# Analysis of an Embedded-Hybridized Discontinuous Galerkin Method for the Time-Dependent Incompressible Navier-Stokes Equations 

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#### Abstract

In this paper, the a prior error estimates of an embedded-hybridized discontinuous Galerkin method for the time-dependent Navier-Stokes equations are presented. It is proved that the velocity error in the $L^{2}(\Omega)$ norm, where the constants are independent of the Reynolds number $\operatorname{Re}\left(\right.$ or $v^{-1}$ ), is quasi-optimal with pre-asymptotic convergence order of $k+1 / 2$ in case of $v \leq C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$, with $k$ the polynomial order of the velocity space. In addition, we also provide a Reynolds-dependent error bound with asymptotic convergence order of $k+1$ for the case of the low mesh Reynolds number $R e_{h}$, which is denoted as $h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)} / v$. Finally, numerical experiments are carried out to confirm the rates of convergence.


Keywords Reynolds-robust • Quasi-optimal • Embedded-hybridized discontinuous Galerkin method • Time-dependent Navier-Stokes equations

## 1 Introduction

In many applications, numerical simulation of high Reynolds number flows is a difficult problem. For this reason, the development of error estimates has been an interesting topic for high Reynolds number flows, in which the constants are independent of the Reynolds number. The robust error bounds have been proved for the velocity error in the $L^{2}(\Omega)$ norm for some finite element methods [18-20, 27, 28], but they are sup-optimal with convergence order of $k$. It is to be remarked that there are very few quasi-optimal error bounds in the literature. In [1], the quasi-optimal error bound is proved for the $L^{2}(\Omega)$ error of the velocity, whereas the analysis in [1] was restricted to piecewise linear approximations in space and in time. In [17],

[^0]based on equal-order approximations for the velocity and the pressure, a continuous interior penalty method is considered, in which the error bound is quasi-optimal. In [2], for one of the local projection stabilization methods with non inf-sup stable finite elements, the robust error bound is proved to be quasi-optimal. Recalling the existing literature, these numerical methods are restricted to the $H^{1}$-conforming finite element methods with equal-order finite element spaces. For getting the quasi-optimal estimates, some stabilization terms must be added, which are the jumps of the gradient of the approximate solutions over the internal faces. In addition, the quasi-optimal error bounds are dependent on the pressure, namely, they aren't pressure-robust.

By giving up $H^{1}$-conforming finite elements, H (div)-conforming discontinuous Galerkin (DG) methods are pressure-robust. Meanwhile, due to the natural incorporation of upwinding at element boundaries, it is ideally suited for convection dominated flows [7, 10, 34]. By means of the Raviart-Thomas interpolation, it is proved that the velocity error in the $L^{2}$-norm has the quasi-optimal error bound for the evolutionary Navier-Stokes equations [25]. Unfortunately, DG methods are known to be computationally expensive. To lower the computational cost of the DG methods, the hybridized discontinuous Galerkin (HDG) methods have been developed in [22], by introducing new trace unknowns defined on cell boundaries. H (div)-conforming HDG methods have been popular for numerically solving the Navier-Stokes equations, see [11, 13, 16, 29]. In [29], the HDG method introduces discontinuous trace velocity and trace pressure approximations. If we use continuous trace velocity and discontinuous trace pressure approximations, this results in an embedded-hybridized discontinuous Galerkin (E-HDG) method [24]. The HDG and E-HDG methods provide an exactly divergence-free and H (div)conforming velocity field, in which the velocity error bounds are pressure-robust. For the E-HDG method, the facet velocity functions are continuous, so it has fewer degrees of freedom than the HDG method on a given mesh, and it is better suited to fast iterative solver [24]. If the continuous facet function spaces for the trace velocity and pressure approximations are used, it was well-known as an embedded discontinuous Galerkin (EDG) method. Unfortunately, the EDG method is not pressure-robust [24]. In [33, 35], the space-time HDG, E-HDG and EDG methods have been introduced for the Navier-Stokes equations on time-dependent domains.

As we can see, the HDG, E-HDG and EDG methods mentioned above were analyzed for the steady-state Stokes, Oseen and Navier-Stokes equations, see [3, 23, 24, 30]. The fully discrete analysis of the space-time HDG method for the Navier-Stokes equations on fixed domains were presented in [31, 32]. In [31], it was proved that the discrete solution converges to a weak solution as the time step and mesh size tend to zero. In [32], it provided a rigourous study of well-posedness for the space-time HDG methods applied to the NavierStokes equations, and the a priori error estimates for the velocity were derived under a small data assumption.

In this paper, we analyze the semi-discrete embedded-hybridized discontinuous Galerkin method for the time-dependent Navier-Stokes equations on fixed domains. Firstly, it is proved that the $L^{2}(\Omega)$ error of the velocity is Reynolds-robust with pre-asymptotic convergence order of $k+1 / 2$ in case of $v \leq C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$. Secondly, we obtain a Reynolds-dependent error bound with optimal convergence order of $k+1$ for the velocity $L^{2}$ error, which is applicable for the case of the low mesh Reynolds number. By a careful inspection, all the results in the paper hold true verbatim for the HDG method for the Navier-Stokes equations [29]. Notice that the space-time HDG method in [32] is based on the spatial discretization of [29]. Comparing our analysis results to that of [32], the velocity $L^{2}$ error derived under a small data assumption in [32] isn't strictly Reynolds-robust, in which the spatial convergence order is only suboptimal with convergence order of $k$. In addition, we can use ODE theory to prove
well-posedness for the semi-discrete case, however well-posedness for the fully-discrete space-time case is more complicated [32].

The structure of the paper is as follows: In Sect. 2, we introduce the E-HDG method for the time-dependent Navier-Stokes equations. Some preliminaries are presented in Sect. 3. The error estimates for the velocity are present in Sect. 4. In Sect. 5, we carry out numerical experiments to verify our analytical results.

## 2 Embedded-Hybridized Discontinuous Galerkin Method

In this section, we present the E-HDG method, which is identical to the HDG method of [29] using the E-HDG spaces of [24], for the time-dependent incompressible Navier-Stokes equations

$$
\left\{\begin{align*}
\partial_{t} u-v \Delta u+(u \cdot \nabla) u+\nabla p & =f, & & (0, T] \times \Omega,  \tag{1}\\
\nabla \cdot u & =0, & & (0, T] \times \Omega, \\
u & =0, & & (0, T] \times \Gamma, \\
u(0, x) & =u_{0}(x), & & \Omega,
\end{align*}\right.
$$

in a polygonal $(d=2)$ or polyhedral $(d=3)$ domain $\Omega$ with boundary $\Gamma$. Introduce

$$
V=\left[H_{0}^{1}(\Omega)\right]^{d}, \quad Q=L_{0}^{2}(\Omega)=\left\{q \in L^{2}(\Omega), \int_{\Omega} q d x=0\right\} .
$$

The weak formulation of the Navier-Stokes equations reads as follows: find $(u, p):(0, T] \rightarrow$ ( $V, Q$ ), satisfying

$$
\left\{\begin{align*}
\left(\partial_{t} u, v\right)+v a(u, v)+o(u, u, v)+b(p, v) & =(f, v), & & \forall v \in V,  \tag{2}\\
b(q, u) & =0, & & \forall q \in Q,
\end{align*}\right.
$$

where

$$
\begin{aligned}
& a(u, v)=\int_{\Omega} \nabla u: \nabla v \mathrm{~d} x, \quad o(u, u, v)=\int_{\Omega}(u \cdot \nabla) u \cdot v \mathrm{~d} x, \\
& b(q, u)=-\int_{\Omega} q(\nabla \cdot u) \mathrm{d} x .
\end{aligned}
$$

In addition, for the well-posedness of (2), we can refer to [4, 5].

### 2.1 Notation

Let $\left\{\mathcal{T}_{h}\right\}_{0<h \leq 1}$ be a family of triangulations of the domain $\Omega$ without hanging nodes. For each triangulation $\mathcal{T}_{h}$, define mesh size $h=\max _{K \in \mathcal{T}_{h}} h_{K}$, where $h_{K}$ denotes the diameter of each element $K \in \mathcal{T}_{h}$. Assume that the family of triangulation $\left\{\mathcal{T}_{h}\right\}_{0<h \leq 1}$ is shape-regular and quasi-uniform, i.e., there exists constants $\varrho$ and $\varrho_{1}$ such that

$$
\frac{h_{K}}{\rho_{K}}<\varrho \text { and } \frac{h}{h_{K}}<\varrho_{1}, \quad \forall K \in \mathcal{T}_{h}, \forall h \in(0,1],
$$

where $\rho_{K}$ is the diameter of the largest ball that can be inscribed in $K$. Let $\mathcal{F}_{h}$ and $\Gamma^{0}$ denote the set of all facets and the mesh skeleton, respectively. $\mathcal{F}_{h}=\mathcal{F}_{I} \cup \mathcal{F}_{B}$, where $\mathcal{F}_{I}$ and $\mathcal{F}_{B}$ are the subset of interior facets and boundary facets, respectively. Let $h_{F}$ denote the diameter of each face $F \in \mathcal{F}_{h}$. We denote the boundary of a cell by $\partial K$, and the outward unit normal
vector on $\partial K$ by $n$. Let $P_{l}(M)(l \geq 0)$ denote the space of all polynomials on a domain $M$ with degree less than or equal to $l$.

Next, we use the following finite element spaces on $\Omega$ :

$$
\begin{aligned}
V_{h} & =\left\{v_{h} \in\left[L^{2}(\Omega)\right]^{d}: v_{h} \in\left[P_{k}(K)\right]^{d}, \forall K \in \mathcal{T}_{h}\right\}, \\
Q_{h} & =\left\{q_{h} \in L^{2}(\Omega): q_{h} \in P_{k-1}(K), \forall K \in \mathcal{T}_{h}\right\},
\end{aligned}
$$

and the following facet finite element spaces on $\Gamma^{0}$ :

$$
\begin{aligned}
& \bar{V}_{h}=\left\{\bar{v}_{h} \in\left[L^{2}\left(\Gamma^{0}\right)\right]^{d}: \bar{v}_{h} \in\left[P_{k}(F)\right]^{d}, \forall F \in \mathcal{F}_{h}, \bar{v}_{h}=0 \text { on } \Gamma\right\} \cap C^{0}\left(\Gamma^{0}\right), \\
& \bar{Q}_{h}=\left\{\bar{q}_{h} \in L^{2}\left(\Gamma^{0}\right): \bar{q}_{h} \in P_{k}(F), \forall F \in \mathcal{F}_{h}\right\} .
\end{aligned}
$$

Here, $k \geq 1$. We set $V_{h}^{\star}=V_{h} \times \bar{V}_{h}, Q_{h}^{\star}=Q_{h} \times \bar{Q}_{h}$ and $X_{h}^{\star}=V_{h}^{\star} \times Q_{h}^{\star}$, and denote function pairs in $V_{h}^{\star}$ and $Q_{h}^{\star}$ by boldface, for example, $\boldsymbol{v}_{h}=\left(v_{h}, \bar{v}_{h}\right) \in V_{h}^{\star}$ and $\boldsymbol{q}_{h}=\left(q_{h}, \bar{q}_{h}\right) \in Q_{h}^{\star}$.

