

RESEARCH ARTICLE**WILEY**

An unconditionally stable artificial compression method for the time-dependent groundwater-surface water flows

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Funding information

Innovative team project of Shaanxi Provincial
Department of Education, Grant/Award Numbers:
21JP019, 21JP013; National Natural Science
Foundation of China, Grant/Award Numbers:
12001347, 11771259, 11971378; Shaanxi
Provincial Joint Laboratory of Artificial
Intelligence, Grant/Award Number:
2022JC-SYS-05; Shaanxi Province Natural
Science basic research program key project,
Grant/Award Number: 2023-JC-ZD-02.

Abstract

In this article, we propose a second order, unconditionally stable artificial compression method for the fully evolutionary Stokes/Darcy and Navier-Stokes/Darcy equations that model the coupling surface and groundwater flows. It uncouples the surface from the groundwater flow by the Crank-Nicolson Leapfrog scheme for the discretization in time, and through the artificial compression method without artificial pressure boundary conditions to decouple the velocity and pressure of the incompressible flow. Finally, we have verified the stability and second-order convergence of the algorithm from theoretical analysis and numerical experiments.

KEY WORDS

artificial compression, Crank-Nicolson Leapfrog scheme,
Unconditionally stable

1 | INTRODUCTION

In recent decades, the mixed Stokes/Darcy model and Navier-Stokes/Darcy model, which simulate the coupling of incompressible and porous media flow, have attracted extensive theoretical and numerical attention due to their important applications in the real world [1-5,7,9,12-16,18,24,27-29,31,34-39]. There are many articles that have studied the time-discrete algorithm of the coupled problem [6,20,22,23,30,33,40]. However, one of the difficulties in solving the coupled problem is the coupling of two different physical processes in two adjacent domains. Currently, partitioned methods are commonly used to solve this problem and allow the physical processes in each subdomain to be solved using existing algorithms that have been optimized [10,11,19,21,32].

In [19], it was shown that by adding appropriate stabilization terms to both the Stokes and the ground-water flow equations, so the Crank-Nicolson Leapfrog scheme results in a second order partitioned method as well as unconditionally stable. In [26], a conservative, second order, unconditionally stable artificial compression method for the Navier-Stokes has been proposed, it uncouples the pressure and velocity and requires no artificial pressure boundary conditions. However, as far as the authors know, CNLF scheme and artificial compression method are not combined to solve the linear Stokes/Darcy model, let alone the nonlinear Navier-Stokes/Darcy model, which not only is unconditionally stable, but also can decouple velocity and pressure to improve the calculation efficiency.

The rest of this article is arranged as follows. In Section 2, we provide a brief introduction to the time-dependent Stokes/Darcy model and Navier-Stokes/Darcy equation. We give the second order, unconditionally stable artificial compression method and the corresponding theoretical derivation for Stokes/Darcy model in Section 3. Finally, The numerical tests show the method is unconditional stability and second order accuracy.

2 | THE TIME-DEPENDENT MODEL

The fluid velocity $\mathbf{u} = \mathbf{u}(x, t)$ and pressure $p = p(x, t)$ are defined in Ω_f , the fluid motion is governed by the time-dependent Navier-Stokes equation

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot (\mathbb{T}_v(\mathbf{u}, p)) = f_1(x, t), & \text{in } \Omega_f \times (0, T), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega_f \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}^0(x), & \text{in } \Omega_f, \end{cases} \quad (1)$$

where

$$\mathbb{T}_v(\mathbf{u}, p) = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{u}), \quad \mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}),$$

are the stress tensor and the deformation rate tensor, $\nu > 0$ denotes the kinetic viscosity and f_1 is the external force.

We also have the linear Stokes equation

$$\begin{cases} \mathbf{u}_t - \nabla \cdot (\mathbb{T}_v(\mathbf{u}, p)) = f_1(x, t), & \text{in } \Omega_f \times (0, T), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega_f \times (0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}^0(x), & \text{in } \Omega_f. \end{cases} \quad (2)$$

The porous medium hydraulic head $\phi = \phi(x, t)$ and velocity $u_p = u_p(x, t)$ in region Ω_p satisfy

$$\begin{cases} u_p = -\mathbf{K}\nabla\phi, & \text{in } \Omega_p \times (0, T), \\ S_0\phi_t + \nabla \cdot u_p = f_2, & \text{in } \Omega_p \times (0, T), \\ \phi(x, 0) = \phi^0, & \text{in } \Omega_p, \end{cases} \quad (3)$$

where \mathbf{K} denotes the hydraulic conductivity in Ω_p , which is a positive symmetric tensor and is allowed to vary in space, S_0 represents the specific mass storativity coefficient and f_2 is a source term.

The first equation is the Darcy's law and the second equation is the saturated flow model. Here $\phi = z + \frac{p_p}{\rho g}$, where p_p represents the dynamic pressure, z the height from a reference level, ρ the density and g the gravitational constant.

Combining the two equations in (3), we get the equation for the piezometric head ϕ , which we will refer to it simply as the Darcy equation:

$$S_0\phi_t - \nabla \cdot (\mathbf{K}\nabla\phi) = f_2(x, t), \quad \text{in } \Omega_p \times (0, T). \quad (4)$$

Note that Ω_f and Ω_p are two disjoint, connected and bounded domains occupied by fluid flow and porous media flow, and assume that they lie across an interface $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$.

The above equations (1) or (2) and (4) are completed and coupled together by the following boundary conditions:

$$\mathbf{u} = 0 \quad \text{on } \Gamma_f \times (0, T) \quad \text{and} \quad \phi = 0 \quad \text{on } \Gamma_p \times (0, T), \quad (5)$$

here, we denote $\Gamma_f = \partial\Omega_f \cap \partial\Omega$ and $\Gamma_p = \partial\Omega_p \cap \partial\Omega$.

We also need the interface conditions on Γ :

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n}_f - \mathbf{K}\nabla\phi \cdot \mathbf{n}_p &= 0, & \text{on } \Gamma \times (0, T), \\ -[\mathbb{T}_v(\mathbf{u}, p) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f &= g\phi, & \text{on } \Gamma \times (0, T), \\ -[\mathbb{T}_v(\mathbf{u}, p) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i &= \frac{\alpha_{BJS}\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} \mathbf{u} \cdot \boldsymbol{\tau}_i, & \text{on } \Gamma \times (0, T), \end{aligned}$$

where $\boldsymbol{\tau}_i, i = 1, \dots, d-1$, are the orthonormal tangential unit vectors along Γ , α_{BJS} is an experimentally determined parameter and $\mathbf{\Pi}$ represents the permeability, which has the following relation with the hydraulic conductivity, $\mathbf{K} = \frac{\Pi_g}{\nu}$. \mathbf{n}_p and \mathbf{n}_f are the unit outward normal vectors on $\partial\Omega_p$ and $\partial\Omega_f$, respectively.

Now let us introduce some Hilbert spaces:

$$\begin{aligned} \mathbf{H}_f &= \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) : \mathbf{v}|_{\Gamma_f} = 0\}, \\ H_p &= \{\psi \in H^1(\Omega_p) : \psi|_{\Gamma_p} = 0\}, \\ Q_f &= L_0^2(\Omega_f) = \left\{ q \in L^2(\Omega_f) : \int_{\Omega_f} q = 0 \right\}. \end{aligned}$$

From now on, we always use $(\cdot, \cdot)_D$ and $\|\cdot\|_D$ to denote the L^2 inner product and the corresponding norm on any given domain D . Furthermore, define the H^1 norm on domain D by $\|\cdot\|_{1,D}$.

The weak formulation of the time-dependent Stokes/Darcy model (2) and (4) reads as follows: find $\mathbf{u} : [0, T] \rightarrow \mathbf{H}_f$, $\phi : [0, T] \rightarrow H_p$ and $p : [0, T] \rightarrow Q_f$ satisfying

$$\begin{aligned} (\mathbf{u}_t, \mathbf{v})_f + a_f(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{v}, \phi) &= (f_1, \mathbf{v})_f, & \forall \mathbf{v} \in \mathbf{H}_f, \\ b(\mathbf{u}, q) &= 0, & \forall q \in Q_f, \\ gS_0(\phi_t, \psi)_p + a_p(\phi, \psi) - c(\mathbf{u}, \psi) &= g(f_2, \psi)_p, & \forall \psi \in H_p, \end{aligned} \quad (6)$$

where

$$\begin{aligned} a_f(\mathbf{u}, \mathbf{v}) &= \nu(\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))_f + \sum_{i=1}^{d-1} \int_{\Gamma} \alpha \sqrt{\frac{\nu gd}{\text{tr}(\mathbf{K})}} (\mathbf{u} \cdot \boldsymbol{\tau}_i)(\mathbf{v} \cdot \boldsymbol{\tau}_i) ds, \\ a_p(\phi, \psi) &= g(\mathbf{K}\nabla\phi, \nabla\psi)_p, \\ b(\mathbf{v}, p) &= -(p, \nabla \cdot \mathbf{v})_f, \\ c(\mathbf{v}, \phi) &= g \int_{\Gamma} \phi \mathbf{v} \cdot \mathbf{n}_f ds. \end{aligned}$$

In fact, The weak formulation of the unsteady Navier-Stokes/Darcy model (1) and (4) is given as follows: find $\mathbf{u} : [0, T] \rightarrow \mathbf{H}_f$, $\phi : [0, T] \rightarrow H_p$ and $p : [0, T] \rightarrow Q_f$ satisfying

$$\begin{aligned} & (\mathbf{u}_t, \mathbf{v})_f + a_f(\mathbf{u}, \mathbf{v}) + a_{f,c}(\mathbf{u}, \mathbf{u}; \mathbf{v}) + b(\mathbf{v}, p) + c(\mathbf{v}, \phi) = (f_1, \mathbf{v})_f, \quad \forall \mathbf{v} \in \mathbf{H}_f, \\ & b(\mathbf{u}, q) = 0, \quad \forall q \in Q_f, \\ & gS_0(\phi_t, \psi)_p + a_p(\phi, \psi) - c(\mathbf{u}, \psi) = g(f_2, \psi)_p, \quad \forall \psi \in H_p, \end{aligned} \quad (7)$$

where

$$a_{f,c}(\mathbf{u}, \mathbf{u}; \mathbf{v}) = ((\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v})_f.$$

In addition, we have the equivalence of the weak problems and the model problems, if we assume:

$$f_1 \in \mathbf{L}^2(\Omega_f), \quad f_2 \in L^2(\Omega_p), \quad \mathbf{K} \in L^\infty(\Omega_p)^{d \times d}, \quad (8)$$

and there exist two constants $\lambda_{\max} > 0$, $\lambda_{\min} > 0$ such that

$$0 < \lambda_{\min}|x|^2 \leq \mathbf{K}x \cdot x \leq \lambda_{\max}|x|^2, \quad \forall x \in \Omega_p. \quad (9)$$

For the purpose of later analysis, we recall the trace inequality and Poincaré inequality:

$$\begin{aligned} \|v\|_{L^2(\partial D)} &\leq c_0 \|v\|_D^{\frac{1}{2}} \|v\|_{1,D}^{\frac{1}{2}}, \quad \forall v \in D, \\ \|v\|_D &\leq c_1 \|\nabla v\|_D, \quad \forall v \in D, \\ \|\nabla \cdot \mathbf{u}\|_f &\leq \sqrt{d} \|\nabla \mathbf{u}\|_f, \quad d = 2 \text{ or } 3. \end{aligned}$$

Here and hereafter, we always use c_i and C_i ($i \in N$) to denote positive constants which are not dependent on the data of the problem.

