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Semirobust analysis of an H(div)-conforming DG method with semi-implicit time-marching for the evolutionary incompressible Navier–Stokes equations

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In this paper, we present a fully discrete analysis of an H(div)-conforming DG method with semi-implicit time-marching for the evolutionary incompressible Navier–Stokes equations. We use a semi-implicit time-discrete scheme in which the convection velocity is treated explicitly for the convection term. A stability analysis and *a priori* error estimates are given, in which the constants are independent of the negative powers of the viscosity. For inf-sup stable H(div)-conforming finite element pairs BDM_k/P_{k-1} and RT_k/P_k, the rate of convergence $k+1/2$ is proved for the L^2 error of the velocity in the case of $\nu \leq Ch$, where k is the degree of the polynomials in the velocity approximation. In particular, for the inf-sup stable finite element pair RT_k/P_k, the convergence rate of the pressure is also $k+1/2$ when $\nu \leq Ch$. The numerical experiments verify the analytical results.

Keywords: semirobust; H(div)-conforming DG method; semi-implicit time-marching; evolutionary Navier–Stokes equations.

1. Introduction

In the paper, we study fully discrete approximations to the evolutionary incompressible Navier–Stokes equations, with H(div)-conforming DG method in space and the semi-implicit Euler method in time. The use of the semi-implicit scheme is a popular approach, which avoids solving a nonlinear problem at each discrete time. We derive the error bound of the velocity, where the constants in the error bound of kinetic energy and dissipative energy are independent of the Reynolds number. These kinds of bounds are called semirobust or quasi-robust in the literature.

For semirobust analysis of the velocity, some works have been done recently. The H^1-conforming mixed finite element method with grad-div stabilization has been analyzed in De Frutos *et al.* (2016, 2018), in which a rate of convergence k is proved for the L^2 error of the velocity. The local projection stabilization method has been widely studied. With inf-sup and non-inf-sup stable mixed finite elements, the rates of convergence k and $k+1/2$ are proved for the L^2 error of the velocity, in Arndt *et al.* (2015) and de Frutos *et al.* (2019), respectively. In addition, for continuous internal penalty (CIP) methods, an error bound of order $k+1/2$ is obtained by using equal-order finite element pairs (Burman & Fernández, 2007). Unfortunately, they are not pressure-robust, that is to say, the error bounds of the velocity depend on the pressure. For divergence-free H^1-conforming finite element method, the error bound of the velocity is pressure-robust and semirobust; however, it is sub-optimal with convergence rate of k (Schroeder & Lube, 2017). An optimal estimate of order $k+1$ is given in Evans & Hughes (2013) for isogeometric finite element method, but it is not semirobust.

In this paper, we consider H(div)-conforming DG method for the evolutionary incompressible Navier–Stokes equations. H(div)-conforming DG method provides exact divergence-free velocity, which of course is also pressure-robust. In the method, the natural upwind stabilization is included as a convection stabilization. In addition, the fact that less stability is required leads to less numerical dissipation. So, it is a very promising approach for high Reynolds number flow (Schroeder & Lube, 2018; Schroeder *et al.*, 2019; Lube & Schroeder, 2020). We particularly focus on numerical analysis of H(div)-conforming DG method for the evolutionary incompressible Navier–Stokes equations. H(div)-conforming DG methods with the central flux have been analyzed for the incompressible Euler equations. However, numerical experiments suggest that the analysis is not sharp for the upwind flux (Guzmán *et al.*, 2017). H(div)-conforming DG methods for the space semidiscrete Navier–Stokes equations is presented in Schroeder *et al.* (2018), which basically follow the ideas from Guzmán *et al.* (2017). The L^2 error of the velocity is pressure-robust and semirobust, but it has a rate of convergence k (Schroeder *et al.*, 2018).

In this paper, we focus on the numerical analysis of the fully discrete H(div)-conforming DG method for the evolutionary incompressible Navier–Stokes equations with semi-implicit time-marching strategy. First, we focus on semirobust analysis of the velocity error bound at high Reynolds number. By introducing the Raviart–Thomas (RT) interpolation operator, we apply some specific techniques to the convection term and prove that the L^2 error of the velocity is semirobust, and has a rate of convergence $k + 1/2$ in the case of $\nu \leq Ch$, which shows the same convergence rate as the CIP method (Burman & Fernández, 2007). Secondly, following the ideas of Ahmed *et al.* (2017) and de Frutos *et al.* (2019), we give the error bound of the L^2 norm of a discrete in time primitive of the pressure, which is not of the stronger discrete in time L^2 norm of the pressure. For the inf-sup stable finite element pair, the convergence rate of the pressure is also obtained. In particular, for the inf-sup stable finite element pair RT_k/P_k, the convergence rate of the pressure is $k + 1/2$ in the case of $\nu \leq Ch$.

The outline of the paper is as follows. In Section 2, the weak form of the continuous Navier–Stokes equations is presented. In Section 3, we introduce H(div)-conforming and inf-sup stable finite element method for the evolutionary Navier–Stokes equations. In Section 4, we give the existence and uniqueness of solutions and stability analysis. In Section 5, by introducing H(div) interpolation operator, we prove that when the condition $\nu \leq Ch$ is satisfied, the error bound of the velocity, which is pressure-robust and semirobust, has a rate of convergence $k + 1/2$. In Section 6, the convergence rate of the pressure is obtained. In Section 7, we provide a comment on alternative time discretizations, full-implicit and implicit-explicit (IMEX) time-marching schemes. Finally, Section 8 presents numerical experiments to verify the analytical results.

2. Navier–Stokes problem

Throughout the paper, for $D \subseteq R^d$ ($d \in \{2, 3\}$), we use the Sobolev spaces $W^{m,p}(D)$ for scalar-valued functions with associated norms $\|\cdot\|_{W^{m,p}(D)}$ and seminorms $|\cdot|_{W^{m,p}(D)}$ for $m \geqslant 0$ and $p \geqslant 1$. In the case $m = 0$, $W^{0,p}(D) = L^p(D)$, and when $p = 2$, $W^{m,2}(D) = H^m(D)$. Spaces for vector- and tensor-valued functions are indicated with bold letters. In addition, for the Bochner space $L^p(0, T; Y)(1 \leq p \leq \infty)$, where Y is a Banach space, the abbreviation $L^p(Y) = L^p(0, T; Y)$ is frequently used. $\|v\|_{\mathcal{L}^p(0,T;Y)}$ represents a discrete approximation of $\|v\|_{L^p(0,T;Y)}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain with Lipschitz boundary $\partial\Omega$. We consider the evolutionary incompressible Navier–Stokes equations

$$
\begin{cases}
\partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & (0, T] \times \Omega, \\
\nabla \cdot \boldsymbol{u} = 0 & (0, T] \times \Omega, \\
\boldsymbol{u} = \boldsymbol{0} & (0, T] \times \partial\Omega, \\
\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) & \Omega,
\end{cases}
\tag{2.1}
$$

where $\boldsymbol{u}$ is the velocity field, p the kinematic pressure, $\nu > 0$ the kinematic viscosity, $\boldsymbol{u}_0$ a given initial velocity and $\boldsymbol{f}$ represents the external body force. Introduce

$$\mathbf{X} = \mathbf{H}_0^1(\Omega), \quad Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The weak formulation of (2.1) takes the form: find $(\boldsymbol{u}, p) : (0, T] \to (\boldsymbol{X}, Q)$, satisfying

$$\begin{aligned} (\partial_t \boldsymbol{u}, \boldsymbol{v}) + \nu a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) \qquad & \forall \boldsymbol{v} \in \boldsymbol{X}, \\ b(\boldsymbol{u}, q) = 0 \qquad & \forall q \in Q. \end{aligned} \tag{2.2}$$

Here, the multilinear forms are given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}, \quad c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x},$$
$$b(\boldsymbol{u}, q) = -\int_\Omega q (\nabla \cdot \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x}.$$

We introduce the space of weakly divergence-free velocities

$$\boldsymbol{V} = \{ \boldsymbol{v} \in \boldsymbol{X} : b(\boldsymbol{v}, q) = 0, \quad \forall \, q \in Q \}.$$

3. H(div)-conforming DG finite element method

Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform simplicial mesh of Ω, and mesh size h_T denotes the diameter of the element $T \in \mathcal{T}_h$. The skeleton $\mathcal{F}_h$ denotes the set of all facets. $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$ where $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are the subset of interior facets and boundary facets, respectively. Then, we define the jump $[\![\cdot]\!]_F$ and average $\{\cdot\}_F$ operator across interior facets $F \in \mathcal{F}_h^i$ by

$$[\![\phi]\!]_F = \phi^+ - \phi^-, \quad \{\phi\}_F = \frac{\phi^+ + \phi^-}{2}.$$

For boundary facets $F \in \mathcal{F}_h^\partial$, we set

$$[\![\phi]\!]_F = \{\phi\}_F = \phi.$$

Define the broken gradient $\nabla_h : \boldsymbol{H}^1(\mathcal{T}_h) \to \boldsymbol{L}^2(\Omega)$ by $(\nabla_h \boldsymbol{w})|_T = \nabla(\boldsymbol{w}|_T)$ and the broken Sobolev space $\boldsymbol{H}^m(\mathcal{T}_h) = \{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_T \in \boldsymbol{H}^m(T), \forall \, T \in \mathcal{T}_h \}$. We introduce the following space:

$$\boldsymbol{H}_0(\mathrm{div}; \Omega) = \left\{ \boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \nabla \cdot \boldsymbol{w} \in L^2(\Omega), \, \boldsymbol{w} \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \right\}.$$

We introduce H(div)-conforming velocity space and pressure space,

$$\boldsymbol{X}_h \subset \boldsymbol{H}_0(\mathrm{div}; \Omega), \quad Q_h \subset L_0^2(\Omega) = Q.$$

For simplicial mesh, we consider RT elements and Brezzi–Douglas–Marini (BDM) elements (Boffi *et al.*, 2013). For real application, BDM elements are computationally more efficient compared with RT elements since they have less degrees of freedom for the same velocity approximation (Boffi *et al.*, 2013).

The global spaces $\boldsymbol{X}_h$ and Q_h, say RT_k/P_k or BDM_k/P_{k-1} pair ($k \geq 1$), form a discretely inf-sup stable FE pair (Schroeder & Lube, 2018). That is, there exists $\beta > 0$, independent of the mesh size h, such that

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}_h, q_h\right)}{|||\boldsymbol{v}_h|||_e \|q_h\|_{L^2}} \geqslant \beta, \tag{3.1}$$

where $|||\cdot|||_e$ is defined in (3.6). The global spaces $\boldsymbol{X}_h$ and Q_h are divergence-conforming, namely

$$\nabla \cdot \boldsymbol{X}_h \subseteq Q_h. \tag{3.2}$$

3.1 *Numerical scheme*

Now, we consider the approximation of (2.2) with the semi-implicit Euler method in time and H(div)-conforming DG method in space. Given $\boldsymbol{u}_h^0$, an approximation to $\boldsymbol{u}_0$ in $\boldsymbol{X}_h$, find $\left(\boldsymbol{u}_h^{n+1}, p_h^{n+1}\right) \in \boldsymbol{X}_h \times Q_h$ such that

$$\begin{aligned} \left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right) + \nu a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + b_h(\boldsymbol{v}_h, p_h^{n+1}) + c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) &= \left(\boldsymbol{f}^{n+1}, \boldsymbol{v}_h\right), \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \\ b_h(\boldsymbol{u}_h^{n+1}, q_h) &= 0, \quad \forall q_h \in Q_h, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) = \int_\Omega \nabla_h \boldsymbol{u}_h^{n+1} : \nabla_h \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \sum_{F \in \mathcal{F}_h} \int_F \Big[&(\{\nabla_h \boldsymbol{u}_h^{n+1}\}\boldsymbol{n}_F \cdot [\![\boldsymbol{v}_h]\!]) + ([\![\boldsymbol{u}_h^{n+1}]\!] \cdot \{\nabla_h \boldsymbol{v}_h\}\boldsymbol{n}_F) \\ &- \left(\frac{\sigma}{h_F}[\![\boldsymbol{u}_h^{n+1}]\!] \cdot [\![\boldsymbol{v}_h]\!]\right)\Big] \,\mathrm{d}\boldsymbol{s}, \end{aligned} \tag{3.4}$$

$$b_h\left(\boldsymbol{u}_h^{n+1}, q_h\right) = -\int_\Omega q_h\left(\nabla_h \cdot \boldsymbol{u}_h^{n+1}\right) \mathrm{d}\boldsymbol{x}, \quad b_h\left(\boldsymbol{v}_h, p_h^{n+1}\right) = -\int_\Omega p_h^{n+1}\left(\nabla_h \cdot \boldsymbol{v}_h\right) \mathrm{d}\boldsymbol{x},$$

and

$$\begin{aligned} c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) = &\int_\Omega (\boldsymbol{u}_h^n \cdot \nabla_h)\boldsymbol{u}_h^{n+1} \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \sum_{F \in \mathcal{F}_h^i} \int_F (\boldsymbol{u}_h^n \cdot \boldsymbol{n}_F)[\![\boldsymbol{u}_h^{n+1}]\!]\{\boldsymbol{v}_h\} \,\mathrm{d}\boldsymbol{s} \\ &+ \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{2}|(\boldsymbol{u}_h^n \cdot \boldsymbol{n}_F)|[\![\boldsymbol{u}_h^{n+1}]\!][\![\boldsymbol{v}_h]\!] \,\mathrm{d}\boldsymbol{s}. \end{aligned} \tag{3.5}$$

