International Portfolio Optimization with Chance Constraints[1]

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Abstract

In this paper, we study an international portfolio selection problem, which allocates wealth in different security markets. We built an international portfolio selection model with a chance constraint to guarantee the portfolio performance over a benchmark in a large probability. We use some market factors to explain the return rates of risky securities. We use a copula model to capture the nonlinear dependence structure of the factors and exchange rates. We design a new efficient algorithm based on partial sampling approximation and sequence convex approximation to solve the chance constrained international portfolio selection problem. Numerical tests in a practical international portfolio management problem illustrate the reasonability and superior out-of-sample performance of the proposed model.

Keywords: Stochastic programming, Chance constraint, International portfolio optimization, Multi-factor model, Partial sampling approximation

1. Introduction

1.1. Research background

In recent years, financial crises have occurred frequently in investment markets, for instance, the financial crisis of the United States (USA) market in 2009, the financial crisis of the Hong Kong market in 2015, and the long depression of European (EURO) market from 2010 to now. Due to the homogeneity of risky assets, portfolio management in a single market could not resist the market's systemic risk in financial crises. On the contrary, international portfolio management is an important investment tool, which can invest in foreign markets when the base market is in recession. It benefits from (a) the prospect for higher profit in the event of favorable performance of foreign markets, (b) wider scope for diversification, (c) reduced exposure to systematic risk due to generally low correlations of international securities [26]. With the globalization of economics, investments in foreign securities are becoming accessible to more investors. Thus, the international portfolio selection problem attracts more attention from both practical investors and researchers.

1.2. Literature review

The significant difference between international portfolio optimization from classical (domestic) portfolio optimization is to consider the impact of exchange rate risk [11]. Adler and

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Dumas (1983) deal with portfolio choices in a unified worldwide capital market. Topaloglou et al. [26] built a dynamic stochastic programming model with CVaR measure for international portfolio management and solved it by the scenario tree approach. Maroua and Prigent [22] analyzed the international portfolio optimization problem by introducing the higher moments of the main financial index returns. Fonseca and Rustem [9] studied multi-period robust portfolio optimization with affine policies. Jiang et al. [15] studied the exchange rate risk based on behavioral portfolio theory in an international portfolio selection model. Topaloglou et al. [27] developed scenario-based stochastic programming models for hedging the risks of international portfolios using options. Luan et al. [21] studied distributionally robust international portfolio optimization with worst-case mean-CVaR. Although these models include random exchange rates, they do not specifically consider the impact of investment benchmarks.

To deal with the investment problems when the investor holds a benchmark, we can use chance constraints to give a probability guarantee on the portfolio return over the benchmark. Chance constraints are widely adopted in financial problems such as index tracking, bankrupt control, and pension fund portfolio[25]. Li et al. [17] studied safety-first multi-period portfolio selection problems, which involve a chance constraint on terminal wealth and find its closed-form solution. Zhu et al. [31] further solved the multi-period portfolio selection problem with bankrupt constraints. Xu et al.[29] presented a novel sparse enhanced indexation model with distributionally robust chance and cardinality constraints. Ji and Lejeune [14] proposed a stochastic risk budgeting multi-portfolio optimization model with marginal risk constraints and applied it to general chance-constrained optimization problems. Chen et al. [3] proposed a sparse portfolio selection model with chance constraints and reformulated it as a difference of convex optimization problem.

The major solution difficulty of a chance-constrained optimization problem comes from the non-smooth indicator function if we formulate the chance constraint in an expected value function form. When the random and decision variables are separable, some individual chance constraints can be reformulated as a deterministic equivalent problem by using the inverse distribution function [24]. Under the assumptions of Gaussian (normal) distribution, Student's T distribution or Q-radial distribution, a linear individual chance constraint is equivalent to a second-order cone constraint [2, 24]. Under the elliptical distributions, logarithmic concave distributions or r-concavity distribution-based assumptions, e.g., Gaussian, Student's T, Wishart, Dirichlet, a linear individual chance constraint is convex when the confidence level is greater than 0.5 [7, 12]. For non-separable problems, the linear individual chance constraint under elliptical distributions is convex when the confidence level is large enough [28]. Henrion and Strugarek [13] used copulas to characterize the nonlinear dependence structure of random variables between different rows. Given the log exp-concave property of the copula function, they obtained a sufficient condition of convexity for a joint chance-constrained problem with separated random vectors. Cheng et al. [5] proposed a semidefinite programming approximation for a joint chance-constrained programming problem with Archimedean's copula dependence structure.

1.3. Difficulty points of applying chance constraints in international portfolio optimization

Although there is a rich amount of research on chance-constrained optimization problems with applications in financial decision-making, to the best of our knowledge, it has not been

successfully applied to international portfolio selection problems. The reason can be illustrated as follows.

First, different from classical portfolio selection in one market, the international portfolio selection model should consider the impact of random exchange rates, bringing some product terms of random variables in a chance constraint. However, reformulations and solution methods for such non-linear chance constraints have not been well studied.

Second, it is well known that random returns in different markets are often non-linear dependent [6]. Thus, the classical solution methods based on independence assumption [4, 19], linear-dependence structure assumption do not work well in the chance-constrained international portfolio optimization model with non-linear dependence structure.

Third, international portfolio selection brings a great number of assets into the investment pool leading to a large dimensional chance constraint. Estimating the non-linear dependent joint distribution of a large number of random variables is a challenging task in statistics.