There are continuity and coercivity of the bilinear forms:

$$\begin{aligned} a_f(\mathbf{u}, \mathbf{v}) &\leq C_1 \|\mathbf{u}\|_{1,f} \|\mathbf{v}\|_{1,f}, \\ a_f(\mathbf{u}, \mathbf{u}) &\geq \nu \|\mathbf{u}\|_{1,f}^2, \\ a_p(\phi, \psi) &\leq g \lambda_{\max} \|\phi\|_{1,p} \|\psi\|_{1,p}, \\ a_p(\phi, \phi) &\geq g \lambda_{\min} \|\phi\|_{1,p}^2. \end{aligned} \quad (10)$$

For the trilinear term $a_{f,c}(\mathbf{u}; \mathbf{u}, \mathbf{v})$, it's easy to get, for the constant $M = M(\Omega)$

$$\begin{aligned} a_{f,c}(\mathbf{u}, \mathbf{v}; \mathbf{v}) &= 0, \\ a_{f,c}(\mathbf{u}, \mathbf{v}; \mathbf{w}) &\leq M \|\nabla \mathbf{u}\|_f \|\nabla \mathbf{v}\|_f \|\nabla \mathbf{w}\|_f. \end{aligned}$$

The property of the interface coupling term $c(\cdot, \cdot)$ plays a key role in our analysis:

$$|c(\mathbf{u}, \phi)| \leq C_2 \|\mathbf{u}\|_{1,f} \|\phi\|_{1,p}, \quad (11)$$

in the special case of a flat interface Γ (see details in [17]).

$$|c(\mathbf{u}, \phi)| \leq g C_3 \|\mathbf{u}\|_{div,f} \|\phi\|_{1,p}, \quad (12)$$

where $\|\mathbf{u}\|_{div,f}^2 := \|\mathbf{u}\|_f^2 + \|\nabla \cdot \mathbf{u}\|_f^2$.

Let's first discretize in space using the finite element method (FEM). For any given small parameter $h > 0$, we construct the regular triangulations \mathcal{T}_h , \mathcal{T}_{fh} and \mathcal{T}_{ph} of Ω , Ω_f and Ω_p such that the mesh aligns with Γ . In addition, \mathbf{H}_{fh} , Q_{fh} , and H_{ph} are selected as finite element subspaces of \mathbf{H}_f , Q_f , and H_p , respectively. And the space pair $(\mathbf{H}_{fh}, Q_{fh})$ satisfies the discrete LBB condition:

$$\inf_{q_h \in Q_{fh}} \sup_{\mathbf{v}_h \in \mathbf{H}_{fh}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)_f}{\|q_h\|_f \|\mathbf{v}_h\|_{1,f}} \geq \beta. \quad (13)$$

Assume that \mathbf{H}_{fh} , Q_{fh} , and H_{ph} satisfy approximation properties:

$$\begin{aligned} \inf_{\mathbf{u}_h \in \mathbf{H}_{fh}} \|\mathbf{u} - \mathbf{u}_h\|_f &\leq Ch^{s+1} \|\mathbf{u}\|_{H^{s+1}(\Omega_f)}, \\ \inf_{\mathbf{u}_h \in \mathbf{H}_{fh}} \|\mathbf{u} - \mathbf{u}_h\|_{1,f} &\leq Ch^s \|\mathbf{u}\|_{H^{s+1}(\Omega_f)}, \\ \inf_{\phi_h \in H_{ph}} \|\phi - \phi_h\|_p &\leq Ch^{s+1} \|\phi\|_{H^{s+1}(\Omega_p)}, \\ \inf_{\phi_h \in H_{ph}} \|\phi - \phi_h\|_{1,p} &\leq Ch^s \|\phi\|_{H^{s+1}(\Omega_p)}, \\ \inf_{p_h \in Q_{fh}} \|p - p_h\|_f &\leq Ch^{s+1} \|\phi\|_{H^{s+1}(\Omega_f)}. \end{aligned} \quad (14)$$

Furthermore, suppose that the following inverse inequality holds:

$$\|\nabla v_h\|_D \leq Ch^{-1} \|v_h\|_D, \quad \forall v_h \in D.$$

Hence, the semi-discretized formulation of the time-dependent Stokes/Darcy model: For $\forall \mathbf{v}_h \in \mathbf{H}_{fh}$, $q_h \in Q_{fh}$, and $\psi_h \in H_{ph}$, given $\mathbf{u}_h(x, 0)$, $p_h(x, 0)$, and $\phi_h(x, 0)$, find $\mathbf{u}_h : [0, T] \rightarrow \mathbf{H}_{fh}$, $\phi_h : [0, T] \rightarrow H_{ph}$ and $p : [0, T] \rightarrow Q_{fh}$ such that

$$\begin{aligned} (\mathbf{u}_{h,t}, \mathbf{v}_h)_f + a_f(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + c(\mathbf{v}_h, \phi_h) &= (f_1, \mathbf{v}_h)_f, \\ b(\mathbf{u}_h, q_h) &= 0, \\ gS_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c(\mathbf{u}_h, \psi_h) &= g(f_2, \psi_h)_p. \end{aligned} \quad (15)$$

The semi-discretized formulation of the time-dependent Navier-Stokes/Darcy model:

$$\begin{aligned} (\mathbf{u}_{h,t}, \mathbf{v}_h)_f + a_f(\mathbf{u}_h, \mathbf{v}_h) + a_{f,c}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + c(\mathbf{v}_h, \phi_h) &= (f_1, \mathbf{v}_h)_f, \\ b(\mathbf{u}_h, q_h) &= 0, \\ gS_0(\phi_{h,t}, \psi_h)_p + a_p(\phi_h, \psi_h) - c(\mathbf{u}_h, \psi_h) &= g(f_2, \psi_h)_p. \end{aligned} \quad (16)$$

3 | CNLF-STABLE ARTIFICIAL COMPRESSION METHOD

The regular CNLF scheme is conditional on the time step, which is usually related to the mesh size h and the specific mass storativity coefficient S_0 . However, the time step condition is sensitive to values of S_0 . In some cases, this may be computationally limited. Therefore, it is necessary to propose the scheme of unconditional stability.

As problems become larger, the computation time becomes more and more important. In order to reduce computation time, it is often to break the coupling between velocity and pressure in various ways, such as adding a small (artificial) compression term, which is studied in this article. This can greatly speed up the calculation and does not require pressure boundary conditions. The method described below is a second-order artificial compression method that explicitly handles pressure.

Let $N \in \mathbb{N}$, we choose a uniform distribution of discrete time levels with $t_m = m\Delta t$ and $T = N\Delta t$. If $T = \infty$ then $N = \infty$. $(\mathbf{u}_h^{m+1}, p_h^{m+1}, \phi_h^{m+1})$ denotes the approximation solution of $(\mathbf{u}_h(t_{m+1}), p_h(t_{m+1}), \phi_h(t_{m+1}))$.

Pick constants $\alpha, \beta > 0$ with $\alpha - \frac{1}{2\beta-1} > 0$, then for $\forall \mathbf{v}_h \in \mathbf{H}_{fh}$, $q_h \in Q_{fh}$ and $\psi_h \in H_{ph}$, given $(\mathbf{u}_h^{m-1}, p_h^{m-1}, \phi_h^{m-1})$ and $(\mathbf{u}_h^m, p_h^m, \phi_h^m)$, find $(\mathbf{u}_h^{m+1}, p_h^{m+1}, \phi_h^{m+1})$ satisfying:

Algorithm 1 (CNLF-stable artificial compression method for Stokes/Darcy model).

$$\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t}, \mathbf{v}_h \right)_f + \beta \left(\nabla \cdot \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t} \right), \nabla \cdot \mathbf{v}_h \right)_f + b(\mathbf{v}_h, p_h^m) \quad (17)$$

$$+ a_f \left(\frac{\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}}{2}, \mathbf{v}_h \right) + c(\mathbf{v}_h, \phi_h^m) = (f_1^m, \mathbf{v}_h)_f,$$

$$\alpha \Delta t (p_h^{m+1} - p_h^{m-1}, q_h)_f - b(\mathbf{u}_h^m, q_h) = 0, \quad (18)$$

$$g S_0 \left(\frac{\phi_h^{m+1} - \phi_h^{m-1}}{2\Delta t}, \psi_h \right)_p + a_p \left(\frac{\phi_h^{m+1} + \phi_h^{m-1}}{2}, \psi_h \right) - c(\mathbf{u}_h^m, \psi_h) \quad (19)$$

$$+ \Delta t g^2 C_3^2 \left\{ (\phi_h^{m+1} - \phi_h^{m-1}, \psi_h)_p + (\nabla (\phi_h^{m+1} - \phi_h^{m-1}), \nabla \psi_h)_p \right\} = g(f_2^m, \psi_h)_p.$$

Algorithm 2 (CNLF-stable artificial compression method for Navier-Stokes/Darcy model).

$$\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t}, \mathbf{v}_h \right)_f + \beta \left(\nabla \cdot \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t} \right), \nabla \cdot \mathbf{v}_h \right)_f + b(\mathbf{v}_h, p_h^m) \quad (20)$$

$$+ a_{f,c} \left(\mathbf{u}_h^m, \frac{\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}}{2}; \mathbf{v}_h \right)_f + \frac{1}{2} < \mathbf{u}_h^m \left(\frac{\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}}{2} \right), \mathbf{v}_h \cdot \mathbf{n}_f >_\Gamma$$

$$+ a_f \left(\frac{\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}}{2}, \mathbf{v}_h \right) + c(\mathbf{v}_h, \phi_h^m) = (f_1^m, \mathbf{v}_h)_f,$$

$$\alpha \Delta t (p_h^{m+1} - p_h^{m-1}, q_h)_f - b(\mathbf{u}_h^m, q_h) = 0, \quad (21)$$

$$g S_0 \left(\frac{\phi_h^{m+1} - \phi_h^{m-1}}{2\Delta t}, \psi_h \right)_p + a_p \left(\frac{\phi_h^{m+1} + \phi_h^{m-1}}{2}, \psi_h \right) - c(\mathbf{u}_h^m, \psi_h) \quad (22)$$

$$+ \Delta t g^2 C_3^2 \left\{ (\phi_h^{m+1} - \phi_h^{m-1}, \psi_h)_p + (\nabla (\phi_h^{m+1} - \phi_h^{m-1}), \nabla \psi_h)_p \right\} = g(f_2^m, \psi_h)_p.$$

CNLF is a 3 level method. The first term $(\mathbf{u}_h^0, p_h^0, \phi_h^0)$ arise from the initial conditions of the problem. We can get $(\mathbf{u}_h^1, p_h^1, \phi_h^1)$ through other higher-order schemes.

3.1 | Unconditional stability

Algorithm 1 is a time discretization of the model

$$(1 - \beta \text{grad div}) \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = f_1(x, t), \\ 2\alpha \Delta t^2 p_t + \nabla \cdot \mathbf{u} = 0, \\ S_0 \phi_t - \nabla \cdot (\mathbf{K} \nabla \phi) + \Delta t g^2 C_3^2 (\phi + \nabla \phi) = f_2(x, t). \quad (23)$$

Note that $2\alpha \Delta t^2 p_t$ is the standard artificial compression that allows the pressure to be advanced explicitly in time. Artificial compressibility methods are closely related to pressure projection methods (See [6,8,25]).

