}_{fh})\|_\Omega^2 \le \mathcal{C}^2$, let us consider the following linear coupled system

$$(\mathrm{LC}_h) \qquad \begin{cases} \tilde{L}([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]) = (\boldsymbol{g}_f, \boldsymbol{v}_{fh})_{\Omega_f} + (g_p, \psi_{ph})_{\Omega_p}, \\ 2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\tilde{\boldsymbol{u}}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma \displaystyle\int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p} \boldsymbol{v}_{p\mu} = 0, \\ \boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma, \end{cases}$$

where $\tilde{\boldsymbol{u}}_p^0 = \gamma^{-1}(\tilde{\boldsymbol{u}}_{fh}|_\Gamma)$ with $\nabla \cdot \tilde{\boldsymbol{u}}_p^0 = 0$ and

$$\begin{aligned} &\tilde{L}([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]) \\ &\quad = a([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]) + b_f(\tilde{\boldsymbol{u}}_{fh}, \boldsymbol{u}_{fh}, \boldsymbol{v}_{fh}) + a_\Gamma([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]). \end{aligned}$$

To diminish the restriction $\boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma$ in (LC_h), we introduce the following trace space on Γ,

$$\boldsymbol{M}_\mu = \boldsymbol{X}_{p\mu}|_\Gamma \subset \boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma),$$

and a bilinear form $b_\mu^h(\cdot,\cdot)$ defined on $\boldsymbol{U}_{h\mu} \times \boldsymbol{M}_\mu'$,

$$\begin{aligned} b_\mu^h(\boldsymbol{\xi}_\mu, [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}]) &= \langle \boldsymbol{\xi}_\mu, \Pi_f^\mu \boldsymbol{v}_{fh} - \boldsymbol{v}_{p\mu} \rangle_{\boldsymbol{M}_\mu'} \\ &\triangleq \langle \boldsymbol{\xi}_\mu, B_\mu^h(\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}) \rangle_{\boldsymbol{M}_\mu'}, \end{aligned}$$

where B_μ^h is a linear mapping from $\boldsymbol{U}_{h\mu}$ into $\boldsymbol{M}_\mu$. Now we define a bilinear form $C_h(\cdot,\cdot)$ on $(\boldsymbol{U}_{h\mu} \times \boldsymbol{M}_\mu') \times (\boldsymbol{U}_{h\mu} \times \boldsymbol{M}_\mu')$:

$$\begin{aligned} &\forall [\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}, \boldsymbol{\xi}_\mu], [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}, \boldsymbol{\mu}_\mu] \in \boldsymbol{U}_{h\mu} \times \boldsymbol{M}_\mu', \\ &C_h([\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}, \boldsymbol{\xi}_\mu], [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}, \boldsymbol{\mu}_\mu]) \\ &\quad = \tilde{L}([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_{fh}, \psi_{ph}]) + 2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\tilde{\boldsymbol{u}}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} \\ &\qquad - \sigma \int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p} \boldsymbol{v}_{p\mu} + b_\mu^h(\boldsymbol{\xi}_\mu, [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}]) + b_\mu^h(\boldsymbol{\mu}_\mu, [\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}]) \\ &\quad \triangleq \tilde{a}_h([\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}], [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}]) \\ &\qquad + b_\mu^h(\boldsymbol{\xi}_\mu, [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}]) + b_\mu^h(\boldsymbol{\mu}_\mu, [\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}]). \end{aligned}$$

Let us consider the following problem: find $[\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}, \boldsymbol{\xi}_\mu] \in \boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$ such that for any $[\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}, \boldsymbol{\mu}_\mu] \in \boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$,

$$(\mathrm{LC}'_h) \qquad C_h([\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}, \boldsymbol{\xi}_\mu], [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}, \boldsymbol{\mu}_\mu]) = (\boldsymbol{g}_f, \boldsymbol{v}_{fh})_{\Omega_f} + (g_p, \psi_{ph})_{\Omega_p}.$$