1.4. Motivation of our approach: copula model, factor model and partial sampling

From the second point above, we find that the key modeling issue in an international portfolio management problem is the nonlinear dependence relationship between different markets. Copulas are a class of connection functions that describe the dependence structure between random variables. It is widely used in finance and insurance. Many researchers use copula for risk management or combine copula and time series for economic forecasting [16, 30]. This paper uses the copula model to describe the dependence between different markets. However, estimating a copula, even for the simplest Archimedes's or Gaussian one, suffers from the curse of dimensional issue, i.e., the difficulty of parameter estimation and computation in applications increases exponentially to the dimensional of the risky assets. However, a large-size asset pool is an inherent feature of international portfolio management. Thus, the traditional approach fitting all random return rates as a copula is not practical and tractable in international portfolio optimization problems.

The factor model provides us with a new approach to address this issue. It is an efficient modeling approach that explains the return rates of risky assets by some common market factors and independent residue errors [1]. According to the capital assets pricing model [18], the stock return rate has a linear relationship with the systematic risk of the entire stock market. Thus, it is natural to use a linear regression model to describe the return rates of risky assets by the factors, and i.i.d residues [8].

After using factors to explain the return rates, we can divide the random variables in the chance-constrained international portfolio selection model into two groups. One group contains factors, exchange rates, and the benchmark, which play a key role in the dependence between different markets. Then we use a copula model to characterize the joint distribution of the random variables in the first group. The other random variables are the Gaussian distributed residuals, independent of each other and the former group.

Then, we borrow the idea from [4], and [20] to make a sample approximation to some random variables with a highly non-linear dependence structure, called partial sampling. At the same time, we keep the other residues still as Gaussian distributions themselves. Thus, we can get a mixture Gaussian distribution approximation, leading to a tractable and asymptotic tight approximation. Then, to address the bi-linear terms in the reformulation, we apply the sequence convex approximation approach, which converges to a stationary point.

By doing so, we can largely reduce the dimensionality of the copula model without reducing the model's accuracy. Thus, we can handle hundreds or thousands of assets in the international portfolio selection model without exponentially increasing the model complexity.

1.5. Structure of the paper and contribution

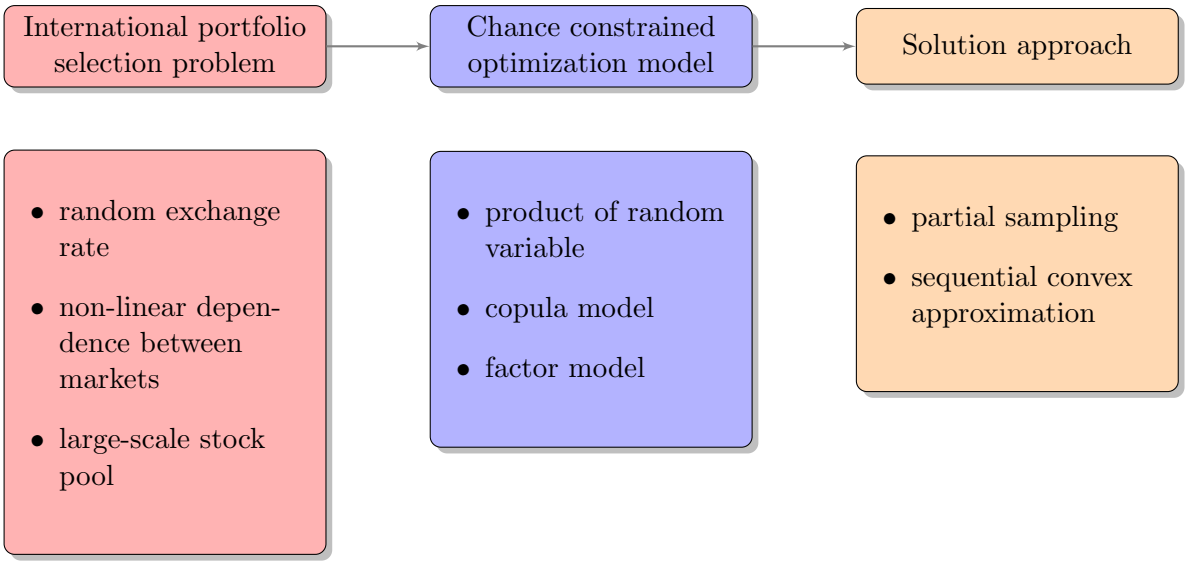


Figure 1: Main modelling approach and solution methods.

Figure 1 illustrates this paper's main ideas and solution methods. We conclude the contribution of this paper with the following three points:

- For the first time, we study the international portfolio optimization problem with a chance constraint considering the random exchange rate and non-linear dependence structure between different markets.
- We use a factor model to explain the random return rates, which largely reduce the computational difficulty of the chance-constrained optimization problem.
- We propose an efficient solution algorithm based on the partial sampling approach and the sequence convex approximation.

The paper is in the following structure: Section 2.1 introduces the international portfolio selection model with chance constraints, where we use market factors to explain the return rates and use a copula model to model the nonlinear dependence structure of the factors, exchange rates, and benchmark. In Section 3, we propose a tractable solution method using the partial sampling approximation and sequential convex approximation algorithm. Section 4 carries out a series of numerical tests in the USA, Hong Kong, and European markets, with empirical market data.

2. Model formulation

This section introduces the international portfolio selection model with chance constraints. We use some market factors to explain the return rates of stocks, introduced in Section 2.2. We

introduce the copula model in Section 2.3 and use a Gaussian or Student's t copula to fit the nonlinear dependence structure of the factors, exchange rates, and benchmark. The marginal distributions of the copula are Student's t distribution which captures the high kurtosis and heavy-tailed property of the factors and exchange rates.

