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Research paper

Error analysis of a linear unconditionally energy-stable Leapfrog scheme for the Swift–Hohenberg equation☆

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ABSTRACT

In this work, we present an energy-stable linear scheme for the Swift–Hohenberg equation based on Leapfrog scheme. We prove rigorously that the scheme satisfies the energy dissipation property. We also prove that our scheme is second-order accurate in time. Moreover, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete scheme. Numerical results are presented to validate our theoretical analysis.

1. Introduction

The Swift–Hohenberg (SH) equation is a widely applied phase-field model and it was originally derived by Swift and Hohenberg [1] to describe Rayleigh–Bénard convection. Related applications can be found in complex pattern formation [2,3], complex fluids and biological tissues [4–6]. The SH equation is derived from the following free energy functional

$$E(u) = \int_{\Omega} \left(\frac{1}{2} u(1+\Delta)^2 u + \frac{\beta}{2} u^2 + F(u) \right) \mathrm{d}x, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^d$ $(d = 1,2,3)$, u is the density field, $F(u) = \frac{1}{4}u^4 - \frac{\epsilon+\beta}{2}u^2$, Δ is the Laplace operator. The SH equation is given by

$$u_t = -\frac{\delta E}{\delta u} = -((1+\Delta)^2 u + \beta u + f(u)), \tag{1.2}$$

with the periodic boundary conditions and initial conditions $u|_{t=0} = u^0$, where $\frac{\delta}{\delta u}$ denotes the variational derivative, $u_t = \frac{\partial u}{\partial t}, f(u) = F'(u) = u^3 - (\epsilon+\beta)u$. It is known that if $\epsilon \le 0$, the SH equation has the trivial solution only (cf. Theorem 9.1.1 in [7]), and hence we consider $\epsilon > 0$ in this paper. A classic example for the SH equation is convection of a thin layer of fluid heated from below for which we can think of the scalar quantity u as representing the temperature of the fluid in the mid plane. The parameter ϵ is the reduced Rayleigh number and is expressed as:

$$\epsilon = \frac{Ra - Ra_c}{Ra_c}$$

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where Ra_c is the critical Rayleigh number at which instability sets in and convection begin [8]. Hence, for $\epsilon > 0$, convection occurs. The free energy is nonincreasing in time:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\delta E}{\delta u}\frac{\partial u}{\partial t}\mathrm{d}x = -\int_\Omega (u_t)^2 \mathrm{d}x \le 0.$$

Here we study the numerical scheme of SH equation with periodic boundary conditions since that is used very frequently in numerical or analytical works of SH equation. If we choose other physical boundary conditions like Neumann type, the analysis is also true.

As a nonlinear fourth-order partial differential equation, the SH equation is difficult to be solved analytically. Hence, various numerical schemes have been proposed in recent years. To design a numerical scheme satisfying energy dissipation law, the linear terms are generally treated implicitly and the nonlinear terms are treated by different approaches. A very efficient approach is the convex splitting method [9]. Based on the convex splitting method, Lee presented a non-iterative scheme for the SH equation with quadratic–cubic nonlinearity without convergence analysis in [10]. Zhang and Ma constructed and analyzed a large time-stepping scheme for the SH equation in [11]. In [12], the authors proposed a second-order energy-stable numerical scheme for the SH equation and presented an optimal error estimate for the scheme. By applying the Crank–Nicolson scheme, a semi-implicit second-order method for the SH equation was given in [13]. In [14], an unconditionally energy-stable, second-order finite element scheme in the mixed formulation is given and analyzed for the SH equation. The convex splitting method is unconditionally energy-stable and uniquely solvable. However, to solve the fully discrete nonlinear systems, these methods generally require the use of an iteration. Hence, the computational costs are often high and the implementations are usually complicated. Another efficient approach is the stabilization method. By introducing artificial stabilization terms, one can alleviate the time step restriction and balance the explicit treatment of the nonlinear term, see [15]. In [16], the authors proposed a stabilized linear predictor–corrector scheme for the SH equation, they also proved rigorously that the scheme satisfies the energy dissipation law and is second-order accurate. In [17], a stabilized linear Crank–Nicolson scheme for the SH equation was proposed and analyzed. Efficiency and simplicity are the main advantages of the stabilization method. The operator splitting method is also an very powerful approach for solving the phase-field models. In [18], based on the operator splitting scheme, the first- and second-order Fourier spectral methods were presented for the SH equation. In [19], a new conservative SH equation was introduced and its first-order and second-order mass conservative operator splitting schemes were proposed. In [20], A fast explicit high-order operator splitting scheme was presented for the SH equation with a nonlocal nonlinearity. There are also various interesting linear approaches that attract the attention of many scholars, such as invariant energy quadratization (IEQ) scheme [21] and scalar auxiliary variable (SAV) scheme [22]. These approaches provide linear numerical schemes and satisfy unconditional energy stability based on a modified energy functional.

In this work, we design and analyze an unconditionally energy-stable linear Leapfrog scheme combined with IEQ approach for the SH equation. Although there exist some works about IEQ type or SAV type schemes for the SH equation, such as [23–28], almost all works only focus on the unconditional energy stability. In view of the absence of error analysis, the main goal of this paper is to derive the error analysis for a second-order IEQ scheme for the SH equation. In [29], Yang and Zhang gave the convergence analysis for the IEQ schemes for solving the Cahn–Hilliard and Allen–Cahn equations with general nonlinear potential, but the authors considered only time discrete schemes in their study and Remark 4.1 was given in their article to indicate, for the fully discrete IEQ scheme of the Cahn–Hilliard equations, their were not clear on how to derive the corresponding error analysis using Galerkin type approximations and this was a challenging work. In this work, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete IEQ scheme, which is not studied in [29]. In order to get optimal error estimate, some reasonable conditions about continuity and boundedness for the nonlinear terms are given. Unconditional energy stability and unique solvability are also rigorously proved. Numerical tests are presented to support our theoretical results.

The rest of the paper is organized as follows. In Section 2, we design the second-order semi-discrete linear energy-stable Leapfrog scheme and prove the scheme satisfies the energy dissipation property. In Section 3, we derive the error estimate of the semi-discrete scheme. In Section 4, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete linear Leapfrog scheme. In Section 5, numerical tests are provided to illustrate the accuracy and energy stability of the proposed scheme. In the end, some conclusions are presented in Section 6.

We introduce some notations which will be used in the analysis. We denote the spaces $L^p(\Omega)$ associated with the L^p norm $\|u\|_{L^p} := (\int_\Omega |u(x)|^p \mathrm{d}x)^{1/p}$. We also introduce the space $L^\infty(\Omega)$ with $\|v\|_{L^\infty} = \sup_{x\in\Omega}|v(x)|$. $W^{k,p}(\Omega)$ stands for the standard Sobolev spaces equipped with the standard Sobolev norms $\|\cdot\|_{k,p}$. For $p = 2$, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and the corresponding norm is $\|\cdot\|_k$. The space $W^{k,p}(0, T; V)$ represents the $W^{k,p}$ space on the interval $(0, T)$ with values in the function space V. We denote by $(\cdot,\cdot)$ the inner product in L^2 and $\|\cdot\|$ the norm in L^2. Let $x \lesssim y$ denote there is a positive constant C that is independent on time step size τ and n such that $x \le Cy$. Let $K > 0$ be any positive integer, T be the final time and set

$$\tau = T/K,\ t_n = n\tau,\ \text{for } n \le K,$$

let u^n be the numerical approximation of $u(t_n)$.

2. The semi-discrete linear Leapfrog scheme for the SH equation

In this section, we develop a second-order semi-discrete time-stepping numerical scheme to solve the SH equation based on the IEQ method. For this purpose, we introduce following auxiliary variable

$$W = \sqrt{F(u) + D},$$

D is a positive constant to make $F(u) + D > 0$ and ensure $W = \sqrt{F(u) + D}$ is well-defined for any $u \in \mathbb{R}$. Since

$$\frac{1}{4}u^4 - \frac{\epsilon + \beta}{2}u^2 + \frac{(\epsilon + \beta)^2}{4} = \left(\frac{1}{2}u^2 - \frac{\epsilon + \beta}{2}\right)^2 \geq 0$$

and $F(u) = \frac{1}{4}u^4 - \frac{\epsilon+\beta}{2}u^2 \geq -\frac{(\epsilon+\beta)^2}{4}$, $F(u)$ is bounded from below. We are able to choose a positive constant D such that $D > \frac{(\epsilon+\beta)^2}{4}$. Thus, the energy functional (1.1) becomes

$$\hat{E}(u, W) = \int_{\Omega} \left(\frac{1}{2}u(1 + \Delta)^2 u + \frac{\beta}{2}u^2 + W^2 - D\right) \mathrm{d}x. \tag{2.1}$$

We denote $H(u) = 2\frac{\mathrm{d}}{\mathrm{d}u}W(u) = \frac{f(u)}{\sqrt{F(u)+D}}$ and we have the following equivalent PDE system:

$$u_t + (1 + \Delta)^2 u + \beta u + H(u)W = 0, \tag{2.2}$$

$$W_t = \frac{1}{2}H(u)u_t, \tag{2.3}$$

with the periodic boundary conditions and initial conditions

$$u|_{t=0} = u^0, \quad W|_{t=0} = \sqrt{F(u^0) + D}.$$

The equivalent PDE system still satisfies the unconditional energy stability. By taking the L^2 inner product of (2.2) with u_t and taking the L^2 inner product of (2.3) with $2W$, summing up the resulting equations, we have the following unconditional energy stability of the equivalent PDE system (2.2)–(2.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}(u, W) = -\|u_t\|^2 \leq 0.$$

We design the second-order semi-discrete scheme based on the Leapfrog scheme as follows,

$$\frac{u^{n+1} - u^{n-1}}{2\tau} + H(u^n)\frac{W^{n+1} + W^{n-1}}{2} + (1 + \Delta)^2\frac{u^{n+1} + u^{n-1}}{2} + \beta\frac{u^{n+1} + u^{n-1}}{2} = 0, \tag{2.4}$$

$$W^{n+1} - W^{n-1} = \frac{1}{2}H(u^n)(u^{n+1} - u^{n-1}). \tag{2.5}$$

Because we tackle the nonlinear coefficient $H(u)$ of the variable W explicitly, we can write Eq. (2.5) as below:

$$W^{n+1} = \frac{1}{2}H(u^n)u^{n+1} + r_1^n, \tag{2.6}$$

with $r_1^n = W^{n-1} - \frac{1}{2}H(u^n)u^{n-1}$. Thus, (2.4) can be written as a linear system as follow

$$\gamma u^{n+1} = -G(u^{n+1}) + r_2^{n-1} - r_3^n, \tag{2.7}$$

with $\gamma = \frac{1}{2\tau}$, $r_2^{n-1} = \frac{1}{2\tau}u^{n-1}$, $G(u^{n+1}) = \frac{1}{4}H(u^n)H(u^n)u^{n+1} + \frac{1}{2}(1+\Delta)^2u^{n+1} + \frac{\beta}{2}u^{n+1}$, $r_3^n = \frac{1}{2}H(u^n)r_1^n + \frac{1}{2}H(u^n)W^{n-1} + \frac{1}{2}(1+\Delta)^2u^{n-1} + \frac{\beta}{2}u^{n-1}$. Actually, u^{n+1} can be solved directly from (2.7). After we get u^{n+1}, W^{n+1} is naturally obtained in (2.6). Moreover, we note that

$$(G(u), v) = \frac{1}{4}(H(u^n)u, H(u^n)v) + \frac{1}{2}((1 + \Delta)u, (1 + \Delta)v) + \frac{\beta}{2}(u, v),$$

if v satisfies the identical boundary conditions as u. Thus, the linear operator $G(\cdot)$ is symmetric. Furthermore, for each u, we derive

$$(G(u), u) = \frac{1}{4}\|H(u^n)u\|^2 + \frac{1}{2}\|(1 + \Delta)u\|^2 + \frac{\beta}{2}\|u\|^2 \geq 0,$$

where "=" is valid if and only if $u \equiv 0$.