If we introduce

$$\boldsymbol{V}_{h\mu} = \{[\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}] \in \boldsymbol{U}_{h\mu} : \ \boldsymbol{v}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{v}_{fh})|_\Gamma\},$$

it is clear that

$$\ker(B_\mu^h) = \boldsymbol{V}_{h\mu}.$$

The bilinear form $\tilde{a}_h(\cdot,\cdot)$ is $\boldsymbol{V}_{h\mu}$-elliptic on $\boldsymbol{V}_{h\mu} \times \boldsymbol{V}_{h\mu}$ for fixed μ and σ described in (4.10), which can be easily showed by the manner in the previous section if we notice $\nu\|\mathbb{D}(\tilde{\boldsymbol{u}}_{fh})\|_\Omega^2 \le \mathcal{C}^2$. In addition,

$$\sup_{[\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}] \in \boldsymbol{U}_{h\mu} \setminus \{0\}} \frac{b_\mu^h(\boldsymbol{\xi}_\mu, [\boldsymbol{v}_{fh}, \boldsymbol{v}_{p\mu}, \psi_{ph}])}{\|\mathbb{D}(\boldsymbol{v}_{fh})\|_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}} \nabla \psi_{ph}\|_{\Omega_p} + \|\mathbb{D}(\boldsymbol{v}_{p\mu})\|_{\Omega_p}}$$
$$\ge \sup_{\boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu} \setminus \{0\}} \frac{\langle \boldsymbol{\xi}_\mu, \boldsymbol{v}_{p\mu} \rangle_{\boldsymbol{M}'_\mu}}{\|\mathbb{D}(\boldsymbol{v}_{p\mu})\|_{\Omega_p}} \ge c\|\boldsymbol{\xi}_\mu\|_{\boldsymbol{M}'_\mu}.$$

The last inequality is valid since for any given $\boldsymbol{v}_{p\mu}|_\Gamma \in \boldsymbol{M}_\mu$, we can always find some $\boldsymbol{v}'_{p\mu} \in \boldsymbol{X}_{p\mu}$ with $\boldsymbol{v}'_{p\mu}|_\Gamma = \boldsymbol{v}_{p\mu}|_\Gamma$ and $\|\mathbb{D}(\boldsymbol{v}'_{p\mu})\|_{\Omega_p} \le c\|\boldsymbol{v}_{p\mu}|_\Gamma\|_{\boldsymbol{M}_\mu}$. Then we assert that the bilinear form $b_\mu^h(\cdot,\cdot)$ satisfies the LBB condition on $\boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$. Together with the $\boldsymbol{V}_{h\mu}$-elliptic of $a_h(\cdot,\cdot)$, we can get the well-posedness of the problem (LC'_h) in $\boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$ (see [8]).

Suppose $[\boldsymbol{u}_{fh}, \boldsymbol{u}_{p\mu}, \phi_{ph}, \boldsymbol{\xi}_\mu] \in \boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$ is the solution to the problem (LC'_h), we can further show that $\boldsymbol{\xi}_\mu = 0$. In fact, taking $\boldsymbol{v}_{fh} = 0$, $\psi_{ph} = 0$, $\boldsymbol{\mu}_\mu = 0$ and $\boldsymbol{v}_{fh} = 0$, $\boldsymbol{v}_{p\mu} = 0$, $\psi_{ph} = 0$ in (LC'_h), respectively, we get

$$\begin{aligned} &2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\tilde{\boldsymbol{u}}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma \int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p} \boldsymbol{v}_{p\mu} \\ &\quad + (\boldsymbol{\xi}_\mu, \boldsymbol{v}_{p\mu})_\Gamma = 0, \quad \forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu}, \\ &\boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma. \end{aligned}$$