2.1. Chance constrained international portfolio selection model

Table 1: Notations

Sets and matrices:	
$\mathcal{M} = \{m_{\mathrm{b}}, m_1, \cdots, m_M\}$	set of markets (synonymously for countries and currencies);
m_{b}	investor's base market (for instance, USA in our numerical test), $m_{\mathrm{b}} \in \mathcal{M}$;
$\mathcal{M}_{\mathrm{f}}$	set of foreign markets, i.e., $\mathcal{M}_{\mathrm{f}} = \mathcal{M} \setminus \{m_{\mathrm{b}}\}$;
M	number of foreign markets, i.e., $M = \lvert\mathcal{M}\rvert - 1$;
$\mathcal{N}_m$	set of risky assets in market $m \in \mathcal{M}$;
N_m	number of risky assets in market $m \in \mathcal{M}$, i.e., $N_m = \lvert\mathcal{N}_m\rvert$;
$\boldsymbol{I}_N$	$N \times N$ dimensional identity matrix;
$\mathbf{1}_N$ or $\mathbf{1}$	N dimensional all-one vector;
$\mathbf{0}_N$ or $\mathbf{0}$	N dimensional all-zero vector;
K_m	number of factors in market $m \in \mathcal{M}$;
Deterministic input parameters:	
h_m^0	initial wealth held in the risk-free asset of market $m \in \mathcal{M}$ (account in local currency);
r_m^{rf}	risk-free return rate in market $m \in \mathcal{M}$;
$\boldsymbol{u}_m^0 = [u_{1,m}^0, \cdots, u_{N_m,m}^0]^\top$	initial portfolio in market $m \in \mathcal{M}$;
$u_{i,m}^0$	the wealth held in the i-th risky asset in market m (account in local currency);
e_m^0	current exchange rate of currency $m \in \mathcal{M}_{\mathrm{f}}$, relative to base currency m_{b};
θ_m	transaction cost rate of buying or selling assets in market $m \in \mathcal{M}$;
θ_{ex}	transaction cost rate for currency exchanges;
Random input parameters:	
$\boldsymbol{r}_m = [r_{1,m}, \cdots, r_{N_m,m}]^\top$	return rate vector of risky assets in market $m \in \mathcal{M}$ over the investment;
e_m	exchange rate of currency $m \in \mathcal{M}_{\mathrm{f}}$, relative to m_{b} over the investment;
y	benchmark;
$f_m = [f_m^1, \cdots, f_m^{K_m}]^\top$	factors vector in market $m \in \mathcal{M}$;
Decision variables:	
w	nominal final wealth of the portfolio (in base currency m_{b});
$\boldsymbol{u}_m = [u_{1,m}, \cdots, u_{N_m,m}]^\top$	portfolio after re-balance in market $m \in \mathcal{M}$;
$\boldsymbol{b}_m = [b_{1,m}, \cdots, b_{N_m,m}]^\top$	vector of cash amounts for buying assets in market $m \in \mathcal{M}$ (in local currency);
$\boldsymbol{s}_m = [s_{1,m}, \cdots, s_{N_m,m}]^\top$	vector of cash amounts for selling assets in market $m \in \mathcal{M}$ (in local currency);
h_m	cash invested in risk-free asset in market $m \in \mathcal{M}$ (in local currency);
g_m	expenditure of base currency m_{b} for purchase of foreign currency $m \in \mathcal{M}_{\mathrm{f}}$;
q_m	expenditure of foreign currency $m \in \mathcal{M}_{\mathrm{f}}$ for purchase of base currency m_{b}.

We consider an international portfolio selection problem for investors such as an international portfolio fund manager. We suppose the investor is in a base market (for instance, the USA in our numerical test). We regard the currency in the base market as the base currency and the other currencies in foreign markets as foreign currencies. The investor may allocate his/her wealth in the base market or several foreign markets. He/she can invest in risky assets or a risk-free asset in a local market. Following Topaloglou et al. [26], no direct exchanges are executed between two foreign currencies to simplify the model. There are only currency exchanges between a foreign currency and the base currency. To reposition the investments from one foreign market m_1 to another foreign market m_2, the investor should first sell some foreign assets in market m_1, then transfer the cash to base currency USD, purchase some foreign currencies in market m_2 and

