自然对流控制方程及其无量纲化方法

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自然对流

由于流体内部存在温度差，使得各部分流体的密度不同，温度高的流体密度小，必然上升，反之下降，由此引起流体内部的流动。这种没有外力作用，仅靠流体内部温差，而使流体流动，称为自然对流，由此产生的换热现象，称为自然对流传热。

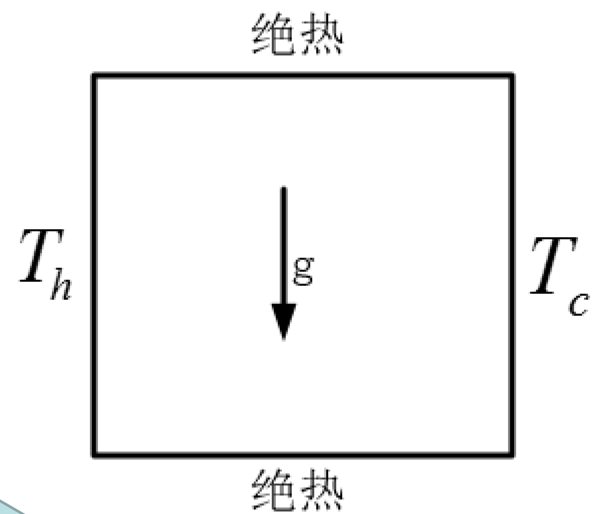
控制方程：

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})+g_y$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\lambda}{\rho C_p}(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2})$$



浮力项

浮力项是密度的函数：$g_y=f(\rho)$

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Boussinesq假设：

1) 除密度外，其他物性为常数。（适用温差约70℃）

2) 对密度，仅考虑动量方程中与体积力有关的项，其余各项中密度为常数。

$$p = p_0 + p'$$

p_0 → 温度为 T_0 时，静止压力

p' → 流动发生时的压力变动。

静止时：$u = v = 0; p = p_0, p' = 0.$

将其代入动量方程：

$$0 = \frac{1}{\rho_0}\frac{\partial p_0}{\partial x}$$

$$0 = \frac{1}{\rho_0}\frac{\partial p_0}{\partial y} + g_y$$

ρ_0 → T_0 时流体的密度

流动时：$u \neq 0, v \neq 0$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\left(\frac{\partial p_0}{\partial x} + \frac{\partial p'}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \quad ①$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y = g_y - \frac{1}{\rho}\left(\frac{\partial p_0}{\partial y} + \frac{\partial p'}{\partial y}\right) = g_y - \frac{\rho_0}{\rho}g_y - \frac{1}{\rho}\frac{\partial p'}{\partial y} \quad ②$$

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此外，对于流体的密度，当 $T \approx T_0$ 时，有如下关系式成立：

$$\frac{1}{\rho} = \frac{1+\beta_0(T-T_0)}{\rho_0} \quad ③$$

$\beta_0 \rightarrow T = T_0$ 时，流体的膨胀系数。

即：$\beta/T_0 = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{\mathrm{p}}\Big|_{T_0} = \frac{1}{1/\rho}\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)\Big|_{T_0} \approx \rho_0 \frac{\beta_0}{\rho_0} = \beta_0$

将③代入②得：

$$g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = g_y - [1+\beta(T-T_0)]g_y - \frac{1}{\rho}\frac{\partial p'}{\partial y} = -g_y\beta(T-T_0) - \frac{1}{\rho}\frac{\partial p'}{\partial y} \quad ④$$

将④代入原动量方程，得：

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g_y\beta_0(T-T_0)$$

将 p' 记作 p，β_0、T_0 为参考点之值，在本例中分别为：$\beta_a, T_a = \frac{T_c + T_h}{2}$

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原方程组化为：

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})+g_y\beta_a(T-T_a)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\lambda}{\rho C_p}(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2})$$