By adding the stabilization terms $\beta \text{grad div} \mathbf{u}_t$ and $\Delta t g^2 C_3^2 (\phi + \nabla \phi)$ that acts through the momentum equation to ensure unconditional stability of the continuity equation. The following theorem shows the unconditional, long time, energy stability of the artificial compression method.

Theorem 3.1 (unconditional stability for Algorithm 1). *Pick constants $\alpha, \beta > 0$ with $\alpha - \frac{1}{2\beta-1} > 0$. For any $N > 1$, the solutions of the Algorithm 1, there holds*

$$\begin{aligned} & \frac{1}{4}(\|\mathbf{u}_h^N\|_f^2 + \|\mathbf{u}_h^{N-1}\|_f^2) + \frac{1}{2}gS_0(\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2) + \left(\alpha - \frac{1}{2\beta-1}\right)\Delta t^2(\|p_h^N\|_f^2 + \|p_h^{N-1}\|_f^2) \\ & + \sum_{m=1}^{N-1} \left(\frac{\Delta t \nu}{4} \|\nabla(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1})\|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \|\nabla(\phi_h^{m+1} + \phi_h^{m-1})\|_p^2 \right) \\ & \leq \frac{3}{4}(\|\mathbf{u}_h^1\|_f^2 + \|\mathbf{u}_h^0\|_f^2) + \left(\frac{2\beta+3}{4}\right)(\|\nabla \cdot \mathbf{u}_h^1\|_f^2 + \|\nabla \cdot \mathbf{u}_h^0\|_f^2) + \frac{1}{2}gS_0(\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2) \\ & + 2\Delta t^2 g^2 C_3^2(\|\phi_h^1\|_{1,p}^2 + \|\phi_h^0\|_{1,p}^2) + \left(\frac{2\alpha+1}{2}\right)\Delta t^2(\|p_h^1\|_f^2 + \|p_h^0\|_f^2) \\ & + \frac{\Delta t}{4} \sum_{m=1}^{N-1} \left(\frac{1}{\nu} \|f_1^m\|_{-1,f}^2 + \frac{g}{\lambda_{\min}} \|f_2^m\|_{-1,p}^2 \right). \end{aligned}$$

Proof. Setting $\mathbf{v}_h = \mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}$, $q_h = p_h^{m+1} + p_h^{m-1}$ and $\phi_h = \phi_h^{m+1} + \phi_h^{m-1}$ in Algorithm 1, then gives

$$\begin{aligned} & \frac{1}{2\Delta t}(\|\mathbf{u}_h^{m+1}\|_f^2 + \beta \|\nabla \cdot \mathbf{u}_h^{m+1}\|_f^2) - \frac{1}{2\Delta t}(\|\mathbf{u}_h^{m-1}\|_f^2 + \beta \|\nabla \cdot \mathbf{u}_h^{m-1}\|_f^2) + b(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, p_h^m) \\ & + \frac{1}{2}a_f(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, \mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}) + c(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, \phi_h^m) = (f_1^m, \mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1})_f, \end{aligned}$$

$$\alpha \Delta t \|p_h^{m+1}\|_f^2 - \alpha \Delta t \|p_h^{m-1}\|_f^2 - b(\mathbf{u}_h^m, p_h^{m+1} + p_h^{m-1}) = 0,$$

and

$$\begin{aligned} & \frac{gS_0}{2\Delta t} \|\phi_h^{m+1}\|_p^2 - \frac{gS_0}{2\Delta t} \|\phi_h^{m-1}\|_p^2 + \Delta t g^2 C_3^2 (\|\phi_h^{m+1}\|_{1,p}^2 - \|\phi_h^{m-1}\|_{1,p}^2) \\ & + \frac{1}{2}a_p(\phi_h^{m+1} + \phi_h^{m-1}, \phi_h^{m+1} + \phi_h^{m-1}) - c(\mathbf{u}_h^m, \phi_h^{m+1} + \phi_h^{m-1}) = g(f_2^m, \phi_h^{m+1} + \phi_h^{m-1})_p. \end{aligned}$$

Multiply all equations by Δt and add the three resulting equations, then using the coercivity of the bilinear forms, the dual norms and Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2}(\|\mathbf{u}_h^{m+1}\|_f^2 + \beta \|\nabla \cdot \mathbf{u}_h^{m+1}\|_f^2 + 2\alpha \Delta t^2 \|p_h^{m+1}\|_f^2 + gS_0 \|\phi_h^{m+1}\|_p^2 + 2\Delta t^2 g^2 C_3^2 \|\phi_h^{m+1}\|_{1,p}^2) \\ & - \frac{1}{2}(\|\mathbf{u}_h^{m-1}\|_f^2 + \beta \|\nabla \cdot \mathbf{u}_h^{m-1}\|_f^2 + 2\alpha \Delta t^2 \|p_h^{m-1}\|_f^2 + gS_0 \|\phi_h^{m-1}\|_p^2 + 2\Delta t^2 g^2 C_3^2 \|\phi_h^{m-1}\|_{1,p}^2) \\ & + \Delta t b(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, p_h^m) - \Delta t b(\mathbf{u}_h^m, p_h^{m+1} + p_h^{m-1}) + \Delta t c(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, \phi_h^m) \\ & - \Delta t c(\mathbf{u}_h^m, \phi_h^{m+1} + \phi_h^{m-1}) + \frac{\Delta t \nu}{4} \|\nabla(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1})\|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \|\nabla(\phi_h^{m+1} + \phi_h^{m-1})\|_p^2 \\ & \leq \frac{\Delta t}{4} \left(\frac{1}{\nu} \|f_1^m\|_{-1,f}^2 + \frac{g}{\lambda_{\min}} \|f_2^m\|_{-1,p}^2 \right). \quad (24) \end{aligned}$$

Note that the terms $\Delta t b(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, p_h^m) - \Delta t b(\mathbf{u}_h^m, p_h^{m+1} + p_h^{m-1})$ in (24) couple the pressure and incompressibility and are the key terms in the stability analysis. Rearrange them into a time difference, as follows

$$\begin{aligned} & \Delta t b(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, p_h^m) - \Delta t b(\mathbf{u}_h^m, p_h^{m+1} + p_h^{m-1}) \\ & = \Delta t (b(\mathbf{u}_h^{m+1}, p_h^m) - b(\mathbf{u}_h^m, p_h^{m+1})) - \Delta t (b(\mathbf{u}_h^m, p_h^{m-1}) - b(\mathbf{u}_h^{m-1}, p_h^m)). \end{aligned}$$

Denote the energy term by

$$\begin{aligned} E^{m+\frac{1}{2}} := & \frac{1}{2} \{ \| \mathbf{u}_h^{m+1} \|_f^2 + \| \mathbf{u}_h^m \|_f^2 + \beta (\| \nabla \cdot \mathbf{u}_h^{m+1} \|_f^2 + \| \nabla \cdot \mathbf{u}_h^m \|_f^2) \\ & + g S_0 (\| \phi_h^{m+1} \|_p^2 + \| \phi_h^m \|_p^2) + 2\Delta t^2 g^2 C_3^2 (\| \phi_h^{m+1} \|_{1,p}^2 + \| \phi_h^m \|_{1,p}^2) \\ & + 2\alpha \Delta t^2 (\| p_h^{m+1} \|_f^2 + \| p_h^m \|_f^2) + 2\Delta t (b(\mathbf{u}_h^{m+1}, p_h^m) - b(\mathbf{u}_h^m, p_h^{m+1})) \}. \end{aligned} \quad (25)$$

Similarly, we let

$$C^{m+\frac{1}{2}} = c(\mathbf{u}_h^{m+1}, \phi_h^m) - c(\mathbf{u}_h^m, \phi_h^{m+1}),$$

then the interface terms in (24) become

$$c(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}, \phi_h^m) - c(\mathbf{u}_h^m, \phi_h^{m+1} + \phi_h^{m-1}) = C^{m+\frac{1}{2}} - C^{m-\frac{1}{2}}.$$

Adding and subtracting $\frac{1}{2}(\| \mathbf{u}_h^m \|_f^2 + \beta \| \nabla \cdot \mathbf{u}_h^m \|_f^2 + g S_0 \| \phi_h^m \|_p^2 + 2\Delta t^2 g^2 C_3^2 \| \phi_h^m \|_{1,p}^2 + 2\alpha \Delta t^2 \| p_h^m \|_f^2)$, then we have

$$\begin{aligned} & E^{m+\frac{1}{2}} - E^{m-\frac{1}{2}} + \frac{\Delta t \nu}{4} \| \nabla (\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}) \|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \| \nabla (\phi_h^{m+1} + \phi_h^{m-1}) \|_p^2 \\ & + \Delta t \left(C^{m+\frac{1}{2}} - C^{m-\frac{1}{2}} \right) \\ & \leq \frac{\Delta t}{4} \left(\frac{1}{\nu} \| f_1^m \|_{-1,f}^2 + \frac{g}{\lambda_{\min}} \| f_2^m \|_{-1,p}^2 \right). \end{aligned} \quad (26)$$

Sum up the inequality from $m = 1$ to $N - 1$, we find

$$\begin{aligned} & E^{N-\frac{1}{2}} + \sum_{m=1}^{N-1} \left(\frac{\Delta t \nu}{4} \| \nabla (\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}) \|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \| \nabla (\phi_h^{m+1} + \phi_h^{m-1}) \|_p^2 \right) + \Delta t C^{N-\frac{1}{2}} \\ & \leq E^{\frac{1}{2}} + \Delta t C^{\frac{1}{2}} + \frac{\Delta t}{4} \sum_{m=1}^{N-1} \left(\frac{1}{\nu} \| f_1^m \|_{-1,f}^2 + \frac{g}{\lambda_{\min}} \| f_2^m \|_{-1,p}^2 \right). \end{aligned} \quad (27)$$

Firstly, we consider the interface terms, applying inequality (12)

$$\begin{aligned} |\Delta t C^{N-\frac{1}{2}}| & \leq \frac{1}{4} (\| \mathbf{u}_h^N \|_{div,f}^2 + \| \mathbf{u}_h^{N-1} \|_{div,f}^2) \\ & + \Delta t^2 g^2 C_3^2 (\| \phi_h^N \|_{1,p}^2 + \| \phi_h^{N-1} \|_{1,p}^2). \end{aligned} \quad (28)$$

Then for the terms that couple the pressure and incompressibility in (25), we obtain

$$\begin{aligned} |\Delta t b(\mathbf{u}_h^N, p_h^{N-1}) - \Delta t b(\mathbf{u}_h^{N-1}, p_h^N)| & \leq \frac{2\beta - 1}{4} (\| \nabla \cdot \mathbf{u}_h^N \|_f^2 + \| \nabla \cdot \mathbf{u}_h^{N-1} \|_f^2) \\ & + \frac{1}{2\beta - 1} \Delta t^2 (\| p_h^N \|_f^2 + \| p_h^{N-1} \|_f^2). \end{aligned} \quad (29)$$

Thus,

$$\begin{aligned} E^{N-\frac{1}{2}} + \Delta t C^{N-\frac{1}{2}} & \geq \frac{1}{4} (\| \mathbf{u}_h^N \|_f^2 + \| \mathbf{u}_h^{N-1} \|_f^2) + \frac{1}{2} g S_0 (\| \phi_h^N \|_p^2 + \| \phi_h^{N-1} \|_p^2) \\ & + (\alpha - \frac{1}{2\beta - 1}) \Delta t^2 (\| p_h^N \|_f^2 + \| p_h^{N-1} \|_f^2). \end{aligned} \quad (30)$$

Therefore, choose the appropriate parameters α and β such that $\alpha - \frac{1}{2\beta - 1} > 0$, we get the result. ■