### 2.2 Weak Formulation

Now, we present the E-HDG method under consideration. The space-semidiscrete weak formulation of (1) reads as follows: given $f \in\left[L^{2}\left(0, T ; L^{2}(\Omega)\right)\right]^{d}$, find $\left(\boldsymbol{u}_{h}, \boldsymbol{p}_{h}\right) \in X_{h}^{\star}$ such that

$$
\begin{align*}
\left(\partial_{t} u_{h}, v_{h}\right)+a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+o_{h}\left(u_{h} ; \boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{p}_{h}, v_{h}\right) & =\left(f, v_{h}\right), & & \forall v_{h} \in V_{h}^{\star},  \tag{3}\\
b_{h}\left(\boldsymbol{q}_{h}, u_{h}\right) & =0, & & \forall \boldsymbol{q}_{h} \in Q_{h}^{\star},
\end{align*}
$$

where

$$
\begin{aligned}
a_{h}(\boldsymbol{u}, \boldsymbol{v})= & \sum_{K \in \mathcal{T}_{h}} \int_{K} v \nabla u: \nabla v \mathrm{~d} x+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{\alpha v}{h_{K}}(u-\bar{u}) \cdot(v-\bar{v}) \mathrm{d} s \\
& -\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left[v(u-\bar{u}) \cdot \partial_{n} v+v \partial_{n} u \cdot(v-\bar{v})\right] \mathrm{d} s, \\
b_{h}(\boldsymbol{p}, v)= & -\sum_{K \in \mathcal{T}_{h}} \int_{K} p \nabla \cdot v \mathrm{~d} x+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} v \cdot n \bar{p} \mathrm{~d} s,
\end{aligned}
$$

and

$$
\begin{align*}
o_{h}(w ; \boldsymbol{u}, \boldsymbol{v})= & -\sum_{K \in \mathcal{T}_{h}} \int_{K}(u \otimes w): \nabla v \mathrm{~d} x+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2} w \cdot n(u+\bar{u}) \cdot(v-\bar{v}) \mathrm{d} s \\
& +\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}|w \cdot n|(u-\bar{u}) \cdot(v-\bar{v}) \mathrm{d} s . \tag{4}
\end{align*}
$$

To ensure stability, we need to choose a sufficiently large penalty parameter $\alpha>0$ in the term $a_{h}$ [3].
Remark 1 Notice that for the HDG method, $\bar{V}_{h}$ is set to be the following discontinuous facet velocity space

$$
\bar{V}_{h}=\left\{\bar{v}_{h} \in\left[L^{2}\left(\Gamma^{0}\right)\right]^{d}: \bar{v}_{h} \in\left[P_{k}(F)\right]^{d}, \forall F \in \mathcal{F}_{h}, \bar{v}_{h}=0 \text { on } \Gamma\right\} .
$$

The E-HDG and HDG formulations yield the approximate velocities that are exactly divergence-free on cells and H (div)-conforming, see [24, 29].

## 3 Preliminaries

Given a domain $M$, for scalar-valued functions $p, q \in L^{2}(M)$, we denote the inner-product $(p, q)_{M}=\int_{M} p q \mathrm{~d} x$ with norm $\|p\|_{M}=\sqrt{(p, p)_{M}}$. Similar definitions hold for vectorvalued and tensor-valued functions. We use the Sobolev spaces $W^{l, p}(M)$ for scalar-valued functions with associated norms $\|\cdot\|_{W^{l, p}(M)}$ and seminorms $|\cdot|_{W^{l, p}(M)}$ for $l \geq 0$ and $p \geq 1$. In the case $l=0, W^{0, p}(M)=L^{p}(M)$, and when $l=2, W^{l, 2}(M)=H^{l}(M) .\|\cdot\|_{W^{l, p}(\Omega)}$ is used to denote the norm both in $W^{l, p}(\Omega)$ or $\left[W^{l, p}(\Omega)\right]^{d} \cdot\|\cdot\|_{l}\left(\right.$ resp. $\left.|\cdot|_{l}\right)$ is used to denote the norm (resp. seminorm) both in $H^{l}(\Omega)$ or $\left[H^{l}(\Omega)\right]^{d} \cdot\|\cdot\|_{L^{p}}$ is ofen used to denote the norm both in $L^{p}(\Omega)$ or $\left[L^{p}(\Omega)\right]^{d}$. The inner product of $L^{2}(\Omega)$ or $\left[L^{2}(\Omega)\right]^{d}$ will be denoted by $(\cdot, \cdot)$. The exact meaning will be clear by the context. Introduce the Bochner space $L^{p}(0, T ; Y)(1 \leq$ $p \leq \infty$ ), where $Y$ is a Banach space, the abbreviation $L^{p}(Y)=L^{p}(0, T ; Y)$ is often used. Introduce the Hilbert space $H_{0}(\operatorname{div}, \Omega)=\{v \in H(\operatorname{div}, \Omega): v \cdot n=0$ on $\Gamma\}$. Define the trace operator $\gamma: H^{l}(\Omega) \rightarrow H^{l-1 / 2}\left(\mathcal{F}_{h}\right)(l \geq 1)$, which restricts functions in $H^{l}(\Omega)$ to $\mathcal{F}_{h}$. Define the broken Sobolev space $H^{l}\left(\mathcal{T}_{h}\right)=\left\{w \in L^{2}(\Omega):\left.w\right|_{K} \in H^{l}(K), \forall K \in \mathcal{T}_{h}\right\}$. Throughout this paper, the broken gradient $\nabla_{h}:\left[H^{1}\left(\mathcal{T}_{h}\right)\right]^{d} \rightarrow\left[L^{p}(\Omega)\right]^{d \times d}$ is defined such that for all $v \in\left[H^{1}\left(\mathcal{T}_{h}\right)\right]^{d}$,

$$
\forall K \in \mathcal{T}_{h},\left.\quad\left(\nabla_{h} v\right)\right|_{K}=\nabla\left(\left.v\right|_{K}\right) .
$$

We will drop the index $h$ in the broken gradient whenever the operator appears inside an integral over a mesh element $K \in \mathcal{T}_{h}$.

Introduce the following extended function spaces

$$
\begin{array}{ll}
V(h)=V_{h}+\left[H_{0}^{1}(\Omega) \cap H^{2}(\Omega)\right]^{d}, & Q(h)=Q_{h}+L_{0}^{2}(\Omega) \cap H^{1}(\Omega), \\
\bar{V}(h)=\bar{V}_{h}+\left[H_{0}^{3 / 2}\left(\Gamma^{0}\right)\right]^{d}, & \bar{Q}(h)=\bar{Q}_{h}+H_{0}^{1 / 2}\left(\Gamma^{0}\right),
\end{array}
$$

in which $\left[H_{0}^{3 / 2}\left(\Gamma^{0}\right)\right]^{d}$ and $H_{0}^{1 / 2}\left(\Gamma^{0}\right)$ are the trace spaces of $\left[H_{0}^{1}(\Omega) \cap H^{2}(\Omega)\right]^{d}$ and $L_{0}^{2}(\Omega) \cap H^{1}(\Omega)$ on $\Gamma^{0}$, respectively. Set $V^{\star}(h)=V(h) \times \bar{V}(h), Q^{\star}(h)=Q(h) \times \bar{Q}(h)$. For $\boldsymbol{\phi}=(\phi, \bar{\phi}) \in V_{h}^{\star}(h)$ or $\phi=(\phi, \bar{\phi}) \in Q_{h}^{\star}(h)$, we define the jump $\mathbb{I} \cdot \mathbb{\rrbracket}$ and average $\{\cdot\}$ operators across the cell boundary $\partial K, \forall K \in \mathcal{T}_{h}$, by

$$
\llbracket \boldsymbol{\phi} \rrbracket=\phi-\bar{\phi}, \quad\{\boldsymbol{\phi}\}=\frac{\phi+\bar{\phi}}{2} .
$$

Let $R T_{k}(K)=\left[P_{k}(K)\right]^{d}+x\left(P_{k}(K) / P_{k-1}(K)\right), \forall K \in \mathcal{T}_{h}$. Define the following space

$$
V_{h}^{d i v}=\left\{v \in H_{0}(\operatorname{div}, \Omega):\left.v\right|_{K} \in R T_{k}(K), \forall K \in \mathcal{T}_{h}\right\} .
$$

We define two norms on $V^{\star}(h)$

$$
\begin{aligned}
& \|\boldsymbol{v}\|_{v}^{2}=\sum_{K \in \mathcal{T}_{h}}\|\nabla v\|_{K}^{2}+\sum_{K \in \mathcal{T}_{h}} \alpha h_{K}^{-1}\|\bar{v}-v\|_{\partial K}^{2}, \\
& \|\boldsymbol{v}\|_{v^{\prime}}^{2}=\|\boldsymbol{v}\|_{v}^{2}+\sum_{K \in \mathcal{T}_{h}} \frac{h_{K}}{\alpha}\left\|\frac{\partial v}{\partial n}\right\|_{\partial K}^{2},
\end{aligned}
$$

where $\|\cdot\|_{v^{\prime}}$ and $\|\cdot\|_{v}$ are equivalent on $V_{h}^{\star}$, namely, $\left\|\boldsymbol{v}_{h}\right\|_{v} \leq\left\|\boldsymbol{v}_{h}\right\|_{v^{\prime}} \leq c\left\|\boldsymbol{v}_{h}\right\|_{v}$, with $c$ independent of $h$, see [3, Eq.(28)]. Define the following norm on $Q_{h}^{\star}$

$$
\|\boldsymbol{q}\|_{p}^{2}=\sum_{K \in \mathcal{T}_{h}}\|q\|_{K}^{2}+|\boldsymbol{q}|_{p}^{2},
$$

with $|\boldsymbol{q}|_{p}^{2}=\sum_{K \in \mathcal{T}_{h}} h_{K}\|\bar{q}-q\|_{\partial K}^{2}$. Moreover, we introduce the following seminorm

$$
|\boldsymbol{v}|_{u, \text { up }}^{2}=\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}|u \cdot n||v-\bar{v}|^{2} \mathrm{~d} s
$$

Let $0 \leq j \leq \ell$ and $1 \leq p, q \leq \infty$, then we have the local inverse inequality [6, Lemma 1.138]

$$
\begin{equation*}
\left\|v_{h}\right\|_{W^{\ell, p}(K)} \leq C_{\mathrm{inv}} h_{K}^{j-\ell+d\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|v_{h}\right\|_{W^{j, q}(K)}, \quad \forall v_{h} \in P_{k}(K), \forall K \in \mathcal{T}_{h} . \tag{5}
\end{equation*}
$$

We will use the following continuous and discrete trace inequalities

$$
\|v\|_{\partial K} \leq C\left(h_{K}^{-\frac{1}{2}}\|v\|_{K}+h_{K}^{\frac{1}{2}}\|\nabla v\|_{K}\right), \quad \forall v \in H^{1}(K), \forall K \in \mathcal{T}_{h},
$$

and

$$
\begin{equation*}
\|v\|_{\partial K} \leq C h_{K}^{-\frac{1}{2}}\|v\|_{K}, \quad \forall v \in P_{k}(K), \forall K \in \mathcal{T}_{h} . \tag{6}
\end{equation*}
$$

The following Sobolev's embedding [21] will be used: For $1 \leq p<d / s$, let $q$ be such that $\frac{1}{q}=\frac{1}{p}-\frac{s}{d}$. There exists a constant $C>0$ such that

$$
\begin{equation*}
\|v\|_{L^{q^{\prime}}(\Omega)} \leq C\|v\|_{W^{s, p}(\Omega)}, \quad \frac{1}{q^{\prime}} \geq \frac{1}{q}, \quad v \in W^{s, p}(\Omega) . \tag{7}
\end{equation*}
$$

If $p>d / s$, the above inequality is valid for $q^{\prime}=\infty$.
Next, we present the stability and boundedness of the multilinear forms, and the consistency of the method.