The penalty parameter $\sigma > 0$ in (3.4) has to be sufficiently large such that the coercivity of a_h is guaranteed (Di Pietro & Ern, 2012, Lemma 4.12). In conjunction with the viscous term a_h, the following norm is used:

$$|||\boldsymbol{v}_h|||_e^2 = \|\nabla_h \boldsymbol{v}_h\|_{L^2}^2 + \sum_{F\in\mathcal{F}_h} \frac{\sigma}{h_F}\|[\![\boldsymbol{v}_h]\!]\|_{L^2(F)}^2. \tag{3.6}$$

In addition, we notice that the above appearance of traces of velocity normal derivatives dictates that the involved velocities at least belong to $\boldsymbol{H}^{\frac{3}{2}+\varepsilon}\left(\mathcal{T}_h\right)$ for some $\varepsilon > 0$. In order to facilitate numerical analysis, we introduce a larger space,

$$\boldsymbol{X}(h) = \boldsymbol{X}_h \oplus \left[\boldsymbol{X} \cap \boldsymbol{H}^{\frac{3}{2}+\varepsilon}\left(\mathcal{T}_h\right)\right].$$

Then, we define a stronger norm in the space $\boldsymbol{X}(h)$

$$|||\boldsymbol{v}|||_{e,\sharp}^2 = |||\boldsymbol{v}|||_e^2 + \sum_{T\in\mathcal{T}_h} h_T\|\nabla_h\boldsymbol{v}\cdot\boldsymbol{n}_T\|_{L^2(\partial T)}^2.$$

Notice that $|||\cdot|||_e$ and $|||\cdot|||_{e,\sharp}$ norms are uniformly equivalent on $\boldsymbol{X}_h$, namely

$$C|||\boldsymbol{v}_h|||_{e,\sharp} \leq |||\boldsymbol{v}_h|||_e \leq |||\boldsymbol{v}_h|||_{e,\sharp}, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_h, \tag{3.7}$$

with C independent of h; cf. (Di Pietro & Ern, 2012, Lemma 4.20) for scalar-valued functions. Assume that $\sigma > 0$ is sufficiently large. Then, there exist constants $C_\sigma > 0$ and $C > 0$, independent of h, such that

$$C_\sigma |||\boldsymbol{v}_h|||_e^2 \leq a_h\big(\boldsymbol{v}_h, \boldsymbol{v}_h\big), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_h \tag{3.8}$$

and

$$a_h\big(\boldsymbol{w}, \boldsymbol{v}_h\big) \leq C|||\boldsymbol{w}|||_{e,\sharp}|||\boldsymbol{v}_h|||_e, \quad \forall\, (\boldsymbol{w}, \boldsymbol{v}_h) \in \boldsymbol{X}(h) \times \boldsymbol{X}_h. \tag{3.9}$$

We introduce the discrete divergence-free space

$$\boldsymbol{V}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{X}_h : b\big(\boldsymbol{v}_h, q_h\big) = 0,\ \forall\, q_h \in Q_h\right\}.$$

Moreover, we introduce the jump seminorm

$$\left|\boldsymbol{v}_h\right|_{\boldsymbol{u}_h,\mathrm{upw}}^2 = \sum_{F\in\mathcal{F}_h^i} \int_F \frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)||[\![\boldsymbol{v}_h]\!]|^2\,\mathrm{d}\boldsymbol{s} \tag{3.10}$$

and the space $P_d^l(\mathcal{T}_h)$

$$P_d^l(\mathcal{T}_h) = \{\boldsymbol{v}_h \in L^2(\Omega) : \boldsymbol{v}_h|_T \in (P_l(T))^d, \forall\, T \in \mathcal{T}_h\},$$

where $l \geq 0$ is an integer.

Let $P^l\boldsymbol{w}$ denote the L^2-projection of $\boldsymbol{w}$ onto $P_d^l(\mathcal{T}_h)$, then there exists C, independent of h, such that, for $0 \leqslant j \leqslant s \leqslant l+1$, $1 \leqslant p \leqslant \infty$, there holds (Di Pietro & Droniou, 2020, Theorem 1.45):

$$|\boldsymbol{w} - P^l\boldsymbol{w}|_{W^{j,p}(T)} \leq C h_T^{s-j} |\boldsymbol{w}|_{W^{s,p}(T)}, \quad \forall T \in \mathcal{T}_h, \forall \boldsymbol{w} \in \boldsymbol{W}^{s,p}(T). \tag{3.11}$$

The space $P_d^l(\mathcal{T}_h)$ satisfies the discrete trace inequality (Di Pietro & Ern, 2012, Remark 1.47)

$$\|\boldsymbol{v}_h\|_{L^2(\partial T)} \leqslant C_{\mathrm{tr}} h_T^{-\frac{1}{2}} \|\boldsymbol{v}_h\|_{L^2(T)}, \quad \forall T \in \mathcal{T}_h, \forall \boldsymbol{v}_h \in P_d^l(\mathcal{T}_h). \tag{3.12}$$

Let $0 \leqslant m \leqslant \ell$ and $1 \leqslant p, q \leqslant \infty$, the space $P_d^l(\mathcal{T}_h)$ satisfies the local inverse inequality (Ern & Guermond, 2004, Lemma 1.138)

$$\|\boldsymbol{v}_h\|_{W^{\ell,p}(T)} \leqslant C_{\mathrm{inv}} h_T^{m-\ell+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|\boldsymbol{v}_h\|_{W^{m,q}(T)}, \quad \forall T \in \mathcal{T}_h, \forall \boldsymbol{v}_h \in P_d^l(\mathcal{T}_h). \tag{3.13}$$

Furthermore, there is C independent of h, such that (Di Pietro & Ern, 2012, Corollary 5.4)

$$\|\boldsymbol{v}_h\|_{L^2} \leq C |||\boldsymbol{v}_h|||_e, \quad \forall \boldsymbol{v}_h \in P_d^l(\mathcal{T}_h). \tag{3.14}$$

In the following numerical analysis, RT interpolation operator I_{Rt}^h (Wang & Ye, 2007; Boffi *et al.*, 2013) plays a crucial role

$$\begin{cases} \displaystyle\int_{\partial T} (\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}) \cdot \boldsymbol{n} p_k \, \mathrm{d}s = 0, \forall p_k \in P_k(\partial T), \\ \displaystyle\int_T (\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}) \cdot \boldsymbol{p}_{k-1} \, \mathrm{d}\boldsymbol{x} = 0, \forall \boldsymbol{p}_{k-1} \in (P_{k-1}(T))^d. \end{cases} \tag{3.15}$$

LEMMA 3.1 (Boffi *et al.*, 2013, Proposition 2.5.2) Let I_{Rt}^h be the interpolation operator $\boldsymbol{H}^1(\Omega) \to RT_k$ and π_0 be the L^2-orthogonal projection on $\nabla \cdot RT_k$. Then, we have, for all $\boldsymbol{q} \in \boldsymbol{H}^1(\Omega)$,

$$\nabla \cdot (I_{\mathrm{Rt}}^h \boldsymbol{q}) = \pi_0 \nabla \cdot \boldsymbol{q}.$$

LEMMA 3.2 Let T be an n-simplicial (triangular or tetrahedral) element. Then, we have

$$BDM_k^0(T) = RT_k^0(T) \subset (P_k(T))^d,$$

where

$$RT_k^0(T) = \{\boldsymbol{q} \in RT_k(T) \mid \nabla \cdot \boldsymbol{q} = 0\},$$
$$BDM_k^0(T) = \{\boldsymbol{q} \in BDM_k(T) \mid \nabla \cdot \boldsymbol{q} = 0\}.$$

Proof. cf. (Boffi *et al.*, 2013, p. 90). □

The RT interpolation operator satisfies the following approximation properties (Guzmán *et al.*, 2017, p. 1737): for $\forall\, T \in \mathcal{T}_h$ and $\forall\, \boldsymbol{w} \in \boldsymbol{H}^m(T)$, with $1 \le m \le k+1$, there holds

$$\|\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^2(T)} + h_T \|\nabla \boldsymbol{w} - \nabla_h I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^2(T)} \leqslant C h_T^m \, |\boldsymbol{w}|_{H^m(T)}, \tag{3.16}$$

and for $\forall\, T \in \mathcal{T}_h$ and $\forall\, \boldsymbol{w} \in \boldsymbol{W}^{1,\infty}(T)$, we also have the following bound:

$$\|\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} + h_T \|\nabla \boldsymbol{w} - \nabla_h I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} \leqslant C h_T \|\nabla \boldsymbol{w}\|_{L^\infty(T)}. \tag{3.17}$$

REMARK 1 For the L^∞ estimate in (3.17), $\|\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} \leqslant C h_T \|\nabla \boldsymbol{w}\|_{L^\infty(T)}$, the detailed proof is similarly proceeded as in the proof of the lowest-order RT interpolation operator $\mathcal{I}_K^{\mathrm{RT}}$ in Ern & Guermond (2004); see (Ern & Guermond, 2004, Lemma 1.113 and Theorem 1.114). In fact, Theorem 1.114 in Ern & Guermond (2004) is valid with $p = \infty$ by means of Lemma 1.113 in Ern & Guermond (2004). In addition, for the stability estimate in (3.17), $\|\nabla \boldsymbol{w} - \nabla_h I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} \leqslant C \|\nabla \boldsymbol{w}\|_{L^\infty(T)}$, it can be proved by using a standard argument below.

$$\begin{aligned}
\|\nabla \boldsymbol{w} - \nabla_h I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} &\le \|\nabla \boldsymbol{w} - \nabla_h P^k \boldsymbol{w}\|_{L^\infty(T)} + \|\nabla_h I_{\mathrm{Rt}}^h \boldsymbol{w} - \nabla_h P^k \boldsymbol{w}\|_{L^\infty(T)} \\
&\le C \|\nabla \boldsymbol{w}\|_{L^\infty(T)} + C h_T^{-1} (\|\boldsymbol{w} - I_{\mathrm{Rt}}^h \boldsymbol{w}\|_{L^\infty(T)} + \|\boldsymbol{w} - P^k \boldsymbol{w}\|_{L^\infty(T)}) \\
&\le C \|\nabla \boldsymbol{w}\|_{L^\infty(T)},
\end{aligned}$$

in which we use the triangle inequality, the inverse inequality, (3.11) and the L^∞ estimate in (3.17).

Next, we introduce two essential lemmas, which are frequently used in fully discrete numerical analysis of the evolutionary incompressible Navier–Stokes equations.

LEMMA 3.3 (Heywood & Rannacher, 1990, Lemma 5.1) Let $k, B, a_j, b_j, c_j, \gamma_j$ be non-negative numbers such that

$$a_n + k \sum_{j=0}^{n} b_j \le k \sum_{j=0}^{n} \gamma_j a_j + k \sum_{j=0}^{n} c_j + B, \quad \textit{for} \quad n \ge 0.$$

Suppose that $k\gamma_j < 1$, for all j, and set $\sigma_j = (1 - k\gamma_j)^{-1}$. Then,

$$a_n + k \sum_{j=0}^{n} b_j \le \exp\left(k \sum_{j=0}^{n} \sigma_j \gamma_j \right) \left\{ k \sum_{j=0}^{n} c_j + B \right\}, \quad \text{for} \quad n \ge 0.$$

LEMMA 3.4 (John, 2016, Lemma 7.67) Let $v, \partial_t v, \partial_{tt} v \in L^2\left(t^n, t^{n+1}; L^2(\Omega)\right)$, then

$$\left\| \partial_t v^{n+1} - \frac{v^{n+1} - v^n}{\Delta t} \right\|_{L^2(\Omega)}^2 \le \Delta t \left\| \partial_{tt} v \right\|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2. \tag{3.18}$$

In addition, integration by parts of different terms in c_h is done in several places. The following equation would help us to understand integration by parts of different terms in c_h. For $\boldsymbol{u}_h, \boldsymbol{w}_h, \boldsymbol{v}_h \in \boldsymbol{X}_h$, and $\nabla \cdot \boldsymbol{u}_h = 0$, we have

$$\sum_{F\in\mathcal{F}_h} \int_F \left(\boldsymbol{u}_h \cdot \boldsymbol{n}_F\right) \left[[\![\boldsymbol{w}_h]\!] \cdot \left\{\boldsymbol{v}_h\right\} + [\![\boldsymbol{v}_h]\!] \cdot \{\boldsymbol{w}_h\}\right] \mathrm{d}s$$
$$= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\boldsymbol{u}_h \cdot \boldsymbol{n}_K\right) \boldsymbol{w}_h \cdot \boldsymbol{v}_h \, \mathrm{d}s = \sum_{K\in\mathcal{T}_h} \int_K \left[(\boldsymbol{u}_h \cdot \nabla_h)\boldsymbol{w}_h \cdot \boldsymbol{v}_h + (\boldsymbol{u}_h \cdot \nabla_h)\boldsymbol{v}_h \cdot \boldsymbol{w}_h \right] \mathrm{d}\boldsymbol{x}, \qquad (3.19)$$

where the first equal sign is due to $[\![\boldsymbol{w}_h \cdot \boldsymbol{v}_h]\!] = [\![\boldsymbol{w}_h]\!] \cdot \left\{\boldsymbol{v}_h\right\} + [\![\boldsymbol{v}_h]\!] \cdot \{\boldsymbol{w}_h\}$ and the second equal sign due to element-wise integration by parts with $\nabla \cdot \boldsymbol{u}_h = 0$.