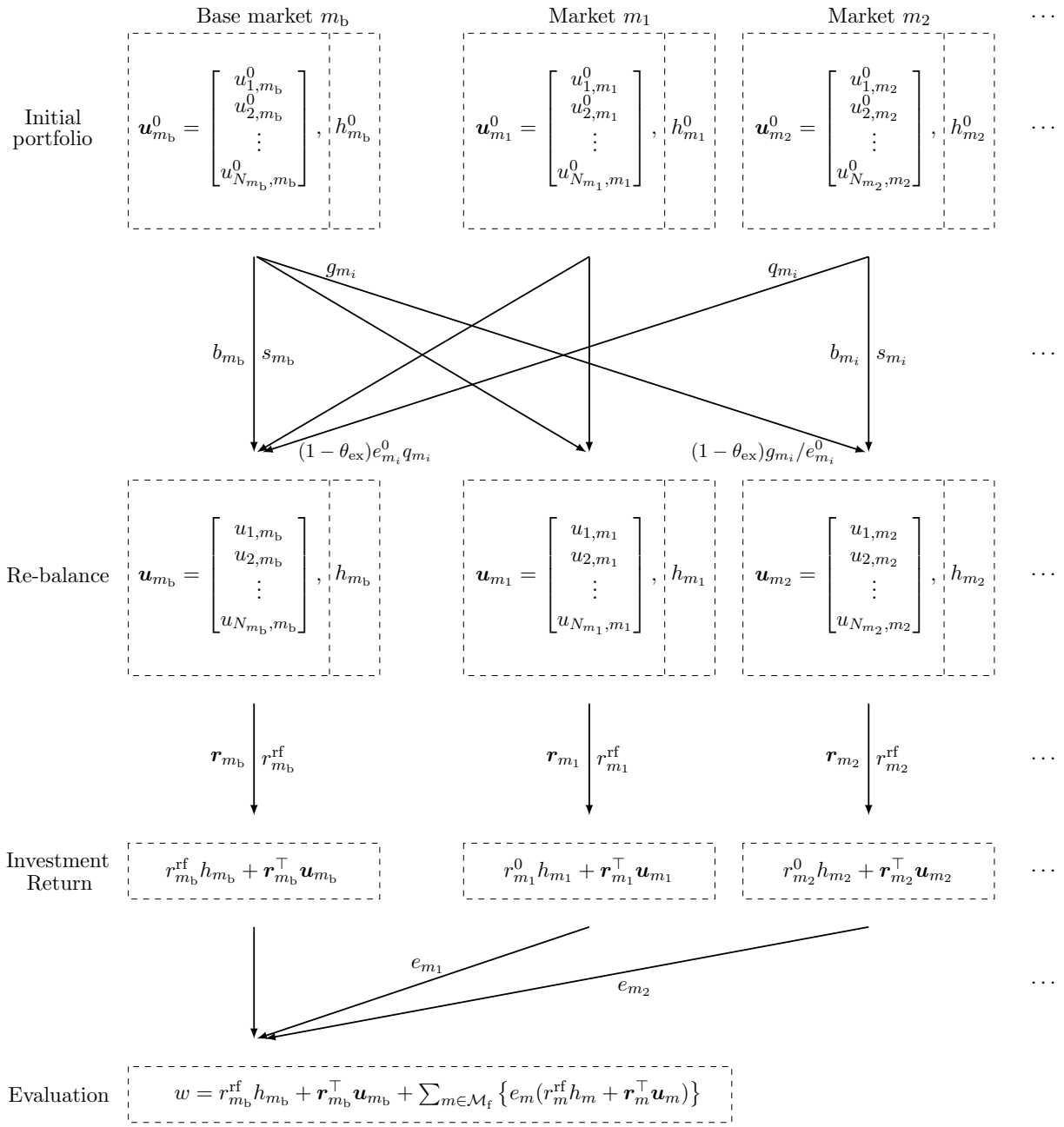


Figure 2: Investment process.

finally by assets in market m_2. To simulate the international portfolio re-balance process more realistically, we consider transaction costs of risky asset buying/selling and currency exchange. Table 1 lists notations used in the model.

The investor joins the international market with an initial portfolio allocated in different markets. The investor forecasts assets' return rates in different markets by some distributions regarding the future state of the economy. He/she can then use an optimization model to determine an optimal portfolio with the forecasted return information of the securities. The portfolio's final wealth (nominal in base currency for evaluation) depends on the realization of random asset returns and random exchange rates after the investment process. Figure 2 illustrates the whole investment process, including initial portfolio, re-balance, investment profit, and final evaluation.

All vectors and matrices are in bold. In some places of the paper, the subscripts of some standard matrices or vectors, for instance, I_N, $\mathbf{1}_N$, or $\mathbf{0}_N$, are omitted when without losing clarity. We list some key assumptions in what follows.

Assumption 1. *Assume that*

- *The investment process is self-financing;*
- *No short selling is allowed;*
- *No direct exchanges are executed between two foreign currencies;*
- *All asset buying/selling and currency exchanges can be realized in real-time;*
- *There are transaction costs of buying or selling risky assets in all markets, linear to the transaction amount;*
- *There are transaction costs for currency exchanges, linear to the transaction amount;*
- *There is no transaction cost of buying or selling risk-free assets in all markets.*

The investor has in her/his mind a random benchmark, which may be the final wealth of a benchmark investment strategy or a benchmark international portfolio management fund. The investor aims to find a portfolio maximizing the expectation of the nominal final wealth; meanwhile, he/she hopes the nominal final wealth exceeds the benchmark in a large probability. We can formulate this international portfolio selection problem as a stochastic optimization model with a chance constraint.

$$
\begin{aligned}
&(IC)\\
&\max_{u,b,s,h,g,q} \quad \mathbb{E}\left[w\right] && \text{(1a)}\\
&\text{s.t.} \quad \boldsymbol{u_m} = \boldsymbol{u_m^0} + \boldsymbol{b_m} - \boldsymbol{s_m},\ m \in \mathcal{M}, && \text{(1b)}\\
&\qquad h^0_{m_\mathrm{b}} + (1-\theta_{m_\mathrm{b}})\mathbf{1}^\top \boldsymbol{s_{m_\mathrm{b}}} - (1+\theta_{m_\mathrm{b}})\mathbf{1}^\top \boldsymbol{b_{m_\mathrm{b}}} + \sum_{m\in\mathcal{M}_\mathrm{f}} (1-\theta_\mathrm{ex}) e^0_m q_m - \sum_{m\in\mathcal{M}_\mathrm{f}} g_m\\
&\qquad\qquad = h_{m_\mathrm{b}}, && \text{(1c)}\\
&\qquad h^0_m + (1-\theta_m)\mathbf{1}^\top \boldsymbol{s_m} - (1+\theta_m)\mathbf{1}^\top \boldsymbol{b_m} + (1-\theta_\mathrm{ex})\frac{g_m}{e^0_m} - q_m = h_m, m \in \mathcal{M}_\mathrm{f}, && \text{(1d)}\\
&\qquad w = r^\mathrm{rf}_{m_\mathrm{b}} h_{m_\mathrm{b}} + \boldsymbol{r}^\top_{m_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m\in\mathcal{M}_\mathrm{f}} \left\{ e_m \left(r^\mathrm{rf}_m h_m + \boldsymbol{r}^\top_m \boldsymbol{u}_m \right) \right\}, && \text{(1e)}\\
&\qquad \mathbb{P}\left(w \geqslant y\right) \geqslant 1-\epsilon, && \text{(1f)}
\end{aligned}
$$

$$\boldsymbol{s_m} \leqslant \boldsymbol{u_m^0},\ m \in \mathcal{M}, \tag{1g}$$

$$g_m \leqslant \hat{g}_m,\ q_m \leqslant \hat{q}_m,\ m \in \mathcal{M}_{\mathrm{f}}, \tag{1h}$$

$$u_{m,i} \leqslant \hat{u}_{m,i}, i = 1, ..., N_m,\ m \in \mathcal{M}, \tag{1i}$$

$$\boldsymbol{u_m}, \boldsymbol{b_m}, \boldsymbol{s_m} \in \mathbb{R}_+^{N_m}, h_m, g_m, q_m \in \mathbb{R}_+, m \in \mathcal{M}. \tag{1j}$$