Theorem 3.2 (unconditional stability for Algorithm 2). *Pick constants $\alpha, \beta > 0$ with $\alpha - \frac{1}{2\beta - 1} > 0$. For any $N > 1$, the solutions of the Algorithm 2, there holds*

$$\begin{aligned}
& \frac{1}{4} (\|\mathbf{u}_h^N\|_f^2 + \|\mathbf{u}_h^{N-1}\|_f^2) + \frac{1}{2} g S_0 (\|\phi_h^N\|_p^2 + \|\phi_h^{N-1}\|_p^2) + \left(\alpha - \frac{1}{2\beta - 1} \right) \Delta t^2 (\|p_h^N\|_f^2 + \|p_h^{N-1}\|_f^2) \\
& + \sum_{m=1}^{N-1} \left(\frac{\Delta t \nu}{4} \|\nabla(\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1})\|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \|\nabla(\phi_h^{m+1} + \phi_h^{m-1})\|_p^2 \right) \\
& \leq \frac{3}{4} (\|\mathbf{u}_h^1\|_f^2 + \|\mathbf{u}_h^0\|_f^2) + \left(\frac{2\beta + 3}{4} \right) (\|\nabla \cdot \mathbf{u}_h^1\|_f^2 + \|\nabla \cdot \mathbf{u}_h^0\|_f^2) + \frac{1}{2} g S_0 (\|\phi_h^1\|_p^2 + \|\phi_h^0\|_p^2) \\
& + 2\Delta t^2 g^2 C_3^2 (\|\phi_h^1\|_{1,p}^2 + \|\phi_h^0\|_{1,p}^2) + \left(\frac{2\alpha + 1}{2} \right) \Delta t^2 (\|p_h^1\|_f^2 + \|p_h^0\|_f^2) \\
& + \frac{\Delta t}{4} \sum_{m=1}^{N-1} \left(\frac{1}{\nu} \|f_1^m\|_{-1,f}^2 + \frac{g}{\lambda_{\min}} \|f_2^m\|_{-1,p}^2 \right).
\end{aligned}$$

Proof. Mainly using the skew symmetrization of nonlinear terms and the proof process similar to Theorem 3.1. ■

3.2 | Error analysis

We establish the method's error analysis over long time intervals. First we introduce the following discrete norms:

$$\begin{aligned}
\|w\|_{L^2(0,T;H^s(\Omega_{f,p}))}^2 &:= \Delta t \sum_{m=1}^N \|w^m\|_{H^s(\Omega_{f,p})}^2 \\
\|w\|_{L^\infty(0,T;H^s(\Omega_{f,p}))}^2 &:= \max_{0 \leq m \leq N} \|w^m\|_{H^s(\Omega_{f,p})}^2.
\end{aligned}$$

The following lemma in [17] will be used for the error analysis.

Lemma 3.1.

$$\Delta t \sum_{m=1}^{N-1} \left\| \mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right\|_f^2 \leq \frac{\Delta t^4}{20} \|\mathbf{u}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega_f))}^2, \quad (31)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| \phi_t^m - \frac{\phi^{m+1} - \phi^{m-1}}{2\Delta t} \right\|_p^2 \leq \frac{\Delta t^4}{20} \|\phi_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2, \quad (32)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| \nabla \left(\mathbf{u}^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2 \leq \frac{7\Delta t^4}{6} \|\mathbf{u}_{tt}\|_{L^2(0,T;\mathbf{H}^1(\Omega_f))}^2, \quad (33)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| \nabla \left(\phi^m - \frac{\phi^{m+1} - \phi^{m-1}}{2\Delta t} \right) \right\|_p^2 \leq \frac{7\Delta t^4}{6} \|\phi_{tt}\|_{L^2(0,T;H^1(\Omega_p))}^2, \quad (34)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| \nabla \left(\mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2 \leq \frac{\Delta t^4}{20} \|\nabla \mathbf{u}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega_f))}^2, \quad (35)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| \phi^{m+1} - \phi^{m-1} \right\|_{1,p}^2 \leq 4\Delta t^2 \|\phi_t\|_{L^2(0,T;H^1(\Omega_p))}^2, \quad (36)$$

$$\Delta t \sum_{m=1}^{N-1} \left\| p^{m+1} - p^{m-1} \right\|_f^2 \leq 4\Delta t^2 \|p_t\|_{L^2(0,T;L^2(\Omega_f))}^2. \quad (37)$$

Denote the error equations

$$\begin{aligned}\mathbf{e}_\mathbf{u}^m &= \mathbf{u}^m - \mathbf{u}_h^m = (\mathbf{u}^m - \tilde{\mathbf{u}}^m) + (\tilde{\mathbf{u}}^m - \mathbf{u}_h^m) = \boldsymbol{\eta}_\mathbf{u}^m + \boldsymbol{\theta}_\mathbf{u}^m, \\ e_p^m &= p^m - p_h^m = (p^m - \tilde{p}^m) + (\tilde{p}^m - p_h^m) = \boldsymbol{\eta}_p^m + \boldsymbol{\theta}_p^m, \\ e_\phi^m &= \phi^m - \phi_h^m = (\phi^m - \tilde{\phi}^m) + (\tilde{\phi}^m - \phi_h^m) = \boldsymbol{\eta}_\phi^m + \boldsymbol{\theta}_\phi^m.\end{aligned}$$

Here, $\tilde{\mathbf{u}}^m \in \mathbf{H}_{fh}$, $\tilde{p}^m \in Q_{fh}$, and $\tilde{\phi}^m \in H_{ph}$ are the approximate solution which are satisfied the property (14).

Theorem 3.3 (error estimate). For any $0 < t_N = T \leq \infty$, consider the Algorithm 1, if \mathbf{u} , p , and ϕ satisfy the regularity conditions

$$\begin{aligned}\mathbf{u} &\in \mathbf{L}^2(0, T; \mathbf{H}^{s+2}(\Omega_f)) \cap \mathbf{L}^\infty(0, T; \mathbf{H}^{s+1}(\Omega_f)) \cap \mathbf{H}^3(0, T; \mathbf{H}^1(\Omega_f)), \\ p &\in L^2(0, T; H^{s+1}(\Omega_f)) \cap H^1(0, T; H^1(\Omega_f)), \\ \phi &\in L^2(0, T; H^{s+2}(\Omega_p)) \cap L^\infty(0, T; H^{s+1}(\Omega_p)) \cap H^3(0, T; H^1(\Omega_p)),\end{aligned}$$

and pick $\frac{\alpha}{2} - \frac{1}{2\beta-1} > 0$, then

$$\begin{aligned}&\frac{1}{4}(\|\mathbf{e}_\mathbf{u}^N\|_f^2 + \|\mathbf{e}_\mathbf{u}^{N-1}\|_f^2) + \frac{1}{2}gS_0(\|e_\phi^N\|_p^2 + \|e_\phi^{N-1}\|_p^2) + \left(\frac{\alpha}{2} - \frac{1}{2\beta-1}\right)\Delta t^2(\|e_p^N\|_f^2 + \|e_p^{N-1}\|_f^2) \\ &+ \frac{\Delta t}{4} \sum_{m=1}^{N-1} (\nu \|\nabla(\mathbf{e}_\mathbf{u}^{m+1} + \mathbf{e}_\mathbf{u}^{m-1})\|_f^2 + g\lambda_{\min} \|\nabla(e_\phi^{m+1} + e_\phi^{m-1})\|_p^2) \\ &\leq C(h^{2s} + \Delta t^4 + \Delta t^{-1}h^{2s} + \Delta t^3),\end{aligned}$$

where $C > 0$ is a constant independent of h and Δt .

Proof. Subtract (17)-(19) from (6) evaluated at time t^m to get

$$\begin{aligned}&\left(\mathbf{u}_t^m - \frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t}, \mathbf{v}_h\right)_f - \beta \left(\nabla \cdot \left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^{m-1}}{2\Delta t}\right), \nabla \cdot \mathbf{v}_h\right)_f + b(\mathbf{v}_h, p^m - p_h^m) \\ &+ a_f \left(\mathbf{u}^m - \frac{\mathbf{u}_h^{m+1} + \mathbf{u}_h^{m-1}}{2}, \mathbf{v}_h\right) + c(\mathbf{v}_h, \phi^m - \phi_h^m) = 0,\end{aligned}\quad (38)$$

$$-\alpha\Delta t(p_h^{m+1} - p_h^{m-1}, q_h)_f - b(\mathbf{u}^m - \mathbf{u}_h^m, q_h) = 0,\quad (39)$$

$$\begin{aligned}&gS_0 \left(\phi_t^m - \frac{\phi_h^{m+1} - \phi_h^{m-1}}{2\Delta t}, \psi_h\right)_p + a_p \left(\phi^m - \frac{\phi_h^{m+1} + \phi_h^{m-1}}{2}, \psi_h\right) - c(\mathbf{u}^m - \mathbf{u}_h^m, \psi_h) \\ &- \Delta t g^2 C_3^2 \left\{(\phi_h^{m+1} - \phi_h^{m-1}, \psi_h)_p + (\nabla(\phi_h^{m+1} - \phi_h^{m-1}), \nabla\psi_h)_p\right\} = 0.\end{aligned}\quad (40)$$

Thus, after using the definition of the error equations and rearranging, we have

$$\begin{aligned}&\left(\frac{\theta_\mathbf{u}^{m+1} - \theta_\mathbf{u}^{m-1}}{2\Delta t}, \mathbf{v}_h\right)_f + \beta \left(\nabla \cdot \left(\frac{\theta_\mathbf{u}^{m+1} - \theta_\mathbf{u}^{m-1}}{2\Delta t}\right), \nabla \cdot \mathbf{v}_h\right)_f + b(\mathbf{v}_h, \theta_p^m) \\ &+ a_f \left(\frac{\theta_\mathbf{u}^{m+1} + \theta_\mathbf{u}^{m-1}}{2}, \mathbf{v}_h\right) + c(\mathbf{v}_h, \theta_\phi^m) \\ &- \left(\frac{\eta_\mathbf{u}^{m+1} - \eta_\mathbf{u}^{m-1}}{2\Delta t}, \mathbf{v}_h\right)_f - \beta \left(\nabla \cdot \left(\frac{\eta_\mathbf{u}^{m+1} - \eta_\mathbf{u}^{m-1}}{2\Delta t}\right), \nabla \cdot \mathbf{v}_h\right)_f - b(\mathbf{v}_h, \eta_p^m) \\ &- a_f \left(\frac{\eta_\mathbf{u}^{m+1} + \eta_\mathbf{u}^{m-1}}{2}, \mathbf{v}_h\right) - c(\mathbf{v}_h, \eta_\phi^m) + e_\mathbf{u}^m(\mathbf{v}_h),\end{aligned}\quad (41)$$

$$\alpha \Delta t (\theta_p^{m+1} - \theta_p^{m-1}, q_h)_f - b(\theta_u^m, q_h) \quad (42)$$