4. Existence and uniqueness of solutions and stability analysis

In this section, we derive the velocity energy estimate, in which the constants are independent of the negative powers of the viscosity, and prove well-posedness of the fully discrete scheme (3.3).

LEMMA 4.1 Let $\boldsymbol{f} \in L^1(0, T; \boldsymbol{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{u}_h^0$ is an approximation of $\boldsymbol{u}_0$. Then, (3.3) has a unique solution, and for all $N \geq 0$, the following stability estimate holds:

$$\frac{1}{2} \left\| \boldsymbol{u}_h^{N+1} \right\|_{L^2}^2 + C_\sigma \nu \Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}_h^n|||_e^2 + \Delta t \sum_{n=1}^{N+1} \left|\boldsymbol{u}_h^n\right|^2_{\boldsymbol{u}_h^{n-1},\mathrm{upw}} \leq \|\boldsymbol{u}_h^0\|_{L^2}^2 + \frac{3}{2} \left(\sum_{n=1}^{N+1} \Delta t \|\boldsymbol{f}^n\|_{L^2} \right)^2 .$$

Proof. First, taking $\left(\boldsymbol{v}_h, q_h\right) = \left(\boldsymbol{u}_h^{n+1}, p_h^{n+1}\right)$ in (3.3) yields

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{u}_h^{n+1} \right) + \nu a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\right) + c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\right) = \left(\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}\right). \qquad (4.1)$$

Due to (3.8) and (3.10), we have

$$\begin{aligned} a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}\right) &\geq C_\sigma |||\boldsymbol{u}_h^{n+1}|||_e^2, \\ c_h(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}) &= \left|\boldsymbol{u}_h^{n+1}\right|^2_{\boldsymbol{u}_h^n,\mathrm{upw}}. \end{aligned} \qquad (4.2)$$

Using (4.1) and (4.2), we have

$$\frac{1}{\Delta t} \left(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1} \right) + \nu C_\sigma |||\boldsymbol{u}_h^{n+1}|||_e^2 + \left|\boldsymbol{u}_h^{n+1}\right|^2_{\boldsymbol{u}_h^n,\mathrm{upw}} \leq (\boldsymbol{f}^{n+1}, \boldsymbol{u}_h^{n+1}). \qquad (4.3)$$

Using (4.3) and Cauchy–Schwarz inequality, we can obtain

$$\|\boldsymbol{u}_h^{n+1}\|_{L^2}\frac{(\|\boldsymbol{u}_h^{n+1}\|_{L^2}-\|\boldsymbol{u}_h^{n}\|_{L^2})}{\Delta t}\le\left(\frac{\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n}}{\Delta t},\boldsymbol{u}_h^{n+1}\right)$$
$$\le(\boldsymbol{f}^{n+1},\boldsymbol{u}_h^{n+1})\le\|\boldsymbol{f}^{n+1}\|_{L^2}\|\boldsymbol{u}_h^{n+1}\|_{L^2}. \tag{4.4}$$

Estimate (4.4) gives

$$\|\boldsymbol{u}_h^{n+1}\|_{L^2}-\|\boldsymbol{u}_h^{n}\|_{L^2}\le\Delta t\|\boldsymbol{f}^{n+1}\|_{L^2}. \tag{4.5}$$

Observing that the left-hand side of (4.5) constitute a telescopic sum, we can get

$$\max_{1\le j\le N+1}\|\boldsymbol{u}_h^{j}\|_{L^2}\le\sum_{n=1}^{N+1}\Delta t\|\boldsymbol{f}^{n}\|_{L^2}+\|\boldsymbol{u}_h^{0}\|_{L^2}. \tag{4.6}$$

Due to

$$\left(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n},\boldsymbol{u}_h^{n+1}\right)=\frac{1}{2}\left(\left\|\boldsymbol{u}_h^{n+1}\right\|_{L^2}^2+\left\|\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n}\right\|_{L^2}^2-\|\boldsymbol{u}_h^{n}\|_{L^2}^2\right). \tag{4.7}$$

Inserting (4.7) in (4.3) yields

$$\frac{1}{2}\left\|\boldsymbol{u}_h^{n+1}\right\|_{L^2}^2+C_\sigma\nu\Delta t|||\boldsymbol{u}_h^{n+1}|||_e^2+\Delta t\left|\boldsymbol{u}_h^{n+1}\right|_{\boldsymbol{u}_h^{n},\mathrm{upw}}^2\le\frac{1}{2}\|\boldsymbol{u}_n^{h}\|_{L^2}^2+\Delta t(\boldsymbol{f}^{n+1},\boldsymbol{u}_h^{n+1}).$$

Observing that the first terms on both sides constitute a telescopic sum, we can obtain

$$\frac{1}{2}\left\|\boldsymbol{u}_h^{N+1}\right\|_{L^2}^2+C_\sigma\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{u}_h^{n}|||_e^2+\Delta t\sum_{n=1}^{N+1}\left|\boldsymbol{u}_h^{n}\right|_{\boldsymbol{u}_h^{n-1},\mathrm{upw}}^2\le\frac{1}{2}\left\|\boldsymbol{u}_h^{0}\right\|_{L^2}^2+\sum_{n=1}^{N+1}\Delta t(\boldsymbol{f}^{n},\boldsymbol{u}_h^{n}).$$

Using Cauchy–Schwarz inequality, (4.6) and Young's inequality, we can get

$$\frac{1}{2}\left\|\boldsymbol{u}_h^{N+1}\right\|_{L^2}^2+C_\sigma\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{u}_h^{n}|||_e^2+\Delta t\sum_{n=1}^{N+1}\left|\boldsymbol{u}_h^{n}\right|_{\boldsymbol{u}_h^{n-1},\mathrm{upw}}^2\le\|\boldsymbol{u}_h^{0}\|_{L^2}^2+\frac{3}{2}\left(\sum_{n=1}^{N+1}\Delta t\|\boldsymbol{f}^{n}\|_{L^2}\right)^2.$$

The existence and uniqueness of the velocity solution is proved with Lax–Milgram theorem, which can be applied since (3.3) is a linear problem. In addition, the existence and uniqueness of the pressure solution can be proved by the inf-sup condition (3.1). □

5. Semirobust analysis for the velocity

For the analysis of the Navier–Stokes problem, the RT interpolation operator (3.15) is used to make the error splitting

$$\boldsymbol{u}-\boldsymbol{u}_h=\left(\boldsymbol{u}-I_{\mathrm{Rt}}^h\boldsymbol{u}\right)-\left(\boldsymbol{u}_h-I_{\mathrm{Rt}}^h\boldsymbol{u}\right)=\boldsymbol{\eta}-\boldsymbol{e}_h. \tag{5.1}$$

REMARK 2 Whether BDM_k or RT_k ($k \geq 1$) elements are chosen to be the velocity space, we introduce $I^h_{\mathrm{Rt}}\boldsymbol{u}$ to make the error splitting, respectively. Because $I^h_{Rt}\boldsymbol{u} \in BDM^0_k(T) = RT^0_k(T), \forall T \in \mathcal{T}_h$, due to Lemmas 3.1 and 3.2.

First, in order to make a better analysis for the convective term, we give an important lemma.

LEMMA 5.1 Assume $\boldsymbol{u}^n \in \boldsymbol{W}^{1,\infty}(\Omega)$. There exists a $C > 0$, independent of h, such that

$$\int_\Omega \left[(\boldsymbol{u}^n \cdot \nabla_h)\boldsymbol{e}^{n+1}_h \cdot \boldsymbol{\eta}^{n+1}\right] \mathrm{d}\boldsymbol{x} \leq C\|\nabla \boldsymbol{u}^n\|_{L^\infty}(\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + \|\boldsymbol{e}^{n+1}_h\|^2_{L^2}).$$

Proof. Let $\langle \boldsymbol{u}^n\rangle_T$ denote the mean value of $\boldsymbol{u}^n$ on each cell $T \in \mathcal{T}_h$

$$\langle \boldsymbol{u}^n\rangle_T = \frac{\int_T \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}{|T|}.$$

On the one hand,

$$\|\boldsymbol{u}^n - \langle \boldsymbol{u}^n\rangle_T\|_{L^\infty(T)} \leqslant Ch_T\|\nabla \boldsymbol{u}^n\|_{L^\infty(T)}, \tag{5.2}$$

since $\boldsymbol{u}^n$ is Lipschitz continuous (Di Pietro & Ern, 2012, p. 59).

On the other hand, $\boldsymbol{e}^{n+1}_h = \boldsymbol{u}^{n+1}_h - I^h_{\mathrm{Rt}}\boldsymbol{u}^{n+1}$, where $I^h_{\mathrm{Rt}}\boldsymbol{u}^{n+1} \in RT^0_k(T)$ from Lemma 3.1 and $\boldsymbol{u}^{n+1}_h \in RT^0_k(T)$ or $\boldsymbol{u}^{n+1}_h \in BDM^0_k(T)$. Because of Lemma 3.2, we have $\boldsymbol{e}^{n+1}_h|_T \in (P_k(T))^d$, so $\big(\langle \boldsymbol{u}^n\rangle_T \cdot \nabla_h\big)\boldsymbol{e}^{n+1}_h|_T \in (P_{k-1}(T))^d$. Using (3.15), we have

$$\int_T \big(\langle \boldsymbol{u}^n\rangle_T \cdot \nabla_h\big)\boldsymbol{e}^{n+1}_h \cdot \boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x} = 0, \quad \forall T \in \mathcal{T}_h. \tag{5.3}$$

Using (5.2) and (5.3), Hölder's inequality, inverse inequality and Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_\Omega (\boldsymbol{u}^n \cdot \nabla_h)\boldsymbol{e}^{n+1}_h \cdot \boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x} &= \sum_{T\in\mathcal{T}_h}\int_T \big((\boldsymbol{u}^n - \langle \boldsymbol{u}^n\rangle_T)\cdot \nabla_h\big)\boldsymbol{e}^{n+1}_h \cdot \boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x}\\ &\leq C\|\nabla \boldsymbol{u}^n\|_{L^\infty}(\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + \|\boldsymbol{e}^{n+1}_h\|^2_{L^2}).\end{aligned}$$

□

Now, we present an error estimate for the convection term that allows for semirobust estimates for the velocity.