It is clear that the above $\boldsymbol{u}_{p\mu}$ satisfies: $\forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu} \cap \boldsymbol{H}_0^1(\Omega_p)$,

$$2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\tilde{\boldsymbol{u}}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} = 0, \quad \boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma.$$

This is also equivalent to the problem: $\forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu}$,

$$\begin{aligned} &2\sigma(\mathbb{D}(\boldsymbol{u}_{p\mu}), \mathbb{D}(\boldsymbol{v}_{p\mu}))_{\Omega_p} + ((\tilde{\boldsymbol{u}}_p^0 \cdot \nabla)\boldsymbol{u}_{p\mu}, \boldsymbol{v}_{p\mu})_{\Omega_p} - \sigma \int_\Gamma \frac{\partial \boldsymbol{u}_{p\mu}}{\partial \boldsymbol{n}_p} \boldsymbol{v}_{p\mu} = 0, \\ &\boldsymbol{u}_{p\mu}|_\Gamma = (\Pi_f^\mu \boldsymbol{u}_{fh})|_\Gamma. \end{aligned}$$

Thus we have

$$\langle \boldsymbol{\xi}_\mu, \boldsymbol{v}_{p\mu} \rangle_{\boldsymbol{M}'_\mu} = 0, \quad \forall \boldsymbol{v}_{p\mu} \in \boldsymbol{X}_{p\mu}.$$

That is $\boldsymbol{\xi}_\mu = 0$. This ensures the problem (LC'_h) is equivalent to the problem (LC_h). And the well-posedness of (LC'_h) in $\boldsymbol{U}_{h\mu} \times \boldsymbol{M}'_\mu$ guarantees the unique solvability of (LC_h) in $\boldsymbol{U}_{h\mu}$.

By the similar procedure for obtaining the a priori estimate in the previous section, if μ and σ satisfy (4.10) and $\nu\|\mathbb{D}(\tilde{\boldsymbol{u}}_{fh})\|_\Omega^2 \le \mathcal{C}^2$, we can get the solution of (LC_h) satisfies

$$\sigma\|\mathbb{D}(\boldsymbol{u}_{p\mu})\|_{\Omega_p}^2 + \nu\|\mathbb{D}(\boldsymbol{u}_{fh})\|_{\Omega_f}^2 + \|\mathbb{K}^{\frac{1}{2}} \nabla \phi_{ph}\|_{\Omega_p}^2 \le \mathcal{C}^2.$$

If we introduce a ball $B^h_{\mathcal{C}} \subset \boldsymbol{V}_{fh} \times X_{ph}$,

$$B^h_{\mathcal{C}} = \{[\boldsymbol{v}_{fh}, \psi_{ph}] \in \boldsymbol{V}_{fh} \times X_{ph} : \nu\|\mathbb{D}(\boldsymbol{v}_{fh})\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{ph}\|^2_{\Omega_p} \le \mathcal{C}^2\},$$

the linear system (LC$_h$) defines a continuous mapping $[\boldsymbol{u}_{fh}, \phi_{ph}] = T(\tilde{\boldsymbol{u}}_{fh})$ from $B^h_{\mathcal{C}}$ into $B^h_{\mathcal{C}}$. Then by the Brouwer's fixed point theorem, (P$_h$) possesses at least one solution $[\boldsymbol{u}_{fh}, \phi_{ph}] \in B^h_{\mathcal{C}}$.

We conclude the above result in the following theorem.

Theorem 5.1 *For any given $h > 0$, the problem* (P$_h$) *has at least one solution $[\boldsymbol{u}_{fh}, \phi_{ph}] \in \boldsymbol{V}_{fh} \times X_{ph}$ satisfying*

$$\nu\|\mathbb{D}(\boldsymbol{u}_{fh})\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_{ph}\|^2_{\Omega_p} \le \mathcal{C}^2,$$

where $\mathcal{C}$ is defined in Theorem 4.1.