$$\begin{aligned} &= -\alpha \Delta t (p^{m+1} - p^{m-1}, q_h)_f - \alpha \Delta t (\eta_p^{m+1} - \eta_p^{m-1}, q_h)_f + b(\eta_u^m, q_h), \\ &g S_0 \left(\frac{\theta_\phi^{m+1} - \theta_\phi^{m-1}}{2 \Delta t}, \psi_h \right)_p + a_p \left(\frac{\theta_\phi^{m+1} + \theta_\phi^{m-1}}{2}, \psi_h \right) - c(\theta_u^m, \psi_h) \quad (43) \\ &- \Delta t g^2 C_3^2 \left\{ (\theta_\phi^{m+1} - \theta_\phi^{m-1}, \psi_h)_p + (\nabla (\theta_\phi^{m+1} - \theta_\phi^{m-1}), \nabla \psi_h)_p \right\} \\ &= -g S_0 \left(\frac{\eta_\phi^{m+1} - \eta_\phi^{m-1}}{2 \Delta t}, \psi_h \right)_p + a_p \left(\frac{\eta_\phi^{m+1} + \eta_\phi^{m-1}}{2}, \psi_h \right) + c(\eta_u^m, \psi_h) \\ &- \Delta t g^2 C_3^2 \left\{ (\eta_\phi^{m+1} - \eta_\phi^{m-1}, \psi_h)_p + (\nabla (\eta_\phi^{m+1} - \eta_\phi^{m-1}), \nabla \psi_h)_p \right\} + \epsilon_\phi^m(\psi_h), \end{aligned}$$

where

$$\begin{aligned} \epsilon_u^m(\psi_h) &= - \left(\mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2 \Delta t}, \mathbf{v}_h \right)_f - \beta \left(\nabla \cdot \left(\mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2 \Delta t} \right), \nabla \cdot \mathbf{v}_h \right)_f \\ &- a_f \left(\mathbf{u}^m - \frac{\mathbf{u}^{m+1} + \mathbf{u}^{m-1}}{2}, \mathbf{v}_h \right), \end{aligned}$$

and

$$\begin{aligned} \epsilon_\phi^m(\psi_h) &= -g S_0 \left(\phi_t^m - \frac{\phi^{m+1} - \phi^{m-1}}{2 \Delta t}, \psi_h \right)_p - a_p \left(\phi^m - \frac{\phi^{m+1} + \phi^{m-1}}{2}, \psi_h \right) \\ &+ \Delta t g^2 C_3^2 \left\{ (\phi^{m+1} - \phi^{m-1}, \psi_h)_p + (\nabla (\phi^{m+1} - \phi^{m-1}), \nabla \psi_h)_p \right\}. \end{aligned}$$

Set $\mathbf{v}_h = \theta_u^{m+1} + \theta_u^{m-1}$, $q_h = \theta_p^{m+1} + \theta_p^{m-1}$, and $\psi_h = \theta_\phi^{m+1} + \theta_\phi^{m-1}$ in the equations above and multiply all equations by Δt and add the three resulting equations to obtain

$$\begin{aligned} &\frac{1}{2} (\|\theta_u^{m+1}\|_f^2 + \beta \|\nabla \cdot \theta_u^{m+1}\|_f^2 + 2\alpha \Delta t^2 \|\theta_p^{m+1}\|_f^2 + g S_0 \|\theta_\phi^{m+1}\|_p^2 + 2\Delta t^2 g^2 C_3^2 \|\theta_\phi^{m+1}\|_{1,p}^2) \\ &- \frac{1}{2} (\|\theta_u^{m-1}\|_f^2 + \beta \|\nabla \cdot \theta_u^{m-1}\|_f^2 + 2\alpha \Delta t^2 \|\theta_p^{m-1}\|_f^2 + g S_0 \|\theta_\phi^{m-1}\|_p^2 + 2\Delta t^2 g^2 C_3^2 \|\theta_\phi^{m-1}\|_{1,p}^2) \\ &+ \frac{\Delta t}{2} (a_f(\theta_u^{m+1} + \theta_u^{m-1}, \theta_u^{m+1} + \theta_u^{m-1}) + a_p(\theta_\phi^{m+1} + \theta_\phi^{m-1}, \theta_\phi^{m+1} + \theta_\phi^{m-1})) \\ &+ \Delta t b(\theta_u^{m+1} + \theta_u^{m-1}, \theta_p^m) - \Delta t b(\theta_u^m, \theta_p^{m+1} + \theta_p^{m-1}) \\ &+ \Delta t c(\theta_u^{m+1} + \theta_u^{m-1}, \theta_\phi^m) - \Delta t c(\theta_u^m, \theta_\phi^{m+1} + \theta_\phi^{m-1}) \\ &= -\frac{1}{2} [(\eta_u^{m+1} - \eta_u^{m-1}, \theta_u^{m+1} + \theta_u^{m-1})_f + \beta (\nabla \cdot (\eta_u^{m+1} - \eta_u^{m-1}), \nabla \cdot (\theta_u^{m+1} + \theta_u^{m-1}))_f] \\ &- \frac{1}{2} [g S_0 (\eta_\phi^{m+1} - \eta_\phi^{m-1}, \theta_\phi^{m+1} + \theta_\phi^{m-1})_p + 2\Delta t^2 g^2 C_3^2 \{(\eta_\phi^{m+1} - \eta_\phi^{m-1}, \theta_\phi^{m+1} + \theta_\phi^{m-1})_p \\ &+ (\nabla (\eta_\phi^{m+1} - \eta_\phi^{m-1}), \nabla (\theta_\phi^{m+1} - \theta_\phi^{m-1}))_p\}] \\ &- \frac{\Delta t}{2} [a_f(\eta_u^{m+1} + \eta_u^{m-1}, \theta_u^{m+1} + \theta_u^{m-1}) + a_p(\eta_\phi^{m+1} + \eta_\phi^{m-1}, \theta_\phi^{m+1} + \theta_\phi^{m-1})] \\ &- \Delta t [b(\theta_u^{m+1} + \theta_u^{m-1}, \eta_p^m) - b(\eta_u^m, \theta_p^{m+1} + \theta_p^{m-1})] \\ &- \Delta t [c(\theta_u^{m+1} + \theta_u^{m-1}, \eta_\phi^m) - c(\eta_u^m, \theta_\phi^{m+1} + \theta_\phi^{m-1})] \\ &- \alpha \Delta t^2 (p^{m+1} - p^{m-1}, \theta_p^{m+1} + \theta_p^{m-1})_f - \alpha \Delta t^2 (\eta_p^{m+1} - \eta_p^{m-1}, \theta_p^{m+1} + \theta_p^{m-1})_f \\ &+ \Delta t \epsilon_u^m(\theta_u^{m+1} + \theta_u^{m-1}) + \Delta t \epsilon_\phi^m(\theta_\phi^{m+1} + \theta_\phi^{m-1}). \end{aligned}$$

Firstly, we bound the first two terms on the right hand

$$\begin{aligned} & \frac{1}{2}[(\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1}, \theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})_f + \beta(\nabla \cdot (\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1}), \nabla \cdot (\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}))_f] \\ & \leq \frac{3c_1^2}{\nu \Delta t} \|\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1}\|_f^2 + \frac{3d^2 \beta^2}{\nu \Delta t} \|\nabla(\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{\nu \Delta t}{24} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2, \end{aligned} \quad (44)$$

and

$$\begin{aligned} & \frac{1}{2}[gS_0(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}, \theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})_p + 2\Delta t^2 g^2 C_3^2 \{(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}, \theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})_p \\ & \quad + (\nabla(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}), \nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1}))_p\}] \\ & \leq \left(\frac{15gS_0^2 c_1^2}{4\lambda_{\min} \Delta t} + \frac{15g^3 c_2^2 \Delta t^3 C_3^4}{\lambda_{\min}} \right) \|\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}\|_p^2 + \frac{15g^3 \Delta t^3 C_3^4}{\lambda_{\min}} \|\nabla(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1})\|_p^2 \\ & \quad + \frac{g \lambda_{\min} \Delta t}{20} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2. \end{aligned} \quad (45)$$

Next, we apply the continuity of the bilinear forms, then

$$\begin{aligned} & \frac{\Delta t}{2}(a_f(\eta_{\mathbf{u}}^{m+1} + \eta_{\mathbf{u}}^{m-1}, \theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}) + a_p(\eta_{\phi}^{m+1} + \eta_{\phi}^{m-1}, \theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})) \\ & \leq \frac{3M^2 \Delta t}{2\nu} \|\nabla(\eta_{\mathbf{u}}^{m+1} + \eta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{\nu \Delta t}{24} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 \\ & \quad + \frac{5g \lambda_{\max}^2 \Delta t}{4\lambda_{\min}} \|\nabla(\eta_{\phi}^{m+1} + \eta_{\phi}^{m-1})\|_p^2 + \frac{g \lambda_{\min} \Delta t}{20} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2. \end{aligned} \quad (46)$$

Considering the pressure terms, we get

$$\begin{aligned} & \alpha \Delta t^2 (p^{m+1} - p^{m-1}, \theta_p^{m+1} + \theta_p^{m-1})_f + \alpha \Delta t^2 (\eta_p^{m+1} - \eta_p^{m-1}, \theta_p^{m+1} + \theta_p^{m-1})_f \\ & \leq 4\alpha \Delta t^2 \|p^{m+1} - p^{m-1}\|_f^2 + 4\alpha \Delta t^2 \|\eta_p^{m+1} - \eta_p^{m-1}\|_f^2 + \frac{\alpha \Delta t^2}{4} \|\theta_p^{m+1}\|_f^2 \\ & \quad + \frac{\alpha \Delta t^2}{4} \|\theta_p^{m-1}\|_f^2, \end{aligned} \quad (47)$$

and

$$\begin{aligned} -\Delta t [b(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}, \eta_p^m) - b(\eta_{\mathbf{u}}^m, \theta_p^{m+1} + \theta_p^{m-1})] & \leq \frac{6dc_1^2 \Delta t}{\nu} \|\nabla \eta_p^m\|_f^2 + \frac{\nu \Delta t}{24} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 \\ & \quad + 2d \|\nabla \eta_{\mathbf{u}}^m\|_f^2 + \frac{\alpha \Delta t^2}{4} \|\theta_p^{m+1}\|_f^2 + \frac{\alpha \Delta t^2}{4} \|\theta_p^{m-1}\|_f^2. \end{aligned} \quad (48)$$

For the interface term, using the (11), this yields

$$\begin{aligned} & \Delta t [c(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}, \eta_{\phi}^m) - c(\eta_{\phi}^m, \theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})] \\ & \leq \frac{\nu \Delta t}{24} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{6C_2^2 \Delta t}{\nu} \|\nabla \eta_{\phi}^m\|_p^2 \\ & \quad + \frac{g \lambda_{\min} \Delta t}{20} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2 + \frac{5C_2^2 \Delta t}{g \lambda_{\min}} \|\nabla \eta_{\mathbf{u}}^m\|_f^2. \end{aligned} \quad (49)$$

Finally, we bound the last two terms

$$\begin{aligned} & \Delta t e_{\mathbf{u}}^m (\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}) \\ & \leq \frac{\nu \Delta t}{24} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{18c_1^2 \Delta t}{\nu} \left\| \mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right\|_f^2 \\ & \quad + \frac{18\beta^2 d^2 \Delta t}{\nu} \left\| \nabla \left(\mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2 + \frac{18M^2 \Delta t}{\nu} \left\| \nabla \left(\mathbf{u}^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2, \end{aligned} \quad (50)$$

and

$$\begin{aligned} & \Delta t \epsilon_{\phi}^m (\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1}) \\ & \leq \frac{g \lambda_{\min} \Delta t}{20} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2 + \frac{20g S_0^2 c_1^2 \Delta t}{\lambda_{\min}} \left\| \phi_t^m - \frac{\phi^{m+1} - \phi^{m-1}}{2\Delta t} \right\|_p^2 \\ & \quad + \frac{20g \lambda_{\max}^2 \Delta t}{\lambda_{\min}} \left\| \nabla(\phi^m - \frac{\phi^{m+1} + \phi^{m-1}}{2}) \right\|_p^2 + \frac{20g^3 C_3^4 c_1^2 \Delta t^3}{\lambda_{\min}} \|\phi^{m+1} - \phi^{m-1}\|_p^2 \\ & \quad + \frac{20g^3 C_3^4 \Delta t^3}{\lambda_{\min}} \|\nabla(\phi^{m+1} - \phi^{m-1})\|_p^2. \end{aligned} \quad (51)$$