Lemma 5.2 There exists $C > 0$, independent of h such that for all $0 \le n \le N$, the following estimate holds:

$$\begin{aligned}
&c_h\Big(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\Big) - c_h\Big(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{e}_h^{n+1}\Big) \\
&\le C\Big\{\|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|_{L^2}^2 + \|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2 + \|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}(\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2 + h^2\|\nabla \boldsymbol{\eta}^{n+1}\|_{L^2}^2) \\
&\quad + \|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|_{L^2}^2 + \|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2 + \|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2 \\
&\quad + \|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2 + h\|\nabla \boldsymbol{\eta}^{n+1}\|_{L^2}^2)\Big\}.
\end{aligned}$$

Proof. First, by using (3.5) and $[\![\boldsymbol{u}]\!]_F = 0$ for $\forall F \in \mathcal{F}_h^i$, we have

$$\begin{aligned}
\boldsymbol{I} &= c_h\Big(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\Big) - c_h\Big(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{e}_h^{n+1}\Big) \\
&= \int_\Omega \Big[\big(\boldsymbol{u}^n\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1} - \big(\boldsymbol{u}_h^n\cdot\nabla_h\big)\boldsymbol{u}_h^{n+1}\cdot\boldsymbol{e}_h^{n+1}\Big]\,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}]\!]\{\boldsymbol{e}_h^{n+1}\}\,\mathrm{d}\boldsymbol{s} \\
&\quad + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s} \\
&= \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3.
\end{aligned} \tag{5.4}$$

By using $\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1} = \boldsymbol{\eta}^{n+1} - \boldsymbol{e}_h^{n+1}$, we make error splitting for $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$, respectively, as shown below.

$$\begin{aligned}
\boldsymbol{I}_1 &= \int_\Omega \Big[\big(\boldsymbol{u}^n\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1} - \big(\boldsymbol{u}_h^n\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1} + \big(\boldsymbol{u}_h^n\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1} - \big(\boldsymbol{u}_h^n\cdot\nabla_h\big)\boldsymbol{u}_h^{n+1}\cdot\boldsymbol{e}_h^{n+1}\Big]\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \Big[\big((\boldsymbol{u}^n - \boldsymbol{u}_h^n)\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1}\Big]\,\mathrm{d}\boldsymbol{x} + \int_\Omega \big(\boldsymbol{u}_h^n\cdot\nabla_h\big)\boldsymbol{\eta}^{n+1}\cdot\boldsymbol{e}_h^{n+1}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \big(\boldsymbol{u}_h^n\cdot\nabla_h\big)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{e}_h^{n+1}\,\mathrm{d}\boldsymbol{x} \\
&= \boldsymbol{I}_{1,1} + \boldsymbol{I}_{1,2} + \boldsymbol{I}_{1,3},
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
\boldsymbol{I}_2 &= -\sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{\eta}^{n+1}]\!]\{\boldsymbol{e}_h^{n+1}\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h^{n+1}]\!]\{\boldsymbol{e}_h^{n+1}\}\,\mathrm{d}\boldsymbol{s} \\
&= \boldsymbol{I}_{2,1} + \boldsymbol{I}_{2,2},
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\boldsymbol{I}_3 &= \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s} - \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{e}_h^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s}\\ &= \boldsymbol{I}_{3,1} - \left|\boldsymbol{e}_h^{n+1}\right|^2_{\boldsymbol{u}_h^n,\mathrm{upw}}.\end{aligned} \tag{5.7}$$

Notice that

$$\boldsymbol{I}_{1,3} + \boldsymbol{I}_{2,2} = 0 \tag{5.8}$$

and

$$\begin{aligned}\boldsymbol{I}_{1,2}+\boldsymbol{I}_{2,1} &= \int_\Omega \left(\boldsymbol{u}_h^n\cdot\nabla_h\right)\boldsymbol{\eta}^{n+1}\cdot\boldsymbol{e}_h^{n+1}\,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{\eta}^{n+1}]\!]\{\boldsymbol{e}_h^{n+1}\}\,\mathrm{d}\boldsymbol{s}\\ &= -\int_\Omega \left(\boldsymbol{u}_h^n\cdot\nabla_h\right)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x} + \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h^{n+1}]\!]\{\boldsymbol{\eta}^{n+1}\}\,\mathrm{d}\boldsymbol{s}.\end{aligned} \tag{5.9}$$

Now, using the above identities (5.4)–(5.9), we can get

$$\begin{aligned}\boldsymbol{I} &= \left\{\int_\Omega\left[\left((\boldsymbol{u}^n-\boldsymbol{u}_h^n)\cdot\nabla\right)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1}\right]\mathrm{d}\boldsymbol{x} - \int_\Omega\left(\boldsymbol{u}_h^n\cdot\nabla_h\right)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x}\right\}\\ &\quad+\left\{\sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h^{n+1}]\!]\{\boldsymbol{\eta}^{n+1}\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s}\right.\\ &\quad\left.-\sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{e}_h^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s}\right\} = \boldsymbol{I}_{vol}+\boldsymbol{I}_{fac}.\end{aligned}$$

For the first and second integral terms in $\boldsymbol{I}_{vol}$, first by writing $\boldsymbol{u}^n - \boldsymbol{u}_h^n = \boldsymbol{\eta}^n - \boldsymbol{e}_h^n$ and $\boldsymbol{u}_h^n = \boldsymbol{e}_h^n + \boldsymbol{u}^n - \boldsymbol{\eta}^n$, respectively, and then applying Hölder's inequality, Cauchy–Schwarz inequality, inverse inequality and Lemma 5.1, we have

$$\begin{aligned}\boldsymbol{I}_{vol} &= \int_\Omega\left[\left(\boldsymbol{\eta}^n\cdot\nabla\right)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1} - \left(\boldsymbol{e}_h^n\cdot\nabla\right)\boldsymbol{u}^{n+1}\cdot\boldsymbol{e}_h^{n+1}\right]\mathrm{d}\boldsymbol{x} - \int_\Omega\left(\boldsymbol{e}_h^n\cdot\nabla_h\right)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x}\\ &\quad-\int_\Omega\left(\boldsymbol{u}^n\cdot\nabla_h\right)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x} + \int_\Omega\left(\boldsymbol{\eta}^n\cdot\nabla_h\right)\boldsymbol{e}_h^{n+1}\cdot\boldsymbol{\eta}^{n+1}\,\mathrm{d}\boldsymbol{x}\\ &\leqslant C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|^2_{L^2} + C\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|^2_{L^2}\\ &\quad+C\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|^2_{L^2} + C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|^2_{L^2}.\end{aligned} \tag{5.10}$$

For the face term,

$$
\begin{aligned}
\boldsymbol{I}_{fac} &= \sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h^{n+1}]\!]\{\boldsymbol{\eta}^{n+1}\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F|[\![\boldsymbol{\eta}^{n+1}]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s}\\
&\quad - \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F||[\![\boldsymbol{e}_h^{n+1}]\!]|^2\,\mathrm{d}\boldsymbol{s}\\
&\leqslant \sum_{F\in\mathcal{F}_h^i}\int_F |(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)||\{\boldsymbol{\eta}^{n+1}\}|^2\,\mathrm{d}\boldsymbol{s} + \frac{1}{4}\sum_{F\in\mathcal{F}_h^i}\int_F |(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)||[\![\boldsymbol{\eta}^{n+1}]\!]|^2\,\mathrm{d}\boldsymbol{s}\\
&= \boldsymbol{I}_{fac1} + \boldsymbol{I}_{fac2},
\end{aligned}
\tag{5.11}
$$

where we apply Cauchy–Schwarz inequality to the first two terms on the right-hand side of the equal sign to cancel out the third term.

For the facet term $\boldsymbol{I}_{fac1}$, we apply $\boldsymbol{u}_h^n = \boldsymbol{e}_h^n + I_{\mathrm{Rt}}^h\boldsymbol{u}^n$, Hölder's inequality, Cauchy–Schwarz inequality and $\|I_{\mathrm{Rt}}^h\boldsymbol{u}^n\|_{L^\infty} \le C\|\boldsymbol{u}^n\|_{W^{1,\infty}}$ from (3.17) to obtain

$$
\begin{aligned}
|\boldsymbol{I}_{fac1}| &\le \sum_{F\in\mathcal{F}_h^i}\int_F |(\boldsymbol{e}_h^n\cdot\boldsymbol{n}_F)||\{\boldsymbol{\eta}^{n+1}\}|^2\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F |(I_{\mathrm{Rt}}^h\boldsymbol{u}^n\cdot\boldsymbol{n}_F)||\{\boldsymbol{\eta}^{n+1}\}|^2\,\mathrm{d}\boldsymbol{s}\\
&\le \|\boldsymbol{\eta}^{n+1}\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\boldsymbol{e}_h^n\|^2_{L^2(F)} + \|\boldsymbol{\eta}^{n+1}\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\{\boldsymbol{\eta}^{n+1}\}\|^2_{L^2(F)}\\
&\quad + \|I_{\mathrm{Rt}}^h\boldsymbol{u}^n\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\{\boldsymbol{\eta}^{n+1}\}\|^2_{L^2(F)}\\
&\le C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|^2_{L^2} + Ch\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + h\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2})\\
&\quad + C\|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + h\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2}),
\end{aligned}
\tag{5.12}
$$

where by applying a continuous trace inequality and quasi-uniformity of the mesh, we have

$$
\begin{aligned}
\sum_{F\in\mathcal{F}_h^i}\|\{\boldsymbol{\eta}^{n+1}\}\|^2_{L^2(F)} &\leqslant 2\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2(\partial T)} \leqslant C\sum_{T\in\mathcal{T}_h}(h_T^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2(T)} + h_T\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2(T)})\\
&\leqslant C(h^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + h\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2}).
\end{aligned}
\tag{5.13}
$$

Similarly, for the facet term $\boldsymbol{I}_{fac2}$, it can be inferred that

$$
\begin{aligned}
|\boldsymbol{I}_{fac2}| &\le C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|^2_{L^2} + Ch\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + h\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2})\\
&\quad + C\|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|^2_{L^2} + h\|\nabla\boldsymbol{\eta}^{n+1}\|^2_{L^2}).
\end{aligned}
\tag{5.14}
$$

Therefore, by (5.12) and (5.14), we have

$$|\boldsymbol{I}_{fac}| \leq C\|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|_{L^2}^2 + Ch\|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2 + h\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2)$$
$$+ C\|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2 + h\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2). \tag{5.15}$$

By combining (5.10) and (5.15), we can finish the proof. □

THEOREM 5.3 Let $\boldsymbol{u}_h(0) = I_{\mathrm{Rt}}^h \boldsymbol{u}_0$, and assume the following regularities for the solution $(\boldsymbol{u}, p)$ of (2.2)

$$\begin{aligned}
&\partial_{tt}\boldsymbol{u} \in L^2\left(0,T;\boldsymbol{H}^1(\Omega)\right), \quad \partial_t\boldsymbol{u} \in L^4\left(0,T;\boldsymbol{L}^2(\Omega)\right),\\
&\partial_t\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^r(\Omega)), \quad p \in L^2\left(0,T;H^r\left(\Omega\right)\right),\\
&\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{H}^r(\Omega)), \quad \boldsymbol{u} \in L^2\left(0,T;\boldsymbol{W}^{1,\infty}\left(\Omega\right)\right).
\end{aligned} \tag{5.16}$$

Then, with $r_{\boldsymbol{u}} = \min\{r, k+1\}$ and a constant C independent of h and ν^{-1}, when time step is sufficiently small, $\alpha_n \Delta t < 1, \forall\, n = 1, \ldots, N+1$, we have the following error estimate:

$$\left\|\boldsymbol{e}_h^{N+1}\right\|_{L^2}^2 + \nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{e}_h^n|||_e^2 \leq \exp\left(\Delta t\sum_{n=1}^{N+1}\frac{\alpha_n}{1-\Delta t\alpha_n}\right)\left(C\Delta t\sum_{n=1}^{N+1}\mathcal{G}_n\right),$$

where

$$\begin{aligned}
\alpha_n &= C(1 + \|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty} + \|\nabla\boldsymbol{u}^n\|_{L^\infty} + \|\nabla\boldsymbol{u}^{n-1}\|_{L^\infty}), \forall\, n = 1, \ldots, N,\\
\alpha_{N+1} &= C(1 + \|\nabla\boldsymbol{u}^{N+1}\|_{L^\infty} + \|\nabla\boldsymbol{u}^N\|_{L^\infty}),
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
\mathcal{G}_n = &\left\|\partial_t\left(\boldsymbol{u}^n - I_{\mathrm{Rt}}^h\boldsymbol{u}^n\right)\right\|_{L^2}^2 + \Delta t\left\|I_{\mathrm{Rt}}^h\left(\partial_{tt}\boldsymbol{u}\right)\right\|_{L^2\left(t^{n-1},t^n;\boldsymbol{L}^2(\Omega)\right)}^2\\
&+ \|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n-1}\|_{L^2}^2 + \|\nabla\boldsymbol{u}^{n-1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|_{L^2}^2 + \|\nabla\boldsymbol{u}^n\|_{L^\infty}(\|\boldsymbol{\eta}^n\|_{L^2}^2 + h^2\|\nabla\boldsymbol{\eta}^n\|_{L^2}^2)\\
&+ \|\boldsymbol{u}^{n-1}\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^n\|_{L^2}^2 + h\|\nabla\boldsymbol{\eta}^n\|_{L^2}^2) + \nu|||\boldsymbol{\eta}^n|||_{e,\sharp}^2\\
&+ (\Delta t)^3\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))}^2 + (\Delta t)^2\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\partial_t\boldsymbol{u}^n\|_{L^2}^2.
\end{aligned} \tag{5.18}$$

Proof. Due to the consistency property, we take arbitrary $\left(\boldsymbol{v}_h, q_h\right) \in \boldsymbol{X}_h \times Q_h$ as test functions in (2.2) and subtract (3.3) from (2.2). One obtains the following error equation in the space $\boldsymbol{V}_h$:

$$\left(\partial_t\boldsymbol{u}^{n+1} - \frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right) + \nu a_h\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + c_h\left(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right)$$
$$- c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) = 0, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{5.19}$$

Next, taking $\boldsymbol{v}_h = \boldsymbol{e}_h^{n+1}$ in (5.19) yields

$$\left(\partial_t \boldsymbol{u}^{n+1} - \frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{e}_h^{n+1}\right) + \nu a_h\left(\boldsymbol{\eta}^{n+1}, \boldsymbol{e}_h^{n+1}\right) - \nu a_h\left(\boldsymbol{e}_h^{n+1}, \boldsymbol{e}_h^{n+1}\right)$$
$$+ c_h\left(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{e}_h^{n+1}\right) = 0. \tag{5.20}$$