Remark 2.1. For the SH equation with quadratic–cubic nonlinearity, that is, the nonlinear term is $f(u) = u^3 - gu^2 - (\epsilon+\beta)u$, $F(u) = \frac{1}{4}u^4 - \frac{g}{3}u^3 - \frac{\epsilon+\beta}{2}u^2$, the current research is also applicable. In this paper, we focus on the SH equation with $\epsilon > 0$,

$g > 0$ and $\beta > 0$, the nonlinear term $F(u)$ has double wells with two local minimal values at $u_\pm = \frac{g\pm\sqrt{g^2+4(\epsilon+\beta)}}{2}$ such that $f(u_\pm) = F'(u_\pm) = 0$, and

$$F(u) \geq \min\{F(u_\pm)\} =: D(g, \epsilon, \beta),$$

where $D(g, \epsilon, \beta)$ is a constant depends on g, ϵ and β only and a similar analysis can be found in [23]. $F(u)$ is bounded from below and we can find a $D_0 > -D(g, \epsilon, \beta)$ such that $F(u) + D_0 > 0$. The analysis in our paper can also be carried out.

Remark 2.2. A symmetric reformulation was introduced in [30], Section 4.3 and [26], Eq. (2.1) for the SH equation. In our work, the SH equation can be written as the following symmetric reformulation:

$$\begin{aligned} &u_t + \mathcal{L}q + \beta u + f(u) = 0, \\ &q = \mathcal{L}u, \end{aligned}$$

where $\mathcal{L} = (\Delta + 1)$. By the IEQ approach, we have the following equivalent system:

$$\begin{aligned} &u_t + \mathcal{L}q + \beta u + H(u)W = 0, \\ &q = \mathcal{L}u, \\ &W_t = \frac{1}{2}H(u)u_t. \end{aligned}$$

The proposed scheme is still applicable to the reformulated equation.

Remark 2.3. The second-order scheme (2.4)–(2.5) involves three time levels and (u^{n+1}, W^{n+1}) can be updated after we obtain the initial values (u^0, W^0) and (u^1, W^1). Obviously, (u^0, W^0) is given by the initial conditions. To get the second-order time accuracy of the scheme, we can calculate $(\tilde{u}^1, \tilde{W}^1)$ by using the first-order scheme (2.8)–(2.9)

$$\frac{\tilde{u}^{n+1} - u^n}{\tau} + (1 + \Delta)^2\tilde{u}^{n+1} + \beta\tilde{u}^{n+1} + H(u^n)\tilde{W}^{n+1} = 0, \tag{2.8}$$

$$\tilde{W}^{n+1} - W^n = \frac{1}{2}H(u^n)(\tilde{u}^{n+1} - u^n). \tag{2.9}$$

then apply the following corrector scheme to get (u^1, W^1),

$$\frac{u^1 - u^0}{\tau} + H(\tilde{u}^1)\frac{W^1 + W^0}{2} + (1 + \Delta)^2\frac{u^1 + u^0}{2} + \beta\frac{u^1 + u^0}{2} = 0, \tag{2.10}$$

$$W^1 - W^0 = \frac{1}{2}H(\tilde{u}^1)(u^1 - u^0), \tag{2.11}$$

with

$$H(\tilde{u}^1) = \frac{f(\tilde{u}^1)}{\sqrt{F(\tilde{u}^1) + D}}.$$

We now prove the well-posedness of the system (2.4)–(2.5) (or (2.7)) as follows.

Theorem 2.1. *The linear system (2.7) can be solved uniquely, and the linear operator is a symmetric positive definite operator.*

Proof. From (2.7), it is obvious that u^{n+1} solves the following system with unknown u,

$$\gamma u + G(u) = r_2^{n-1} - r_3^n. \tag{2.12}$$

Let us denote the above linear system (2.12) by $\mathbb{T}u = y$.

1. For each u_1 and u_2 in $H^2(\Omega)$ with the periodic boundary conditions, applying integration by parts, we obtain

$$\begin{aligned} (\mathbb{T}(u_1), u_2) &= \gamma(u_1, u_2) + (G(u_1), u_2) \\ &\leq C_1(\|u_1\|\|u_2\| + \|\nabla u_1\|\|\nabla u_2\| + \|\Delta u_1\|\|\Delta u_2\|) \\ &\leq C_1\|u_1\|_2\|u_2\|_2. \end{aligned} \tag{2.13}$$

Hence, the boundedness of the bilinear form $(\mathbb{T}(\cdot), \cdot)$ is proved.

2. For each u in $H^2(\Omega)$ with the periodic boundary conditions, using the H^2-regularity of the second-order elliptic equation $-\Delta u = \bar{f}$ (cf. Chapter 6 of [31]), we have

$$\begin{aligned} \|u\|_2 &\leq C(\|\bar{f}\| + \|u\|) \\ &= C(\|\Delta u\| + \|u\|) \end{aligned}$$

$$= C(\|(1+\Delta)u - u\| + \|u\|)$$
$$= C(\|(1+\Delta)u\| + 2\|u\|),$$

it is not hard to obtain that

$$(\mathbb{T}(u), u) = \gamma\|u\|^2 + \frac{1}{4}\|H(u^n)u\|^2 + \frac{1}{2}\|(1+\Delta)u\|^2 + \frac{\beta}{2}\|u\|^2 \geq C_2\|u\|_2^2. \tag{2.14}$$

Consequently, the coercivity of bilinear form $(\mathbb{T}(\cdot), \cdot)$ is proved. Here, C_1 and C_2 are positive constants.

In this way, the well-posedness of the system $\mathbb{T}u = y$ is obtained from the Lax–Milgram theorem, that is, the linear system (2.12) has an unique solution in $H^2(\Omega)$. Moreover, we can easily obtain

$$(\mathbb{T}(u_1), u_2) = (u_1, \mathbb{T}(u_2)). \tag{2.15}$$

From this, $\mathbb{T}$ is symmetric. At the same time, the positive definiteness of $\mathbb{T}$ comes from coercivity in (2.14). Hence, $\mathbb{T}$ is a symmetric positive definite operator. □

Theorem 2.2. *The scheme (2.4)–(2.5) (or (2.7)) satisfies the discrete energy dissipation law as follows:*

$$E_{LF}^{n+1,n} - E_{LF}^{n,n-1} = -\frac{1}{4\tau}\|u^{n+1} - u^{n-1}\|^2 \leq 0, \tag{2.16}$$

where

$$E_{LF}^{n+1,n} = \frac{1}{2}(\|W^{n+1}\|^2 + \|W^n\|^2) + \frac{1}{4}(\|(1+\Delta)u^{n+1}\|^2 + \|(1+\Delta)u^n\|^2) + \frac{\beta}{4}(\|u^{n+1}\|^2 + \|u^n\|^2) - D|\Omega|.$$

Proof. First of all, taking the L^2-inner product of $\frac{1}{2}(u^{n+1} - u^{n-1})$ with (2.4), and using the following identity

$$(x - 2y + z, x - z) = (x - y)^2 - (y - z)^2,$$

we have

$$-\frac{1}{4\tau}\|u^{n+1} - u^{n-1}\|^2 = \left(H(u^n)\frac{W^{n+1} + W^{n-1}}{2}, \frac{u^{n+1} - u^{n-1}}{2}\right) + \frac{1}{4}(\|(1+\Delta)u^{n+1}\|^2 - \|(1+\Delta)u^{n-1}\|^2)$$
$$+ \frac{\beta}{4}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2). \tag{2.17}$$

Secondly, taking the L^2-inner product of $\frac{1}{2}(W^{n+1} + W^{n-1})$ with (2.5), we derive

$$\frac{1}{2}\|W^{n+1}\|^2 - \frac{1}{2}\|W^{n-1}\|^2 = \frac{1}{4}\left(H(u^n)(u^{n+1} - u^{n-1}), W^{n+1} + W^{n-1}\right). \tag{2.18}$$

In the end, combining (2.17) and (2.18), we derive

$$\frac{1}{2}(\|W^{n+1}\|^2 + \|W^n\|^2) - \frac{1}{2}(\|W^n\|^2 + \|W^{n-1}\|^2) + \frac{1}{4}(\|(1+\Delta)u^{n+1}\|^2 + \|(1+\Delta)u^n\|^2$$
$$- \|(1+\Delta)u^n\|^2 - \|(1+\Delta)u^{n-1}\|^2) + \frac{\beta}{4}(\|u^{n+1}\|^2 + \|u^n\|^2 - \|u^n\|^2 - \|u^{n-1}\|^2) = -\frac{1}{4\tau}\|u^{n+1} - u^{n-1}\|^2,$$

which implies the desired result (2.16) is hold. This completes the proof. □

3. Error analysis of the semi-discrete scheme

We now give the error analysis for the second-order scheme (2.4)–(2.5). First, we formulate a truncation form for the SH system (2.4)–(2.5) as follows:

$$\frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} + (1+\Delta)^2\frac{u(t_{n+1}) + u(t_{n-1})}{2} + \beta\frac{u(t_{n+1}) + u(t_{n-1})}{2} + H(u(t_n))\frac{W(t_{n+1}) + W(t_{n-1})}{2} = G_u^n, \tag{3.1}$$

$$W(t_{n+1}) - W(t_{n-1}) = \frac{1}{2}H(u(t_n))(u(t_{n+1}) - u(t_{n-1})) + 2\tau G_W^n, \tag{3.2}$$

where

$$G_u^n = \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u_t(t_n) + (1+\Delta)^2\frac{u(t_{n+1}) + u(t_{n-1})}{2} - (1+\Delta)^2 u(t_n)$$
$$+ \beta\frac{u(t_{n+1}) + u(t_{n-1})}{2} - \beta u(t_n) + H(u(t_n))\frac{W(t_{n+1}) + W(t_{n-1})}{2} - H(u(t_n))W(t_n),$$
$$G_W^n = \frac{W(t_{n+1}) - W(t_{n-1})}{2\tau} - W_t(t_n) + \frac{1}{2}H(u(t_n))u_t(t_n) - \frac{1}{2}H(u(t_n))\frac{u(t_{n+1}) - u(t_{n-1})}{2\tau}.$$