Lemma 1 [3, Lemmas 4.2 and 4.3](Coercivity and boundedness of $a_{h}$ ) For sufficiently large $\alpha$, there exist constants $C_{a}^{c}>0$ and $C_{a}^{b}>0$, independent of $h$ and $\nu$, such that for all $v_{h} \in V_{h}^{\star}$ and $\boldsymbol{u} \in V^{\star}(h)$,

$$
\begin{equation*}
a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq v C_{a}^{c}\left\|\boldsymbol{v}_{h}\right\|_{v}^{2} \quad \text { and } \quad\left|a_{h}\left(\boldsymbol{u}, \boldsymbol{v}_{h}\right)\right| \leq \nu C_{a}^{b}\|\boldsymbol{u}\|_{v^{\prime}}\left\|\boldsymbol{v}_{h}\right\|_{v} . \tag{8}
\end{equation*}
$$

Lemma 2 [24, Lemma 8](Stability of $b_{h}$ ) There exists a constant $\beta_{p}>0$, independent of $h$, such that for all $\boldsymbol{q}_{h} \in Q_{h}^{\star}$,

$$
\begin{equation*}
\beta_{p}\|\boldsymbol{q}\|_{p} \leq \sup _{\boldsymbol{w}_{h} \in V_{h}^{\star}} \frac{b_{h}\left(\boldsymbol{q}_{h}, w_{h}\right)}{\left\|\boldsymbol{w}_{h}\right\|_{v}} \tag{9}
\end{equation*}
$$

Lemma 3 [12, Proposition 3.6](Stability of $\left.o_{h}\right)$ For all $w_{h} \in V_{h}$ and $\boldsymbol{v}_{h} \in V_{h}^{\star}$, then we have

$$
\begin{equation*}
o_{h}\left(w_{h} ; \boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right)=\frac{1}{2}\left|\boldsymbol{v}_{h}\right|_{w_{h}, \text { up }}^{2} . \tag{10}
\end{equation*}
$$

Lemma 4 (Consistency) If $(u, p) \in\left(\left[H_{0}^{1}(\Omega) \cap H^{2}(\Omega)\right]^{d}\right) \times\left(L_{0}^{2}(\Omega) \cap H^{1}(\Omega)\right)$, letting $\boldsymbol{u}=(u, \gamma(u))$ and $\boldsymbol{p}=(p, \gamma(p))$, then

$$
\begin{align*}
\left(\partial_{t} u, v_{h}\right)+a_{h}\left(\boldsymbol{u}, \boldsymbol{v}_{h}\right)+o_{h}\left(u ; \boldsymbol{u}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{p}, v_{h}\right) & =\left(f, v_{h}\right), & & \forall v_{h} \in V_{h}^{\star},  \tag{11}\\
b_{h}\left(\boldsymbol{q}_{h}, u\right) & =0, & & \forall \boldsymbol{q}_{h} \in Q_{h}^{\star} .
\end{align*}
$$

For the proofs of Lemmas 3 and 4, we can follow (18) and (20) in [23], respectively.

For $w, u, v \in\left[H^{1}\left(\mathcal{T}_{h}\right)\right]^{d}$ and $\bar{u}, \bar{v} \in\left[L^{2}\left(\Gamma^{0}\right)\right]^{d}$, let $\boldsymbol{u}=(u, \bar{u})$ and $\boldsymbol{v}=(v, \bar{v})$, then we have

$$
\begin{align*}
& \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}(w \cdot n)[\llbracket \boldsymbol{u} \rrbracket \cdot\{\boldsymbol{v}\}+\llbracket \boldsymbol{v} \rrbracket \cdot\{\boldsymbol{u}\}+\bar{u} \cdot \bar{v}] \mathrm{d} s \\
& =\sum_{K \in \mathcal{T}_{h}} \int(w \cdot \nabla) u \cdot v+(w \cdot \nabla) v \cdot u+(\nabla \cdot w) u v \mathrm{~d} x, \tag{12}
\end{align*}
$$

where the equal sign is due to $u \cdot v-\bar{u} \cdot \bar{v}=\llbracket u \rrbracket \cdot\{\boldsymbol{v}\}+\llbracket \boldsymbol{v} \rrbracket \cdot\{\boldsymbol{u}\}$ and elementwise integration by parts. Provided $w \in H(\operatorname{div}, \Omega), \nabla \cdot w=0$ and $\bar{v}=0$ on $\Gamma$, by using (12), we can give an equivalent form of $o_{h}(w ; \boldsymbol{u}, \boldsymbol{v})$ :

$$
\begin{align*}
o_{h}(w ; \boldsymbol{u}, \boldsymbol{v})= & \sum_{K \in \mathcal{T}_{h}} \int_{K}(w \cdot \nabla) u \cdot v \mathrm{~d} x-\sum_{K \in \mathcal{I}_{h}} \int_{\partial K} \frac{1}{2} w \cdot n(u-\bar{u}) \cdot(v+\bar{v}) \mathrm{d} s \\
& +\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}|w \cdot n|(u-\bar{u}) \cdot(v-\bar{v}) \mathrm{d} s, \tag{13}
\end{align*}
$$

which will be used in the following analysis.
Let $\mathcal{I}_{h} u$ denote the Lagrange interpolant of order $k$ of a continuous function $u$. We have the following bound [9, Theorem 4.4.4]:

$$
\begin{equation*}
\left|u-\mathcal{I}_{h} u\right|_{W^{j, p}(K)} \leq C h^{s-j}|u|_{W^{s, p}(K)}, \quad 0 \leq j \leq s \leq k+1, \tag{14}
\end{equation*}
$$

where $s>d / p$ when $1<p \leq \infty$ and $s \geq d$ when $p=1$.
In what follows, we define the broken polynomial space

$$
P_{k}\left(\mathcal{T}_{h}\right)=\left\{v \in L^{2}(\Omega):\left.v\right|_{K} \in P_{k}(K), \forall K \in \mathcal{T}_{h}\right\},
$$

and $\pi_{h}^{k}$ is the corresponding $L^{2}$-orthogonal projector on $P_{k}\left(\mathcal{T}_{h}\right)$. There exists a constant $C>0$, independent of $h$, such that for $0 \leq l \leq k+1$ and $1 \leq p \leq \infty$ [6, Proposition 1.135]

$$
\begin{equation*}
\left\|w-\pi_{h}^{k} w\right\|_{L^{p}(\Omega)} \leq C h^{l}|w|_{W^{l, p}(\Omega)}, \quad \forall w \in W^{l, p}(\Omega) . \tag{15}
\end{equation*}
$$

The Raviart-Thomas interpolation operator will be used in the sequel. It is defined as follows: $\Pi_{d i v}:\left[H^{1}(\Omega)\right]^{d} \cap H_{0}(\operatorname{div}, \Omega) \rightarrow V_{h}^{\text {div }}$ where $\Pi_{d i v} v$ is the unique function of $V_{h}^{\text {div }}$ satisfying

$$
\begin{align*}
& \int_{K}\left(\Pi_{d i v} v-v\right) \cdot w d x=0, \quad \text { for all } w \in\left[P_{k-1}(K)\right]^{d}, \text { and all } K \in \mathcal{T}_{h},  \tag{16}\\
& \int_{F}\left(\Pi_{d i v} v-v\right) \cdot n w d s=0, \quad \text { for all } w \in P_{k}(F), \text { and all } F \in \mathcal{F}_{h} .
\end{align*}
$$

Remark 2 For the following analysis, it is crucial that the term $u-\Pi_{d i v} u$ is $L^{2}$-orthogonal to the polynomial space $\left[P_{k-1}(K)\right]^{d}, \forall K \in \mathcal{I}_{h}$, see (29) and (30). This is also why the Raviart-Thomas interpolation of the velocity is used instead of the Brezzi-Douglas-Marini interpolation [15].

The operator $\Pi_{d i v}$ satisfies the following commutative property [15, Proposition 2.5.2]

$$
\operatorname{div} \Pi_{d i v} v=\pi_{h}^{k} \operatorname{div} v
$$

Let $\Pi_{d i v} v \in V_{h}^{\text {div }}$ with div $\Pi_{d i v} v=0$ on $\Omega$, then $\left.\Pi_{d i v} v\right|_{K} \in\left[P_{k}(K)\right]^{d}$ [15, Corollary 2.3.1]. Thus, for $\Pi_{d i v} v$ with div $v=0$, we have div $\Pi_{d i v} v=0$ and $\left.\Pi_{d i v} v\right|_{K} \in\left[P_{k}(K)\right]^{d}$.

The Raviart-Thomas interpolation operator satisfies the following approximation properties. Let $j$ and $k$ be integers such that $0 \leq j \leq k+1$. Then there exists $C>0$ independent of $h$ such that [14, Lemma 3.16]

$$
\left|w-\Pi_{d i v} w\right|_{j, K} \leq C h_{K}^{k+1-j}|w|_{k+1, K}, \quad \forall w \in\left[H^{k+1}(K)\right]^{d} .
$$

And, the following maximum norm bounds hold [34, (2.9)]:

$$
\begin{equation*}
\left\|w-\Pi_{d i v} w\right\|_{L^{\infty}(K)}+h_{K}\left\|\nabla\left(w-\Pi_{d i v} w\right)\right\|_{L^{\infty}(K)} \leq C h_{K}\|\nabla w\|_{L^{\infty}(K)} . \tag{17}
\end{equation*}
$$

For each fixed time $t \in[0, T]$, consider a Stokes problem with right-hand side $-v \Delta u+$ $\nabla p$, where $(u, p)$ is the solution of (1). We will denote by $\left(s_{h}, \boldsymbol{\psi}_{h}\right) \in X_{h}^{\star}$ with $\boldsymbol{s}_{h}=\left(s_{h}, \bar{s}_{h}\right)$ and $\boldsymbol{\psi}_{h}=\left(\psi_{h}, \bar{\psi}_{h}\right)$, the E-HDG approximation satisfying

$$
\begin{align*}
a_{h}\left(\boldsymbol{s}_{h}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{\psi}_{h}, v_{h}\right) & =\left(-v \Delta u+\nabla p, v_{h}\right), & & \forall \boldsymbol{v}_{h} \in V_{h}^{\star}, \\
b_{h}\left(\boldsymbol{q}_{h}, s_{h}\right) & =0, & & \forall \boldsymbol{q}_{h} \in Q_{h}^{\star} . \tag{18}
\end{align*}
$$

Then, the following bounds hold [3, 24]:

$$
\begin{align*}
\left\|u-s_{h}\right\|_{L^{2}}+h\| \| \boldsymbol{u}-\boldsymbol{s}_{h} \|_{v} & \leq C h^{j}\|u\|_{j}, \quad 1 \leq j \leq k+1,  \tag{19}\\
\left\|\boldsymbol{p}-\boldsymbol{\psi}_{h}\right\|_{p} & \leq C h^{j-1}\left(\|u\|_{j}+\|p\|_{j-1}\right), \quad 1 \leq j \leq k+1,
\end{align*}
$$

where $\boldsymbol{u}=(u, \gamma(u))$ and $\boldsymbol{p}=(p, \gamma(p))$.
Lemma 5 Assume that $u \in\left[H_{0}^{1}(\Omega) \cap H^{2}(\Omega)\right]^{d}$ is the velocity solution of (1). Let $\boldsymbol{s}_{h}=$ $\left(s_{h}, \bar{s}_{h}\right)$ be the velocity solution of (18). Then, there exists a constant $C$ independent of $h$ such that

$$
\begin{equation*}
\left\|u-s_{h}\right\|_{L^{\infty}(\Omega)}+\left\|u-\bar{s}_{h}\right\|_{L^{\infty}\left(\mathcal{F}_{h}\right)} \leq C\|u\|_{2} . \tag{20}
\end{equation*}
$$

Remark 3 Lemma 5 will be used to prove the optimal error estimates for the velocity, which is applicable for the case of the low mesh Reynolds number.