We expand the left argument of the first term of (5.20) in the form

$$\begin{aligned}
&\partial_t \boldsymbol{u}^{n+1} - \frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t} \\
&= \partial_t \boldsymbol{u}^{n+1} - \partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} + \partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t} \\
&\quad + \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t} - \frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t} \\
&= \partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1}\right) + \partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t} - \frac{\boldsymbol{e}_h^{n+1} - \boldsymbol{e}_h^n}{\Delta t}.
\end{aligned} \tag{5.21}$$

Using (5.21), one can obtain the error equation

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2}^2 + \left\|\boldsymbol{e}_h^{n+1} - \boldsymbol{e}_h^n\right\|_{L^2}^2 - \left\|\boldsymbol{e}_h^n\right\|_{L^2}^2\right) + \nu a_h(\boldsymbol{e}_h^{n+1}, \boldsymbol{e}_h^{n+1}) \\
&= \left(\partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1}\right), \boldsymbol{e}_h^{n+1}\right) + \left(\partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t}, \boldsymbol{e}_h^{n+1}\right) + \nu a_h\left(\boldsymbol{\eta}^{n+1}, \boldsymbol{e}_h^{n+1}\right) \\
&\quad + c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}^n - \boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right).
\end{aligned} \tag{5.22}$$

We estimate all terms on the right-hand side of (5.22). The first term on the right-hand side of (5.22) is bounded by using Cauchy–Schwarz inequality

$$\begin{aligned}
\left(\partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1}\right), \boldsymbol{e}_h^{n+1}\right) &\le \left\|\partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1}\right)\right\|_{L^2} \left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2} \\
&\le \left\|\partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1}\right)\right\|_{L^2}^2 + \left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2}^2.
\end{aligned}$$

For the estimate of the second term, we use the Cauchy–Schwarz inequality, Lemma 3.4 and the commutation of temporal derivative and RT projection

$$\begin{aligned}
&\left(\partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - \tfrac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t}, \boldsymbol{e}_h^{n+1}\right) \\
&\le \left\|\partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - \tfrac{I_{\mathrm{Rt}}^h \boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h \boldsymbol{u}^n}{\Delta t}\right\|_{L^2} \left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2} \\
&\le C(\Delta t)^{1/2} \left\|\partial_{tt} I_{\mathrm{Rt}}^h \boldsymbol{u}\right\|_{L^2\left(t^n, t^{n+1}; \boldsymbol{L}^2(\Omega)\right)} \left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2} \\
&\le C\Delta t \left\|I_{\mathrm{Rt}}^h\left(\partial_{tt} \boldsymbol{u}\right)\right\|_{L^2\left(t^n, t^{n+1}; \boldsymbol{L}^2(\Omega)\right)}^2 + \left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2}^2.
\end{aligned}$$

Then, the triangle inequality and Lemma 3.4 are applied to get

$$\begin{aligned}\|(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\|_{L^2}^2 &\le 2\|\Delta t\partial_t\boldsymbol{u}^{n+1}-(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\|_{L^2}^2+2(\Delta t)^2\|\partial_t\boldsymbol{u}^{n+1}\|_{L^2}^2\\ &\le 2(\Delta t)^3\|\partial_{tt}\boldsymbol{u}\|^2_{L^2(t^n,t^{n+1};\boldsymbol{L}^2(\Omega))}+2(\Delta t)^2\|\partial_t\boldsymbol{u}^{n+1}\|_{L^2}^2. \end{aligned}\tag{5.23}$$

So, for the sixth term on the right-hand side of (5.22), we have

$$\begin{aligned}&c_h\left(\boldsymbol{u}^n-\boldsymbol{u}^{n+1},\boldsymbol{u}^{n+1},\boldsymbol{e}_h^{n+1}\right)\\ &\le \|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}^{n+1}\|_{L^2}^2+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\|_{L^2}^2\\ &\le \|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}^{n+1}\|_{L^2}^2+2(\Delta t)^3\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\partial_{tt}\boldsymbol{u}\|^2_{L^2(t^n,t^{n+1};\boldsymbol{L}^2(\Omega))}\\ &\quad+2(\Delta t)^2\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\partial_t\boldsymbol{u}^{n+1}\|_{L^2}^2. \end{aligned}\tag{5.24}$$

By the boundedness of a_h and Young's inequality, we have

$$\nu a_h\left(\boldsymbol{\eta}^{n+1},\boldsymbol{e}_h^{n+1}\right)\le C\nu|||\boldsymbol{\eta}^{n+1}|||_{e,\sharp}^2+\frac{C_\sigma\nu}{2}|||\boldsymbol{e}_h^{n+1}|||_e^2.$$

From Lemma 5.2, we have

$$\begin{aligned}&c_h\left(\boldsymbol{u}^n,\boldsymbol{u}^{n+1},\boldsymbol{e}_h^{n+1}\right)-c_h\left(\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1},\boldsymbol{e}_h^{n+1}\right)\\ &\le C\Big\{\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|_{L^2}^2+\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+h^2\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2)\\ &\quad+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|_{L^2}^2+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2+\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2\\ &\quad+\|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+h\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2)\Big\}. \end{aligned}$$

Inserting the above estimates in the right-hand side of (5.22), and using (3.8), one gets

$$\begin{aligned}&\frac{1}{2\Delta t}\left\|\boldsymbol{e}_h^{n+1}\right\|_{L^2}^2+\frac{C_\sigma\nu}{2}|||\boldsymbol{e}_h^{n+1}|||_e^2\\ &\le\frac{1}{2\Delta t}\|\boldsymbol{e}_h^n\|_{L^2}^2+C\Big[\left\|\partial_t\left(\boldsymbol{u}^{n+1}-I_{\mathrm{Rt}}^h\boldsymbol{u}^{n+1}\right)\right\|_{L^2}^2+\Delta t\left\|I_{\mathrm{Rt}}^h\left(\partial_{tt}\boldsymbol{u}\right)\right\|^2_{L^2\left(t^n,t^{n+1};\boldsymbol{L}^2(\Omega)\right)}\\ &\quad+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|_{L^2}^2+\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+h^2\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2)\\ &\quad+\|\boldsymbol{u}^n\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^{n+1}\|_{L^2}^2+h\|\nabla\boldsymbol{\eta}^{n+1}\|_{L^2}^2)+\nu|||\boldsymbol{\eta}^{n+1}|||_{e,\sharp}^2\\ &\quad+(\Delta t)^3\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\partial_{tt}\boldsymbol{u}\|^2_{L^2(t^n,t^{n+1};\boldsymbol{L}^2(\Omega))}+(\Delta t)^2\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\partial_t\boldsymbol{u}^{n+1}\|_{L^2}^2\Big]\\ &\quad+C\Big[\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|_{L^2}^2+(1+\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty})\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2+\|\nabla\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}^2\Big]. \end{aligned}$$

Summing over all discrete times, and by $\|\boldsymbol{e}_h^0\|_{L^2}^2 = 0$, we can get

$$
\begin{aligned}
&\left\|\boldsymbol{e}_h^{N+1}\right\|_{L^2}^2 + \nu\Delta t \sum_{n=1}^{N+1} |||\boldsymbol{e}_h^n|||_e^2 \\
&\leq C\Delta t \sum_{n=1}^{N+1} \Bigg[\left\|\partial_t \left(\boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^n\right)\right\|_{L^2}^2 + \Delta t \left\| I_{\mathrm{Rt}}^h \left(\partial_{tt}\boldsymbol{u}\right)\right\|_{L^2\left(t^{n-1},t^n;\boldsymbol{L}^2(\Omega)\right)}^2 \\
&\quad + \|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{\eta}^{n-1}\|_{L^2}^2 + \|\nabla \boldsymbol{u}^{n-1}\|_{L^\infty}\|\boldsymbol{\eta}^n\|_{L^2}^2 + \|\nabla \boldsymbol{u}^n\|_{L^\infty}(\|\boldsymbol{\eta}^n\|_{L^2}^2 + h^2\|\nabla\boldsymbol{\eta}^n\|_{L^2}^2) \\
&\quad + \|\boldsymbol{u}^{n-1}\|_{W^{1,\infty}}(h^{-1}\|\boldsymbol{\eta}^n\|_{L^2}^2 + h\|\nabla\boldsymbol{\eta}^n\|_{L^2}^2) + \nu|||\boldsymbol{\eta}^n|||_{e,\sharp}^2 \\
&\quad + (\Delta t)^3\|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))}^2 + (\Delta t)^2\|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\partial_t \boldsymbol{u}^n\|_{L^2}^2 \Bigg] \\
&\quad + C\Delta t \sum_{n=1}^{N+1} \Big[\|\nabla \boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}_h^{n-1}\|_{L^2}^2 + (1 + \|\nabla \boldsymbol{u}^n\|_{L^\infty})\|\boldsymbol{e}_h^n\|_{L^2}^2 + \|\nabla \boldsymbol{u}^{n-1}\|_{L^\infty}\|\boldsymbol{e}_h^n\|_{L^2}^2 \Big].
\end{aligned}
$$

So, we can have

$$
\begin{aligned}
&\left\|\boldsymbol{e}_h^{N+1}\right\|_{L^2}^2 + \nu\Delta t \sum_{n=1}^{N+1} |||\boldsymbol{e}_h^n|||_e^2 \\
&\quad \leq C\Delta t \sum_{n=1}^{N+1} \mathcal{G}_n + \Delta t \sum_{n=1}^{N+1} \alpha_n \|\boldsymbol{e}_h^n\|_{L^2}^2.
\end{aligned}
$$

Thus, when $\alpha_n \Delta t < 1, \forall\, n = 1, \ldots, N+1$, the discrete Gronwall Lemma 3.3 can be applied and one gets

$$
\begin{aligned}
&\left\|\boldsymbol{e}_h^{N+1}\right\|_{L^2}^2 + \nu\Delta t \sum_{n=1}^{N+1} |||\boldsymbol{e}_h^n|||_e^2 \\
&\quad \leq \exp\left(\Delta t \sum_{n=1}^{N+1} \frac{\alpha_n}{1-\Delta t\alpha_n}\right)\left(C\Delta t \sum_{n=1}^{N+1} \mathcal{G}_n\right).
\end{aligned}
$$

□

COROLLARY 5.4 Under the assumptions of the previous Theorem 5.3, with $r_{\boldsymbol{u}} = \min\{r, k+1\}$ and a constant C independent of h and ν, we have the following estimate:

$$
\begin{aligned}
&\left\|\boldsymbol{u}^{N+1} - \boldsymbol{u}_h^{N+1}\right\|_{L^2}^2 + \nu\Delta t \sum_{n=1}^{N+1} |||(\boldsymbol{u}^n - \boldsymbol{u}_h^n)|||_{e,\sharp}^2 \\
&\quad \leqslant h^{2r_{\boldsymbol{u}}-2}\Big(A(\boldsymbol{u})B_1(\boldsymbol{u}) + C_1(\boldsymbol{u})\Big) + (\Delta t)^2 A(\boldsymbol{u})D(\boldsymbol{u}), \qquad (5.25)
\end{aligned}
$$

and when $\nu \leq Ch$, then we have

$$\begin{aligned}
&\left\|\boldsymbol{u}^{N+1}-\boldsymbol{u}_h^{N+1}\right\|_{L^2}^2+\nu\Delta t\sum_{n=1}^{N+1}|||(\boldsymbol{u}^n-\boldsymbol{u}_h^n)|||_{e,\sharp}^2 \\
&\quad\leqslant h^{2r_{\boldsymbol{u}}-1}\Big(A(\boldsymbol{u})B_2(\boldsymbol{u})+C_2(\boldsymbol{u})\Big)+(\Delta t)^2A(\boldsymbol{u})D(\boldsymbol{u}),
\end{aligned}\tag{5.26}$$

where

$$\begin{aligned}
A(\boldsymbol{u}) &= C\exp\left(\Delta t\sum_{n=1}^{N+1}\frac{\alpha_n}{1-\Delta t\alpha_n}\right),\\
B_1(\boldsymbol{u}) &= h^2\left\|\partial_t\boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{H}^{r_{\boldsymbol{u}}}(\Omega))}^2+\Big(\nu T+(h^2+h)\|\boldsymbol{u}\|_{\mathcal{L}^1(\boldsymbol{W}^{1,\infty})}\Big)\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}^2,\\
B_2(\boldsymbol{u}) &= h\left\|\partial_t\boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{H}^{r_{\boldsymbol{u}}}(\Omega))}^2+\Big(T+(h+1)\|\boldsymbol{u}\|_{\mathcal{L}^1(\boldsymbol{W}^{1,\infty})}\Big)\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}^2,\\
C_1(\boldsymbol{u}) &= C(h^2+\nu T)\,\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}^2,\\
C_2(\boldsymbol{u}) &= C(h+T)\,\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}^2,\\
D(\boldsymbol{u}) &= \|\partial_{tt}\boldsymbol{u}\|_{L^2(0,t^{N+1};\boldsymbol{H}^1(\Omega))}^2+\left\|\partial_t\boldsymbol{u}\right\|_{\mathcal{L}^4(0,t^{N+1};\boldsymbol{L}^2(\Omega))}^2\|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))}\\
&\quad+(\Delta t)^{\frac{3}{2}}\left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))}^2\|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))},
\end{aligned}$$

and α_n is defined by (5.17).