To derive the error estimate, we assume that the analytic solution of the system (2.2)–(2.3) satisfies the following regularity conditions

$$u \in L^\infty(0,T;H^4(\Omega)),\quad W \in L^\infty(0,T;L^\infty(\Omega)),\quad u_t \in L^\infty(0,T;L^\infty(\Omega)), \tag{3.3}$$

$$W_{tt} \in L^2(0,T;L^2(\Omega)),\quad u_{tt} \in L^2(0,T;H^4(\Omega)),\quad u_{ttt}, W_{ttt} \in L^2(0,T;L^2(\Omega)). \tag{3.4}$$

We define the error functions for $n = 0, 1, 2, \ldots, K$ as

$$e_u^n = u(t_n) - u^n,\ e_H^n = H(u(t_n)) - H(u^n),\ e_W^n = W(t_n) - W^n.$$

Subtracting (2.4)–(2.5) from (3.1)–(3.2), respectively, we get the following error equations for $n \geq 1$,

$$\frac{e_u^{n+1} - e_u^{n-1}}{2\tau} + (1+\Delta)^2\frac{e_u^{n+1} + e_u^{n-1}}{2} + \beta\frac{e_u^{n+1} + e_u^{n-1}}{2} + e_H^n\frac{W(t_{n+1}) + W(t_{n-1})}{2} + H(u^n)\frac{e_W^{n+1} + e_W^{n-1}}{2} = G_u^n, \tag{3.5}$$

$$e_W^{n+1} - e_W^{n-1} = \frac{1}{2}e_H^n(u(t_{n+1}) - u(t_{n-1})) + \frac{1}{2}H(u^n)(e_u^{n+1} - e_u^{n-1}) + 2\tau G_W^n. \tag{3.6}$$

Before further investigation, we introduce the following lemmas.

Lemma 3.1 ([29]). *Suppose (i) $F(x)$ is uniformly bounded from below: $F(x) > -D$ for any $x \in \mathbb{R}$; (ii) $F(x) \in C^3(\mathbb{R})$; and (iii) there exists a positive constant D_0 such that*

$$\max_{0\leq n\leq K}(\|u(t_n)\|_{L^\infty}, \|\nabla u(t_n)\|_{L^3}, \|u^n\|_{L^\infty}) \leq D_0,$$

then we have

$$\|H(u(t_n)) - H(u^n)\| \leq \hat{C}_0\|u(t_n) - u^n\|,$$
$$\|\nabla H(u(t_n)) - \nabla H(u^n)\| \leq \hat{D}_0(\|u(t_n) - u^n\| + \|\nabla(u(t_n) - u^n)\|),$$

for $n \leq K$, where $\hat{C}_0$ and $\hat{D}_0$ are positive constants dependent on Ω, D_0 and D.

Lemma 3.2. *Under the regularity conditions (3.3)–(3.4), the truncation errors satisfy*

$$\tau\sum_{n=1}^{K-1}(\|G_u^n\|^2 + \|G_W^n\|^2) \lesssim \tau^4.$$

Proof. Since the proof is rather straight forward, we omit the details. □

Let $\nu = \max_{0\leq t\leq T}\|u(t)\|_{L^\infty} + 1$, we now prove the L^∞ stability of solution u^n.

Lemma 3.3. *Under the regularity conditions (3.3)–(3.4), there exists a positive constant r (which is given in the proof), such that when $\tau \leq r$, the numerical solution u^n of (2.4)–(2.5) satisfies the following uniformly boundedness*

$$\|u^n\|_{L^\infty} \leq \nu,\quad n = 0, 1, 2, \ldots, K.$$

Proof. We prove this lemma by mathematical induction. Because $u^0 = u(t_0)$, $\|u^0\|_{L^\infty} \leq \nu$ holds naturally. Assuming that $\|u^n\|_{L^\infty} \leq \nu$ is true for $0 \leq n \leq M$, we derive $\|u^{M+1}\|_{L^\infty} \leq \nu$ is also true by the following two steps.

(i) Taking the L^2-inner product of (3.5) with $e_u^{n+1} - e_u^{n-1}$, we get

$$\frac{1}{2\tau}\|e_u^{n+1} - e_u^{n-1}\|^2 + \frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2 - \|(1+\Delta)e_u^{n-1}\|^2) + \frac{\beta}{2}(\|e_u^{n+1}\|^2 - \|e_u^{n-1}\|^2)$$
$$+ \left(e_H^n\frac{W(t_{n+1}) + W(t_{n-1})}{2}, e_u^{n+1} - e_u^{n-1}\right) + \left(H(u^n)\frac{e_W^{n+1} + e_W^{n-1}}{2}, e_u^{n+1} - e_u^{n-1}\right) = (G_u^n, e_u^{n+1} - e_u^{n-1}). \tag{3.7}$$

Taking the L^2-inner product of (3.6) with $e_W^{n+1} + e_W^{n-1}$, we have

$$\|e_W^{n+1}\|^2 - \|e_W^{n-1}\|^2 = \frac{1}{2}(e_H^n(u(t_{n+1}) - u(t_{n-1})), e_W^{n+1} + e_W^{n-1})$$
$$+ \frac{1}{2}(H(u^n)(e_u^{n+1} - e_u^{n-1}), e_W^{n+1} + e_W^{n-1}) + 2\tau(G_W^n, e_W^{n+1} + e_W^{n-1}). \tag{3.8}$$

Taking the L^2-inner product of (3.5) with $\tau(e_u^{n+1}+e_u^{n-1})$, we get

$$\frac{1}{2}(\|e_u^{n+1}\|^2-\|e_u^{n-1}\|^2)+\frac{1}{2}\tau\|(1+\Delta)(e_u^{n+1}+e_u^{n-1})\|^2+\frac{1}{2}\beta\tau\|e_u^{n+1}+e_u^{n-1}\|^2$$
$$+\ \tau\left(e_H^n\frac{W(t_{n+1})+W(t_{n-1})}{2},e_u^{n+1}+e_u^{n-1}\right)+\tau\left(H(u^n)\frac{e_W^{n+1}+e_W^{n-1}}{2},e_u^{n+1}+e_u^{n-1}\right)=\tau(G_u^n,e_u^{n+1}+e_u^{n-1}). \tag{3.9}$$

Combining (3.7)–(3.9), we derive

$$\begin{aligned}&\frac{1}{2}(\|e_u^{n+1}\|^2-\|e_u^{n-1}\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2-\|(1+\Delta)e_u^{n-1}\|^2)+\|e_W^{n+1}\|^2-\|e_W^{n-1}\|^2\\&+\frac{\beta}{2}(\|e_u^{n+1}\|^2-\|e_u^{n-1}\|^2)+\frac{1}{2}\beta\tau\|e_u^{n+1}+e_u^{n-1}\|^2+\frac{1}{2\tau}\|e_u^{n+1}-e_u^{n-1}\|^2+\frac{1}{2}\tau\|(1+\Delta)(e_u^{n+1}+e_u^{n-1})\|^2\\=&-\left(e_H^n\frac{W(t_{n+1})+W(t_{n-1})}{2},e_u^{n+1}-e_u^{n-1}\right)+(G_u^n,e_u^{n+1}-e_u^{n-1})+\frac{1}{2}(e_H^n(u(t_{n+1})-u(t_{n-1})),e_W^{n+1}+e_W^{n-1})\\&+2\tau(G_W^n,e_W^{n+1}+e_W^{n-1})-\tau\left(e_H^n\frac{W(t_{n+1})+W(t_{n-1})}{2},e_u^{n+1}+e_u^{n-1}\right)-\tau\left(H(u^n)\frac{e_W^{n+1}+e_W^{n-1}}{2},e_u^{n+1}+e_u^{n-1}\right)\\&+\tau(G_u^n,e_u^{n+1}+e_u^{n-1}).\end{aligned} \tag{3.10}$$

Using Lemma 3.1 and regularity conditions (3.3)–(3.4), we estimate each terms on the right hand side of (3.10).

$$\begin{aligned}\left|\left(e_H^n\frac{W(t_{n+1})+W(t_{n-1})}{2},e_u^{n+1}-e_u^{n-1}\right)\right|&\le\|e_H^n\|(\|W(t_{n+1})\|_{L^\infty}+\|W(t_{n-1})\|_{L^\infty})\|e_u^{n+1}-e_u^{n-1}\|\\&\lesssim\|e_H^n\|\|e_u^{n+1}-e_u^{n-1}\|\\&\lesssim\frac{1}{4\tau}\|e_u^{n+1}-e_u^{n-1}\|^2+\tau\|e_H^n\|^2\\&\lesssim\frac{1}{4\tau}\|e_u^{n+1}-e_u^{n-1}\|^2+\tau\|e_u^n\|^2.\end{aligned} \tag{3.11}$$