We denote the buying, and selling amounts vector of risky assets in market m as $\boldsymbol{b}_m$ and $\boldsymbol{s}_m$. Thus we have the portfolio balance equations, which is formulated by constraint (1b). Constraint (1c) is the risk-free asset re-balance equation in the base market. Constraint (1d) is the risk-free asset re-balancing equation in foreign markets. As there is no transaction cost for risk-free asset buying/selling, we can view the risk-free asset as cash. Thus, we can view (1c)-(1d) as usual cash re-balance constraints with transaction costs of both risky asset buying/selling and currency exchanges. Constraint (1e) defines the nominal final wealth, which is the summation of wealth in all markets accounted in base currency. Constraint (1f) is a chance constraint guaranteeing a large probability that the international portfolio return exceeds the benchmark. Constraint (1g) guarantees that the amount of selling a risky asset cannot exceed the initial holding. Constraints (1h) limits the maximal amount of daily flow-in and flow-out of the international transfer between a foreign market m and the base market, where $\hat{g}_m$ and $\hat{h}_m$ are the upper bounds of the cash amount maximal transferred in/out the market m from/to the base market. Constraint (1i) limits the upper bound of the amount invested in each risky asset. A classical choice of the upper bound is a proportion to current wealth in a market $\hat{\boldsymbol{u}}_{m,i} = \lambda \left(\sum_{i=1}^{N_m} u_{m,i}^0 + h_m^0 \right)$, $i = 1, \ldots, N_m$, $m \in \mathcal{M}$.

Model (1a)-(1j) is a stochastic optimization problem with a chance constraint, where the randomness arises from the return rate $\boldsymbol{r}_m$, the currency exchange rate e_m, and the benchmark y. In the next three subsections, we will introduce the statistical description of the random vector/variables $\boldsymbol{r}_m$, e_m, and y.

The key modeling issue in an international portfolio management problem is the nonlinear dependence structure of random returns/exchange rates in different markets. We first use a factor model [8] to explain the return rates of risky securities by some common market factors and independent residue errors. Then we use a copula function to describe the dependence structure of the market factors, the exchange rate, and the benchmark and then explain the return rates by the factors.

2.2. Factor model for $\boldsymbol{r}_m$

We assume that there are K_m market factors in market $m \in \mathcal{M}$. We use a multi-factor model to explain the return rate of risky assets

$$r_{i,m} - r_m^{\mathrm{rf}} = \sum_{k=1}^{K_m} \beta_{i,m}^k f_m^k + \delta_{i,m},\ i = 1, ..., N_m,\ m \in \mathcal{M}, \tag{2}$$

where $r_{i,m}$ is the return rate of the ith asset in market m; r_m^{rf} is the risk-free return rate of market m; f_m^k is the kth market factor in market m and $\beta_{i,m}^k$ is its loading parameter to the ith asset, $\delta_{i,m}$ is the residue error. The residue errors $\delta_{i,m}, i = 1, ..., N_m$ are independent to each other and the factors as well. We suppose that the residue errors of risky assets in market m follows a multivariate Gaussian distribution $N(\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m)$, where $\boldsymbol{\mu}_m = [\mu_{1,m}, ..., \mu_{N_m,m}]^\top$ is the

mean value vector and $\mathbf{\Sigma}_m = \{\sigma_{i,m}^2\}_{i=1,...,N_m}$ is the diagonal covariance matrix, $m \in \mathcal{M}$.

A popular choice of the market factors is the Fama-French three-factor model, which sets f_m^1 as Mkt-RF (the excess return factor on the market), f_m^2 as SMB (small minus big factor), and f_m^3 as HML (high minus low factor).

2.3. Copula model for factors, exchange rates and benchmark

To reduce complexity, we model the non-linear dependence structure between market factors rather than the return rates of risky assets. For this sake, we use a copula model to describe the non-linear dependence structure between factors, exchange rates, and the benchmark. We denote all the factors, exchange rates and benchmark as a vector $\boldsymbol{\xi} = [\boldsymbol{f}_{m_b}^\top, \boldsymbol{f}_{m_1}^\top, ..., \boldsymbol{f}_{m_M}^\top, e_1, ..., e_M, y]^\top$, where $\boldsymbol{f}_m = [f_m^1, \ldots, f_m^{K_m}]^\top, m \in \mathcal{M}$. We introduce the definition and basic properties of the copula model.

Definition 1. (Copula [23]) *An n-dimensional copula, denoted by $C(u) = C(u_1, ..., u_n) : [0,1]^n \to [0,1]$, is a distribution function on $[0,1]^n$ with standard uniform marginal distributions, which satisfies the following three properties:*

(1) $C(u_1, ..., u_n)$ is increasing in each component u_i.

(2) $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for all $i \in 1, ..., n$, $u_i \in [0,1]$.