Proof. The application of the triangle inequality and the norm equivalence (3.7) of $|||\cdot|||_e$ and $|||\cdot|||_{e,\sharp}$ on $\boldsymbol{X}_h$ gives

$$\begin{aligned}
&\left\|\boldsymbol{u}^{N+1}-\boldsymbol{u}_h^{N+1}\right\|_{L^2}^2+\nu\Delta t\sum_{n=1}^{N+1}|||(\boldsymbol{u}^n-\boldsymbol{u}_h^n)|||_{e,\sharp}^2\\
&\leq 2\left(\left\|\boldsymbol{\eta}^{N+1}\right\|_{L^2}^2+\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{\eta}^n|||_{e,\sharp}^2+\left\|\boldsymbol{e}_h^{N+1}\right\|_{L^2}^2+C\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{e}_h^n|||_e^2\right)\\
&\leq 2\left\|\boldsymbol{\eta}^{N+1}\right\|_{L^2}^2+2\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{\eta}^n|||_{e,\sharp}^2+\exp\left(\Delta t\sum_{n=1}^{N+1}\frac{\alpha_n}{1-\Delta t\alpha_n}\right)\left(C\Delta t\sum_{n=1}^{N+1}\mathcal{G}_n\right).
\end{aligned}$$

The stability estimate of the RT projection yields

$$\begin{aligned}
\sum_{n=1}^{N+1}\left\|I_{\mathrm{Rt}}^h\left(\partial_{tt}\boldsymbol{u}\right)\right\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))}^2 &= \left\|I_{\mathrm{Rt}}^h\left(\partial_{tt}\boldsymbol{u}\right)\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))}^2\\
&\leq C\left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{H}^1(\Omega))}^2.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\Delta t \sum_{n=1}^{N+1} \left\|\partial_t \boldsymbol{u}^n\right\|_{L^2}^2 \|\nabla \boldsymbol{u}^n\|_{L^\infty(\Omega)} \\
&\leq \left(\sum_{n=1}^{N+1} \Delta t \left\|\partial_t \boldsymbol{u}^n\right\|_{L^2}^4\right)^{1/2} \left(\sum_{n=1}^{N+1} \Delta t \|\boldsymbol{u}^n\|_{W^{1,\infty}(\Omega)}^2\right)^{1/2} \\
&= \left\|\partial_t \boldsymbol{u}\right\|_{\mathcal{L}^4\left(0,t^{N+1};\boldsymbol{L}^2(\Omega)\right)}^2 \|\boldsymbol{u}\|_{\mathcal{L}^2\left(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega)\right)}
\end{aligned}$$

and

$$\begin{aligned}
&(\Delta t)^{1/2} \sum_{n=1}^{N+1} \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2\left(t^{n-1},t^n;\boldsymbol{L}^2(\Omega)\right)}^2 \|\nabla \boldsymbol{u}^n\|_{L^\infty(\Omega)} \\
&\leq \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2\left(0,t^{N+1};\boldsymbol{L}^2(\Omega)\right)}^2 \left(\sum_{n=1}^{N+1} \Delta t \|\boldsymbol{u}^n\|_{W^{1,\infty}(\Omega)}^2\right)^{1/2} \\
&= \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2\left(0,t^{N+1};\boldsymbol{L}^2(\Omega)\right)}^2 \|\boldsymbol{u}\|_{\mathcal{L}^2\left(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega)\right)}.
\end{aligned}$$

At last, by using (3.16), the above stability estimates and (5.16), we can finish the proof. □

6. Error analysis for the pressure

Now, based on the previous error analysis for the velocity, we give error estimates for the pressure. To this end, we firstly give the following two lemmas, which are crucial to proving the convergence rate of the pressure. First, we give maximum norm estimate for the discrete velocity $\boldsymbol{u}_h$.

LEMMA 6.1 (Maximum norm estimate) Under the hypothesis of the previous corollary, the following error estimate holds: for any $1 \leq n \leq N+1$,

$$\begin{aligned}
&\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)} \\
&\quad \leqslant C(h\|\nabla \boldsymbol{u}^n\|_{L^\infty(\Omega)} + h^{r_{\boldsymbol{u}}-1-d/2}\sqrt{A(\boldsymbol{u})B_1(\boldsymbol{u}) + C_1(\boldsymbol{u})} + \Delta t h^{-d/2}\sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}).
\end{aligned}$$

In particular, for $\nu \leq Ch$,

$$\begin{aligned}
&\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)} \\
&\quad \leqslant C(h\|\nabla \boldsymbol{u}^n\|_{L^\infty(\Omega)} + h^{r_{\boldsymbol{u}}-1/2-d/2}\sqrt{A(\boldsymbol{u})B_2(\boldsymbol{u}) + C_2(\boldsymbol{u})} + \Delta t h^{-d/2}\sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}).
\end{aligned}$$

Proof. Using the triangle inequality, we have

$$\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)} \leq \|\boldsymbol{u}^n - P^k\boldsymbol{u}^n\|_{L^\infty(\Omega)} + \|P^k\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)}, \tag{6.1}$$

where $P^k\boldsymbol{u}^n$ denotes the L^2-projection of $\boldsymbol{u}^n$ onto $P_d^k(\mathcal{T}_h)$. Due to (3.11), we can get

$$\|\boldsymbol{u}^n - P^k\boldsymbol{u}^n\|_{L^\infty(\Omega)} \leqslant Ch\|\nabla \boldsymbol{u}^n\|_{L^\infty(\Omega)}. \tag{6.2}$$

For the second term on the right-hand side of (6.1), we use the inverse inequality (3.13), which yields

$$\|P^k\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)} \leq Ch^{-\frac{d}{2}}\|P^k\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^2}. \tag{6.3}$$

Using again the triangle inequality, (5.25) and $\|\boldsymbol{u}^n - P^k\boldsymbol{u}^n\|_{L^2} \leq Ch^{r_{\boldsymbol{u}}}|\boldsymbol{u}^n|_{H^{r_{\boldsymbol{u}}}(\Omega)}$, we get

$$\begin{aligned}\|P^k\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^2} &\leq \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^2} + \|\boldsymbol{u}^n - P^k\boldsymbol{u}^n\|_{L^2}\\ &\leq Ch^{r_{\boldsymbol{u}}-1}\sqrt{A(\boldsymbol{u})B_1(\boldsymbol{u}) + C_1(\boldsymbol{u})} + \Delta t\sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}.\end{aligned} \tag{6.4}$$

Using (6.1)–(6.4), we can obtain

$$\begin{aligned}&\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^\infty(\Omega)}\\ &\quad\leqslant C(h\|\nabla\boldsymbol{u}^n\|_{L^\infty(\Omega)} + h^{r_{\boldsymbol{u}}-1-d/2}\sqrt{A(\boldsymbol{u})B_1(\boldsymbol{u}) + C_1(\boldsymbol{u})} + \Delta t h^{-d/2}\sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}).\end{aligned}$$

□

REMARK 3 We remark that the L^2 projection P^k can be also replaced by the RT interpolation operator I_{Rt}^h in the proof of Lemma 6.1.

Next, we present an estimate for the difference of the convective terms.

LEMMA 6.2 For the difference of the convective terms, the following estimate holds true:

$$\begin{aligned}&\frac{c_h(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right)}{|||\boldsymbol{v}_h|||_e}\\ &\leq C\|\boldsymbol{u}_h^n\|_{L^\infty}(\|\boldsymbol{e}_h^{n+1}\|_{L^2} + h\|\nabla_h\boldsymbol{\eta}^{n+1}\|_{L^2} + \|\boldsymbol{\eta}^{n+1}\|_{L^2})\\ &\quad + C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(\|\boldsymbol{e}_h^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2})\\ &\quad + C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\left((\Delta t)^{\frac{3}{2}}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t^n,t^{n+1};\boldsymbol{L}^2(\Omega))} + \Delta t\|\partial_t\boldsymbol{u}^{n+1}\|_{L^2}\right).\end{aligned}$$

Proof. First, let us denote $\boldsymbol{K} = c_h(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right)$, then we have

$$\begin{aligned}\boldsymbol{K} &= \left(c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right)\right) - c_h\left(\boldsymbol{u}^n - \boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right)\\ &= \boldsymbol{K}_1 + \boldsymbol{K}_2.\end{aligned}$$

For $\boldsymbol{K}_1$, we add or subtract one term and use integration by parts to obtain

$$\begin{aligned}\boldsymbol{K}_1 &= c_h\Big(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\Big) - c_h\Big(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\Big)\\ &= \int_\Omega \Big[\big((\boldsymbol{u}^n - \boldsymbol{u}_h^n)\cdot\nabla\big)\boldsymbol{u}^{n+1}\cdot\boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \big(\boldsymbol{u}_h^n\cdot\nabla_h\big)\boldsymbol{v}_h\cdot(\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1})\Big]\,\mathrm{d}\boldsymbol{x}\\ &\quad + \sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{v}_h]\!]\{\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|(\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)|[\![\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}]\!][\![\boldsymbol{v}_h]\!]\,\mathrm{d}\boldsymbol{s}. \qquad (6.5)\end{aligned}$$

Applying Hölder's inequality and using $\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1} = \boldsymbol{\eta}^{n+1} - \boldsymbol{e}_h^{n+1}$ in (6.5), we can get

$$\begin{aligned}\boldsymbol{K}_1 \le\ & \|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^2}\|\boldsymbol{v}_h\|_{L^2} + \|\boldsymbol{u}_h^n\|_{L^\infty}\|\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}\|_{L^2}\|\nabla\boldsymbol{v}_h\|_{L^2}\\ & + \|\boldsymbol{u}_h^n\|_{L^\infty}\left(\sum_{F\in\mathcal{F}_h^i} h_F\|\{\boldsymbol{\eta}^{n+1}\}\|^2_{L^2(F)}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|[\![\boldsymbol{v}_h]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}\\ & + \|\boldsymbol{u}_h^n\|_{L^\infty}\left(\sum_{F\in\mathcal{F}_h^i} h_F\|\{\boldsymbol{e}_h^{n+1}\}\|^2_{L^2(F)}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|[\![\boldsymbol{v}_h]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}\\ & + \|\boldsymbol{u}_h^n\|_{L^\infty}\left(\sum_{F\in\mathcal{F}_h^i} h_F\|[\![\boldsymbol{\eta}^{n+1}]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|[\![\boldsymbol{v}_h]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}\\ & + \|\boldsymbol{u}_h^n\|_{L^\infty}\left(\sum_{F\in\mathcal{F}_h^i} h_F\|[\![\boldsymbol{e}_h^{n+1}]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}_h^i} h_F^{-1}\|[\![\boldsymbol{v}_h]\!]\|^2_{L^2(F)}\right)^{\frac{1}{2}}.\end{aligned}$$

Applying trace inequality, (3.6) and $\|\boldsymbol{v}_h\|_{L^2} \le C|\!|\!|\boldsymbol{v}_h|\!|\!|_e$ from (3.14), we have

$$\begin{aligned}\frac{\boldsymbol{K}_1}{|\!|\!|\boldsymbol{v}_h|\!|\!|_e} &\le C\Big\{\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}\|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{L^2} + \|\boldsymbol{u}_h^n\|_{L^\infty}\|\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}\|_{L^2}\\ &\quad + \|\boldsymbol{u}_h^n\|_{L^\infty}(h\|\nabla_h\boldsymbol{\eta}^{n+1}\|_{L^2} + \|\boldsymbol{\eta}^{n+1}\|_{L^2}) + \|\boldsymbol{u}_h^n\|_{L^\infty}\|\boldsymbol{e}_h^{n+1}\|_{L^2}\Big\}\\ &\le C\|\boldsymbol{u}_h^n\|_{L^\infty}(\|\boldsymbol{e}_h^{n+1}\|_{L^2} + h\|\nabla_h\boldsymbol{\eta}^{n+1}\|_{L^2} + \|\boldsymbol{\eta}^{n+1}\|_{L^2})\\ &\quad + C\|\nabla\boldsymbol{u}^{n+1}\|_{L^\infty}(\|\boldsymbol{e}_h^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2}).\end{aligned}$$

So, applying Hölder's inequality, $\|\boldsymbol{v}_h\|_{L^2} \le C|||\boldsymbol{v}_h|||_e$ and (5.23) yields

$$\frac{\boldsymbol{K}_2}{|||\boldsymbol{v}_h|||_e} \le C\|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty}\Big((\Delta t)^{\frac{3}{2}}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t^n,t^{n+1};\boldsymbol{L}^2(\Omega))} + \Delta t\|\partial_t \boldsymbol{u}^{n+1}\|_{L^2}\Big).$$

Then, the proof is finished by collecting the above estimates. $\square$

Now, we end this section with an *a priori* error estimate for the pressure. First, we make error splitting as follows:

$$p - p_h = \big(p - P_o p\big) - \big(p_h - P_o p\big) = \eta_p - e_p,$$

where $P_o p$ denotes the L^2-projection of p onto Q_h.