Since

$$\begin{aligned}\|\nabla e_u^n\|^2=|(\nabla e_u^n,\nabla e_u^n)|&=|(e_u^n,\Delta e_u^n)|\lesssim\|e_u^n\|^2+\|\Delta e_u^n\|^2\\&=\|e_u^n\|^2+\|(1+\Delta)e_u^n-e_u^n\|^2\le\|e_u^n\|^2+(\|(1+\Delta)e_u^n\|+\|e_u^n\|)^2\\&\lesssim\|e_u^n\|^2+\|(1+\Delta)e_u^n\|^2,\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}|(e_H^n(u(t_{n+1})-u(t_{n-1})),e_W^{n+1}+e_W^{n-1})|\\&\le\|e_{H^n}\|_{L^4}\|u(t_{n+1})-u(t_{n-1})\|_{L^4}\|e_W^{n+1}+e_W^{n-1}\|\\&\lesssim\|e_H^n\|_{L^4}\|e_W^{n+1}+e_W^{n-1}\|\\&\lesssim\tau\|e_W^{n+1}+e_W^{n-1}\|^2+\tau\|e_H^n\|_{L^4}^2\\&\lesssim\tau(\|e_W^{n+1}\|^2+\|e_W^{n-1}\|^2)+\tau(\|e_H^n\|^2+\|\nabla e_H^n\|^2)\\&\lesssim\tau(\|e_W^{n+1}\|^2+\|e_W^{n-1}\|^2)+\tau(\|e_u^n\|^2+\|(1+\Delta)e_u^n\|^2).\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\tau\left|\left(e_H^n\frac{W(t_{n+1})+W(t_{n-1})}{2},e_u^{n+1}+e_u^{n-1}\right)\right|\\&\lesssim\tau\|e_H^n\|(\|W(t_{n+1})\|_{L^\infty}+\|W(t_{n-1})\|_{L^\infty})\|e_u^{n+1}+e_u^{n-1}\|\\&\lesssim\tau\|e_H^n\|\|e_u^{n+1}+e_u^{n-1}\|\\&\lesssim\tau\|e_H^n\|^2+\tau\|e_u^{n+1}+e_u^{n-1}\|^2\\&\lesssim\tau\|e_u^n\|^2+\tau\|e_u^{n+1}+e_u^{n-1}\|^2.\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\tau\left|\left(H(u^n)\frac{e_W^{n+1}+e_W^{n-1}}{2},e_u^{n+1}+e_u^{n-1}\right)\right|\\&\lesssim\tau\|H(u^n)\|_{L^\infty}\left\|\frac{e_W^{n+1}+e_W^{n-1}}{2}\right\|\|e_u^{n+1}+e_u^{n-1}\|\end{aligned}$$

$$\lesssim \tau\|e_W^{n+1}+e_W^{n-1}\|^2+\tau\|e_u^{n+1}+e_u^{n-1}\|^2$$
$$\lesssim \tau(\|e_W^{n+1}\|^2+\|e_W^{n-1}\|^2+\|e_u^{n+1}\|^2+\|e_u^{n-1}\|^2). \tag{3.14}$$

$$\begin{aligned}&\tau|2(G_W^n, e_W^{n+1}+e_W^{n-1})+(G_u^n, e_u^{n+1}+e_u^{n-1})|\\ &\lesssim \tau(\|G_W^n\|\|e_W^{n+1}+e_W^{n-1}\|+\|G_u^n\|\|e_u^{n+1}+e_u^{n-1}\|)\\ &\lesssim \tau(\|G_W^n\|^2+\|e_W^{n+1}+e_W^{n-1}\|^2+\|G_u^n\|^2+\|e_u^{n+1}+e_u^{n-1}\|^2)\\ &\lesssim \tau(\|G_W^n\|^2+\|G_u^n\|^2+\|e_W^{n+1}\|^2+\|e_W^{n-1}\|^2+\|e_u^{n+1}\|^2+\|e_u^{n-1}\|^2).\end{aligned} \tag{3.15}$$

$$|(G_u^n, e_u^{n+1}-e_u^{n-1})| \le \|G_u^n\|\|e_u^{n+1}-e_u^{n-1}\| \le \frac{1}{4\tau}\|e_u^{n+1}-e_u^{n-1}\|^2+\tau\|G_u^n\|^2. \tag{3.16}$$

Combining (3.11)–(3.16) with (3.10) and dropping some positive terms, we obtain

$$\begin{aligned}&\frac{1+\beta}{2}(\|e_u^{n+1}\|^2-\|e_u^{n-1}\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{n+1}\|^2-\|(1+\Delta)e_u^{n-1}\|^2)+\|e_W^{n+1}\|^2-\|e_W^{n-1}\|^2\\ &\lesssim \tau(\|e_u^{n+1}\|^2+\|e_u^n\|^2+\|e_u^{n-1}\|^2+\|(1+\Delta)e_u^n\|^2+\|e_W^{n+1}\|^2+\|e_W^{n-1}\|^2+\|G_W^n\|^2+\|G_u^n\|^2).\end{aligned}$$

Summing up for n from 0 to m ($m \le M$) and using Lemma 3.2, we have

$$\begin{aligned}&\frac{1+\beta}{2}(\|e_u^{m+1}\|^2+\|e_u^m\|^2)+\frac{1}{2}(\|(1+\Delta)e_u^{m+1}\|^2+\|(1+\Delta)e_u^m\|^2)+\|e_W^{m+1}\|^2+\|e_W^m\|^2\\ &\lesssim \tau\sum_{n=0}^{m+1}(\|e_u^n\|^2+\|(1+\Delta)e_u^n\|^2+\|e_W^n\|^2)+\tau^4.\end{aligned}$$

Applying the Grönwall's inequality, there exist two positive constants r_1 and r_2 such that when $\tau \le r_1$,

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2 \le r_2\tau^4. \tag{3.17}$$

(ii) Since

$$\begin{aligned}\|e_u^{M+1}\|_1^2 &= \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2\\ &\lesssim r_2\tau^4,\end{aligned}$$

$$\begin{aligned}\|e_u^{M+1}\|_2^2 &= \|e_u^{M+1}\|^2+\|\nabla e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|\Delta e_u^{M+1}\|^2\\ &\lesssim \|e_u^{M+1}\|^2+\|(1+\Delta)e_u^{M+1}\|^2\\ &\lesssim r_2\tau^4,\end{aligned}$$

we have

$$\begin{aligned}\|u^{M+1}\|_{L^\infty} &\le \|e_u^{M+1}\|_{L^\infty}+\|u(t_{M+1})\|_{L^\infty}\\ &\le C_\Omega\|e_u^{M+1}\|_1^{\frac{1}{2}}\|e_u^{M+1}\|_2^{\frac{1}{2}}+\|u(t_{M+1})\|_{L^\infty}\\ &\le C_\Omega\sqrt{r_2}\tau^2+\|u(t_{M+1})\|_{L^\infty} \le \nu,\end{aligned}$$

if $\tau \le \frac{1}{\sqrt{C_\Omega\sqrt{r_2}}}$. Thus the proof is completed by setting $r=\min\{r_1, \frac{1}{\sqrt{C_\Omega\sqrt{r_2}}}\}$. □

Theorem 3.1. *Under the regularity conditions (3.3)–(3.4), the numerical solution u^n of (2.4)–(2.5) satisfies the following estimate:*

$$\|e_u^{m+1}\|^2+\|(1+\Delta)e_u^{m+1}\|^2+\|e_W^{m+1}\|^2 \lesssim \tau^4, \quad 0 \le m \le K-1. \tag{3.18}$$

Proof. If $\tau \le r$, we have $\|u^n\|_{L^\infty} \le \nu$ for $0 \le n \le K$. Hence, following the proof of Lemma 3.3, we get the result (3.18). □

4. The fully discrete scheme and its error analysis

In this section, we adopt a spectral-Galerkin approximation for the spatial variables and establish error estimates for the fully discrete linear Leapfrog scheme. Let $\xi=(1+\Delta)u$, the system (2.2)–(2.3) can be rewritten as

$$u_t+(1+\Delta)\xi+\beta u+H(u)W=0, \tag{4.1}$$

$$\xi = (1+\Delta)u, \tag{4.2}$$

$$W_t = \frac{1}{2}H(u)u_t, \tag{4.3}$$

with $u(t_0) = u^0$, $\xi(t_0) = \xi^0 := (1+\Delta)u^0$ and $W(t_0) = W^0 := \sqrt{F(u^0)+D}$. The weak form of the above system (4.1)–(4.3) is

$$(u_t, \psi) + (\xi, \psi) - (\nabla\xi, \nabla\psi) + \beta(u, \psi) + (H(u)W, \psi) = 0, \ \forall\ \psi \in H^1(\Omega) \tag{4.4}$$

$$(\xi, v) = (u, v) - (\nabla u, \nabla v), \ \forall\ v \in H^1(\Omega) \tag{4.5}$$

$$(W_t, \zeta) = \frac{1}{2}(H(u)u_t, \zeta), \ \forall\ \zeta \in L^2(\Omega). \tag{4.6}$$

We denote by V_N the space of polynomials of degree $\leq N$ in each direction, and for any $\varphi \in H^k(\Omega)$, we define a projection $\Pi_N : H^k(\Omega) \to V_N$ by

$$(\Pi_N\varphi - \varphi, 1) = 0, \ (\nabla(\Pi_N\varphi - \varphi), \nabla\psi) = 0, \ \forall\ \psi \in V_N. \tag{4.7}$$

It is well known that the following estimate holds [32]:

$$\|\varphi - \Pi_N\varphi\|_s \lesssim N^{s-k}\|\varphi\|_k, \ s = 0, 1, \ \forall\ \varphi \in H^k(\Omega), \ k \geq 1. \tag{4.8}$$

Let $L_0^2(\Omega) = \{v \in L^2(\Omega) : (v, 1) = 0\}$. The discrete Laplacian $\Delta_N : V_N \cap L_0^2 \to V_N \cap L_0^2$ is defined as follows: for any $\psi_N \in V_N \cap L_0^2$, let $\Delta_N\psi_N$ be the unique solution to

$$(\Delta_N\psi_N, \chi) = -(\nabla\psi_N, \nabla\chi), \ \forall\ \chi \in V_N. \tag{4.9}$$

The fully discrete form of (4.1)–(4.3) is

$$\frac{u_N^{n+1} - u_N^{n-1}}{2\tau} + (1+\Delta_N)\frac{\xi_N^{n+1} + \xi_N^{n-1}}{2} + \beta\frac{u_N^{n+1} + u_N^{n-1}}{2} + H(u_N^n)\frac{W_N^{n+1} + W_N^{n-1}}{2} = 0, \tag{4.10}$$

$$\xi_N^n = (1+\Delta_N)u_N^n, \tag{4.11}$$

$$\frac{W_N^{n+1} - W_N^{n-1}}{2\tau} = \frac{1}{2}H(u_N^n)\frac{u_N^{n+1} - u_N^{n-1}}{2\tau}. \tag{4.12}$$

The spectral-Galerkin method for the scheme (4.10)–(4.12) reads: given $u_N^0 = \Pi_N u^0$, $\xi_N^0 = \Pi_N\xi^0$ and $W_N^0 = \Pi_N W^0$, find $u_N^{n+1} \in V_N$ such that

$$\left(\frac{u_N^{n+1} - u_N^{n-1}}{2\tau}, \psi\right) + \left(\frac{\xi_N^{n+1} + \xi_N^{n-1}}{2}, \psi\right) - \left(\frac{\nabla\xi_N^{n+1} + \nabla\xi_N^{n-1}}{2}, \nabla\psi\right) + \beta\left(\frac{u_N^{n+1} + u_N^{n-1}}{2}, \psi\right)$$
$$+ \left(H(u_N^n)\frac{W_N^{n+1} + W_N^{n-1}}{2}, \psi\right) = 0, \ \forall\ \psi \in V_N \tag{4.13}$$

$$(\xi_N^n, v) = (u_N^n, v) - (\nabla u_N^n, \nabla v), \ \forall\ v \in V_N, \tag{4.14}$$

$$\left(\frac{W_N^{n+1} - W_N^{n-1}}{2\tau}, \zeta\right) = \frac{1}{2}\left(H(u_N^n)\frac{u_N^{n+1} - u_N^{n-1}}{2\tau}, \zeta\right), \ \forall\ \zeta \in V_N. \tag{4.15}$$

Remark 4.1. Since the above scheme involves three time levels, we need to apply the spectral-Galerkin method to the initialization step (2.8)–(2.11) to calculate (u_N^1, ξ_N^1, W_N^1) and then start the above scheme.