Proof By using the triangle inequality, shape-regular and quasi-uniformity of the mesh, the inverse inequality (5), (15), (19) and (7), we obtain

$$
\begin{align*}
\left\|u-s_{h}\right\|_{L^{\infty}(\Omega)} & \leq\left\|u-\pi_{h}^{k} u\right\|_{L^{\infty}(\Omega)}+\left\|\pi_{h}^{k} u-s_{h}\right\|_{L^{\infty}(\Omega)} \\
& \leq C\|u\|_{L^{\infty}(\Omega)}+C h^{-\frac{d}{2}}\left\|\pi_{h}^{k} u-s_{h}\right\|_{L^{2}(\Omega)} \\
& \leq C\|u\|_{L^{\infty}(\Omega)}+C h^{-\frac{d}{2}}\left(\left\|u-s_{h}\right\|_{L^{2}(\Omega)}+\left\|u-\pi_{h}^{k} u\right\|_{L^{2}(\Omega)}\right)  \tag{21}\\
& \leq C\|u\|_{2},
\end{align*}
$$

and

$$
\begin{aligned}
\left\|u-\bar{s}_{h}\right\|_{L^{\infty}\left(\mathcal{F}_{h}\right)} & \leq\left\|u-s_{h}^{+}\right\|_{L^{\infty}(F)}+\left\|s_{h}^{+}-\bar{s}_{h}\right\|_{L^{\infty}(F)} \\
& \leq\left\|u-s_{h}\right\|_{L^{\infty}(K)}+C h_{F}^{-\frac{d-1}{2}}\left\|s_{h}^{+}-\bar{s}_{h}\right\|_{L^{2}(F)} \\
& \leq\left\|u-s_{h}\right\|_{L^{\infty}(\Omega)}+C h_{F}^{-\frac{d-1}{2}}\left\|s_{h}^{+}-\bar{s}_{h}\right\|_{L^{2}(F)} \\
& \leq\left\|u-s_{h}\right\|_{L^{\infty}(\Omega)}+C h^{-\frac{d}{2}+1}\left(\sum_{K \in \mathcal{T}_{h}} h_{K}^{-1}\left\|\bar{s}_{h}-s_{h}\right\|_{\partial K}^{2}\right)^{\frac{1}{2}} \\
& \leq C\|u\|_{2},
\end{aligned}
$$

where $F \in \partial K$ stands for the face where the maximum value is taken. Here, notice that

$$
\begin{equation*}
\left\|u-s_{h}^{+}\right\|_{L^{\infty}(F)} \leq\left\|u-s_{h}\right\|_{L^{\infty}(K)}, \tag{22}
\end{equation*}
$$

where the well-known embedding $H^{2}(K) \subset \mathcal{C}^{0}(\bar{K})$ is used, such that $u-s_{h} \in\left[\mathcal{C}^{0}(\bar{K})\right]^{d}$, see Theorem B. 46 in [6].

Lemma 6 (Well-posedness and velocity energy estimate) Assume that $u_{0 h}$ is an approximation of $u_{0}$. Then, there exists an unique solution $\left(\boldsymbol{u}_{h}, \boldsymbol{p}_{h}\right)$ to (3), which satisfies the following energy estimate:

$$
\begin{equation*}
\frac{1}{2}\left\|u_{h}(T)\right\|_{L^{2}}^{2}+\int_{0}^{T} \nu C_{a}^{c}\left\|\boldsymbol{u}_{h}\right\|_{v}^{2}+\frac{1}{2}\left|\boldsymbol{u}_{h}\right|_{u_{h}, u p}^{2} \mathrm{~d} t \leq\left\|u_{0 h}\right\|_{L^{2}}^{2}+\frac{3}{2}\|f\|_{L^{1}\left(0, T ; L^{2}(\Omega)\right)}^{2} \tag{23}
\end{equation*}
$$

Proof First, we basically follow the proof of Lemma 3.1 in [26] to obtain the energy estimate. The well-posedness of the velocity and the pressure solutions follows from the theorem of Carathéodory [4, Theorem A.50] and the discrete inf-sup condition (9), respectively.

Remark 4 Here and in what follows, we shall set $u_{0 h}=s_{0 h}$, where $s_{0 h}$ is the E-HDG approximation of a Stokes problem (18) with the right-hand side $-v \Delta u+\nabla p$ replaced by $-v \Delta u_{0}$.

## 4 Error Estimates for the Velocity

We introduce the following approximation and discretization errors for the velocity and the pressure, respectively:

$$
\begin{array}{llll}
\zeta_{u}=u-\Pi_{d i v} u, & \omega_{u}=u_{h}-\Pi_{d i v} u, & \bar{\zeta}_{u}=\gamma(u)-\overline{\mathcal{I}}_{h} u, & \bar{\omega}_{u}=\bar{u}_{h}-\overline{\mathcal{I}}_{h} u \\
\zeta_{p}=p-\Pi_{Q} p, & \omega_{p}=p_{h}-\Pi_{Q} p, & \bar{\zeta}_{p}=\gamma(p)-\bar{\Pi}_{Q} p, & \bar{\omega}_{p}=\bar{p}_{h}-\bar{\Pi}_{Q} p \tag{24}
\end{array}
$$

where $\Pi_{Q}$ and $\bar{\Pi}_{Q}$ are the standard $L^{2}$-projection operators onto $Q_{h}$ and $\bar{Q}_{h}$, respectively, and $\overline{\mathcal{I}}_{h} u=\left.\mathcal{I}_{h} u\right|_{\mathcal{F}_{h}} \in \bar{V}_{h}$. Set $\zeta_{u}=\left(\zeta_{u}, \bar{\zeta}_{u}\right), \omega_{u}=\left(\omega_{u}, \bar{\omega}_{u}\right), \zeta_{p}=\left(\zeta_{p}, \bar{\zeta}_{p}\right)$ and $\omega_{p}=\left(\omega_{p}, \bar{\omega}_{p}\right)$.

Next, the result that the $L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$ error of the velocity is Reynolds-robust with pre-asymptotic convergence order of $k+1 / 2$ in case of $v \leq C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$, is presented in the following theorem.

Theorem 1 Assume that $(u, p)$ is the solution of (1), and $\left(\boldsymbol{u}_{h}, \boldsymbol{p}_{h}\right) \in X_{h}^{\star}$ the solution of (3). Set $\boldsymbol{u}=(u, \gamma(u))$ and $\boldsymbol{p}=(p, \gamma(p))$, and $u_{0 h}$ is an approximation of $u_{0}$. Let $u \in$ $\left[L^{1}\left(0, T ; W^{1, \infty}(\Omega)\right) \cap L^{\infty}\left(0, T ; H^{r}(\Omega)\right)\right]^{d}, \partial_{t} u \in\left[L^{2}\left(0, T ; H^{r}(\Omega)\right)\right]^{d}$, with $2 \leq r \leq$ $k+1$, and assume that $v \leq C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$, then we have the following estimate:

$$
\begin{aligned}
& \left\|u-u_{h}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\int_{0}^{T} \nu\| \| \boldsymbol{u}-\boldsymbol{u}_{h}\| \|_{v}^{2} \mathrm{~d} t \\
& \quad \leq \Pi_{1}(u)\left\|\omega_{u}(0)\right\|_{L^{2}}^{2}+h^{2 r-1}\left(\Pi_{1}(u) \Pi_{2}(u)+\Pi_{3}(u)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& \Pi_{1}(u)=C e^{\Upsilon(u)}, \\
& \Pi_{2}(u)=h\left\|\partial_{t} u\right\|_{L^{2}\left(0, T ; H^{r}(\Omega)\right)}^{2}+\left(T+(h+1)\|u\|_{L^{1}\left(0, T ; W^{1, \infty}(\Omega)\right)}\right)\|u\|_{L^{\infty}\left(0, T ; H^{r}(\Omega)\right)}^{2}, \\
& \Pi_{3}(u)=C(h+T)\|u\|_{L^{\infty}\left(0, T ; H^{r}(\Omega)\right)}^{2},
\end{aligned}
$$

with $\omega_{u}(0)=u_{0}-\Pi_{d i v} u_{0}$ and $\Upsilon(u)=\int_{0}^{T} C\left(1+\|\nabla u\|_{L^{\infty}}\right) \mathrm{d} t$.

Proof Firstly, by subtracting (3) from (11),

$$
\begin{align*}
& \left(\partial_{t}\left(u-u_{h}\right), v_{h}\right)+a_{h}\left(\boldsymbol{u}-\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{p}-\boldsymbol{p}_{h}, v_{h}\right)-b_{h}\left(\boldsymbol{q}_{h}, u-u_{h}\right)  \tag{25}\\
& \quad+o_{h}\left(u, \boldsymbol{u}, \boldsymbol{v}_{h}\right)-o_{h}\left(u_{h}, \boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)=0 .
\end{align*}
$$

By using (24) and taking $\left(\boldsymbol{v}_{h}, \boldsymbol{q}_{h}\right)=\left(\boldsymbol{\omega}_{u}, \boldsymbol{\omega}_{p}\right)$ in (25), we have

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|\omega_{u}\right\|_{L^{2}}^{2}+a_{h}\left(\boldsymbol{\omega}_{u}, \boldsymbol{\omega}_{u}\right) \\
& \quad=\left(\partial_{t} \zeta_{u}, \omega_{u}\right)+a_{h}\left(\zeta_{u}, \boldsymbol{\omega}_{u}\right)+b_{h}\left(\boldsymbol{\zeta}_{p}, \omega_{u}\right)-b_{h}\left(\boldsymbol{\omega}_{p}, \zeta_{u}\right)  \tag{26}\\
& \quad+o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u_{h} ; \zeta_{u}, \boldsymbol{\omega}_{u}\right)-o_{h}\left(u_{h} ; \boldsymbol{\omega}_{u}, \boldsymbol{\omega}_{u}\right)
\end{align*}
$$