THEOREM 6.3 Assume that $\Delta t \le Ch^{d/2}$ and $r_{\boldsymbol{u}} \ge \frac{d+2}{2}$ ($r_{\boldsymbol{u}} \ge \frac{d+1}{2}$ when $\nu \le Ch$), and under the assumptions of Theorem 5.3, then the following error estimate holds:

$$\begin{aligned}
&\|\Delta t\sum_{n=1}^{N+1}(p^n - p_h^n)\|_{L^2} \\
&\le C\Big\{\Big(1 + \Delta t\sum_{n=1}^{N+1}K(\boldsymbol{u}^n)\Big)\max_{1\le n\le N+1}\|\boldsymbol{e}_h^n\|_{L^2} + T^{1/2}\left\|\partial_t \eta\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \\
&\quad + \Delta t\sum_{n=1}^{N+1}K(\boldsymbol{u}^n)(h\|\nabla_h \boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^{n-1}\|_{L^2}) \\
&\quad + T^{1/2}\left\|\eta_p\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} + \nu^{1/2}T^{1/2}\left(\nu\Delta t\sum_{n=1}^{N+1}|||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp}^2\right)^{1/2} \\
&\quad + \Delta t\left\|\partial_t \boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))}\|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))} + (\Delta t)^{3/2}\left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{H}^1(\Omega))} \\
&\quad + (\Delta t)^2\left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))}\|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))}\Big\},
\end{aligned}$$

where $K(\boldsymbol{u}^n) = K_1(\boldsymbol{u}^n)$ ($K(\boldsymbol{u}^n) = K_2(\boldsymbol{u}^n)$ when $\nu \le Ch$), $K_1(\boldsymbol{u}^n)$ and $K_2(\boldsymbol{u}^n)$ given by (6.8) and (6.9), respectively.

Proof. First, we have

$$\begin{aligned}
&b_h(\boldsymbol{v}_h, e_p^{n+1}) \\
&= \Big(\partial_t\left(\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h\boldsymbol{u}^{n+1}\right), \boldsymbol{v}_h\Big) + \left(\partial_t I_{\mathrm{Rt}}^h\boldsymbol{u}^{n+1} - \frac{I_{\mathrm{Rt}}^h\boldsymbol{u}^{n+1} - I_{\mathrm{Rt}}^h\boldsymbol{u}^n}{\Delta t}, \boldsymbol{v}_h\right) - \left(\frac{\boldsymbol{e}_h^{n+1} - \boldsymbol{e}_h^n}{\Delta t}, \boldsymbol{v}_h\right) \\
&\quad + \nu a_h\Big(\boldsymbol{u}^{n+1} - \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\Big) + c_h\left(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{v}_h\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + b_h(\boldsymbol{v}_h, \eta_p^{n+1}).
\end{aligned} \tag{6.6}$$

Let us denote $\Pi_h^{N+1} = \Delta t \sum\limits_{n=1}^{N+1} e_p^n$ and $\ddot{\Pi}_h^{N+1} = \Delta t \sum\limits_{n=1}^{N+1} \eta_p^n$. Then, we have the following error equation:

$$
\begin{aligned}
& b_h(\boldsymbol{v}_h, \Pi_h^{N+1}) \\
&= \Delta t \sum_{n=1}^{N+1} \left(\partial_t \left(\boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^n \right), \boldsymbol{v}_h \right) + \Delta t \sum_{n=1}^{N+1} \left(\partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^n - \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n-1}}{\Delta t}, \boldsymbol{v}_h \right) \\
&\quad - (\boldsymbol{e}_h^{N+1} - \boldsymbol{e}_h^0, \boldsymbol{v}_h) + \nu \Delta t \sum_{n=1}^{N+1} a_h(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v}_h) \\
&\quad + \Delta t \sum_{n=1}^{N+1} \left(c_h \left(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}_h \right) - c_h \left(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h \right) \right) + b_h(\boldsymbol{v}_h, \ddot{\Pi}_h^{N+1}).
\end{aligned}
$$

Applying the discrete inf-sup condition (3.1), Cauchy–Schwarz inequality, boundedness of a_h, $\|\boldsymbol{e}_h^0\|_{L^2} = 0$ and $\|\boldsymbol{v}_h^{n+1}\|_{L^2} \le C \|\boldsymbol{v}_h^{n+1}\|_e$ from (3.14), we can get

$$
\begin{aligned}
\beta \|\Pi_h^{N+1}\|_{L^2} &\le \sup_{\boldsymbol{v}_h \in \boldsymbol{X}_h \setminus 0} \frac{b_h(\boldsymbol{v}_h, \Pi_h^{N+1})}{|||\boldsymbol{v}_h|||_e} \\
&\le C \Bigg\{ \|\boldsymbol{e}_h^{N+1}\|_{L^2} + \|\ddot{\Pi}_h^{N+1}\|_{L^2} + C \nu \Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp} \\
&\quad + \Delta t \sum_{n=1}^{N+1} \frac{\left| c_h \left(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}_h \right) - c_h \left(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h \right) \right|}{|||\boldsymbol{v}_h|||_e} \\
&\quad + \Delta t \sum_{n=1}^{N+1} \left\| \partial_t \left(\boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^n \right) \right\|_{L^2} + \Delta t \sum_{n=1}^{N+1} \left\| \partial_t I_{\mathrm{Rt}}^h \boldsymbol{u}^n - \frac{I_{\mathrm{Rt}}^h \boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^{n-1}}{\Delta t} \right\|_{L^2} \Bigg\} \\
&= C \Big\{ T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \Big\}.
\end{aligned} \tag{6.7}
$$

Next, we estimate all the terms on the right-hand side of (6.7). For the first term T_1, it was already bounded in the derivation of the velocity error bound. Using Cauchy–Schwarz inequality, we can obtain

$$
\begin{aligned}
T_2 + T_3 + T_5 \ \le\ & T^{1/2} \left(\Delta t \sum_{n=1}^{N+1} \|\eta_p^n\|_{L^2}^2 \right)^{1/2} + C \nu^{1/2} T^{1/2} \left(\nu \Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp}^2 \right)^{1/2} \\
& + T^{1/2} \left(\Delta t \sum_{n=1}^{N+1} \|\partial_t \left(\boldsymbol{u}^n - I_{\mathrm{Rt}}^h \boldsymbol{u}^n \right)\|_{L^2}^2 \right)^{1/2} \\
=\ & T^{1/2} \left\| \eta_p \right\|_{\mathcal{L}^2(0, t^{N+1}; \boldsymbol{L}^2(\Omega))} + C \nu^{1/2} T^{1/2} \left(\nu \Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp}^2 \right)^{1/2} \\
& + T^{1/2} \left\| \partial_t \eta \right\|_{\mathcal{L}^2(0, t^{N+1}; \boldsymbol{L}^2(\Omega))}.
\end{aligned}
$$

From Lemma 6.2, we can get

$$
\begin{aligned}
T_4 \leq\ & C\Delta t \sum_{n=1}^{N+1} \|\nabla \boldsymbol{u}^n\|_{L^\infty} (\|\boldsymbol{e}_h^{n-1}\|_{L^2} + \|\boldsymbol{\eta}^{n-1}\|_{L^2}) \\
& + C\Delta t \sum_{n=1}^{N+1} \|\nabla \boldsymbol{u}^n\|_{L^\infty} \Big((\Delta t)^{\frac{3}{2}} \|\partial_{tt} \boldsymbol{u}\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))} + \Delta t \|\partial_t \boldsymbol{u}^n\|_{L^2} \Big) \\
& + C\Delta t \sum_{n=1}^{N+1} \|\boldsymbol{u}_h^{n-1}\|_{L^\infty} (\|\boldsymbol{e}_h^n\|_{L^2} + h\|\nabla_h \boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2}) \\
\leq\ & C\Delta t \max_{1\leq n\leq N+1} \|\boldsymbol{e}_h^n\|_{L^2} \sum_{n=1}^{N+1} (\|\nabla \boldsymbol{u}^n\|_{L^\infty} + \|\boldsymbol{u}_h^{n-1}\|_{L^\infty}) \\
& + C\Delta t \sum_{n=1}^{N+1} (\|\nabla \boldsymbol{u}^n\|_{L^\infty} + \|\boldsymbol{u}_h^{n-1}\|_{L^\infty})(h\|\nabla_h \boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^{n-1}\|_{L^2}) \\
& + C\Delta t \sum_{n=1}^{N+1} \|\nabla \boldsymbol{u}^n\|_{L^\infty} \Big((\Delta t)^{\frac{3}{2}} \|\partial_{tt} \boldsymbol{u}\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))} + \Delta t \|\partial_t \boldsymbol{u}^n\|_{L^2} \Big).
\end{aligned}
$$

By Lemma 6.1, $r_{\boldsymbol{u}} \geq \frac{d+2}{2}$ and $\Delta t \leq Ch^{d/2}$, we have

$$
\begin{aligned}
& \|\nabla \boldsymbol{u}^n\|_{L^\infty} + \|\boldsymbol{u}_h^{n-1}\|_{L^\infty} \\
& \leq \|\nabla \boldsymbol{u}^n\|_{L^\infty} + C(\|\boldsymbol{u}^{n-1}\|_{W^{1,\infty}(\Omega)} + \sqrt{A(\boldsymbol{u})B_1(\boldsymbol{u}) + C_1(\boldsymbol{u})} + \sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}) \\
& = K_1(\boldsymbol{u}^n).
\end{aligned} \tag{6.8}
$$

In particular, when $\nu \leq Ch$, $r_{\boldsymbol{u}} \geq \frac{d+1}{2}$ and $\Delta t \leq Ch^{d/2}$, we similarly have

$$
\begin{aligned}
& \|\nabla \boldsymbol{u}^n\|_{L^\infty} + \|\boldsymbol{u}_h^{n-1}\|_{L^\infty} \\
& \leq \|\nabla \boldsymbol{u}^n\|_{L^\infty} + C(\|\boldsymbol{u}^{n-1}\|_{W^{1,\infty}(\Omega)} + \sqrt{A(\boldsymbol{u})B_2(\boldsymbol{u}) + C_2(\boldsymbol{u})} + \sqrt{A(\boldsymbol{u})D(\boldsymbol{u})}) \\
& = K_2(\boldsymbol{u}^n).
\end{aligned} \tag{6.9}
$$

In addition,

$$
\begin{aligned}
& \Delta t \sum_{n=1}^{N+1} \|\nabla \boldsymbol{u}^n\|_{L^\infty} \Big((\Delta t)^{\frac{3}{2}} \|\partial_{tt} \boldsymbol{u}\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))} + \Delta t \|\partial_t \boldsymbol{u}^n\|_{L^2} \Big) \\
& \leq \Delta t \left(\sum_{n=1}^{N+1} \Delta t \left\|\partial_t \boldsymbol{u}^n\right\|_{L^2}^2 \right)^{1/2} \left(\sum_{n=1}^{N+1} \Delta t \left\|\nabla \boldsymbol{u}^n\right\|_{L^\infty(\Omega)}^2 \right)^{1/2} \\
& \quad + (\Delta t)^2 \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \left(\sum_{n=1}^{N+1} \Delta t \left\|\nabla \boldsymbol{u}^n\right\|_{L^\infty(\Omega)}^2 \right)^{1/2} \\
& \leq \Delta t \left\|\partial_t \boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))} \\
& \quad + (\Delta t)^2 \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))}.
\end{aligned} \tag{6.10}
$$

Now, defining $K(\boldsymbol{u}^n) = K_1(\boldsymbol{u}^n)$ ($K(\boldsymbol{u}^n) = K_2(\boldsymbol{u}^n)$ when $\nu \le Ch$), we can have

$$\begin{aligned}
T_4 \ \le\ & C\Delta t \max_{1\le n\le N+1} \|\boldsymbol{e}_h^n\|_{L^2} \sum_{n=1}^{N+1} K(\boldsymbol{u}^n) \\
& + C\Delta t \sum_{n=1}^{N+1} K(\boldsymbol{u}^n)(h\|\nabla_h \boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^{n-1}\|_{L^2}) \\
& + C\Delta t \left\|\partial_t \boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))} \\
& + C(\Delta t)^2 \left\|\partial_{tt} \boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))}.
\end{aligned}$$

Using Lemma 3.4, the commutation of temporal derivative and RT projection, the stability estimate of the RT projection yields

$$\begin{aligned}
T_6 \ &\le\ (\Delta t)^{3/2} \sum_{n=1}^{N+1} \left\|\partial_{tt} I_{\mathrm{Rt}}^h \boldsymbol{u}\right\|_{L^2(t^{n-1},t^n;\boldsymbol{L}^2(\Omega))} \\
&=\ (\Delta t)^{3/2} \left\|I_{\mathrm{Rt}}^h\left(\partial_{tt}\boldsymbol{u}\right)\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \\
&\le\ C(\Delta t)^{3/2} \left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{H}^1(\Omega))}.
\end{aligned}$$