5.2 Well-posedness

Thanks to Theorem 5.1, we derive a uniformly bounded sequence $\{[\boldsymbol{u}_{fh}, \phi_{ph}]\}_{h>0}$ in $\boldsymbol{X}_f \times X_p$. Since $\boldsymbol{X}_f \times X_p$ is compactly embedded in $\boldsymbol{L}^2(\Omega_f) \times L^2(\Omega_p)$ for bounded domains Ω_f and Ω_p, we can extract a subsequence, which is still denoted by h, such that as $h \to 0$ there exists $[\boldsymbol{u}_f, \phi_p] \in \boldsymbol{V}_f \times X_p$ such that

$$[\boldsymbol{u}_{fh}, \phi_{ph}] \to [\boldsymbol{u}_f, \phi_p] \quad \text{weakly in } \boldsymbol{X}_f \times X_p, \tag{5.1}$$

$$[\boldsymbol{u}_{fh}, \phi_{ph}] \to [\boldsymbol{u}_f, \phi_p] \quad \text{strongly in } \boldsymbol{L}^2(\Omega_f) \times L^2(\Omega_p). \tag{5.2}$$

Lemma 5.2 *For $[\boldsymbol{u}_f, \phi_p]$ defined in* (5.1) *and* (5.2), *we have $\forall[\boldsymbol{v}_f, \psi_p] \in \boldsymbol{X}_f \times X_p$,*

$$\begin{aligned}
&\lim_{h\to 0} a([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_f, \psi_p]) = a([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p]),\\
&\lim_{h\to 0} a_\Gamma([\boldsymbol{u}_{fh}, \phi_{ph}], [\boldsymbol{v}_f, \psi_p] = a_\Gamma([\boldsymbol{u}_f, \phi_p], [\boldsymbol{v}_f, \psi_p],\\
&\lim_{h\to 0} b_f(\boldsymbol{u}_{fh}, \boldsymbol{u}_{fh}, \boldsymbol{v}_f) = b_f(\boldsymbol{u}_f, \boldsymbol{u}_f, \boldsymbol{v}_f) = b_{1f}(\boldsymbol{u}_f, \boldsymbol{u}_f, \boldsymbol{v}_f).
\end{aligned}$$

Proof The proof of this lemma is trivial and we omit it. □

Theorem 5.3 *The problem* (Q) *has at least one solution $[\boldsymbol{u}_f, p_f, \phi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$ with the following bounds*

$$\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} \le \mathcal{C}^2, \quad \|p_f\|_{\Omega_f} \le c\beta^{-1}\mathcal{C}(1+\mathcal{C}),$$

where $\mathcal{C}$ is defined in Theorem 4.1.

Proof Taking $h \to 0$ in the problem (P$_h$) and being aware of the results in Lemma 5.2, we know that the limit $[\boldsymbol{u}_f, \phi_p]$ in (5.1) and (5.2) is a solution to the problem (P) and the bound for $\nu\|\mathbb{D}(\boldsymbol{u}_f)\|^2_{\Omega_f} + \|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p}$ is obvious thanks to Theorem 5.1.

Thanks to the LBB condition (3.3), we know that there exists a unique $p_f \in Q_f$ such that $[\boldsymbol{u}_f, p_f, \phi_p] \in \boldsymbol{X}_f \times Q_f \times X_p$ is a solution to problem (Q). And it is easily obtained by using (3.3) that

$$\|p_f\|_{\Omega_f} \le c\beta^{-1}\mathcal{C}(1+\mathcal{C}).$$ □

Finally, we can also get the following global uniqueness of the weak solution.

Theorem 5.4 *Assume the data of the mixed Navier–Stokes/Darcy model with* BJS *interface condition satisfy*

$$c(\nu^{-2}\|\boldsymbol{g}_f\|_{\boldsymbol{X}'_f} + \nu^{-\frac{3}{2}}\lambda_{\min}^{-\frac{1}{2}}\|g_p\|_{X'_p}) < 1. \tag{5.3}$$

The problem (P) (*and* (Q)) *has only one solution in* $\boldsymbol{V}_f \times X_p$ (*and* $\boldsymbol{X}_f \times Q_f \times X_p$).