(3) For all $(a_1, ..., a_n), (b_1, ..., b_n) \in [0,1]^n$ with $a_i \leqslant b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1,i_1}, ..., u_{n,i_n}) \geqslant 0, \tag{3}$$

where $u_{j,1} = a_j$ and $u_{j,2} = b_j$ for all $j \in \{1, ..., n\}$.

Proposition 1. (Sklar's Theorem [23]) *Let F be a joint distribution function with margins $F_1, ..., F_n$. Then there exists a copula $C : [0,1]^n \to [0,1]$ such that, for all $x_1, ..., x_n$ in $\bar{\mathbb{R}} = [-\infty, \infty]$,*

$$F(x_1, ..., x_n) = C\left(F_1(x_1), ..., F_n(x_n)\right). \tag{4}$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on $\mathrm{Ran}F_1 \times \mathrm{Ran}F_2 \times \cdots \times \mathrm{Ran}F_n$, where $\mathrm{Ran}F_i = F_i(\bar{\mathbb{R}})$ denotes the range of F_i. Conversely, if C is a copula and $F_1, ..., F_n$ are univariate distribution functions, then the function F defined in (4) is a joint distribution function with margins $F_1, ..., F_n$.

The idea of Sklar's theorem comes from the fact that an n-dimensional joint distribution function can be decomposed into n marginal distributions and the dependence between these marginal distributions. The copula function proposed by Sklar is such a distribution function that can fully capture the dependence structure between marginal distributions. This theorem provides an efficient way to build suitable multivariate distributions from known marginal distributions.

The copula distribution function also has a density form:

$$c\left(u_1, \cdots, u_n\right) = \frac{f\left(x_1, \cdots, x_n\right)}{\prod_{i=1}^{n} f_i\left(x_i\right)}, \tag{5}$$

where $f(\cdot)$ is the joint density function with marginal densities $f_i(\cdot)$, $i = 1, \ldots, n$.

Another important advantage of copulas is that they do not require the similarity of marginal distributions, and the choice of the copula does not depend on the marginal distributions. Thus, the dependence structure of a copula can be considered independent of the marginal distributions.

2.4. Gaussian copula and Student's t copula

The Gaussian copula is the most basic and widely used class of copula functions, which can capture the location and dispersion feature of the dependence. The Student's t copula can capture fat/light-tailed dependence structure.

Definition 2. *An n-dimensional Gaussian copula with correlation matrix $\boldsymbol{R}$ can be described as following:*

$$C_{\boldsymbol{R}}^{Gaussian}\left(u_1, \cdots, u_n; \boldsymbol{R}\right) = \Phi_{\boldsymbol{R}}\left(\Phi^{-1}\left(u_1\right), \cdots, \Phi^{-1}\left(u_n\right)\right). \tag{6}$$

The density function of the Gaussian copula can be calculated as:

$$c_{\boldsymbol{R}}^{Gaussian}\left(u_1, \cdots, u_n; \boldsymbol{R}\right) = |\boldsymbol{R}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\boldsymbol{Z}^{\top}\left(\boldsymbol{R}^{-1} - I\right)\boldsymbol{Z}\right\}, \tag{7}$$

where $\boldsymbol{Z} = \left(Z_1, \cdots, Z_n\right)^{\top} = \left(\Phi^{-1}\left(u_1\right), \cdots, \Phi^{-1}\left(u_n\right)\right)^{\top}$, $u_i = F_i\left(x_i\right)$, $i = 1, \ldots, n$, Φ_R is the c.d.f. of an n-dimensional Gaussian distribution with mean at zero and correlation matrix R, Φ^{-1} is the inverse distribution function of the standard Gaussian distribution $N(0, 1)$.

Definition 3. *An n-dimensional Student's t copula with degrees of freedom ν and correlation matrix R can be described as following:*

$$C_{\boldsymbol{R},\nu}^{T}\left(u_1, \cdots, u_n; \boldsymbol{R}, \nu\right) = T_{\boldsymbol{R},\nu}\left(T_{\nu}^{-1}\left(u_1\right), \cdots, T_{\nu}^{-1}\left(u_n\right)\right). \tag{8}$$

The density function can be calculated as:

$$c_{\boldsymbol{R},\nu}^{T}\left(u_1, \cdots, u_n; \boldsymbol{R}, \nu\right) = |\boldsymbol{R}|^{\frac{1}{2}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right)^{n} \frac{\left(1 + \frac{1}{\nu}\boldsymbol{Z}^{\top}R^{-1}\boldsymbol{Z}\right)^{-\frac{\nu+n}{2}}}{\prod_{j=1}^{n}\left(1 + \frac{Z_j^2}{\nu}\right)^{\frac{\nu+1}{2}}}, \tag{9}$$

where $Z = \left(Z_1, \cdots, Z_n\right)^{\top} = \left(T_{\nu}^{-1}\left(u_1\right), \cdots, T_{\nu}^{-1}\left(u_n\right)\right)^{\top}$, $T_{\boldsymbol{R},\nu}$ is c.d.f of an n-dimensional t distribution $t_n(\nu, 0, R)$ with mean value 0, correlation matrix $\boldsymbol{R}$ and degree of freedom ν, T_v^{-1} is the quantile function of univariate standard Student's t distribution $t_n(\nu, \mathbf{0}, \boldsymbol{R})$ with degrees of freedom ν.

In practice, it is not easy to specify the best copula function in the model. In our numerical tests, we consider both Gaussian and Student's t copulas as two instances to fit the dependence structure between factors, exchange rates, and the benchmark and then compare their practical performances.

It is worth mentioning that changing the Gaussian/t copula into an arbitrary copula does not affect the tractability of the proposed international portfolio selection model.