Using (4.11) and the same proof as Theorem 2.2, we can obtain the following theorem.

Theorem 4.1. *The fully scheme (4.10)–(4.12) satisfies the discrete energy dissipation law as follows*

$$\mathcal{E}_{LF}^{n+1,n} - \mathcal{E}_{LF}^{n,n-1} = -\frac{1}{4\tau}\|u_N^{n+1} - u_N^{n-1}\|^2, \tag{4.16}$$

where

$$\mathcal{E}_{LF}^{n+1,n} = \frac{1}{2}(\|W_N^{n+1}\|^2 + \|W_N^n\|^2) + \frac{1}{4}(\|\xi_N^{n+1}\|^2 + \|\xi_N^n\|^2) + \frac{\beta}{4}(\|u_N^{n+1}\|^2 + \|u_N^n\|^2) - D|\Omega|.$$

In this work, we assume that the initial data satisfies the following stability:

$$\mathcal{E}_{LF}^{1,0} = \frac{1}{2}(\|W_N^1\|^2 + \|W_N^0\|^2) + \frac{1}{4}(\|\xi_N^1\|^2 + \|\xi_N^0\|^2) + \frac{\beta}{4}(\|u_N^1\|^2 + \|u_N^0\|^2) - D|\Omega| \leq C. \tag{4.17}$$

We define the discrete H^2-norm as

$$\|\phi_N^n\|_{H^2} = \|\phi_N^n\| + \|\nabla\phi_N^n\| + \|\Delta_h\phi_N^n\|, \ \forall\ \phi_N^n \in V_N.$$

To derive the error estimate, we first give the H^2-boundedness for the numerical solution.

Lemma 4.1. *Assuming that u_N^n is the solution of the scheme (4.10)–(4.12), there is a constant $C > 0$ such that*

$$\|u_N^n\|_{H^2} \le C. \tag{4.18}$$

Proof. From Theorem 4.1 and Eq. (4.17), we know that there is constant $C \ge 0$ such that

$$\frac{1}{2}\|W_N^n\|^2 + \frac{1}{4}\|\xi_N^n\|^2 + \frac{\beta}{4}\|u_N^n\|^2 \le C, \ \ 0 \le n \le K,$$

hence, we have

$$\|u_N^n\| \le C, \quad \|\xi_N^n\| \le C. \tag{4.19}$$

By applying (4.14), we have

$$(\nabla u_N^n, \nabla u_N^n) = (u_N^n, u_N^n) - (\xi_N^n, u_N^n),$$
$$(\nabla u_N^n, \nabla\Delta_N u_N^n) = (u_N^n, \Delta_N u_N^n) - (\xi_N^n, \Delta_N u_N^n).$$

Hence, it holds that

$$\|\nabla u_N^n\|^2 \le \|u_N^n\|^2 + \|u_N^n\|\|\xi_N^n\| \le C, \tag{4.20}$$
$$\|\Delta_N u_N^n\|^2 \le \|u_N^n\|\|\Delta_N u_N^n\| + \|\xi_N^n\|\|\Delta_N u_N^n\|,$$
$$\|\Delta_N u_N^n\| \le \|u_N^n\| + \|\xi_N^n\| \le C. \tag{4.21}$$

From (4.19), (4.20) and (4.21), we can deduce (4.18). □

Let us denote

$$\sigma_u^n := u_N^n - \Pi_N u(t_n), \quad \rho_u^n := \Pi_N u(t_n) - u(t_n),$$
$$\sigma_\xi^n := \xi_N^n - \Pi_N \xi(t_n), \quad \rho_\xi^n := \Pi_N \xi(t_n) - \xi(t_n),$$
$$\sigma_W^n := W_N^n - \Pi_N W(t_n), \quad \rho_W^n := \Pi_N W(t_n) - W(t_n),$$

thus,

$$\varepsilon_u^n := u_N^n - u(t_n) = u_N^n - \Pi_N u(t_n) + \Pi_N u(t_n) - u(t_n) = \sigma_u^n + \rho_u^n,$$
$$\varepsilon_\xi^n := \xi_N^n - \xi(t_n) = \xi_N^n - \Pi_N \xi(t_n) + \Pi_N \xi(t_n) - \xi(t_n) = \sigma_\xi^n + \rho_\xi^n,$$
$$\varepsilon_W^n := W_N^n - W(t_n) = W_N^n - \Pi_N W(t_n) + \Pi_N W(t_n) - W(t_n) = \sigma_W^n + \rho_W^n.$$

By the definition of the projection Π_N, we have

$$(\nabla\rho_u^n, \nabla\psi) = (\nabla\rho_\xi^n, \nabla\psi) = 0, \quad \forall\ \psi \in V_N.$$

We also denote

$$T_1^n = \frac{u(t_{n+1}) - u(t_{n-1})}{2\tau} - u_t(t_n), \quad T_2^n = \frac{\xi(t_{n+1}) + \xi(t_{n-1})}{2} - \xi(t_n),$$
$$T_3^n = \frac{u(t_{n+1}) + u(t_{n-1})}{2} - u(t_n), \quad T_4^n = \frac{W(t_{n+1}) - W(t_{n-1})}{2\tau} - W_t(t_n),$$
$$T_5^n = \frac{W(t_{n+1}) + W(t_{n-1})}{2} - W(t_n).$$

By using the Taylor expansion, we can easily derive the following estimates:

$$\|T_1^n\| \le \frac{1}{6}\|u\|_{W^{3,\infty}(0,T;L^2(\Omega))}\tau^2, \quad \|T_2^n\| \le \frac{1}{2}\|\xi\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2, \tag{4.22}$$
$$\|T_3^n\| \le \frac{1}{2}\|u\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2, \quad \|T_4^n\| \le \frac{1}{6}\|W\|_{W^{3,\infty}(0,T;L^2(\Omega))}\tau^2, \tag{4.23}$$
$$\|T_5^n\| \le \frac{1}{2}\|W\|_{W^{2,\infty}(0,T;L^2(\Omega))}\tau^2. \tag{4.24}$$

Theorem 4.2. *Assuming that $u \in W^{3,\infty}(0,T;H^k(\Omega))$, $\xi \in W^{2,\infty}(0,T;H^{k+2}(\Omega))$ and $W \in W^{3,\infty}(0,T;H^k(\Omega))$, then we have the following error estimate*

$$\|u(t_n) - u_N^n\| \le CN^{-k} + C(\|u\|_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^2, \quad 0 \le n \le K.$$

Proof. Subtracting (4.4)–(4.6) from (4.13)–(4.15) at t_n, we get

$$\begin{aligned}
&\left(\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau},\psi\right) + \left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau},\psi\right) + (T_1^n,\psi) + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2},\psi\right) + \left(\frac{\rho_\xi^{n+1}+\rho_\xi^{n-1}}{2},\psi\right)\\
&+ (T_2^n,\psi) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^{n-1}}{2},\nabla\psi\right) - (\nabla T_2^n,\nabla\psi) + \beta\left(\frac{\sigma_u^{n+1}+\sigma_u^{n-1}}{2},\psi\right)\\
&+ \beta\left(\frac{\rho_u^{n+1}+\rho_u^{n-1}}{2},\psi\right) + \beta(T_3^n,\psi) + \left(H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n),\psi\right) = 0,
\end{aligned} \tag{4.25}$$

$$(\sigma_\xi^n, v) - (\sigma_u^n, v) + (\nabla\sigma_u^n, \nabla v) = (\rho_u^n, v) - (\rho_\xi^n, v), \tag{4.26}$$

$$\left(\frac{\varepsilon_W^{n+1}-\varepsilon_W^{n-1}}{2\tau},\zeta\right) + (T_4^n,\zeta) = \frac{1}{2}\left(H(u_N^n)\frac{u_N^{n+1}-u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n),\zeta\right). \tag{4.27}$$

Arranging (4.25), we have

$$\begin{aligned}
&\left(\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau},\psi\right) + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2},\psi\right) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^{n-1}}{2},\nabla\psi\right) + \beta\left(\frac{\sigma_u^{n+1}+\sigma_u^{n-1}}{2},\psi\right)\\
&= -\left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau},\psi\right) - (T_1^n,\psi) - \left(\frac{\rho_\xi^{n+1}+\rho_\xi^{n-1}}{2},\psi\right) - (T_2^n,\psi) + (\nabla T_2^n,\nabla\psi)\\
&\quad - \beta\left(\frac{\rho_u^{n+1}+\rho_u^{n-1}}{2},\psi\right) - \beta(T_3^n,\psi) - \left(H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n),\psi\right).
\end{aligned} \tag{4.28}$$

From (4.26), we have

$$\begin{aligned}
&\left(\frac{\sigma_\xi^{n+1}-\sigma_\xi^{n-1}}{2\tau},v\right) - \left(\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau},v\right) + \left(\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^{n-1}}{2\tau},\nabla v\right)\\
&= \left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau},v\right) - \left(\frac{\rho_\xi^{n+1}-\rho_\xi^{n-1}}{2\tau},v\right).
\end{aligned} \tag{4.29}$$