Furthermore, we note that $b_{h}\left(\zeta_{p}, \omega_{u}\right)=b_{h}\left(\boldsymbol{\omega}_{p}, \zeta_{u}\right)=0$ with $\omega_{u}$ and $\zeta_{u}$, which are pointwise divergence-free and H (div)-conforming. Then, we have

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|\omega_{u}\right\|_{L^{2}}^{2}+a_{h}\left(\boldsymbol{\omega}_{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u_{h} ; \boldsymbol{\omega}_{u}, \boldsymbol{\omega}_{u}\right)  \tag{27}\\
& \quad=\left(\partial_{t} \zeta_{u}, \omega_{u}\right)+a_{h}\left(\boldsymbol{\zeta}_{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u_{h} ; \zeta_{u}, \boldsymbol{\omega}_{u}\right)
\end{align*}
$$

On the left-hand side of (27), we apply the discrete coercivity of $a_{h}$ in (8) and the stability of $o_{h}$ (10). On the right-hand side of (27), applying the boundedness of $a_{h}$ in (8) and CauchySchwarz inequality, we have

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\|\omega_{u}\right\|_{L^{2}}^{2}+v C_{a}^{c}\| \| \boldsymbol{\omega}_{u}\| \|_{v}^{2}+\left|\omega_{h}\right|_{u_{h}, \text { up }}^{2}  \tag{28}\\
& \quad \leq\left\|\partial_{t} \zeta_{u}\right\|_{L^{2}}^{2}+\left\|\omega_{u}\right\|_{L^{2}}^{2}+v C\| \| \zeta_{u}\| \|_{v^{\prime}}^{2}+2 \Lambda,
\end{align*}
$$

with $\Lambda=o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u_{h} ; \boldsymbol{\zeta}_{u}, \boldsymbol{\omega}_{u}\right)$. Next, we give a bound for the term $\Lambda$, which is crucial to derive the Reynolds-robust error bound for the velocity. By using the equivalent forms (13) and (4) of the convective term, we have

$$
\begin{aligned}
\Lambda= & o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}\right)+o_{h}\left(u_{h} ; \boldsymbol{\zeta}_{u}, \boldsymbol{\omega}_{u}\right) \\
= & \underbrace{\int_{\Omega}\left(\left(u-u_{h}\right) \cdot \nabla\right) u \cdot \omega_{u} \mathrm{~d} x-\int_{\Omega}\left(u_{h} \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x}_{\Lambda_{1}} \\
& +\underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(u_{h} \cdot n\right) \llbracket \omega_{u} \rrbracket\left\{\zeta_{u}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|\left(u_{h} \cdot n\right)\right| \llbracket \zeta_{u} \rrbracket \llbracket \omega_{u} \rrbracket \mathrm{~d} s .}_{\Lambda_{2}}
\end{aligned}
$$

Recall $\omega_{u}=u_{h}-\Pi_{d i v} u$ where $\left.\Pi_{d i v} u\right|_{K} \in\left[P_{k}(K)\right]^{d}, \forall K \in \mathcal{T}_{h}$. Then we have $\left.\omega_{u}\right|_{K} \in$ $\left[_{k}(K)\right]^{d}$ and $\left.\left(\pi_{h}^{0} u \cdot \nabla_{h}\right) \omega_{u}\right|_{K} \in\left[P_{k-1}(K)\right]^{d}$. Then, by using (16), we have

$$
\begin{equation*}
\int_{K}\left(\pi_{h}^{0} u \cdot \nabla\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x=0, \quad \forall K \in \mathcal{T}_{h} \tag{29}
\end{equation*}
$$

For the term $\Lambda_{1}$, inserting $u_{h}=\omega_{u}+u-\zeta_{u}$, (29) and applying Hölder's inequality, (17), inverse inequality and Cauchy-Schwartz inequality, we have

$$
\begin{align*}
\Lambda_{1}= & \int_{\Omega}\left(\zeta_{u} \cdot \nabla\right) u \cdot \omega_{u}-\left(\omega_{u} \cdot \nabla\right) u \cdot \omega_{u} \mathrm{~d} x-\int_{\Omega}\left(\omega_{h} \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x-\int_{\Omega}\left(u \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x \\
& +\int_{\Omega}\left(\zeta_{u} \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x \\
= & \int_{\Omega}\left(\zeta_{u} \cdot \nabla\right) u \cdot \omega_{u}-\left(\omega_{u} \cdot \nabla\right) u \cdot \omega_{u} \mathrm{~d} x-\int_{\Omega}\left(\omega_{h} \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x+\int_{\Omega}\left(\zeta_{u} \cdot \nabla_{h}\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x \\
& -\sum_{K \in \mathcal{T}_{h}} \int_{K}\left(\left(u-\pi_{h}^{0} u\right) \cdot \nabla\right) \omega_{u} \cdot \zeta_{u} \mathrm{~d} x \\
\leq & C\|\nabla u\|_{L^{\infty}}\left\|\zeta_{u}\right\|_{L^{2}}^{2}+C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2} . \tag{30}
\end{align*}
$$

For $\Lambda_{2}$, we apply Young's inequality to obtain

$$
\begin{aligned}
\Lambda_{2}= & \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(u_{h} \cdot n\right) \llbracket \omega_{u} \rrbracket\left\{\zeta_{u}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|u_{h} \cdot n\right| \llbracket \zeta_{u} \rrbracket \llbracket \omega_{u} \rrbracket \mathrm{~d} s \\
\leq & \underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left|u_{h} \cdot n \|\left\{\zeta_{u}\right\}\right|^{2} \mathrm{~d} s}_{\Lambda_{21}}+\underbrace{\frac{1}{2} \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left|u_{h} \cdot n \|\left|\llbracket \zeta_{u} \rrbracket\right|^{2} \mathrm{~d} s\right.}_{\Lambda_{22}} \\
& +\frac{3}{8} \underbrace{}_{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left|u_{h} \cdot n\right|\left|\llbracket \omega_{u} \rrbracket\right|^{2} \mathrm{~d} s}
\end{aligned}
$$

Applying $u_{h}=\omega_{u}+u-\zeta_{u}$, Hölder's inequality and the trace inequality, we have

$$
\begin{align*}
\Lambda_{21} \leq & \left.\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \mid \omega_{u} \cdot n\| \| \zeta_{u}\right\}\left.\right|^{2} \mathrm{~d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left|\zeta_{u} \cdot n \|\left\{\zeta_{u}\right\}\right|^{2} \mathrm{~d} s \\
& +\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left|u \cdot n \|\left\{\zeta_{u}\right\}\right|^{2} \mathrm{~d} s \\
\leq & \frac{1}{2} \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{\infty}(\partial K)}\left\|\omega_{u}\right\|_{L^{2}(\partial K)}^{2}+\frac{1}{2} \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{\infty}(\partial K)}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}  \tag{31}\\
& +\left\|\zeta_{u}\right\|_{L^{\infty}} \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+\|u\|_{L^{\infty}} \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2} \\
\leq & C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2}+C\left(\|u\|_{L^{\infty}}+h\|\nabla u\|_{\left.L^{\infty}\right)} \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}\right.
\end{align*}
$$

Here, notice that the inequalities (14) with $p=\infty$ and (17) are used. For $\Lambda_{22}$, we similarly have

$$
\begin{equation*}
\Lambda_{22} \leq C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2}+C\left(\|u\|_{L^{\infty}}+h\|\nabla u\|_{L^{\infty}}\right) \sum_{K \in \mathcal{T}_{h}}\left\|\llbracket \zeta_{u}\right\| \|_{L^{2}(\partial K)}^{2} . \tag{32}
\end{equation*}
$$

Then, by combining (31) and (32), we obtain

$$
\begin{align*}
\Lambda_{2} \leq & C\left(\|u\|_{L^{\infty}}+h\|\nabla u\|_{L^{\infty}}\right)\left(\sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+\sum_{K \in \mathcal{T}_{h}}\left\|\llbracket \zeta_{u}\right\| \|_{L^{2}(\partial K)}^{2}\right)  \tag{33}\\
& +C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2}+\frac{3}{8}\left|\omega_{h}\right|_{u_{h}, \text { up }}^{2} .
\end{align*}
$$

Collecting the above estimates, we can obtain

$$
\begin{align*}
\Lambda \leq & C\|\nabla u\|_{L^{\infty}}\left\|\zeta_{u}\right\|_{L^{2}}^{2}+C\left(\|u\|_{L^{\infty}}+h\|\nabla u\|_{L^{\infty}}\right)\left(\sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+\sum_{K \in \mathcal{T}_{h}}\| \| \zeta_{u}\| \|_{L^{2}(\partial K)}^{2}\right) \\
& +C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2}+\frac{3}{8}\left|\omega_{h}\right|_{u_{h}, \text { up }}^{2} \\
\leq & C\left(1+h^{-1}\right)\|u\|_{W^{1, \infty}(\Omega)}\left(\left\|\zeta_{u}\right\|_{L^{2}}^{2}+h \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+h \sum_{K \in \mathcal{T}_{h}}\| \| \zeta_{u}\| \|_{L^{2}(\partial K)}^{2}\right) \\
& +C\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}\right\|_{L^{2}}^{2}+\frac{3}{8}\left|\omega_{h}\right|_{u_{h}, \text { up }}^{2} . \tag{34}
\end{align*}
$$

Using (34) on the right-hand side of (28), we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\|\omega_{u}\right\|_{L^{2}}^{2}+v C_{a}^{c}\| \| \omega_{u}\| \|_{v}^{2}+\left|\omega_{h}\right|_{u_{h}, \text { up }}^{2} \\
& \quad \leq\left\|\partial_{t} \zeta_{u}\right\|_{L^{2}}^{2}+\left.v C\left|\left\|\zeta_{u}\right\|\left\|_{v^{\prime}}^{2}+C\left(1+\|\nabla u\|_{L^{\infty}}\right)\right\| \omega_{u} \|_{L^{2}}^{2}+\frac{3}{4}\right| \omega_{h}\right|_{u_{h}, \text { up }} ^{2} \\
& \quad+C\left(1+h^{-1}\right)\|u\|_{W^{1, \infty(\Omega)}}\left(\left\|\zeta_{u}\right\|_{L^{2}}^{2}+h \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+h \sum_{K \in \mathcal{T}_{h}}\| \| \zeta_{u}\| \|_{L^{2}(\partial K)}^{2}\right) .
\end{aligned}
$$

By applying Gronwall's Lemma, we can obtain

$$
\begin{align*}
& \left\|\omega_{u}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\int_{0}^{T} C_{a}^{c} \nu\| \| \omega_{u}\| \|_{v}^{2}+\frac{1}{4}\left|\boldsymbol{\omega}_{h}\right|_{u_{h}, \mathrm{up}}^{2} \mathrm{~d} t \\
& \quad \leq e^{\Upsilon(u)}\left\|\omega_{u}(0)\right\|_{L^{2}}^{2}+C e^{\Upsilon(u)} \int_{0}^{T}\left\|\partial_{t} \zeta_{u}\right\|_{L^{2}}^{2}+v\| \| \zeta_{u}\| \|_{v^{\prime}}^{2} \\
& \quad+\left(1+h^{-1}\right)\|u\|_{W^{1, \infty}(\Omega)}\left(\left\|\zeta_{u}\right\|_{L^{2}}^{2}+h \sum_{K \in \mathcal{T}_{h}}\left\|\left\{\zeta_{u}\right\}\right\|_{L^{2}(\partial K)}^{2}+h \sum_{K \in \mathcal{T}_{h}}\| \| \zeta_{u} \rrbracket \|_{L^{2}(\partial K)}^{2}\right) \mathrm{d} t \tag{35}
\end{align*}
$$

with $\Upsilon(u)=\int_{0}^{T} C\left(1+\|\nabla u\|_{L^{\infty}}\right) \mathrm{d} t$. By means of triangle inequality and (35), we can conclude the proof.