Similar to Theorem 3.1, we denote the θ energy term

$$\begin{aligned} E_{\theta}^{m+\frac{1}{2}} := & \frac{1}{2} \{ \|\theta_{\mathbf{u}}^{m+1}\|_f^2 + \|\theta_{\mathbf{u}}^m\|_f^2 + \beta (\|\nabla \cdot \theta_{\mathbf{u}}^{m+1}\|_f^2 + \|\nabla \cdot \theta_{\mathbf{u}}^m\|_f^2) \\ & + g S_0 (\|\theta_{\phi}^{m+1}\|_p^2 + \|\theta_{\phi}^m\|_p^2) + 2\Delta t^2 g^2 C_3^2 (\|\theta_{\phi}^{m+1}\|_{1,p}^2 + \|\theta_{\phi}^m\|_{1,p}^2) \\ & + \alpha \Delta t^2 (\|\theta_p^{m+1}\|_f^2 + \|\theta_p^m\|_f^2) + 2\Delta t (b(\theta_{\mathbf{u}}^{m+1}, \theta_p^m) - b(\theta_{\mathbf{u}}^m, \theta_p^{m+1})) \}, \end{aligned}$$

and define

$$C_{\theta}^{m+\frac{1}{2}} = c(\theta_{\mathbf{u}}^{m+1}, \theta_{\phi}^m) - c(\theta_{\mathbf{u}}^m, \theta_{\phi}^{m+1}),$$

then the interface terms become

$$c(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1}, \theta_{\phi}^m) - c(\theta_{\mathbf{u}}^m, \theta_{\phi}^{m+1} + \theta_{\phi}^{m-1}) = C_{\theta}^{m+\frac{1}{2}} - C_{\theta}^{m-\frac{1}{2}}.$$

Therefore, after absorbing all the resulting θ terms into the left-hand side of the energy inequality, we group together the remaining terms and sum the inequality over $m = 1, \dots, N-1$.

$$\begin{aligned} & E_{\theta}^{N-\frac{1}{2}} - E_{\theta}^{\frac{1}{2}} + \Delta t \left(C_{\theta}^{N-\frac{1}{2}} - C_{\theta}^{\frac{1}{2}} \right) + \frac{\Delta t}{4} \sum_{m=1}^{N-1} \left(v \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 + g \lambda_{\min} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2 \right. \\ & \leq \sum_{m=1}^{N-1} \left(\frac{3c_1^2}{v \Delta t} \|\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1}\|_f^2 + \frac{3d^2 \beta^2}{v \Delta t} \|\nabla(\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1})\|_f^2 \right. \\ & \quad + \left(\frac{15g S_0^2 c_1^2}{4 \lambda_{\min} \Delta t} + \frac{15g^3 c_1^2 \Delta t^3 C_3^4}{\lambda_{\min}} \right) \|\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}\|_p^2 \\ & \quad \left. + \frac{15g^3 \Delta t^3 C_3^4}{\lambda_{\min} \Delta t} \|\nabla(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1})\|_p^2 + 4\alpha \Delta t^2 \|\eta_p^{m+1} - \eta_p^{m-1}\|_f^2 \right) \\ & \quad + \sum_{m=1}^{N-1} \left(\frac{3M^2 \Delta t}{2v} \|\nabla(\eta_{\mathbf{u}}^{m+1} + \eta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{5g \lambda_{\max}^2 \Delta t}{4 \lambda_{\min}} \|\nabla(\eta_{\phi}^{m+1} + \eta_{\phi}^{m-1})\|_p^2 \right. \\ & \quad + \frac{5C \Delta t}{g \lambda_{\min}} \|\nabla \eta_{\mathbf{u}}^m\|_f^2 + 2d \|\nabla \eta_{\mathbf{u}}^m\|_f^2 + \frac{6C \Delta t}{v} \|\nabla \eta_{\phi}^m\|_p^2 + \frac{6dc_1^2 \Delta t}{v} \|\nabla \eta_p^m\|_f^2 \\ & \quad + \frac{18c_1^2 \Delta t}{v} \left\| \mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right\|_f^2 + \frac{18\beta^2 d^2 \Delta t}{v} \left\| \nabla \left(\mathbf{u}_t^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2 \\ & \quad + \frac{18M^2 \Delta t}{v} \left\| \nabla \left(\mathbf{u}^m - \frac{\mathbf{u}^{m+1} - \mathbf{u}^{m-1}}{2\Delta t} \right) \right\|_f^2 + 4\alpha \Delta t^2 \|\eta_p^{m+1} - \eta_p^{m-1}\|_f^2 \\ & \quad \left. + \frac{20g S_0^2 c_1^2 \Delta t}{\lambda_{\min}} \left\| \phi_t^m - \frac{\phi^{m+1} - \phi^{m-1}}{2\Delta t} \right\|_p^2 + \frac{20g \lambda_{\max}^2 \Delta t}{\lambda_{\min}} \left\| \nabla \left(\phi^m - \frac{\phi^{m+1} + \phi^{m-1}}{2} \right) \right\|_p^2 \right. \\ & \quad \left. + \frac{20g^3 C_3^4 c_1^2 \Delta t^3}{\lambda_{\min}} \|\phi^{m+1} - \phi^{m-1}\|_p^2 + \frac{20g^3 C_3^4 \Delta t^3}{\lambda_{\min}} \|\nabla(\phi^{m+1} - \phi^{m-1})\|_p^2 \right). \end{aligned}$$

Note that, using the Cauchy-Schwarz and other basic inequalities, we have

$$\begin{aligned} \sum_{m=1}^{N-1} \|\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1}\|_f^2 &= \sum_{m=1}^{N-1} \left\| \int_{t^{m-1}}^{t^{m+1}} \eta_{\mathbf{u},t} dt \right\|_f^2 \\ &\leq \sum_{m=1}^{N-1} \int_{\Omega_f} (2\Delta t) \int_{t^{m-1}}^{t^{m+1}} |\eta_{\mathbf{u},t}|^2 dt dx \\ &\leq 4\Delta t \|\eta_{\mathbf{u},t}\|_{L^2(0,T;L^2(\Omega_f))}^2, \end{aligned}$$

similarly,

$$\begin{aligned} \sum_{m=1}^{N-1} \|\eta_p^{m+1} - \eta_p^{m-1}\|_f^2 &\leq 4\Delta t \|\eta_{p,t}\|_{L^2(0,T;L^2(\Omega_f))}^2, \\ \sum_{m=1}^{N-1} \|\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1}\|_p^2 &\leq 4\Delta t \|\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2, \\ \sum_{m=1}^{N-1} \|\nabla(\eta_{\mathbf{u}}^{m+1} - \eta_{\mathbf{u}}^{m-1})\|_f^2 &\leq 4\Delta t \|\nabla \eta_{\mathbf{u},t}\|_{L^2(0,T;L^2(\Omega_f))}^2, \\ \sum_{m=1}^{N-1} \|\nabla(\eta_{\phi}^{m+1} - \eta_{\phi}^{m-1})\|_p^2 &\leq 4\Delta t \|\nabla \eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2, \\ \sum_{m=1}^{N-1} \|\nabla(\eta_{\mathbf{u}}^{m+1} + \eta_{\mathbf{u}}^{m-1})\|_f^2 &\leq 2 \sum_{m=1}^{N-1} (\|\nabla \eta_{\mathbf{u}}^{m+1}\|_f^2 + \|\nabla \eta_{\mathbf{u}}^{m-1}\|_f^2) \\ &\leq 4 \sum_{m=0}^N \|\nabla \eta_{\mathbf{u}}^m\|_f^2 \\ &\leq 4(\Delta t)^{-1} \|\nabla \eta_{\mathbf{u}}\|_{L^2(0,T;L^2(\Omega_f))}^2, \\ \sum_{m=1}^{N-1} \|\nabla(\eta_{\phi}^{m+1} + \eta_{\phi}^{m-1})\|_p^2 &\leq 4(\Delta t)^{-1} \|\nabla \eta_{\phi}\|_{L^2(0,T;L^2(\Omega_p))}^2, \\ \sum_{m=1}^{N-1} \|\nabla \eta_{\mathbf{u}}^m\|_f^2 &\leq (\Delta t)^{-1} \|\nabla \eta_{\mathbf{u}}\|_{L^2(0,T;L^2(\Omega_f))}^2, \\ \sum_{m=1}^{N-1} \|\nabla \eta_p^m\|_f^2 &\leq (\Delta t)^{-1} \|\nabla \eta_p\|_{L^2(0,T;L^2(\Omega_f))}^2, \\ \sum_{m=1}^{N-1} \|\nabla \eta_{\phi}^m\|_p^2 &\leq (\Delta t)^{-1} \|\nabla \eta_{\phi}\|_{L^2(0,T;L^2(\Omega_p))}^2. \end{aligned}$$

After applying the above bounds, Lemma 3.1 and the bound from the stability proof, we have the important inequality

$$\begin{aligned} \frac{1}{4} (\|\theta_{\mathbf{u}}^N\|_f^2 + \|\theta_{\mathbf{u}}^{N-1}\|_f^2) + \frac{1}{2} g S_0 (\|\theta_{\phi}^N\|_p^2 + \|\theta_{\phi}^{N-1}\|_p^2) + \left(\frac{\alpha}{2} - \frac{1}{2\beta - 1} \right) \Delta t^2 (\|\theta_p^N\|_f^2 + \|\theta_p^{N-1}\|_f^2) \\ + \sum_{m=1}^{N-1} \left(\frac{\Delta t \nu}{4} \|\nabla(\theta_{\mathbf{u}}^{m+1} + \theta_{\mathbf{u}}^{m-1})\|_f^2 + \frac{\Delta t g \lambda_{\min}}{4} \|\nabla(\theta_{\phi}^{m+1} + \theta_{\phi}^{m-1})\|_p^2 \right) \end{aligned}$$

$$\begin{aligned}
&\leq C(\|\eta_{\mathbf{u},t}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\nabla\eta_{\mathbf{u},t}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2 \\
&\quad + \Delta t^4 \|\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2 + \|\nabla\eta_{\phi,t}\|_{L^2(0,T;L^2(\Omega_p))}^2 + \Delta t^3 \|\eta_{p,t}\|_{L^2(0,T;L^2(\Omega_f))}^2 \\
&\quad + \|\nabla\eta_{\mathbf{u}}\|_{L^2(0,T;L^2(\Omega_f))}^2 + (\Delta t)^{-1} \|\nabla\eta_{\mathbf{u}}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\nabla\eta_{\phi}\|_{L^2(0,T;L^2(\Omega_p))}^2 \\
&\quad + \|\nabla\eta_p\|_{L^2(0,T;L^2(\Omega_f))}^2 + \Delta t^4 \|\mathbf{u}_{ttt}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \Delta t^4 \|\nabla\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2 \\
&\quad + \Delta t^4 \|\mathbf{u}_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \Delta t^4 \|\phi_{ttt}\|_{L^2(0,T;L^2(\Omega_p))}^2 + \Delta t^4 \|\phi_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2 \\
&\quad + \Delta t^4 \|\phi_t\|_{L^2(0,T;L^2(\Omega_p))}^2 + \Delta t^3 \|p_t\|_{L^2(0,T;L^2(\Omega_f))}^2) + E_\theta^{\frac{1}{2}} + \Delta t C_\theta^{\frac{1}{2}}.
\end{aligned}$$

By using the approximation properties (14) and the triangle inequality, we can get the final result. ■

Remark. The artificial compression method accelerates the computation speed but introduces extra numerical errors and new physical flow behaviors associated with compressibility. These include non-physical fast pressure oscillations (acoustics) analyzed in [26].