Taking $\psi = (\sigma_u^{n+1} - \sigma_u^{n-1})/2\tau$ in (4.28), we get

$$\begin{aligned}
&\left\|\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right\|^2 + \left(\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2},\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) - \left(\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^{n-1}}{2},\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^{n-1}}{2\tau}\right)\\
&+ \frac{\beta}{4\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^{n-1}\|^2)\\
&= -\left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau},\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) - \left(T_1^n,\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) - \left(\frac{\rho_\xi^{n+1}+\rho_\xi^{n-1}}{2},\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right)\\
&\quad - \left(T_2^n,\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) + \left(\nabla T_2^n,\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^{n-1}}{2\tau}\right) - \beta\left(\frac{\rho_u^{n+1}+\rho_u^{n-1}}{2},\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right)\\
&\quad - \beta\left(T_3^n,\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) - \left(H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n),\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right).
\end{aligned} \tag{4.30}$$

Taking $v = (\sigma_\xi^{n+1} + \sigma_\xi^{n-1})/2$ in (4.29), we have

$$\begin{aligned}
&\left(\frac{\sigma_\xi^{n+1}-\sigma_\xi^{n-1}}{2\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right) - \left(\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right) + \left(\frac{\nabla\sigma_u^{n+1}-\nabla\sigma_u^{n-1}}{2\tau},\frac{\nabla\sigma_\xi^{n+1}+\nabla\sigma_\xi^{n-1}}{2}\right)\\
&= \left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right) - \left(\frac{\rho_\xi^{n+1}-\rho_\xi^{n-1}}{2\tau},\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right).
\end{aligned} \tag{4.31}$$

Taking $\zeta = (\varepsilon_W^{n+1} + \varepsilon_W^{n-1})/2$ in (4.27), we obtain

$$\frac{1}{4\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^{n-1}\|^2) = -\left(T_4^n, \frac{\varepsilon_W^{n+1} + \varepsilon_W^{n-1}}{2}\right) + \frac{1}{2}\left(H(u_N^n)\frac{u_N^{n+1} - u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n), \frac{\varepsilon_W^{n+1} + \varepsilon_W^{n-1}}{2}\right). \tag{4.32}$$

Combining (4.30), (4.31) and (4.32), we have

$$\begin{aligned}
&\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2 + \frac{\beta}{4\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^{n-1}\|^2) + \frac{1}{4\tau}(\|\sigma_\xi^{n+1}\|^2 - \|\sigma_\xi^{n-1}\|^2) + \frac{1}{4\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^{n-1}\|^2)\\
= &-\left(\frac{\rho_u^{n+1} - \rho_u^{n-1}}{2\tau}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) - \left(T_1^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) - \left(\frac{\rho_\xi^{n+1} + \rho_\xi^{n-1}}{2}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right)\\
&- \left(T_2^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) + \left(\nabla T_2^n, \frac{\nabla\sigma_u^{n+1} - \nabla\sigma_u^{n-1}}{2\tau}\right) - \beta\left(\frac{\rho_u^{n+1} + \rho_u^{n-1}}{2}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right)\\
&- \beta\left(T_3^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) - \left(H(u_N^n)\frac{W_N^{n+1} + W_N^{n-1}}{2} - H(u(t_n))W(t_n), \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right)\\
&+ \left(\frac{\rho_u^{n+1} - \rho_u^{n-1}}{2\tau}, \frac{\sigma_\xi^{n+1} + \sigma_\xi^{n-1}}{2}\right) - \left(\frac{\rho_\xi^{n+1} - \rho_\xi^{n-1}}{2\tau}, \frac{\sigma_\xi^{n+1} + \sigma_\xi^{n-1}}{2}\right) - \left(T_4^n, \frac{\varepsilon_W^{n+1} + \varepsilon_W^{n-1}}{2}\right)\\
&+ \frac{1}{2}\left(H(u_N^n)\frac{u_N^{n+1} - u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n), \frac{\varepsilon_W^{n+1} + \varepsilon_W^{n-1}}{2}\right).
\end{aligned} \tag{4.33}$$

Now, we estimate each term at the right hand side of (4.33).

$$-\left(\frac{\rho_u^{n+1} - \rho_u^{n-1}}{2\tau}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2, \tag{4.34}$$

$$\begin{aligned}
-\left(T_1^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) &\le 4\|T_1^n\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2\\
&\le \frac{1}{9}\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2,
\end{aligned} \tag{4.35}$$

$$-\left(\frac{\rho_\xi^{n+1} + \rho_\xi^{n-1}}{2}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2, \tag{4.36}$$

$$\begin{aligned}
-\left(T_2^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) &\le 4\|T_2^n\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2\\
&\le \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2,
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\left(\nabla T_2^n, \frac{\nabla\sigma_u^{n+1} - \nabla\sigma_u^{n-1}}{2\tau}\right) &\le \left|\left(\Delta T_2^n, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right)\right|\\
&\le 4\|\Delta T_2^n\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2\\
&\le \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2,
\end{aligned} \tag{4.38}$$

$$-\beta\left(\frac{\rho_u^{n+1} + \rho_u^{n-1}}{2}, \frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right) \le CN^{-2k} + \frac{1}{16}\left\|\frac{\sigma_u^{n+1} - \sigma_u^{n-1}}{2\tau}\right\|^2, \tag{4.39}$$

$$-\beta\left(T_3^n, \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) \le 4\beta^2\|T_3^n\|^2 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right\|^2$$
$$\le \beta^2\|u\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\|\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right\|^2, \tag{4.40}$$

$$\left(\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau}, \frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right) \le \frac{1}{8}\left\|\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau}\right\|^2 + 2\left\|\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right\|^2$$
$$\le CN^{-2k} + \|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^{n-1}\|^2, \tag{4.41}$$

$$\left(\frac{\rho_\xi^{n+1}-\rho_\xi^{n-1}}{2\tau}, \frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right) \le \frac{1}{8}\left\|\frac{\rho_\xi^{n+1}-\rho_\xi^{n-1}}{2\tau}\right\|^2 + 2\left\|\frac{\sigma_\xi^{n+1}+\sigma_\xi^{n-1}}{2}\right\|^2$$
$$\le CN^{-2k} + \|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^{n-1}\|^2, \tag{4.42}$$

$$-\left(T_4^n, \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2}\right) \le \frac{1}{2}\|T_4^n\|^2 + \frac{1}{2}\left\|\frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2}\right\|^2 \tag{4.43}$$
$$\le \frac{1}{72}\|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{4}(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^n\|^2). \tag{4.44}$$

Since $F(u) \ge -\frac{(\epsilon+\beta)^2}{4}$, we can choose $D > \frac{5}{4}(\epsilon+\beta)^2$ such that $\sqrt{F(u)+D} > \epsilon+\beta$. Thus, we have

$$\|H(u)\| = \left\|\frac{f(u)}{\sqrt{F(u)+D}}\right\| \le \frac{1}{\epsilon+\beta}\|u^3-(\epsilon+\beta)u\| \tag{4.45}$$
$$\le \frac{1}{\epsilon+\beta}(\|u\|^3_{L^\infty(0,T;L^2(\Omega))} + (\epsilon+\beta)\|u\|_{L^\infty(0,T;L^2(\Omega))}) \le C. \tag{4.46}$$

Using (4.18), we also have

$$\|H(u_N^n)\| \le \frac{1}{\epsilon+\beta}\|(u_N^n)^3-(\epsilon+\beta)u_N^n\| \le \frac{1}{\epsilon+\beta}(\|u_N^n\|^3 + (\epsilon+\beta)\|u_N^n\|) \le C. \tag{4.47}$$

Applying Lemma 3.1, we have

$$\|H(u_N^n) - H(u(t_n))\| \le \hat{C}_0\|u_N^n - u(t_n)\| \le \hat{C}_0(\|\sigma_u^n\| + \|\rho_u^n\|). \tag{4.48}$$

Since

$$\begin{aligned}
&H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n)\\
&= H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u_N^n)\frac{W(t_{n+1})+W(t_{n-1})}{2}\\
&\quad + H(u_N^n)\frac{W(t_{n+1})+W(t_{n-1})}{2} - H(u(t_n))\frac{W(t_{n+1})+W(t_{n-1})}{2}\\
&\quad + H(u(t_n))\frac{W(t_{n+1})+W(t_{n-1})}{2} - H(u(t_n))W(t_n)\\
&= H(u_N^n)\frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} + (H(u_N^n) - H(u(t_n)))\frac{W(t_{n+1})+W(t_{n-1})}{2} + H(u(t_n))T_5^n,
\end{aligned}$$

we have

$$\begin{aligned}
&\left(H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n), \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right)\\
&= \left(H(u_N^n)\frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2}, \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right)\\
&\quad + \left((H(u_N^n) - H(u(t_n)))\frac{W(t_{n+1})+W(t_{n-1})}{2}, \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right) + \left(H(u(t_n))T_5^n, \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}\right)\\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{4.49}$$

Using (4.45)–(4.48), we have the following estimates:

$$
\begin{aligned}
I_1 &\le 8\|H(u_N^n)\|^2 \left\| \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right\|^2 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \\
&\le C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2) + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2, \qquad (4.50)\\
I_2 &\le 8\|H(u_N^n) - H(u(t_n))\|^2 \left\| \frac{W(t_{n+1})+W(t_{n-1})}{2} \right\|^2 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \\
&\le C(\|\sigma_u^n\| + \|\rho_u^n\|)^2 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \\
&\le C\|\sigma_u^n\|^2 + CN^{-2k} + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \qquad (4.51)\\
I_3 &\le 8\|H(u(t_n))\|^2\|T_5^n\|^2 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \\
&\le C\|W\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2. \qquad (4.52)
\end{aligned}
$$

Combining (4.49)–(4.52), we get

$$
\begin{aligned}
&\left(H(u_N^n)\frac{W_N^{n+1}+W_N^{n-1}}{2} - H(u(t_n))W(t_n), \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right) \\
&\le \frac{1}{4}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2 + \|\sigma_u^n\|^2) \\
&\quad + C\|W\|^2_{W^{2,\infty}(0,T;H^k(\Omega))}\tau^4 + CN^{-2k}. \qquad (4.53)
\end{aligned}
$$