Next, based on the above analysis framework, a Reynolds-dependent error bound is easily obtained here, which has an optimal convergence order $k+1$. It is applicable for the case of the low mesh Reynolds number. For completeness, we present and prove the optimal results. To this end, we introduce the following approximation and discretization errors for the velocity and the pressure, respectively:

$$
\begin{array}{lll}
\zeta_{u}^{\prime}=u-s_{h}, & \omega_{u}^{\prime}=u_{h}-s_{h}, & \bar{\zeta}_{u}^{\prime}=\gamma(u)-\bar{s}_{h}, \\
\zeta_{p}^{\prime}=p-\bar{\omega}_{h}^{\prime}=\bar{u}_{h}-\bar{s}_{h} & \omega_{p}^{\prime}=p_{h}-\psi_{h}, & \bar{\zeta}_{p}^{\prime}=\gamma(p)-\bar{\psi}_{h},  \tag{36}\\
\bar{\omega}_{p}^{\prime}=\bar{p}_{h}-\bar{\psi}_{h},
\end{array}
$$

and set $\zeta_{u}^{\prime}=\left(\zeta_{u}^{\prime}, \bar{\zeta}_{u}^{\prime}\right), \omega_{u}^{\prime}=\left(\omega_{u}^{\prime}, \bar{\omega}_{u}^{\prime}\right), \zeta_{p}^{\prime}=\left(\zeta_{p}^{\prime}, \bar{\zeta}_{p}^{\prime}\right)$ and $\boldsymbol{\omega}_{p}^{\prime}=\left(\omega_{p}^{\prime}, \bar{\omega}_{p}^{\prime}\right)$.

Remark 5 It is interesting to explain why one can choose $\boldsymbol{s}_{h}=\left(s_{h}, \bar{s}_{h}\right)$ and $\boldsymbol{\psi}_{h}=\left(\psi_{h}, \bar{\psi}_{h}\right)$ as the discrete approximations for the velocity and the pressure, respectivly, to get the optimal convergence order. It allows us to obtain the super-convergence between $u_{h}$ and $s_{h}$ in some norms, by avoiding some terms in the truncation error, see (39) and (44).

Theorem 2 Assume that $(u, p)$ is the solution of (1), and $\left(\boldsymbol{u}_{h}, \boldsymbol{p}_{h}\right) \in X_{h}^{\star}$ the solution of (3). Set $\boldsymbol{u}=(u, \gamma(u))$ and $\boldsymbol{p}=(p, \gamma(p))$. Let $u \in\left[L^{1}\left(0, T ; W^{1, \infty}(\Omega)\right) \cap L^{\infty}\left(0, T ; H^{r}(\Omega)\right)\right]^{d}$, $\partial_{t} u \in\left[L^{2}\left(0, T ; H^{r}(\Omega)\right)\right]^{d}$, with $2 \leq r \leq k+1$ and $u_{0} h$ be an approximation of $u_{0}$. Then the following error estimates hold:

$$
\begin{align*}
\left\|u-u_{h}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}^{2} & \leq \Pi_{1}^{\prime}(u)\left\|\omega_{u}^{\prime}(0)\right\|_{L^{2}}^{2}+h^{2 r}\left(\Pi_{1}^{\prime}(u) \Pi_{2}^{\prime}(u)+\Pi_{3}^{\prime}(u)\right) \\
\int_{0}^{T} \nu\| \| \boldsymbol{u}-\boldsymbol{u}_{h}\| \|_{v}^{2} \mathrm{~d} t & \leq \Pi_{1}^{\prime}(u)\left\|\omega_{u}^{\prime}(0)\right\|_{L^{2}}^{2}+h^{2 r-2}\left(h^{2} \Pi_{1}^{\prime}(u) \Pi_{2}^{\prime}(u)+v T \Pi_{3}^{\prime}(u)\right), \tag{37}
\end{align*}
$$

with $\omega_{u}^{\prime}(0)=u_{0 h}-s_{h}(0), \Upsilon^{\prime}(u)=\int_{0}^{T} C\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}+1\right) \mathrm{d} t$, and

$$
\begin{aligned}
\Pi_{1}^{\prime}(u)= & C e^{\Upsilon^{\prime}(u)}, \\
\Pi_{2}^{\prime}(u)= & \left(\frac{1}{v}\|u\|_{L^{2}\left(0, T ; H^{2}(\Omega)\right)}^{2}+\|u\|_{L^{1}\left(0, T ; W^{1, \infty}(\Omega)\right)}\right)\|u\|_{L^{\infty}\left(0, T ; H^{r}(\Omega)\right)}^{2} \\
& +\left\|\partial_{t} u\right\|_{L^{2}\left(0, T ; H^{r}(\Omega)\right)}^{2} \\
\Pi_{3}^{\prime}(u) & =C\|u\|_{L^{\infty}\left(0, T ; H^{r}(\Omega)\right)}^{2} .
\end{aligned}
$$

Proof Firstly, by using (36) and taking $\left(\boldsymbol{v}_{h}, \boldsymbol{q}_{h}\right)=\left(\boldsymbol{\omega}_{u}^{\prime}, \boldsymbol{\omega}_{p}^{\prime}\right)$, we have

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+a_{h}\left(\omega_{u}^{\prime}, \omega_{u}^{\prime}\right)+o_{h}\left(u_{h} ; \boldsymbol{\omega}_{u}^{\prime}, \boldsymbol{\omega}_{u}^{\prime}\right)  \tag{38}\\
& \quad=\left(\partial_{t} \zeta_{u}^{\prime}, \omega_{u}^{\prime}\right)+o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}^{\prime}\right)+o_{h}\left(u_{h} ; \zeta_{u}^{\prime}, \boldsymbol{\omega}_{u}^{\prime}\right),
\end{align*}
$$

where, by (18), we have

$$
\begin{equation*}
a_{h}\left(\zeta_{u}^{\prime}, \omega_{u}^{\prime}\right)+b_{h}\left(\zeta_{p}^{\prime}, \omega_{u}^{\prime}\right)-b_{h}\left(\boldsymbol{\omega}_{p}^{\prime}, \zeta_{u}^{\prime}\right)=0 \tag{39}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+2 v C_{a}^{c}\| \| \omega_{u}^{\prime}\| \|_{v}^{2}+\left|\omega_{h}^{\prime}\right|_{u_{h}, \text { up }}^{2} \leq\left\|\partial_{t} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+2 \Xi \tag{40}
\end{equation*}
$$

with $\Xi=o_{h}\left(u-u_{h} ; \boldsymbol{u}, \boldsymbol{\omega}_{u}^{\prime}\right)+o_{h}\left(u_{h} ; \boldsymbol{\zeta}_{u}^{\prime}, \boldsymbol{\omega}_{u}^{\prime}\right)$. By using the forms (4) and (13) of the convective term, we have

$$
\begin{aligned}
\Xi= & \underbrace{\int_{\Omega}\left(\left(u-u_{h}\right) \cdot \nabla\right) u \cdot \omega_{u}^{\prime} \mathrm{d} x-\int_{\Omega}\left(u_{h} \cdot \nabla_{h}\right) \omega_{u}^{\prime} \cdot \zeta_{u}^{\prime} \mathrm{d} x}_{\Xi_{1}} \\
& +\underbrace{\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(u_{h} \cdot n\right) \llbracket \omega_{u}^{\prime} \rrbracket\left\{\zeta_{u}^{\prime}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|\left(u_{h} \cdot n\right)\right| \llbracket \zeta_{u}^{\prime} \rrbracket \llbracket\left[\omega_{u}^{\prime} \rrbracket \mathrm{d} s\right.}_{\Xi_{2}}
\end{aligned}
$$

For the term $\Xi_{1}$, inserting $u_{h}=\omega_{u}^{\prime}+s_{h}$ and applying Hölder's inequality, (20) and Young's inequality, we have

$$
\begin{align*}
\Xi_{1}= & \int_{\Omega}\left[\left(\zeta_{u}^{\prime} \cdot \nabla\right) u \cdot \omega_{u}^{\prime}-\left(\omega_{u}^{\prime} \cdot \nabla\right) u \cdot \omega_{u}^{\prime}\right] \mathrm{d} x-\int_{\Omega}\left(\omega_{u}^{\prime} \cdot \nabla_{h}\right) \omega_{u}^{\prime} \cdot \zeta_{u}^{\prime} \mathrm{d} x-\int_{\Omega}\left(s_{h} \cdot \nabla_{h}\right) \omega_{u}^{\prime} \cdot \zeta_{u}^{\prime} \mathrm{d} x \\
\leq & \|\nabla u\|_{L^{\infty}}\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\|\nabla u\|_{L^{\infty}}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+C \frac{1}{v}\left\|s_{h}\right\|_{L^{\infty}}^{2}\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+C \frac{1}{v}\|u\|_{2}^{2}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2} \\
& +\frac{C_{a}^{c} v}{4}\left\|\nabla_{h} \omega_{u}^{\prime}\right\|_{L^{2}}^{2} . \tag{41}
\end{align*}
$$

For the term $\Xi_{2}$, applying $u_{h}=\omega_{u}^{\prime}+s_{h}$, Hölder's inequality, Young's inequality, (20) and the trace inequality, we can obtain

$$
\begin{aligned}
\Xi_{2}= & \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(u_{h} \cdot n\right) \llbracket \omega_{u}^{\prime} \rrbracket\left\{\zeta_{u}^{\prime}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|\left(u_{h} \cdot n\right)\right| \llbracket \zeta_{u}^{\prime} \rrbracket \llbracket \omega_{u}^{\prime} \rrbracket \mathrm{d} s \\
\leq & \sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(\omega_{u}^{\prime} \cdot n\right) \llbracket \omega_{u}^{\prime} \rrbracket\left\{\zeta_{u}^{\prime}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|\left(\omega_{u}^{\prime} \cdot n\right)\right| \llbracket \zeta_{u}^{\prime} \rrbracket \llbracket \omega_{u}^{\prime} \rrbracket \mathrm{d} s \\
& +\sum_{K \in \mathcal{T}_{h}} \int_{\partial K}\left(s_{h} \cdot n\right) \llbracket \omega_{u}^{\prime} \rrbracket\left\{\zeta_{u}^{\prime}\right\} \mathrm{d} s+\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} \frac{1}{2}\left|\left(s_{h} \cdot n\right)\right| \llbracket \zeta_{u}^{\prime} \rrbracket \llbracket \omega_{u}^{\prime} \rrbracket \mathrm{d} s \\
\leq & C \sum_{K \in \mathcal{T}_{h}} \frac{h_{K}}{v} \int_{\partial K}\left|\omega_{u}^{\prime} \cdot n\right|^{2}\left|\left\{\zeta_{u}^{\prime}\right\}\right|^{2} \mathrm{~d} s+C \sum_{K \in \mathcal{T}_{h}} \frac{h_{K}}{v} \int_{\partial K}\left|\omega_{u}^{\prime} \cdot n\right|^{2}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s \\
& +C \sum_{K \in \mathcal{T}_{h}} \frac{h_{K}}{v} \int_{\partial K}\left|\left(s_{h} \cdot n\right)\right|^{2}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s+C \sum_{K \in \mathcal{T}_{h}} \frac{h_{K}}{v} \int_{\partial K}\left|\left(s_{h} \cdot n\right)\right|^{2}\left|\left\{\zeta_{u}^{\prime}\right\}\right|^{2} \mathrm{~d} s \\
& +\frac{1}{4} \sum_{K \in \mathcal{T}_{h}} \frac{C_{a}^{c} \nu \alpha}{h_{K}} \int_{\partial K}\left|\llbracket \omega_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s \\
\leq & C \frac{1}{v}\|u\|_{2}^{2}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+C \frac{1}{v}\left\|s_{h}\right\|_{L^{\infty}}^{2}\left(\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\left\{\zeta_{u}^{\prime}\right\}\right|^{2} \mathrm{~d} s+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s\right) \\
& +\frac{1}{4} \sum_{K \in \mathcal{T}_{h}} \frac{C_{a}^{c} \nu \alpha}{h_{K}}\left\|\llbracket \omega_{u}^{\prime} \rrbracket\right\|_{L^{2}(\partial K)}^{2} .
\end{aligned}
$$