Proof Suppose $(\boldsymbol{u}_f^i, \phi_p^i) \in \boldsymbol{V}_f \times X_p$, $i = 1, 2$, be two solutions to (P). Their difference satisfies

$$\begin{aligned}
2\nu(\mathbb{D}(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2), \mathbb{D}(\boldsymbol{v}_f))_{\Omega_f} + (((\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2) \cdot \nabla)\boldsymbol{u}_f^1, \boldsymbol{v}_f)_{\Omega_f} \\
+((\boldsymbol{u}_f^2 \cdot \nabla)(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2), \boldsymbol{v}_f)_{\Omega_f} + (\mathbb{K}\nabla(\phi_p^1 - \phi_p^2), \nabla\psi_p)_{\Omega_p} \\
+(\phi_p^1 - \phi_p^2, \boldsymbol{v}_f \cdot \boldsymbol{n}_f)_\Gamma - (\psi_p, (\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2) \cdot \boldsymbol{n}_f)_\Gamma \\
+\sum_{i=1}^{d-1} G_i \int_\Gamma ((\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2) \cdot \boldsymbol{\tau}_i)(\boldsymbol{v}_f \cdot \boldsymbol{\tau}_i) = 0.
\end{aligned}$$

Taking $\boldsymbol{v}_f = \boldsymbol{u}_f^1 - \boldsymbol{u}_f^2$ and $\psi_p = \phi_p^1 - \phi_p^2$, using (3.7) and taking the result in Theorem 5.3 into account, we obtain

$$\begin{aligned}
&2\nu\|\mathbb{D}(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2)\|_{\Omega_f}^2 + \|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p^1 - \phi_p^2)\|_{\Omega_p}^2 \\
&\quad \le c(\|\mathbb{D}(\boldsymbol{u}_f^1)\|_{\Omega_f} + \|\mathbb{D}(\boldsymbol{u}_f^2)\|_{\Omega_f})\|\mathbb{D}(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2)\|_{\Omega_f}^2 \\
&\quad \le \frac{c\mathcal{C}}{\nu^{\frac{1}{2}}}\|\mathbb{D}(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2)\|_{\Omega_f}^2.
\end{aligned}$$

Thanks to the definition of $\mathcal{C}$ in Theorem 4.1 and (5.3), we have

$$c\nu\|\mathbb{D}(\boldsymbol{u}_f^1 - \boldsymbol{u}_f^2)\|_{\Omega_f}^2 + c\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p^1 - \phi_p^2)\|_{\Omega_p}^2 \le 0.$$

This leads to the global uniqueness of the weak solution of (P) in $\boldsymbol{V}_f \times X_p$. The uniqueness of the solution of (Q) in $\boldsymbol{X}_f \times Q_f \times X_p$ is obvious thanks to (3.3). □

6 Conclusion

By means of expanding the Navier–Stokes/Darcy model with BJS interface condition to a more large coupled system, we establish an a priori estimate of the weak solutions of the original problem. Then an existence result of weak solution to this coupled system is obtained without the restriction of the small data and/or the large viscosity for the first time. Finally, a global uniqueness result of the weak solution is derived which solves the open problem raised in [9]. The contribution of this paper is to overcome the difficulty caused by the nonlinear convection in the Navier–Stokes equations. However, if we consider the Navier–Stokes/Darcy problem with BJ interface condition, the difficulty pointed out in [4] and [11] is still there. Whether one can get the a priori estimate of the weak solution and obtain the unique solvability globally is still unclear. It is obvious that the uniqueness holds true for small data, while for the problem with general large data, only weaker results can be proved.

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