Similarly, since

$$
\begin{aligned}
&H(u_N^n)\frac{u_N^{n+1}-u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n) \\
&= H(u_N^n)\frac{u_N^{n+1}-u_N^{n-1}}{2\tau} - H(u_N^n)\frac{\Pi_N u(t_{n+1}) - \Pi_N u(t_{n-1})}{2\tau} + H(u_N^n)\frac{\Pi_N u(t_{n+1}) - \Pi_N u(t_{n-1})}{2\tau} \\
&\quad - H(u_N^n)\frac{u(t_{n+1})-u(t_{n-1})}{2\tau} + H(u_N^n)\frac{u(t_{n+1})-u(t_{n-1})}{2\tau} - H(u(t_n))\frac{u(t_{n+1})-u(t_{n-1})}{2\tau} \\
&\quad + H(u(t_n))\frac{u(t_{n+1})-u(t_{n-1})}{2\tau} - H(u(t_n))u_t(t_n) \\
&= H(u_N^n)\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} + H(u_N^n)\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau} + (H(u_N^n) - H(u(t_n)))\frac{u(t_{n+1})-u(t_{n-1})}{2\tau} \\
&\quad + H(u(t_n))T_1^n,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{2}\left(H(u_N^n)\frac{u_N^{n+1}-u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n), \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) \\
&= \frac{1}{2}\left(H(u_N^n)\frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) + \frac{1}{2}\left(H(u_N^n)\frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) \\
&\quad + \frac{1}{2}\left((H(u_N^n) - H(u(t_n)))\frac{u(t_{n+1})-u(t_{n-1})}{2\tau}, \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) \\
&\quad + \frac{1}{2}\left(H(u(t_n))T_1^n, \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) \\
&=: J_1 + J_2 + J_3 + J_4. \qquad (4.54)
\end{aligned}
$$

Using (4.45)–(4.48), we have the following estimates:

$$
\begin{aligned}
J_1 &\le \|H(u_N^n)\|^2 \left\| \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right\|^2 + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 \\
&\le C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2) + \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2, \qquad (4.55)\\
J_2 &\le \frac{1}{4}\|H(u_N^n)\|^2 \left\| \frac{\rho_u^{n+1}-\rho_u^{n-1}}{2\tau} \right\|^2 + \frac{1}{4}\left\| \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right\|^2 \\
&\le CN^{-2k} + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2), \qquad (4.56)\\
J_3 &\le \frac{1}{4}\|H(u_N^n) - H(u(t_n))\|^2 \left\| \frac{u(t_{n+1})-u(t_{n-1})}{2\tau} \right\|^2 + \frac{1}{4}\left\| \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right\|^2 \\
&\le C(\|\sigma_u^n\|^2 + \|\rho_u^n\|^2) + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2) \\
&\le C(\|\sigma_u^n\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2) + CN^{-2k}, \qquad (4.57)\\
J_4 &\le \frac{1}{4}\|H(u(t_n))\|^2 \|T_1^n\|^2 + \frac{1}{4}\left\| \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right\|^2 \\
&\le C\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2). \qquad (4.58)
\end{aligned}
$$

Combining (4.54)–(4.58), we get

$$
\begin{aligned}
&\left(H(u_N^n)\frac{u_N^{n+1}-u_N^{n-1}}{2\tau} - H(u(t_n))u_t(t_n),\ \frac{\varepsilon_W^{n+1}+\varepsilon_W^{n-1}}{2} \right) \\
&\le \frac{1}{16}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 + C(\|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2 + \|\sigma_u^n\|^2) \\
&\quad + C\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))}\tau^4 + CN^{-2k}. \qquad (4.59)
\end{aligned}
$$

Substituting (4.34)–(4.43), (4.53) and (4.59) into (4.33), we get

$$
\begin{aligned}
&\frac{1}{4}\left\| \frac{\sigma_u^{n+1}-\sigma_u^{n-1}}{2\tau} \right\|^2 + \frac{\beta}{2\tau}(\|\sigma_u^{n+1}\|^2 - \|\sigma_u^{n-1}\|^2) + \frac{1}{4\tau}(\|\sigma_\xi^{n+1}\|^2 - \|\sigma_\xi^{n-1}\|^2) + \frac{1}{4\tau}(\|\varepsilon_W^{n+1}\|^2 - \|\varepsilon_W^{n-1}\|^2) \\
&\le CN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4 \\
&\quad + C(\|\sigma_\xi^{n+1}\|^2 + \|\sigma_\xi^{n-1}\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2 + \|\sigma_u^n\|^2). \qquad (4.60)
\end{aligned}
$$

Multiplying (4.60) by τ, dropping some nonnegative terms, summing n from 0 to l $(0 < l \le K-1)$, we get

$$
\begin{aligned}
&\frac{\beta}{2}\|\sigma_u^{l+1}\|^2 + \frac{1}{4}\|\sigma_\xi^{l+1}\|^2 + \frac{1}{4}\|\varepsilon_W^{l+1}\|^2 \le C\tau\sum_{n=0}^{l}(\|\sigma_u^n\|^2 + \|\sigma_\xi^{n-1}\|^2 + \|\sigma_\xi^{n+1}\|^2 + \|\varepsilon_W^{n+1}\|^2 + \|\varepsilon_W^{n-1}\|^2) \qquad (4.61)\\
&+ CTN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4 \qquad (4.62)\\
&+ \|\sigma_u^1\|^2 + \|\sigma_\xi^1\|^2 + \|\varepsilon_W^1\|^2. \qquad (4.63)
\end{aligned}
$$

Applying the same approach to the initialization step (2.8)–(2.11) and noting that $\sigma_u^0 = \sigma_\xi^0 = \sigma_W^0$=0, we have

$$
\|\sigma_u^1\|^2 + \|\sigma_\xi^1\|^2 + \|\varepsilon_W^1\|^2 \le CTN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4. \qquad (4.64)
$$

Combining (4.61) and (4.64) and using the discrete Grönwall's inequality, we obtain

$$
\|\sigma_u^{l+1}\|^2 + \|\sigma_\xi^{l+1}\|^2 + \|\varepsilon_W^{l+1}\|^2 \le CN^{-2k} + C(\|u\|^2_{W^{3,\infty}(0,T;H^k(\Omega))} + \|\xi\|^2_{W^{2,\infty}(0,T;H^{k+2}(\Omega))} + \|W\|^2_{W^{3,\infty}(0,T;H^k(\Omega))})\tau^4.
$$

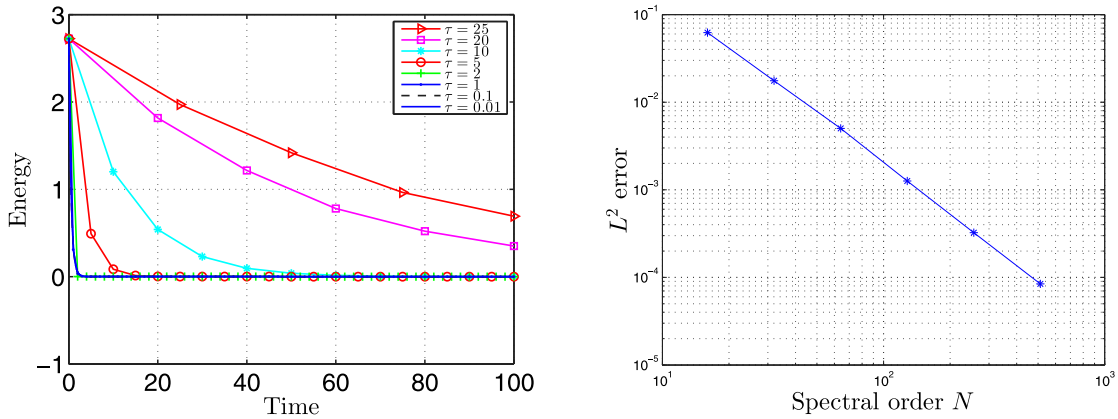
In addition, because $\|u_N^l - u(t_l)\| \le \|\sigma_u^l\| + \|\rho_u^l\|$, $\|\xi_N^l - \xi(t_l)\| \le \|\sigma_\xi^l\| + \|\rho_\xi^l\|$ and (4.8), we get the desired result. □

5. Numerical experiments

In this section, we give numerical experiments for the SH equation to verify the accuracy and energy stability of the proposed scheme.

Table 1
The errors and rate of convergence at $T = 10$ for the phase variable u that are computed by the Leapfrog scheme, using different time step sizes and $N = 256$. The physical parameter is $\epsilon = 0.025$.

τ	L^2 error	Rate
1/16	6.5927e−03	4.64
1/32	1.0632e−03	2.63
1/64	2.5910e−04	2.04
1/128	6.4808e−05	2.00
1/256	1.6206e−05	2.00
1/512	4.0519e−06	2.00
1/1024	1.0130e−06	2.00



(a) Evolution of the energy with different time step size (b) Spatial L^2 errors

Fig. 1. (a) Evolution of the energy with different time step size of τ=0.01, 0.1, 1, 2, 5, 10, 20, 25 using the Leapfrog scheme, where $\epsilon = 0.025$. (b) The spatial L^2 errors.

5.1. Accuracy test

We first test the convergence rate of the proposed scheme. The parameter is $\epsilon = 0.025$, $\beta = 1$, $D = 50$. Because it is difficult to obtain the analytical solution for SH equation, We add a suitable source term such that the exact solution is

$$u(x, y, t) = \cos(t)\sin(\frac{2\pi}{64}x)\cos(\frac{2\pi}{64}y).$$

Set the computational domain to be $\Omega = [0, 128] \times [0, 128]$. We set $N = 256$ so that the spatial discretization errors are negligible compared with the time discretization errors. In Table 1, we show the L^2 errors of the phase variable between the analytical solution and numerical solution with different time step sizes at $T = 10$. From Table 1, we can observe that the scheme gives desired rate of accuracy in time. The spatial L^2 errors are plotted in Fig. 1(b).

5.2. Energy stability test

In this subsection, we consider the smooth initial condition (5.1) to verify the energy stability of our scheme.

$$\begin{aligned} u(x, y, 0) =& 0.07 - 0.02\cos(\frac{\pi(x-12)}{16})\sin(\frac{\pi(y-1)}{16}) - 0.01\sin^2(\frac{\pi x}{8})\sin^2(\frac{\pi(y-6)}{8}) \\ &+ 0.02\cos^2(\frac{\pi(x+10)}{32})\cos^2(\frac{\pi(y+3)}{32}). \end{aligned} \tag{5.1}$$

The parameters are $\epsilon = 0.025$, $\beta = 1$, $D = 50$, $T = 100$, $N = 64$ and $\Omega = [0, 32]^2$. In Fig. 1(a), we present the evolution of the discrete energy with different time step sizes of τ = 0.01, 0.1, 1, 2, 5, 10, 20, 25 using the Leapfrog scheme. We see that the energy is nonincreasing, which validates that our scheme satisfies the unconditional energy stability.