The result of the Theorem 3.3 is not optimal, which is mainly affected by the pressure term, but the numerical experiment proves that the optimal order can be reached.

4 | NUMERICAL EXPERIMENTS

In this section, we give some numerical tests to demonstrate the effectiveness and efficiency of the unconditionally stable artificial compression method. We use the well-known Taylor-Hood elements (P2-P1) for the fluid equation and the piecewise quadratics (P2) for the porous equation. Let the parameters $\alpha = 2.2$ and $\beta = 0.6$. Furthermore, we implement the codes by using the software package FreeFEM++.

Test 1. For the Stokes/Darcy model, let the computational domain Ω be composed of $\Omega_f = (0, 1) \times (1, 2)$ and $\Omega_p = (0, 1) \times (0, 1)$ with the interface $\Gamma = (0, 1) \times \{1\}$. We take the exact solution:

$$\begin{aligned}
\mathbf{u} &= ((x^2(y-1)^2 + y)\cos(t), -\frac{2}{3}x(y-1)^3\cos(t) + (2 - \pi \sin(\pi x))\cos(t)), \\
p &= (2 - \pi \sin(\pi x))\sin(\frac{1}{2}\pi y)\cos(t), \\
\phi &= (2 - \pi \sin(\pi x))(1 - y - \cos(\pi y))\cos(t).
\end{aligned}$$

■

All the physical parameters n , ρ , g , v , K , S_0 , and α_{BJS} are simply set to 1, and the initial conditions, boundary conditions and the source terms follow from the exact solution.

To confirm the second-order accuracy, we set $h = \Delta t$ and calculate the errors and convergence rates for the variables \mathbf{u} , p , and ϕ with the following discrete norms

$$\begin{aligned}
e(\mathbf{u}) &= \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega_f))}, \\
e(p) &= \|p - p_h\|_{L^\infty(0,T;L^2(\Omega_f))}, \\
e(\phi) &= \|\phi - \phi_h\|_{L^\infty(0,T;L^2(\Omega_p))}.
\end{aligned}$$

Let $r_{\mathbf{u},p,\phi}$ denote the calculated order of convergence, which is given by $r = \log 2(e(N)/e(2N))$. Table 1 presents the second order for the Stokes velocity \mathbf{u} , Darcy hydraulic head ϕ and first order for the pressure p .

Now, we consider the efficiency of artificial compression algorithm, as shown in Table 2. With the change of mesh size, the calculation times of CNLF scheme and the artificial compressed CNLF scheme are proposed in this article. It indicates that the artificial compression algorithm takes less time, and the time gap between the two algorithms becomes larger and larger as the mesh size becomes smaller and smaller.

Test 2. In order to verify the unconditional stability and second order convergence of the algorithm when parameters change in the model. The true solution is given by

$$\begin{aligned}\mathbf{u} &= ((y - 1)^2 \cos(t), (x^2 - x) \cos(t)), \\ p &= \left(2\nu(x + y + 1) + \frac{\rho gn}{3k_{\min}} \right) \cos(t), \\ \phi &= \left(\frac{n}{k_{\min}}(x(1 - x)(y - 1) + \frac{1}{3}y^3 - y^2 + y) + \frac{2\nu}{g}x \right) \cos(t).\end{aligned}$$

■

Here, the hydraulic conductivity tensor in a porous medium flow $\mathbf{K} = k_{\min} \mathbf{I}$. Firstly, we still verify the second-order convergence of the algorithm. Here, we set the parameter $S_0 = 10^{-4}$, $k_{\min} = 10^{-1}$, and the other parameters are still equal to one, as shown in the Table 3.

Then, we set the external force terms $f_1 = 0$ and $f_2 = 0$, and consider the homogeneous boundary conditions. Figures 1–4 show the changes of energy with time when $S_0 = k_{\min} = 1.0 \times 10^{-4}$,

TABLE 1 The convergence orders at time $T = 1$ for the Test 1.

| $h = \Delta t$ | $e(\mathbf{u})$ | $r_{\mathbf{u}}$ | $e(p)$ | r_p | $e(\phi)$ | r_{ϕ} |
|-----------------|-----------------|------------------|------------|---------|-------------|------------|
| $\frac{1}{8}$ | 0.00968658 | - | 0.178708 | - | 0.0339505 | - |
| $\frac{1}{16}$ | 0.00166856 | 2.53738 | 0.0518993 | 1.78382 | 0.0089169 | 1.92882 |
| $\frac{1}{32}$ | 0.000338434 | 2.30166 | 0.0141453 | 1.87539 | 0.00221755 | 2.00757 |
| $\frac{1}{64}$ | 9.02258e-05 | 1.90726 | 0.00389015 | 1.86243 | 0.000553375 | 2.00264 |
| $\frac{1}{128}$ | 2.24514e-05 | 2.00674 | 0.00114547 | 1.76388 | 0.000138324 | 2.00021 |

TABLE 2 The calculation time of two algorithms.

| $h = \Delta t$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{128}$ |
|----------------|---------------|----------------|----------------|----------------|-----------------|
| CNLF(S) | 0.226422 | 1.4648 | 13.2021 | 112.065 | 1056.78 |
| AC+CNLF(S) | 0.222182 | 1.4264 | 12.219 | 103.255 | 944.971 |

TABLE 3 The convergence orders at time $T = 1$ for the Test 2.

| $h = \Delta t$ | $e(\mathbf{u})$ | $r_{\mathbf{u}}$ | $e(p)$ | r_p | $e(\phi)$ | r_{ϕ} |
|-----------------|-----------------|------------------|------------|---------|-------------|------------|
| $\frac{1}{16}$ | 0.0444772 | - | 0.787235 | - | 0.108311 | - |
| $\frac{1}{32}$ | 0.0123904 | 1.84384 | 0.215425 | 1.86961 | 0.0312734 | 1.79218 |
| $\frac{1}{64}$ | 0.00314853 | 1.97647 | 0.0546863 | 1.97794 | 0.00803146 | 1.9612 |
| $\frac{1}{128}$ | 0.000788867 | 1.99682 | 0.0137122 | 1.99572 | 0.0020142 | 1.99546 |
| $\frac{1}{256}$ | 0.000197228 | 1.99992 | 0.00343742 | 1.99606 | 0.000504013 | 1.99867 |

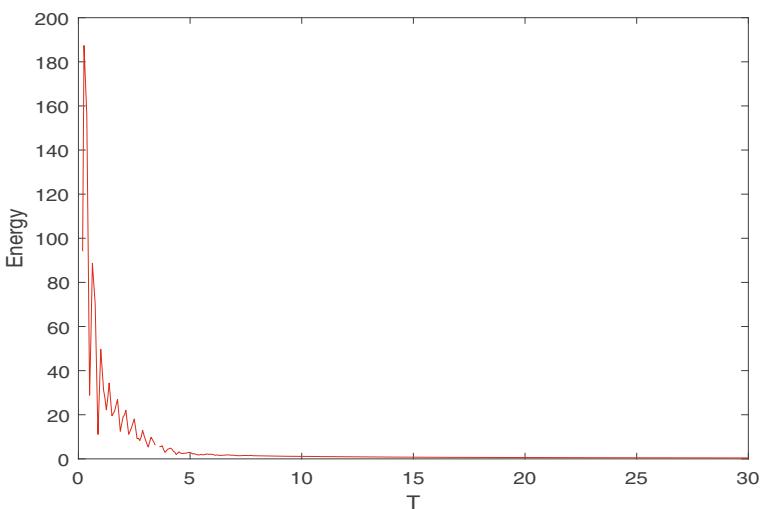


FIGURE 1 The change in energy at parameter $S_0 = k_{\min} = 1.0 \times 10^{-4}$.

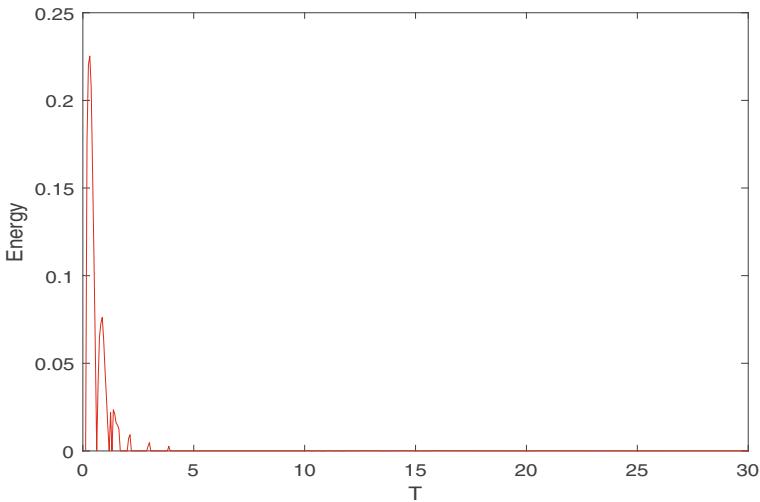


FIGURE 2 The change in energy at parameter $S_0 = 1.0 \times 10^{-4}$ and $k_{\min} = 1.0$.

$S_0 = 1.0 \times 10^{-4}, k_{\min} = 1.0, S_0 = 1.0 \times 10^{-2}, k_{\min} = 1.0$, and $S_0 = 1.0, k_{\min} = 1.0 \times 10^{-2}$ in the model, respectively. It can be obtained that the energy finally approaches 0, which is consistent with the fact, indicating the unconditional stability of the algorithm.

Test 3. Finally, a test is given to verify the convergence of the algorithm for Navier-Stokes/Darcy model. The true solution in the Navier-Stokes/Darcy is

$$\begin{aligned}\mathbf{u} &= ((x^2(y-1)^2 + y) \cos(t), -\frac{2}{3}x(y-1)^3 \cos(t) + (2 - \pi \sin(\pi x)) \cos(t)), \\ p &= (2 - \pi \sin(\pi x)) \sin(\frac{1}{2}\pi y) \cos(t), \\ \phi &= (2 - \pi \sin(\pi x))(1 - y - \cos(\pi y)) \cos(t) + \frac{1}{2}[1 + (2 - \pi \sin(\pi x))^2][(y-1)^2 + 1] \cos(t)^2.\end{aligned}$$

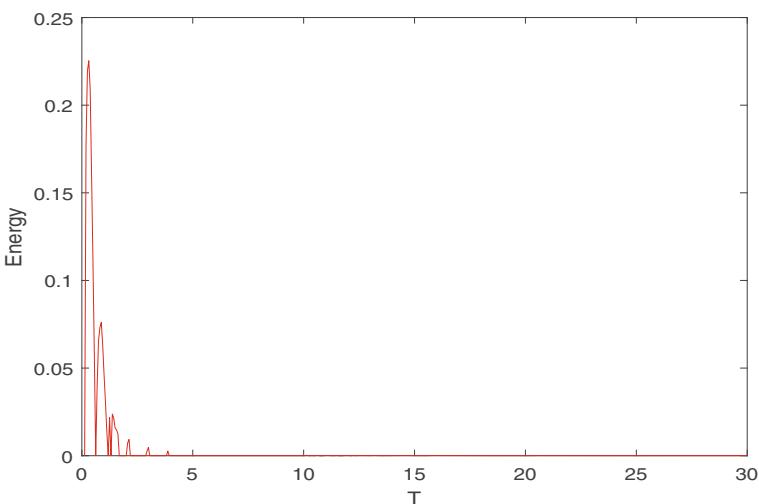


FIGURE 3 The change in energy at parameter $S_0 = 1.0 \times 10^{-2}$ and $k_{\min} = 1.0$.