无量纲化：

$$X=\frac{x}{x^*},Y=\frac{y}{y^*},x^*=y^*=L^*$$

$$U=\frac{u}{u^*},V=\frac{v}{u^*},P=\frac{p}{p^*}$$

$$\tau=\frac{t}{t^*},\widetilde{T}=\frac{T-T^{**}}{T^*}$$

以下：$T^*=T_h-T_c,T^{**}=\frac{T_h+T_c}{2}$

一般来说，$T^*=T_h-T_c$，T^{**}构造原则：

当 $T=T_h$ 时，使得：$\widetilde{T}=0.5$

当 $T=T_c$ 时，使得：$\widetilde{T}=-0.5$

或 ：当 $T=T_h$ 时，使得：$\widetilde{T}=1$

当 $T=T_c$ 时，使得：$\widetilde{T}=0$

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代入上述方程组，得：

$$\frac{u^*}{L^*}\frac{\partial U}{\partial X}+\frac{u^*}{L^*}\frac{\partial V}{\partial Y}=0\Rightarrow\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$

$$\frac{u^*}{t^*}\frac{\partial U}{\partial \tau}+\frac{u^{*2}}{L^*}U\frac{\partial U}{\partial X}+\frac{u^{*2}}{L^*}V\frac{\partial U}{\partial Y}=-\frac{1}{\rho}\frac{p^*}{L^*}\frac{\partial P}{\partial X}+\frac{\upsilon u^*}{L^{*2}}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$

两边同乘 $\frac{L^*}{U^{*2}}$，得：

$$\underbrace{\frac{L^*}{t^* u^*}}_{①}\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\underbrace{\frac{1}{\rho}\frac{p^*}{u^{*2}}}_{②}\frac{\partial P}{\partial X}+\underbrace{\frac{\upsilon}{L^* u^*}}_{③}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$

Y方向上：

$$-g_y\beta_a(T-T_a)=-g_y\beta_a[(\tilde{T}T^*+T^{**})-(\tilde{T}_aT^*+T^{**})]$$

$$\because g_y=-g\Rightarrow \text{上式}=g\beta_aT^*(\tilde{T}-\tilde{T}_a)$$

$\tilde{T}_a=(\tilde{T}_h+\tilde{T}_c)/2=(0.5-0.5)/2=0$，记 $\beta_a=\beta$

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Y方向上变为：

$$\overset{①}{\boxed{\frac{L^*}{t^*u^*}}}\frac{\partial V}{\partial \tau}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\overset{②}{\boxed{\frac{1}{\rho}\frac{p^*}{u^{*2}}}}\frac{\partial P}{\partial X}+\overset{③}{\boxed{\frac{\upsilon}{L^*u^*}}}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)+\overset{④}{\boxed{\frac{L^*}{u^{*2}}g\beta T^*}}\tilde{T}$$

能量方程：

$$\frac{\partial(\tilde{T}T^*+T^{**})}{t^*\partial\tau}+\frac{u^*T^*}{L^*}U\frac{\partial\tilde{T}}{\partial X}+\frac{u^*T^*}{L^*}V\frac{\partial\tilde{T}}{\partial Y}=\frac{\alpha T^*}{L^{*2}}\left(\frac{\partial^2\tilde{T}}{\partial X^2}+\frac{\partial^2\tilde{T}}{\partial Y^2}\right)$$

两边同乘 $\frac{L^*}{U^*T^*}$，得：

$$\overset{①}{\boxed{\frac{L^*}{t^*u^*}}}\frac{\partial\tilde{T}}{\partial \tau}+U\frac{\partial\tilde{T}}{\partial X}+V\frac{\partial\tilde{T}}{\partial Y}=\overset{⑤}{\boxed{\frac{\alpha}{L^*u^*}}}\left(\frac{\partial^2\tilde{T}}{\partial X^2}+\frac{\partial^2\tilde{T}}{\partial Y^2}\right)$$