Remark 5.1. The energy in Fig. 1(a) goes below 0, since for any constant C,

$$u_t = -\frac{\delta}{\delta u}E(u) = -\frac{\delta}{\delta u}(E(u) + C),$$

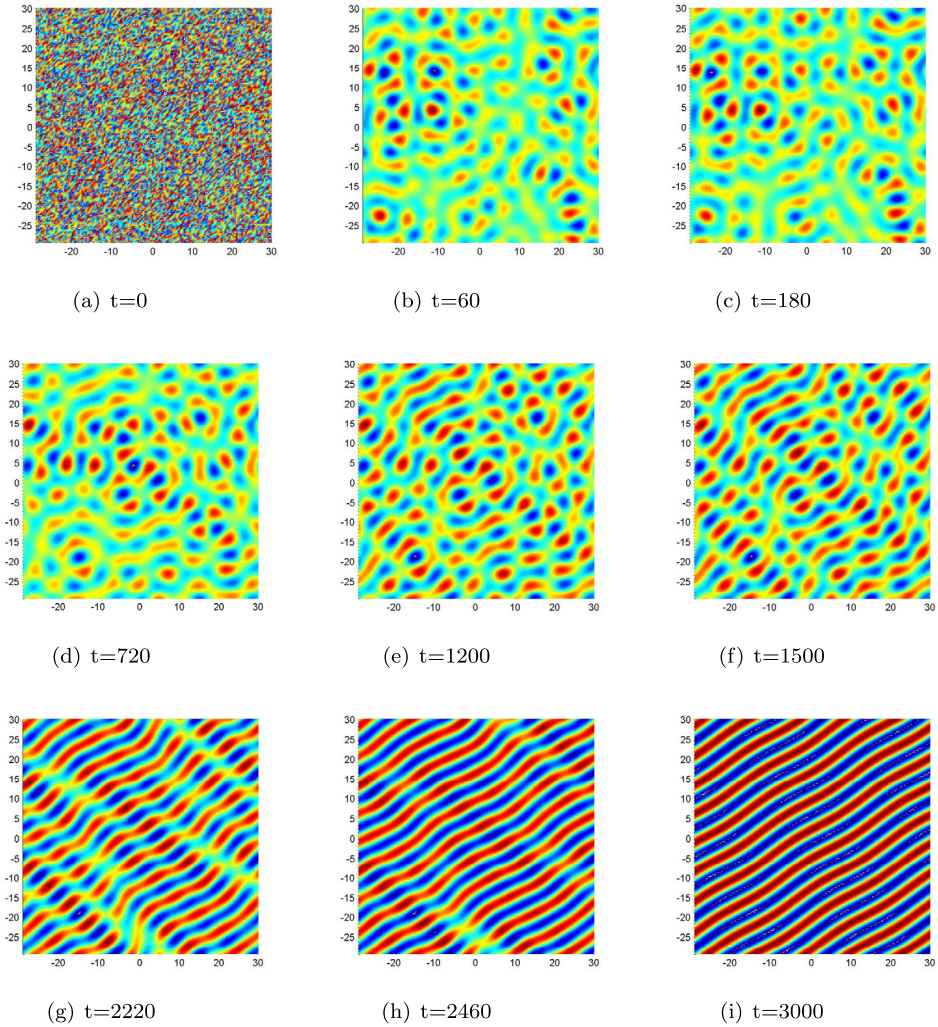


Fig. 2. The evolution of the phase transition behavior in $2D$ with $\overline{u} = -0.2$. Snapshots of the numerical approximation of the density field u are taken at $t = 0$, 60, 180, 720, 1200, 1500, 2220, 2460, 3000. The computational domain is $[-30, 30]^2$. The parameters are $\epsilon = 0.025$, $T = 3000$, $N = 128$, the time step is $\tau = 1$.

the SH equation can be derived from any free energy functional $E(u) + C$, you can choose an appropriate constant C to make the energy nonnegative.

5.3. Phase transition behaviors

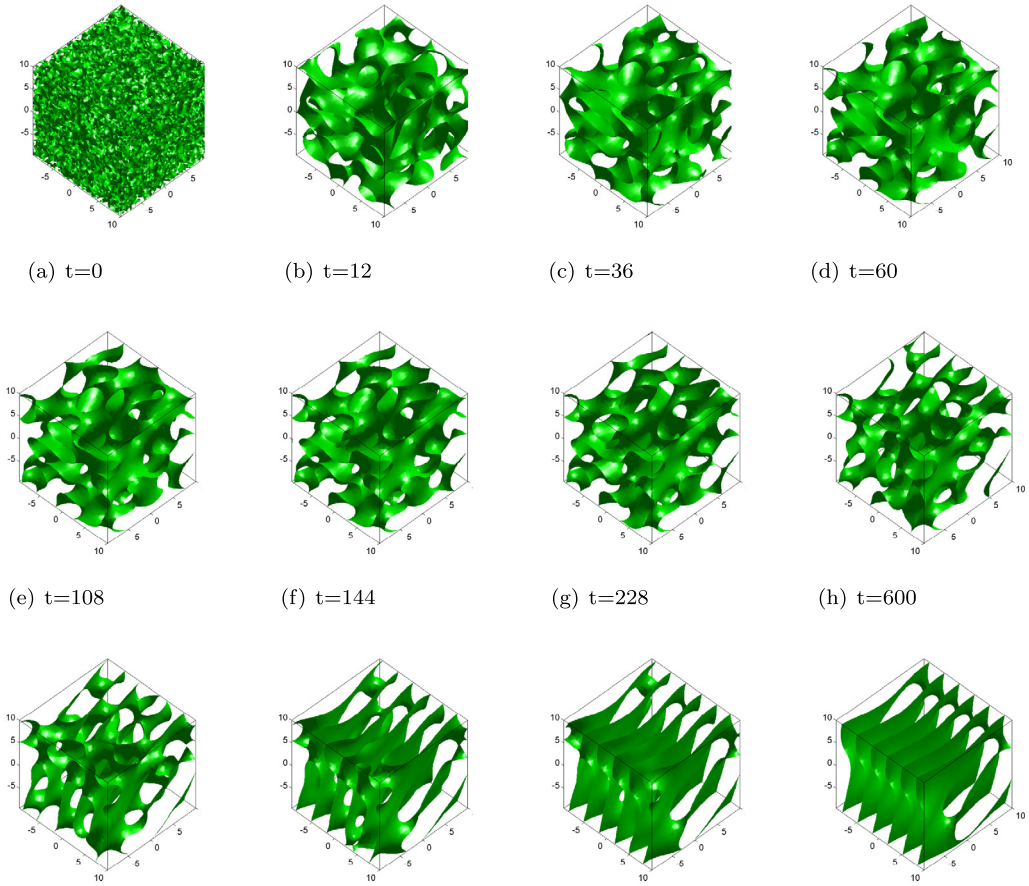
In this subsection, we apply the Leapfrog scheme to check the evolution from a randomly perturbed non-equilibrium state to a steady state.

5.3.1. 2D case

With the initial condition $u^0 = \overline{u}$+rand, where $\overline{u} = -0.2$ and rand is a randomly chosen number between -0.4 and 0.4 at the grid points, we set $N = 128$ to discrete the $2D$ space on the computational domain of $[-30, 30]^2$. Let the time step be $\tau = 1$ and the parameter be $\epsilon = 0.025$, $\beta = 1$, $D = 50$. Fig. 2 shows the time evolution of the phase transition behavior, which validates that our scheme does lead to the expected states. Fig. 4(a) shows the energy evolution with the random initial condition.

5.3.2. 3D case

With the initial condition $u^0 = \overline{u}$+rand, where $\overline{u} = -0.2$ and rand is a randomly chosen number between -0.4 and 0.4 at the grid points, we set $N = 40$ to discrete the $3D$ space on the computational domain of $[-10, 10]^3$. Let the time

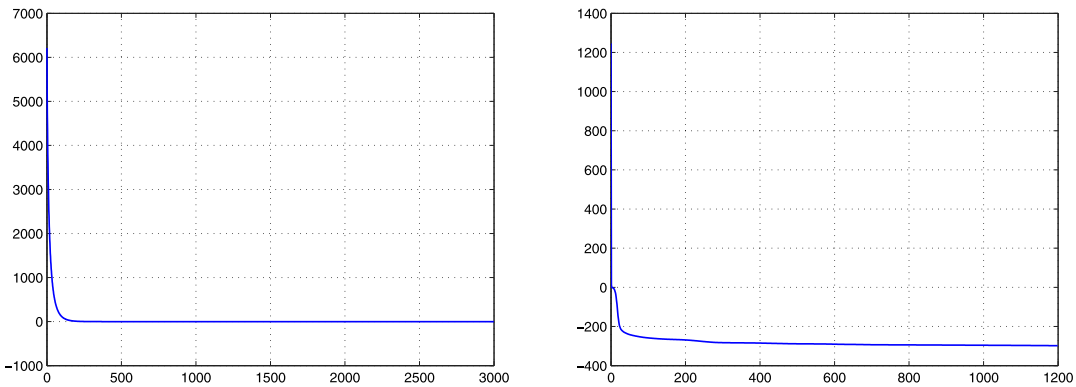


(a) t=0 (b) t=12 (c) t=36 (d) t=60

(e) t=108 (f) t=144 (g) t=228 (h) t=600

(i) t=900 (j) t=996 (k) t=1020 (l) t=1200

Fig. 3. The evolution of the phase transition behavior in 3D with $\overline{u} = -0.2$. Snapshots of the numerical approximation of the density field u are taken at $t = 0$, 12, 36, 60, 108, 144, 228, 600, 900, 996, 1020, 1200. The computational domain is $[-10, 10]^3$. The parameters are $\epsilon = 0.5$, $T = 1200$, $N = 40$, the time step is $\tau = 1$.



(a) 2D (b) 3D

Fig. 4. Evolution of the energy with the random initial condition.

step be $\tau = 1$ and the parameter be $\epsilon = 0.5$, $\beta = 1$, $D = 50$. Fig. 3 shows the time evolution of the phase transition behavior, which validates that our scheme does lead to the expected states. Fig. 4(b) shows the energy evolution with the random initial condition.

6. Conclusions

In this paper, we propose and analyze a second-order linear energy-stable Leapfrog scheme for the SH equation. We prove rigorously that the scheme satisfies the energy dissipation property and derive the error estimate. Numerical tests are given to show the accuracy and energy stability of the proposed scheme.

CRediT authorship contribution statement

Longzhao Qi: Conceptualization, Methodology, Code, Visualization, Writing – original draft. **Yanren Hou:** Supervision, Conceptualization, Methodology, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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