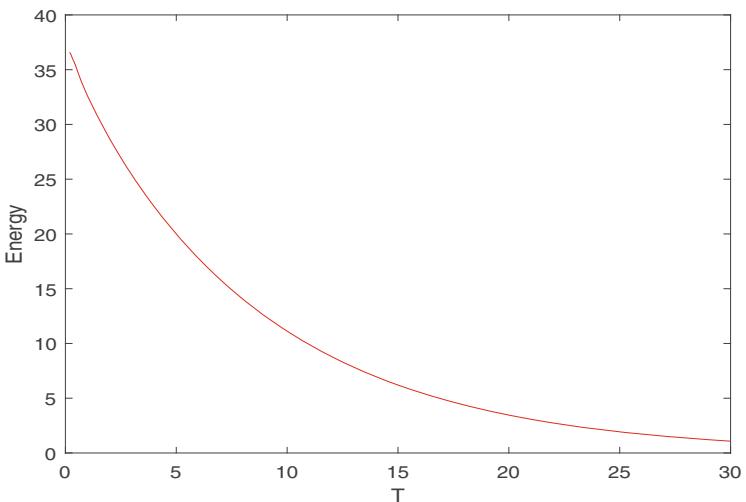


FIGURE 4 The change in energy at parameter $S_0 = 1.0$ and $k_{\min} = 1.0 \times 10^{-2}$.

TABLE 4 The convergence orders at time $T=1$ for the Navier-Stokes/Darcy

| $h = \Delta t$ | $e(\mathbf{u})$ | r_u | $e(p)$ | r_p | $e(\phi)$ | r_ϕ |
|----------------|-----------------|---------|------------|---------|------------|----------|
| $\frac{1}{8}$ | 0.00952175 | - | 0.249692 | - | 0.0779601 | - |
| $\frac{1}{16}$ | 0.00144726 | 2.7179 | 0.0704306 | 1.82588 | 0.0197743 | 1.97911 |
| $\frac{1}{32}$ | 0.000203285 | 2.83175 | 0.0176727 | 1.99468 | 0.00491406 | 2.00864 |
| $\frac{1}{64}$ | 4.98674e-05 | 2.02734 | 0.00487887 | 1.8569 | 0.00122886 | 1.9996 |

The calculated region and other parameters are consistent with the first experiment. Then we have the Table 4 which verify the second order convergence. ■

ACKNOWLEDGMENTS

The authors of this article would like to thank the editor and anonymous referees for their helpful comments and suggestions.

FUNDING INFORMATION

The work of the first author was supported by the National Natural Science Foundation of China (NSFC) grant 12001347 and Innovative team project of Shaanxi Provincial Department of Education (21JP019). The work of the third author was supported by NSFC grant 11971378. The work of the fourth author was supported by NSFC grant 11771259, Shaanxi Provincial Joint Laboratory of Artificial Intelligence (2022JC-SYS-05), Innovative team project of Shaanxi Provincial Department of Education (21JP013) and Shaanxi Province Natural Science basic research program key project (2023-JC-ZD-02).

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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REFERENCES

- [1] A. Çeşmelioglu and B. Rivière, *Analysis of time-dependent Navier-Stokes flow coupled with Darcy flow*, *J. Numer. Math.* 16 (2008), no. 4, 249–280.
- [2] A. Çeşmelioglu and B. Rivière, *Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow*, *J. Sci. Comput.* 40 (2009), no. 1–3, 115–140.
- [3] B. Rivière and I. Yotov, *Locally conservative coupling of Stokes and Darcy flows*, *SIAM J. Numer. Anal.* 42 (2005), 1959–1977.
- [4] G. Kanschat and B. Rivière, *A strongly conservative finite element method for the coupling of Stokes and Darcy flow*, *J. Comput. Phys.* 229 (2010), 5933–5943.
- [5] G. Z. Du and L. Y. Zuo, *Local and parallel finite element methods for the coupled Stokes/Darcy model*, *Numer. Algorithms* (2021), 1593–1611. <https://doi.org/10.1007/s11075-020-01021-5>.
- [6] J. L. Guermond and P. Minev, *High-order time stepping for the Navier-Stokes equations with minimal computational complexity*, *J. Comput. Appl. Math.* 310 (2017), 92–103.
- [7] J. M. Urquiza, D. N'Dri, A. Garon, and M. C. Delfour, *Coupling Stokes and Darcy equations*, *Appl. Numer. Math.* 58 (2008), 525–538.
- [8] J. Shen, *On error estimates of the projection method for the Navier-Stokes equations: Second order schemes*, *Math. Comput.* 65 (1996), 1039–1065.
- [9] L. Badea, M. Discacciati, and A. Quarteroni, *Numerical analysis of the Navier-Stokes/Darcy coupling*, *Numer. Math.* 115 (2010), no. 2, 195–227.
- [10] L. Shan, H. B. Zheng, and W. J. Layton, *A decoupling method with different subdomain time steps for the nonstationary Stokes-Darcy model*, *Numer. Methods Partial Differ. Equ.* 29 (2013), 549–583.
- [11] L. Shan and N. B. Zheng, *Partitioned time stepping method for fully evolutionary Stokes-Darcy flow with the Beavers-Joseph interface conditions*, *SIAM J. Numer. Anal.* 51 (2013), 813–839.
- [12] G. Z. Du and L. Y. Zuo, *Local and parallel finite element method for the mixed Navier-Stokes/Darcy model with Beavers-Joseph interface conditions*, *Acta Math. Sci.* 37 (2017), no. 5, 1331–1347.
- [13] L. Y. Zuo and Y. R. Hou, *A two-grid decoupling method for the mixed Stokes-Darcy model*, *J. Comput. Appl. Math.* 275 (2015), 139–147.
- [14] G. Z. Du, *Expandable parallel finite element methods for linear elliptic problems*, *Acta Math. Sci.* 40B (2020), no. 2, 572–588.

- [15] G. Z. Du, Q. T. Li, and Y. H. Zhang, *A two-grid method with backtracking for the mixed Navier-Stokes/Darcy model*, Numer. Meth. Partial Differ. Equ. 36 (2020), no. 6, 1601–1610.
- [16] Y. Qin and Y. R. Hou, *Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Navier-Stokes/Darcy model*, Acta Math. Sci. 38 (2018), no. 4, 1361–1369.
- [17] M. Discacciati, E. Miglio, and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, Appl. Numer. Math. 43 (2002), 57–74.
- [18] M. Discacciati and A. Quarteroni, *Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations*, Comput. Vis. Sci. 6 (2004), 93–103.
- [19] M. Kubacki and M. Moraiti, *Analysis of a second-order, unconditionally stable, partitioned method for the evolutionary Stokes-Darcy model*, Int. J. Numer. Anal. Model. 12 (2015), no. 4, 704–730.
- [20] M. Mu and J. C. Xu, *A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow*, SIAM J. Numer. Anal. 45 (2007), 1801–1813.
- [21] Y. Li and Y. R. Hou, *A second-order partitioned method with different subdomain time steps for the evolutionary Stokes-Darcy system*, Math. Meth. Appl. Sci. 41 (2018), 2178–2208.
- [22] M. Mu and X. H. Zhu, *Decoupled schemes for a non-stationary mixed Stokes-Darcy model*, Math. Comput. 79 (2010), no. 270, 707–731.
- [23] Y. Rong, Y. R. Hou, and Y. H. Zhang, *Numerical analysis of a second order algorithm for simplified magnetohydrodynamic flows*, Adv. Comput. Math. 43 (2017), 823–848.
- [24] T. Arbogast and M. S. Gomez, *A discretization and multigrid solver for a Darcy-Stokes system of three dimensional vuggy porous media*, Comput. Geosci. 13 (2009), no. 3, 331–348.
- [25] T. Ohwada and P. Asinari, *Artificial compressibility method revisited: Asymptotic numerical method for incompressible Navier-Stokes equations*, J. Comput. Phys. 229 (2010), 1698–1723.
- [26] V. DeCaria, W. J. Layton, and M. McLaughlin, *A conservative, second order, unconditionally stable artificial compression method*, Comput. Methods Appl. Mech. Eng. 325 (2017), 733–747.
- [27] V. Girault and P. A. Raviart, *Finite element methods for Navier-Stokes equations, theory and algorithms*, Springer-Verlag, Berlin, 1986.
- [28] V. Girault and B. Rivière, *DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition*, SIAM J. Numer. Anal. 47 (2009), no. 3, 2052–2089.
- [29] V. J. Ervin, E. W. Jenkins, and S. Sun, *Coupled generalized nonlinear Stokes flow with flow through a porous medium*, SIAM J. Numer. Anal. 47 (2009), no. 2, 929–952.
- [30] W. J. Layton and C. Trenchea, *Stability of two IMEX methods, CNLF and BDF2-AB2, for uncoupling systems of evolution equations*, Appl. Numer. Math. 62 (2002), no. 2, 112–120.
- [31] W. J. Layton, F. Schieweck, and I. Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal. 40 (2003), 2195–2218.
- [32] W. J. Layton, H. Tran, and C. Trenchea, *Analysis of long time stability and errors of two partitioned methods for uncoupling evolutionary groundwater? Surface water flows*, SIAM J. Numer. Anal. 51 (2013), no. 1, 248–272.
- [33] W. B. Chen, B. Gunzburger, D. Sun, and X. Wang, *An efficient and long-time accurate third-order algorithm for the Stokes-Darcy system*, Numer. Math. 134 (2016), no. 4, 1–23.
- [34] Y. Z. Cao, M. Gunzburger, F. Hua, and X. Wang, *Coupled Stokes-Darcy model with Beavers-Joseph interface boundary condition*, Commun. Math. Sci. 8 (2010), no. 1, 1–25.
- [35] Y. R. Hou, *Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Stokes-Darcy model*, Appl. Math. Lett. 57 (2016), 90–96.
- [36] Y. R. Hou and Y. Qin, *On the solution of coupled Stokes-Darcy model with Beavers-Joseph interface condition*, Comput. Math. Appl. 77 (2019), 50–65.
- [37] J. Li, *Numerical methods for the incompressible Navier-Stokes equations*, Science Press, Beijing, 2019, (in Chinese).
- [38] J. Li, Y. Bai, and X. Zhao, *Modern numerical methods for mathematical physics equations*, Science Press, Beijing, 2022, (in Chinese).
- [39] J. Li, X. L. Lin, and Z. X. Chen, *Finite volume methods for the incompressible Navier-Stokes equations*, Springer-Verlag, Berlin, Heidelberg, 2021.
- [40] Y. Qin, Y. S. Wang, and J. Li, *A variable time step time filter algorithm for the geothermal system*, SIAM J. Numer. Anal. 60 (2022), 2781–2806.

How to cite this article: Y. Qin, Y. Wang, Y. Hou, and J. Li, *An unconditionally stable artificial compression method for the time-dependent groundwater-surface water flows*, Numer. Methods Partial Differ. Eq. 39 (2023), 3705–3724. <https://doi.org/10.1002/num.23022>