By using the triangle inequality, we have

$$
\begin{align*}
& \sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\left\{\zeta_{u}^{\prime}\right\}\right|^{2} \mathrm{~d} s+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s \\
& \quad \leq \frac{3}{2} \sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s+2 \sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\zeta_{u}^{\prime}\right|^{2} \mathrm{~d} s . \tag{42}
\end{align*}
$$

Then, by using (42) and the trace inequality, we have

$$
\begin{aligned}
\Xi_{2} \leq & C \frac{1}{v}\|u\|_{2}^{2}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+C \frac{1}{v}\left\|s_{h}\right\|_{L^{\infty}}^{2}\left(\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+h^{2}\left\|\nabla_{h} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \rrbracket\right|^{2} \mathrm{~d} s\right) \\
& +\frac{1}{4} \sum_{K \in \mathcal{T}_{h}} \frac{C_{a}^{c} v \alpha}{h_{K}}\left\|\llbracket \omega_{u}^{\prime}\right\| \|_{L^{2}(\partial K)}^{2} .
\end{aligned}
$$

Collecting the above estimates, we can obtain

$$
\begin{align*}
\Xi \leq & C\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}\right)\left(\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+h^{2}\left\|\nabla_{h} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \mathbb{\|}\right|^{2} \mathrm{~d} s\right)  \tag{43}\\
& +C\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}\right)\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+\frac{1}{2} \nu C_{a}^{c}\| \| \omega_{u}^{\prime}\| \|_{v}^{2} .
\end{align*}
$$

Inserting (43) into (40), we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+v C_{a}^{c}\| \| \omega_{u}^{\prime}\| \|_{v}^{2}+\left|\omega_{h}^{\prime}\right|_{u_{h}, \text { up }}^{2} \\
& \quad \leq \\
& \quad C\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}\right)\left(\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+h^{2}\left\|\nabla_{h} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \|\right|^{2} \mathrm{~d} s\right) \\
& \quad+C\left(1+\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}\right)\left\|\omega_{u}^{\prime}\right\|_{L^{2}}^{2}+\left\|\partial_{t} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2} .
\end{aligned}
$$

Then, applying Gronwall's Lemma, we can obtain

$$
\begin{align*}
& \left\|\omega_{u}^{\prime}\right\|_{L^{\infty}\left(0, T ; L^{2}(\Omega)\right)}^{2}+\int_{0}^{T} \nu C_{a}^{c}\| \| \omega_{u}^{\prime}\| \|_{v}^{2} \mathrm{~d} t+\left|\omega_{h}^{\prime}\right|_{u_{h}, \text { up }}^{2} \mathrm{~d} t \\
& \quad \leq e^{\Upsilon^{\prime}(u)}\left\|\omega_{u}^{\prime}(0)\right\|_{L^{2}}^{2}+C e^{\Upsilon^{\prime}(u)} \int_{0}^{T}\left\|\partial_{t} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}  \tag{44}\\
& \quad+\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}\right)\left(\left\|\zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+h^{2}\left\|\nabla_{h} \zeta_{u}^{\prime}\right\|_{L^{2}}^{2}+\sum_{K \in \mathcal{T}_{h}} h_{K} \int_{\partial K}\left|\llbracket \zeta_{u}^{\prime} \|\right|^{2} \mathrm{~d} s\right) \mathrm{d} t,
\end{align*}
$$

with $\Upsilon^{\prime}(u)=\int_{0}^{T} C\left(\frac{1}{v}\|u\|_{2}^{2}+\|\nabla u\|_{L^{\infty}}+1\right) \mathrm{d} t$. Finally, we can obtain the velocity errors by combining (44) and the triangle inequality.

Remark 6 Notice that Theorem 1 only provides pre-asymptotic rate of convergence $k+1 / 2$ in case of $v \leq C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$. For a given viscosity, the relationship between $v$ and $C h\|u\|_{L^{\infty}\left(L^{\infty}(\Omega)\right)}$ reflects the size of the local mesh Reynolds number. When $h$ tends to zero, from Theorem 2, the velocity error has the asymptotic convergence order $k+1$, in which the constants are dependent on the Reynolds number. Thus, for small values of $v$, it would be interesting to see the transition from pre-asymptotic to asymptotic rate of convergence as the mesh size tends to 0 .

## 5 Numerical Studies

In this section, we present a numerical example with a known solution to check the analytical results of the previous section, which is implemented in the NGSolve software [8]. For other numerical performences of these types of numerical methods, we can refer to some literature [24, 29]. In the implementation, after cellwise static condensation, only the degrees of freedom related to the facet spaces appear in the global coupled system. The velocity penalty parameter $\alpha$ is set to be $10 k^{2}$.

Table 1 Velocity errors in the $L^{2}$-norm with varying $v, k=1$ and $T=2.5$

| $\mathrm{E}-\mathrm{HDG}$ <br> ndof | $v=10^{0}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-2}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-4}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-6}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-8}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-10}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 564 | $2.89 \mathrm{E}-02$ | $8.15 \mathrm{E}-02$ | $2.52 \mathrm{E}-01$ | $2.58 \mathrm{E}-01$ | $2.58 \mathrm{E}-01$ | $2.58 \mathrm{E}-01$ |
| 2208 | $7.59 \mathrm{E}-03$ | $1.86 \mathrm{E}-02$ | $1.02 \mathrm{E}-01$ | $1.09 \mathrm{E}-01$ | $1.09 \mathrm{E}-01$ | $1.09 \mathrm{E}-01$ |
| 8700 | $1.86 \mathrm{E}-03$ | $4.17 \mathrm{E}-03$ | $2.85 \mathrm{E}-02$ | $3.19 \mathrm{E}-02$ | $3.19 \mathrm{E}-02$ | $3.19 \mathrm{E}-02$ |
| 33,870 | $4.65 \mathrm{E}-04$ | $1.01 \mathrm{E}-03$ | $7.24 \mathrm{E}-03$ | $8.81 \mathrm{E}-03$ | $8.83 \mathrm{E}-03$ | $8.83 \mathrm{E}-03$ |
| EOC | 1.99 | 2.11 | 1.71 | 1.62 | 1.62 | 1.62 |
| 266 | $5.66 \mathrm{E}-02$ | $1.41 \mathrm{E}-01$ | $3.01 \mathrm{E}-01$ | $3.08 \mathrm{E}-01$ | $3.08 \mathrm{E}-01$ | $3.08 \mathrm{E}-01$ |
| 1010 | $1.55 \mathrm{E}-02$ | $3.91 \mathrm{E}-02$ | $9.99 \mathrm{E}-02$ | $1.05 \mathrm{E}-01$ | $1.05 \mathrm{E}-01$ | $1.05 \mathrm{E}-01$ |
| 3922 | $3.81 \mathrm{E}-03$ | $9.36 \mathrm{E}-03$ | $2.95 \mathrm{E}-02$ | $3.45 \mathrm{E}-02$ | $3.45 \mathrm{E}-02$ | $3.45 \mathrm{E}-02$ |
| 15,162 | $9.52 \mathrm{E}-04$ | $2.32 \mathrm{E}-03$ | $8.74 \mathrm{E}-03$ | $1.21 \mathrm{E}-02$ | $1.21 \mathrm{E}-02$ | $1.21 \mathrm{E}-02$ |
| EOC | 1.98 | 1.97 | 1.73 | 1.56 | 1.56 | 1.56 |

Table 2 Velocity errors in the $L^{2}$-norm with $v=10^{-8}, k=1$ and $T=2.5$

| $h$ | ndof | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | Rate |
| :--- | :--- | :--- | :--- |
| $1 / 5$ | 266 | $3.08 \mathrm{E}-01$ | - |
| $1 / 10$ | 1010 | $1.05 \mathrm{E}-01$ | 1.55 |
| $1 / 20$ | 3922 | $3.45 \mathrm{E}-02$ | 1.61 |
| $1 / 40$ | 15,162 | $1.21 \mathrm{E}-02$ | 1.51 |
| $1 / 80$ | 60,042 | $3.36 \mathrm{E}-03$ | 1.85 |
| $1 / 160$ | 238,386 | $8.97 \mathrm{E}-04$ | 1.91 |
| $1 / 320$ | 949,954 | $2.28 \mathrm{E}-04$ | 1.98 |