By applying the triangle inequality and inserting the above inequalities in (6.7), we obtain

$$\begin{aligned}
&\left\|\Delta t \sum_{n=1}^{N+1} (p^n - p_h^n)\right\|_{L^2} \le \|\Pi_h^{N+1}\|_{L^2} + \|\ddot{\Pi}_h^{N+1}\|_{L^2} \\
&\le C\Bigg\{\left(1 + \Delta t \sum_{n=1}^{N+1} K(\boldsymbol{u}^n)\right) \max_{1\le n\le N+1} \|\boldsymbol{e}_h^n\|_{L^2} + T^{1/2}\left\|\partial_t \eta\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \\
&\quad + \Delta t \sum_{n=1}^{N+1} K(\boldsymbol{u}^n)(h\|\nabla_h \boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^n\|_{L^2} + \|\boldsymbol{\eta}^{n-1}\|_{L^2}) \\
&\quad + T^{1/2}\left\|\eta_p\right\|_{\mathcal{L}^2(0,t^{N+1};L^2(\Omega))} + \nu^{1/2} T^{1/2}\left(\nu\Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp}^2\right)^{1/2} \\
&\quad + \Delta t \left\|\partial_t \boldsymbol{u}\right\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))} + (\Delta t)^{3/2}\left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{H}^1(\Omega))} \\
&\quad + (\Delta t)^2 \left\|\partial_{tt}\boldsymbol{u}\right\|_{L^2(0,t^{N+1};\boldsymbol{L}^2(\Omega))} \|\boldsymbol{u}\|_{\mathcal{L}^2(0,t^{N+1};\boldsymbol{W}^{1,\infty}(\Omega))}\Bigg\}.
\end{aligned}$$

□

Remark 4 For estimating the pressure, we need to use the two error bounds $\max_{1\le n\le N+1} \|\boldsymbol{e}_h^n\|_{L^2}$ and $\nu\Delta t \sum_{n=1}^{N+1} |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{e,\sharp}^2$ in the error analysis for the velocity. Assume the solution $(\boldsymbol{u}, p)$ of (2.2) is sufficiently smooth. From Theorem 6.3, when polynomials of degree k_u and k_p are used to approximate the velocity and pressure, respectively, then we have

$$\left\|\Delta t \sum_{n=1}^{N+1} (p^n - p_h^n)\right\|_{L^2} \le C\left(\boldsymbol{u}, \partial_t \boldsymbol{u}, \partial_{tt}\boldsymbol{u}, p, T\right)\left(\nu^{\frac{1}{2}} h^{k_u} + h^{k_u+\frac{1}{2}} + h^{k_p+1} + \Delta t\right). \tag{6.11}$$

Corollary 5.4 shows that whether RT_k/P_k or BDM_k/P_{k-1} elements are used for the velocity–pressure approximation, the L^2 error bound of the velocity has a rate of convergence $k+1/2$ in the case of $\nu \le Ch$. Theorem 6.3 shows that for the RT_k/P_k, the convergence rate of the pressure is $k + 1/2$ when $\nu \le Ch$, while an optimal convergence rate k is obtained for the BDM_k/P_{k-1}.

7. Extension to full-implicit and IMEX time-marching schemes

For the time integration, we also consider the full-implicit and IMEX Euler methods in time, respectively. For the L^2 error of the velocity, we will comment on whether the current analysis still works for the full-implicit and IMEX Euler methods.

Full-implicit Euler method:

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right) + \nu a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + b_h(\boldsymbol{v}_h, p_h^{n+1}) + c_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) = \left(\boldsymbol{f}^{n+1}, \boldsymbol{v}_h\right),$$
$$b_h(\boldsymbol{u}_h^{n+1}, q_h) = 0.$$

IMEX Euler method:

$$\left(\frac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^n}{\Delta t}, \boldsymbol{v}_h\right) + \nu a_h\left(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h\right) + b_h(\boldsymbol{v}_h, p_h^{n+1}) = \left(\boldsymbol{f}^{n+1}, \boldsymbol{v}_h\right) - c_h(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{v}_h),$$
$$b_h(\boldsymbol{u}_h^{n+1}, q_h) = 0.$$

For the full-implicit Euler method, it is easy to prove that the L^2 error bound of the velocity has a convergence rate of $k + 1/2$ by following the proofs of this paper. For the IMEX Euler method, we notice that in the error analysis, the most important term to deal with is

$$\begin{aligned}
&c_h\left(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}_h^{n+1}\right)\\
&= \left[c_h\left(\boldsymbol{u}^{n+1}, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}_h^{n+1}\right)\right] + \left[c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}_h^{n+1}\right) - c_h\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}_h^{n+1}\right)\right]\\
&= \Theta_1 + \Theta_2.
\end{aligned}$$

For Θ_1, by using (5.24), we have

$$\begin{aligned}
\Theta_1 &= c_h\left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{u}^{n+1}, \boldsymbol{e}_h^{n+1}\right) + c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1} - \boldsymbol{u}^n, \boldsymbol{e}_h^{n+1}\right)\\
&\le (\|\nabla \boldsymbol{u}^{n+1}\|_{L^\infty} + \|\boldsymbol{u}^n\|_{L^\infty})\Big[\|\boldsymbol{e}^{n+1}\|_{L^2}^2 + 2(\Delta t)^3\|\partial_{tt}\boldsymbol{u}\|^2_{L^2(t^n,t^{n+1};\boldsymbol{H}^1(\Omega))}\\
&\quad + 2(\Delta t)^2\|\partial_t \boldsymbol{u}^{n+1}\|_{H^1}^2\Big],
\end{aligned}$$

where for the second term of Θ_1, similar to (5.24), we have

$$\begin{aligned} & c_h\left(\boldsymbol{u}^n, \boldsymbol{u}^{n+1}-\boldsymbol{u}^n, \boldsymbol{e}_h^{n+1}\right) \\ & \leq \|\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}^{n+1}\|_{L^2}^2 + \|\boldsymbol{u}^n\|_{L^\infty}\|\nabla(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\|_{L^2}^2 \\ & \leq \|\boldsymbol{u}^n\|_{L^\infty}\|\boldsymbol{e}^{n+1}\|_{L^2}^2 + 2(\Delta t)^3\|\boldsymbol{u}^n\|_{L^\infty}\|\partial_{tt}\boldsymbol{u}\|_{L^2(t^n,t^{n+1};\boldsymbol{H}^1(\Omega))}^2 \\ & \quad + 2(\Delta t)^2\|\boldsymbol{u}^n\|_{L^\infty}\|\partial_t\boldsymbol{u}^{n+1}\|_{H^1}^2. \end{aligned}$$

For Θ_2, we follow the proof of Lemma 5.2 word by word, and we can remark that the corresponding term $\boldsymbol{I}_{fac}$ (see (5.11)) can be written in the following form:

$$\begin{aligned} \boldsymbol{I}_{fac} = & \sum_{F\in\mathcal{F}_h^i}\int_F (\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h^{n+1}]\!]\{\boldsymbol{\eta}^n\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F|[\![\boldsymbol{\eta}^n]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s} \\ & - \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|\boldsymbol{u}_h^n\cdot\boldsymbol{n}_F|[\![\boldsymbol{e}_h^n]\!][\![\boldsymbol{e}_h^{n+1}]\!]\,\mathrm{d}\boldsymbol{s}. \end{aligned} \tag{7.1}$$

The third term of (7.1) prevents the error bound of order $k+1/2$ for the facet terms, so the L^2 error bound of the velocity does not have a convergence rate of $k+1/2$. For the higher-order full-implicit/semi-implicit/IMEX time-marching schemes, similar results can be obtained in the analytical framework. From the point of view of practical application, we can refer to some recent literature (Guzmán *et al.*, 2017; Schroeder *et al.*, 2018; Schroeder & Lube, 2018; Schroeder *et al.*, 2019; Lube & Schroeder, 2020) about the related performance of the H(div)-DG method with different time discrete schemes for incompressible flows.

8. Numerical studies

In this section, we carry out numerical experiments to support our analytical results. We will concentrate on the convergence with respect to the mesh size and test the rate of convergence. Numerical simulations were performed at a problem defined in the domain $\Omega = (0,1)^2$ with the exact solution

$$\begin{aligned} \boldsymbol{u} &= \cos(2\pi t)\begin{pmatrix} \sin(\pi x - 0.7)\sin(\pi y + 0.2) \\ \cos(\pi x - 0.7)\cos(\pi y + 0.2) \end{pmatrix}, \\ p &= \cos(2\pi t)(\sin(x)\cos(y) + (\cos(1)-1)\sin(1)). \end{aligned}$$

The right-hand side, the Dirichlet boundary condition and the initial condition are derived from the exact solution.

In our implementation, the penalty parameter σ is equal to $10k^2$. As for temporal discretization, the semi-implicit Euler scheme (3.3) was applied. We set the time interval small enough to ensure that the spatial error dominates over the temporal error. We choose the small time step $\triangle t = 5E-4$ and the final time $T = 1.0$. All numerical examples have been run on regular triangulations with diagonals (from bottom right to top left), with the same number of subdivisions on each coordinate direction using the high-order finite element library NGSolve (Schöberl, 2014).

TABLE 1 *BDM_1 /P_0 pair of finite element spaces, $T = 1.0$, behavior of errors with respect to ν*

ν	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2}$	$\|p-p_h\|_{L^2}$	$\|\nabla\cdot\boldsymbol{u}_h\|_{L^2}$
10^0	1.05E–2	4.24E–1	8.14E–1	2.32E–15
10^{-2}	8.23E–3	4.28E–1	2.28E–2	1.89E–15
10^{-4}	1.13E–2	4.61E–1	2.07E–2	1.85E–15
10^{-6}	1.14E–2	4.63E–1	2.07E–2	1.85E–15
10^{-8}	1.14E–2	4.63E–1	2.07E–2	1.86E–15
10^{-10}	1.14E–2	4.63E–1	2.07E–2	1.87E–15

TABLE 2 *RT_1 /P_1 pair of finite element spaces, $T = 1.0$, behavior of errors with respect to h*

$N\times N$	ndof	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\boldsymbol{L}^2(\Omega))}$	Rate	$\|p-p_h\|_{L^2(L^2(\Omega))}$	Rate
$\nu = 1$							
4×4	272	5.30E–2	—	7.19E–1	—	1.43E+0	—
8×8	1056	1.58E–2	1.75	3.73E–1	0.95	8.24E–1	0.80
16×16	4160	4.25E–3	1.89	1.88E–1	0.99	4.34E–1	0.92
32×32	16512	1.10E–3	1.95	9.42E–2	1.00	2.21E–1	0.97
$\nu = 10^{-8}$							
4×4	272	5.48E–2	—	7.96E–1	—	1.70E–2	—
8×8	1056	1.70E–2	1.69	4.15E–1	0.94	5.47E–3	1.64
16×16	4160	4.69E–3	1.86	2.11E–1	0.98	1.54E–3	1.83
32×32	16512	1.23E–3	1.93	1.07E–1	0.98	4.29E–4	1.84

We use the mesh with $N = 10$ subdivisions in each coordinate direction to observe the variation of the error with respect to ν. We use BDM_1/P_0 pair for the velocity and the pressure. The mesh in space consisted of 200 mesh cells (640 velocity degrees of freedom, 200 pressure degrees of freedom). From Table 1, we can observe that when the viscosity is small enough, the velocity and pressure errors hold unchanged, that is to say, they are independent of the viscosity. This is consistent with our theoretical results.

Next, we test the rates of convergence for the velocity and pressure for $\nu = 1$ and $1E-8$, respectively. The meshes with $N = 4, 8, 16$ and 32 subdivisions in each coordinate direction are used. We use RT_1/P_1 pair for the velocity and the pressure. From Table 2, we observe that both $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$ and $\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\boldsymbol{L}^2(\Omega))}$ are optimal. For the pressure, the convergence rate of the pressure error in Theorem 6.3 is similar to that of $\|p-p_h\|_{L^2(L^2(\Omega))}$ in Table 2, so we omit it here. We observe that the convergence rate of the pressure error increases by one order at small viscosity $\nu = 1E-8$. We notice that if the optimal and semirobust L^2 error bound of the velocity can be proved at small viscosity, the optimal convergence rate of the pressure is easily proved from Theorem 6.3. It may be an open question whether the optimal and semirobust L^2 error bound for the velocity can be proved at small viscosity.

9. Conclusions

In this paper, we present a fully discrete analysis of H(div)-conforming finite element method with semi-implicit time-marching for the evolutionary incompressible Navier–Stokes equations. The stability analysis and *a priori* error estimates are given. For inf-sup stable finite element pairs, the L^2 error bound of the velocity is pressure-robust, semirobust and has a convergence rate of $k + 1/2$ ($\nu \leq Ch$). In particular, for the inf-sup stable finite element pair RT_k/P_k, the convergence rate of the pressure is also $k + 1/2$ in the case of $\nu \leq Ch$.

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