2.5. Marginal distributions of factors, exchange rates and benchmark

In order to reflect the high kurtosis and fat-tailed property of the factors, exchange rates, and the benchmark, we select the Student's t distribution as marginal distribution. The density function of a Student's t distribution $t_1(\nu, \mu, \sigma)$ can be written as

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sigma\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left[\frac{\nu + (\frac{x-\mu}{\sigma})^2}{\nu}\right]^{-\frac{\nu+1}{2}},$$

where $\Gamma(\cdot)$ is the Gamma function, μ is the location parameter, σ is the scale parameter and ν is the degree of freedom. In practical use, we can change the Student's t distribution into an arbitrary distribution without reducing the tractability of the proposed international portfolio selection model.

3. Tractable approximation and solution algorithm

In this section, we investigate the solution approach of the international portfolio selection model with chance constraint (IC) by using the partial sampling approximation and the sequence convex approximation approaches.

3.1. Partial sampling approximation

The major difficulty in solving the international portfolio selection problem arises from the chance constraint (1f). First, the chance constraint involves a large number of random variables with different kinds of dependence structures. The curse of dimensionality in distribution estimation and computation makes the classical sampling method inefficient, for instance, the sample approximation approach (SAA). Second, the model contains a product of non-Gaussian distributed exchange rates and return rates. Thus, we can not find a tractable reformulation, like classical chance-constrained portfolio optimization with Gaussian distributed return rates.

We observe that after using factors to explain the return rates, the random variables in the model can be divided into two groups. $\boldsymbol{\xi} = [\boldsymbol{f}, \boldsymbol{e}, y]$ contains factors, exchange rates, and the benchmark which has a copula dependence and t marginal distributions; $\boldsymbol{\delta}$ are the Gaussian distributed residuals, which are independent to each other and $\boldsymbol{\xi}$.

Then, we borrow the idea from [4] and [20] to make a sample approximation to some of the random variables with a highly non-linear dependence structure. At the same time, we keep the other residues still as Gaussian distributions. Thus, we can get a mixture Gaussian distribution approximation, leading to a tractable reformulation.

In detail, according to equation (2), the final wealth w defined in equation (1e) can be expressed as

$$\begin{aligned} w =& r^{\mathrm{rf}}_{m_{\mathrm{b}}} h_{m_{\mathrm{b}}} + \sum_{i=1}^{N_{m_{\mathrm{b}}}} \left(\sum_{k=1}^{K_{m_{\mathrm{b}}}} \beta^k_{i,m_{\mathrm{b}}} f^k_{m_{\mathrm{b}}} + \delta_{i,m_{\mathrm{b}}} + r^{\mathrm{rf}}_{m_{\mathrm{b}}} \right) u_{i,m_{\mathrm{b}}} \\ &+ \sum_{m \in \mathcal{M}_{\mathrm{f}}} \left\{ e_m \left[r^{\mathrm{rf}}_m h_m + \sum_{i=1}^{N_m} \left(\sum_{k=1}^{K_m} \beta^k_{i,m} f^k_m + \delta_{i,m} + r^{\mathrm{rf}}_m \right) u_{i,m} \right] \right\}, \end{aligned} \tag{10}$$

which is constituted by random variables $\boldsymbol{\xi} = [\boldsymbol{f}, \boldsymbol{e}, y]$ and random residuals $\boldsymbol{\delta}$. We separate the random variables into two groups and rewrite the individual chance constraint (1f) as

$$\mathbb{E}_{\boldsymbol{\xi}} \left[\mathbb{P}_{\delta}(w \geq y \mid \boldsymbol{\xi})\right] \geq 1 - \epsilon,$$

by the tower property of the expected value function.

Then, we approximate the distribution of $\boldsymbol{\xi}$ with L i.i.d samples, $\boldsymbol{\xi}(1), \boldsymbol{\xi}(2), ..., \boldsymbol{\xi}(L)$, where

$$\boldsymbol{\xi}(l) = [\boldsymbol{f}_{m_b}(l)^\top, \boldsymbol{f}_{m_1}(l)^\top, \boldsymbol{f}_{m_2}(l)^\top, ..., e_1(l), ..., e_M(l), y(l)]^\top, l = 1, ..., L,$$

are drawn from the copula function with marginal distributions defined in Sections 2.3 and 2.5 by using the Monte-Carlo method.

By using the samples of $\boldsymbol{\xi}$, we get a sample average approximation of the outer-level expected value function

$$\frac{1}{L} \sum_{l=1}^{L} \left[\mathbb{P}(w \geq y(l) \mid \boldsymbol{\xi} = \boldsymbol{\xi}(l))\right] \geq 1 - \epsilon, \tag{11}$$

which is equivalent to the following group of constraints

$$\sum_{l=1}^{L} z_l \geq L(1 - \epsilon), \tag{12}$$

$$\mathbb{P}(w \geq y(l) \mid \boldsymbol{\xi} = \boldsymbol{\xi}(l)) \geq z_l, l = 1, ..., L, \tag{13}$$

with auxiliary variables $\boldsymbol{z} = [z_1, z_2, \ldots, z_L]^\top \in [0, 1]^L$.