Table 3 Velocity errors in the $L^{2}$-norm with varying $v, k=2$ and $T=6$

| HDG <br> ndof | $v=10^{0}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-2}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-4}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-6}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-8}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $v=10^{-10}$ <br> $\left\\|u-u_{h}\right\\|_{L^{2}}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1269 | $6.90 \mathrm{E}-03$ | $6.16 \mathrm{E}-03$ | $5.26 \mathrm{E}-01$ | $5.52 \mathrm{E}-01$ | $5.53 \mathrm{E}-01$ | $5.53 \mathrm{E}-01$ |
| 4968 | $6.33 \mathrm{E}-04$ | $6.26 \mathrm{E}-04$ | $6.73 \mathrm{E}-02$ | $8.65 \mathrm{E}-02$ | $8.68 \mathrm{E}-02$ | $8.68 \mathrm{E}-02$ |
| 18,603 | $7.38 \mathrm{E}-05$ | $7.41 \mathrm{E}-05$ | $5.92 \mathrm{E}-03$ | $1.53 \mathrm{E}-02$ | $1.55 \mathrm{E}-02$ | $1.55 \mathrm{E}-02$ |
| 72,900 | $8.74 \mathrm{E}-06$ | $8.90 \mathrm{E}-06$ | $3.23 \mathrm{E}-04$ | $1.93 \mathrm{E}-03$ | $1.98 \mathrm{E}-03$ | $1.98 \mathrm{E}-03$ |
| EOC | 3.21 | 3.15 | 3.56 | 2.72 | 2.71 | 2.71 |
| E-HDG | $v=10^{0}$ | $v=10^{-2}$ | $v=10^{-4}$ | $v=10^{-6}$ | $v=10^{-8}$ | $v=10^{-10}$ |
| ndof | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $\left\\|u-u_{h}\right\\|_{L^{2}}$ | $\left\\|u-u_{h}\right\\|_{L^{2}}$ |
| 817 | $9.61 \mathrm{E}-03$ | $8.47 \mathrm{E}-03$ | $7.76 \mathrm{E}-01$ | $7.94 \mathrm{E}-01$ | $7.94 \mathrm{E}-01$ | $7.94 \mathrm{E}-01$ |
| 3162 | $8.51 \mathrm{E}-04$ | $8.44 \mathrm{E}-04$ | $1.22 \mathrm{E}-01$ | $2.06 \mathrm{E}-01$ | $2.07 \mathrm{E}-01$ | $2.07 \mathrm{E}-01$ |
| 11,779 | $9.53 \mathrm{E}-05$ | $9.60 \mathrm{E}-05$ | $1.15 \mathrm{E}-02$ | $3.96 \mathrm{E}-02$ | $4.01 \mathrm{E}-02$ | $4.01 \mathrm{E}-02$ |
| 46,030 | $1.11 \mathrm{E}-05$ | $1.12 \mathrm{E}-05$ | $5.71 \mathrm{E}-04$ | $6.28 \mathrm{E}-03$ | $6.46 \mathrm{E}-03$ | $6.46 \mathrm{E}-03$ |
| EOC | 3.25 | 3.19 | 3.57 | 2.33 | 2.32 | 2.32 |

Let the domain $\Omega=(0,1)^{2}$ and choose the exact solution given by

$$
\begin{align*}
& u(x, y, t)=\frac{6+4 \cos (4 t)}{10}\left[\begin{array}{c}
8 \sin ^{2}(\pi x)(2 y(1-y)(1-2 y)) \\
-8 \pi \sin (2 \pi x)(y(1-y))^{2}
\end{array}\right],  \tag{45}\\
& p(x, y, t)=\frac{6+4 \cos (4 t)}{10} \sin (\pi x) \cos (\pi y) .
\end{align*}
$$

We derive the initial condition and the Dirichlet boundary condition from the exact solution.
For temporal discretization, we use an implicit/explicit (IMEX) BDF2 scheme, in which $o_{h}\left(2 u_{h}^{n-1}-u_{h}^{n-2}, \boldsymbol{u}_{h}^{n}, \boldsymbol{v}_{h}\right)$ is used in the convection term, except using $o_{h}\left(u_{h}^{n-1}, \boldsymbol{u}_{h}^{n}, \boldsymbol{v}_{h}\right)$ in the first time step. We take the small time step $\Delta t=10^{-3}$, which can ensure that the spatial error is dominant. Here, 'EOC' represents the average estimated orders of convergence. We test the convergence orders of the velocity errors in the $L^{2}$-norm for the HDG and E-HDG methods with varying viscosity, respectively.

We take the polynomial order $k=1$ and the final time $T=2.5$. The quasi-uniform unstructured triangular meshes are used with mesh size $h=1 / 5,1 / 10,1 / 20,1 / 40$. From Table 1, for the E-HDG method, the velocity error has the optimal and quasi-optimal convergence rates for large values and small values of $v$, respectively. In addition, by fixing the mesh size, it can be observed that the velocity errors become larger and larger as the viscosity decreases, and when the viscosity is small enough, the velocity errors are independent of the small viscosity. These are consitent with the theoretical estimates, see Theorems 1 and 2. For the HDG method, we can observe the similar convergence and error behaviors as that of the E-HDG method.

Notice that for small values of $v$, we present the above results, in which the finest grid is too big. Because Theorem 1 only provides pre-asymptotic rate of convergence for large values of $h$, it would still be useful to see the transition from pre-asymptotic to asymptotic rate of convergence as the mesh size tends to zero. To the end, we contiue to take the smaller mesh size $h=1 / 80,1 / 160,1 / 320$ for the E-HDG method with $v=10^{-8}$. To ensure that the spatial error is dominant, we take the smaller time step $\Delta t=10^{-4}$. From Table 2, we can see the transition, as expected in Remark 6.

In addition, we take the higher polynomial order $k=2$ and the final time $T=6$. We use the quasi-uniform unstructured triangular meshes with mesh size $h=1 / 6,1 / 12,1 / 24,1 / 48$. Similar numerical results can be also obtained from Table 3. Finally, from Tables 1 and 3, we can notice that the HDG method has been better in terms of accuracy and convergence rate than the E-HDG method for small values of $v$.

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Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

## Declarations

Competing Interests The authors have not disclosed any competing interests.

## References

1. Hansbo, P., Szepessy, A.: A velocity-pressure streamline diffusion finite element method for the incompressible Navier-Stokes equations. Comput. Methods Appl. Mech. Eng. 84(2), 175-192 (1990)
2. De Frutos, J., García-Archilla, B., John, V., et al.: Error analysis of non inf-sup stable discretizations of the time-dependent Navier-Stokes equations with local projection stabilization[J]. IMA J. Numer. Anal. 39(4), 1747-1786 (2019)
3. Rhebergen, S., Wells, G.: Analysis of a hybridized/interface stabilized finite element method for the Stokes equations. SIAM J. Numer. Anal. 55(4), 1982-2003 (2017)
4. John, V.: Finite element methods for incompressible flow problems. Springer, Cham (2016)
5. Boyer, F., Fabrie, P.: Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models. Springer, Berlin (2012)
6. Ern, A., Guermond, J.L.: Theory and practice of finite elements. Springer, New York (2004)
7. Schroeder, P.W., Lube, G.: Divergence-free H(div)-FEM for time-dependent incompressible flows with applications to high Reynolds number vortex dynamics[J]. J. Sci. Comput. 75(2), 830-858 (2018)
8. Schöberl, J.: C++ 11 implementation of finite elements in NGSolve[J]. Vienna University of Technology, Institute for analysis and scientific computing (2014)
9. Brenner, S., Scott, R.: The mathematical theory of finite element methods. Springer, Berlin (2007)
10. Cockburn, B., Kanschat, G., Schötzau, D.: A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations. J. Sci. Comput. 31(1), 61-73 (2007)
11. Lehrenfeld, C., Schöberl, J.: High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows. Comput. Methods Appl. Mech. Eng. 307, 339-361 (2016)
12. Cesmelioglu, A., Cockburn, B., Qiu, W.: Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations. Math. Comput. 86(306), 1643-1670 (2017)
13. Lederer, P.L., Lehrenfeld, C., Schöberl, J.: Hybrid Discontinuous Galerkin methods with relaxed H(div)conformity for incompressible flows. Part II. ESAIM: Math. Model. Numer. Anal. 53(2), 503-522 (2019)
14. Gatica, G.N.: A Simple Introduction to the Mixed Finite Element Method: Theory and Applications[M]. Springer, Berlin (2014)
15. Boffi, D., Brezzi, F., Fortin, M.: Mixed finite element methods and applications. Springer, Heidelberg (2013)
16. Fu, G.: An explicit divergence-free DG method for incompressible flow. Comput. Methods Appl. Mech. Eng. 345, 502-517 (2019)
17. Burman, E., Fernández, M.A.: Continuous interior penalty finite element method for the time-dependent Navier-Stokes equations: space discretization and convergence[J]. Numer. Math. 107(1), 39-77 (2007)
18. Schroeder, P.W., Lehrenfeld, C., Linke, A., et al.: Towards computable flows and robust estimates for inf-sup stable FEM applied to the time-dependent incompressible Navier-Stokes equations[J]. SeMA J. 75(4), 629-653 (2018)
19. Schroeder, P.W., Lube, G.: Pressure-robust analysis of divergence-free and conforming FEM for evolutionary incompressible Navier-Stokes flows[J]. J. Numer. Math. 25(4), 249-276 (2017)
20. De Frutos, J., García-Archilla, B., Novo, J.: Fully Discrete Approximations to the Time-Dependent Navier-Stokes Equations with a Projection Method in Time and Grad-Div Stabilization[J]. J. Sci. Comput. 80(2), 1330-1368 (2019)
21. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces. Elsevier, Amsterdam (2003)
22. Cockburn, B., Gopalakrishnan, J., Lazarov, R.: Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems[J]. SIAM J. Numer. Anal. 47(2), 1319-1365 (2009)
23. Kirk, K.L.A., Rhebergen, S.: Analysis of a pressure-robust hybridized discontinuous Galerkin method for the stationary Navier-Stokes equations. J. Sci. Comput. 81(2), 881-897 (2019)
24. Rhebergen, S., Wells, G.N.: An embedded-hybridized discontinuous Galerkin finite element method for the Stokes equations. Comput. Methods Appl. Mech. Eng. 358, 112619 (2020)
25. Han, Y., Hou, Y.: Semirobust analysis of an H(div)-conforming DG method with semi-implicit timemarching for the evolutionary incompressible Navier-Stokes equations. IMA J. Numer. Anal. 42(2), 1568-1597 (2022)
26. Dallmann, H., Arndt, D., Lube, G.: Local projection stabilization for the Oseen problem. IMA J. Numer. Anal. 36(2), 796-823 (2016)
27. De Frutos, J., García-Archilla, B., John, V., et al.: Analysis of the grad-div stabilization for the timedependent Navier-Stokes equations with inf-sup stable finite elements. Adv. Comput. Math. 44(1), 195225 (2018)
28. De Frutos, J., García-Archilla, B., John, V., et al.: Grad-div stabilization for the evolutionary Oseen problem with inf-sup stable finite elements. J. Sci. Comput. 66(3), 991-1024 (2016)
29. Rhebergen, S., Wells, G.N.: A hybridizable discontinuous Galerkin method for the Navier-Stokes equations with pointwise divergence-free velocity field. J. Sci. Comput. 76(3), 1484-1501 (2018)
30. Han, Y., Hou, Y.: An embedded discontinuous Galerkin method for the Oseen equations. ESAIM: Math. Model. Numer. Anal. 55(5), 2349-2364 (2021)
31. Kirk, K., Çeşmelioğ̆lu, A., Rhebergen, S.: Convergence to weak solutions of a space-time hybridized discontinuous Galerkin method for the incompressible Navier-Stokes equations. Math. Comput. 92(339), 147-174 (2023)
32. Kirk, K., Horváth, T., Rhebergen, S.: Analysis of an exactly mass conserving space-time hybridized discontinuous Galerkin method for the time-dependent Navier-Stokes equations. Math. Comput. 92 (340), 525-556 (2023)
33. Horváth, T.L., Rhebergen, S.: An exactly mass conserving space-time embedded-hybridized discontinuous Galerkin method for the Navier-Stokes equations on moving domains. J. Comput. Phys. 417, 109577 (2020)
34. Guzmán, J., Shu, C.W., Sequeira, F.A.: H(div) conforming and DG methods for incompressible Euler's equations. IMA J. Numer. Anal. 37(4), 1733-1771 (2017)
35. Horváth, T.L., Rhebergen, S.: A locally conservative and energy-stable finite-element method for the Navier-Stokes problem on time-dependent domains[J]. Int. J. Numer. Meth. Fluids 89(12), 519-532 (2019)

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