$$\mathrm{Pr}=\frac{\text{粘性扩散}}{\text{热扩散}}$$

$$① =1 \Rightarrow L^*=t^*u^*$$

$$⑤ =1 \Rightarrow u^*=\frac{\alpha}{L^*}=\frac{\alpha}{H}$$

$$② =1 \Rightarrow p^*=\rho u^{*2}$$

$$③ =\frac{\upsilon}{L^*u^*}=\frac{\upsilon}{H\frac{\alpha}{H}}=\frac{\upsilon}{\alpha}=\mathrm{Pr}$$

$$\Rightarrow \quad t^*=\frac{L^*}{u^*}=\frac{H^2}{\alpha}$$

$$p^*=\rho\left(\frac{\alpha}{H}\right)^2$$

Pr是表示流体中动量迁移和能量迁移相对大小的无量纲数，表明了流动边界层与温度边界层的关系，是物性参数，反映了流体物理性质对流动传热过程的影响。

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$$④\frac{L^*}{u^{*2}}g\beta T^* = \frac{H}{\alpha^2/H^2}g\beta(T_h - T_c) = \frac{g\beta(T_h - T_c)H^3}{\alpha\cdot\alpha}\frac{\upsilon}{\upsilon} = \frac{g\beta(T_h - T_c)H^3}{\alpha\cdot\upsilon}\frac{\upsilon}{\alpha} = Ra\cdot\Pr = Gr\cdot\Pr^2$$

$$Ra = \frac{g\beta(T_h - T_c)H^3}{\alpha\upsilon}$$

Ra：瑞利数

自然对流传热中，传热系数关联的无量纲参数。

$$Ra = Gr\cdot\Pr$$

$$Gr = \frac{g\beta(T_h - T_c)H^3}{\upsilon^2}$$

Gr：格拉晓夫数

浮力与粘性力的比值，自然对流重要参数。

无量纲化结果：

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\frac{\partial U}{\partial \tau} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \Pr\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right)$$

$$\frac{\partial V}{\partial \tau} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{\partial P}{\partial X} + \Pr\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + Gr\Pr^2\tilde{T}$$

$$\frac{\partial \tilde{T}}{\partial \tau} + U\frac{\partial \tilde{T}}{\partial X} + V\frac{\partial \tilde{T}}{\partial Y} = \left(\frac{\partial^2 \tilde{T}}{\partial X^2} + \frac{\partial^2 \tilde{T}}{\partial Y^2}\right)$$

边界条件：

$$
\begin{aligned}
&X=0, && \widetilde{T}=0.5, && U=V=0\\
&X=1, && \widetilde{T}=-0.5, && U=V=0\\
&Y=0,1 && \frac{\partial \widetilde{T}}{\partial Y}=0, && U=V=0
\end{aligned}
$$

两点说明：

（1）若 $\widetilde{T}_c=0, \widetilde{T}_h=1$（见第5页， T^{**} 构造原则）

则V方程源项为： $Gr\,\mathrm{Pr}^2(\widetilde{T}-0.5)$

（2）若令

$③=1, u^*=\frac{\upsilon}{H}$

$⑤式=\frac{\alpha}{u^*L^*}=\frac{\alpha}{\frac{\upsilon}{H}H}=\frac{\alpha}{\upsilon}=\frac{1}{\mathrm{Pr}}$

则方程无量纲化结果为：

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$

$$\frac{\partial U}{\partial \tau}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2})$$

$$\frac{\partial V}{\partial \tau}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial X}+(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2})+Gr\widetilde{T}$$

$$\frac{\partial \widetilde{T}}{\partial \tau}+U\frac{\partial \widetilde{T}}{\partial X}+V\frac{\partial \widetilde{T}}{\partial Y}=\frac{1}{\mathrm{Pr}}(\frac{\partial^2 \widetilde{T}}{\partial X^2}+\frac{\partial^2 \widetilde{T}}{\partial Y^2})$$