We can find from (10) that w is linear to the Gaussian distributed residuals $\delta_{i,m}, i = 1, ...N_m$, $m \in \mathcal{M}$. And $\delta_{i,m}$ is assumed to follow i.i.d. Gaussian distribution in Section 2.2. Thus, conditional on the sample value $\boldsymbol{\xi} = \boldsymbol{\xi}(l)$, w follows a univariate Gaussian distribution with mean value

$$\begin{aligned}\mu(l) =& r^{\mathrm{rf}}_{m_\mathrm{b}} h_{m_\mathrm{b}} + \boldsymbol{u}^\top_{m_\mathrm{b}} \boldsymbol{B}_{m_\mathrm{b}} \boldsymbol{f}_{m_\mathrm{b}}(l) + \boldsymbol{\mu}^\top_{m_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + r^{\mathrm{rf}}_{m_\mathrm{b}} \mathbf{1}^\top \boldsymbol{u}_{m_\mathrm{b}} \\ &+ \sum_{m \in \mathcal{M}_\mathrm{f}} e_m(l)(r^{\mathrm{rf}}_m h_m + \boldsymbol{u}^\top_m \boldsymbol{B}_m \boldsymbol{f}_m(l) + \boldsymbol{\mu}^\top_m \boldsymbol{u}_m + r^{\mathrm{rf}}_m \mathbf{1}^\top \boldsymbol{u}_m),\ l = 1, \ldots, L,\end{aligned} \tag{14}$$

and variance

$$\sigma^2(l) = \boldsymbol{u}^\top_{m_\mathrm{b}} \boldsymbol{\Sigma}_{\boldsymbol{m}_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m \in \mathcal{M}_\mathrm{f}} e_m(l)^2 \boldsymbol{u}^\top_m \boldsymbol{\Sigma}_{\boldsymbol{m}} \boldsymbol{u}_m,\ l = 1, \ldots, L, \tag{15}$$

where $\boldsymbol{B}_m = [\boldsymbol{\beta}_{1,m}, \boldsymbol{\beta}_{2,m}, ..., \boldsymbol{\beta}_{N_m,m}]^\top$, $\boldsymbol{\beta}_{i,m} = [\beta^1_{i,m}, \beta^2_{i,m}, ..., \beta^{K_m}_{i,m}]^\top$, $i = 1, ..., N_m$, $m \in \mathcal{M}$.

We have from the property of the Gaussian distribution that

$$\mathbb{P}(w \geq y(l) \mid \boldsymbol{\xi} = \boldsymbol{\xi}(l)) = \mathbb{P}\left(\frac{-w + \mu(l)}{\sigma(l)} \leq \frac{-y(l) + \mu(l)}{\sigma(l)} \mid \boldsymbol{\xi} = \boldsymbol{\xi}(l)\right) = \Phi\left(\frac{-y(l) + \mu(l)}{\sigma(l)}\right),$$

where $\frac{-w+\mu(l)}{\sigma(l)}$ follows a standard Gaussian distribution $N(0, 1)$ conditional on $\boldsymbol{\xi} = \boldsymbol{\xi}(l)$, $l = 1, 2, \ldots, L$. $\Phi(\cdot)$ is the distribution function of $N(0, 1)$.

Then we can reformulate (13) as the following SOCP constraints

$$\Phi^{-1}(z_l)\sqrt{\boldsymbol{u}_{m_\mathrm{b}}^\top \Sigma_{m_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)^2 \boldsymbol{u}_m^\top \boldsymbol{\Sigma_m} \boldsymbol{u}_m} - r_{m_\mathrm{b}}^{\mathrm{rf}} h_{m_\mathrm{b}} - \boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{B}_{m_\mathrm{b}} \boldsymbol{f}_{m_\mathrm{b}}(l) - \boldsymbol{\mu}_{m_\mathrm{b}}^\top \boldsymbol{u}_{m_\mathrm{b}}$$
$$- r_{m_\mathrm{b}}^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_{m_\mathrm{b}} - \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)(r_m^{\mathrm{rf}} h_m + \boldsymbol{u}_m^\top \boldsymbol{B}_m \boldsymbol{f}_m(l) + \boldsymbol{\mu}_m^\top \boldsymbol{u}_m + r_m^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_m) + y(l) \le 0,$$
$$l = 1, ..., L, \tag{16}$$

where $\Phi^{-1}(\cdot)$ is the inverse distribution (quantile) function of $N(0,1)$.

Then we have an approximation of (IC),

$$\begin{aligned}
(IC_L) \quad \max_{u,b,s,h,g,q,z} \quad & \frac{1}{L}\sum_{l=1}^{L}\Bigg(r_{m_\mathrm{b}}^{\mathrm{rf}} h_{m_\mathrm{b}} + \boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{B}_{m_\mathrm{b}} \boldsymbol{f}_{m_\mathrm{b}}(l) + \boldsymbol{\mu}_{m_\mathrm{b}}^\top \boldsymbol{u}_{m_\mathrm{b}} + r_{m_\mathrm{b}}^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_{m_\mathrm{b}} \\
& + \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)(r_m^{\mathrm{rf}} h_m + \boldsymbol{u}_m^\top \boldsymbol{B}_m \boldsymbol{f}_m(l) + \boldsymbol{\mu}_m^\top \boldsymbol{u}_m + r_m^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_m)\Bigg) \\
\text{s.t.} \quad & (1\mathrm{b}) - (1\mathrm{d}), (1\mathrm{g}) - (1\mathrm{j}), (12), (16), 0 \le z_l \le 1, l = 1, ..., L.
\end{aligned}$$

Compared with the SAA method, we only generate samples for a small subset of random variables. The SAA method would face the dimensional curse problem as it generates samples for all random variables when we apply both methods to a large-scale problem.

3.2. Convergence analysis of the partial SAA

Applying the convergence results of the SAA method to the outer-level approximation, we can show that (IC_L) converges to (IC) when we use many enough i.i.d. samples.

Theorem 1. *Suppose some constraint qualification for (IC) holds such that for some optimal solution x of (IC), there exists a sequence of points x_L in feasible set of (IC_L) with L i.i.d samples such that $x_L \to x$ w.p.1. The optimal value of (IC_L) converges to the optimal value of (IC) when $L \to +\infty$.*

Proof. We denote

$$C(x, \boldsymbol{\xi}) = \mathbb{P}(w(x) \le y \mid \boldsymbol{\xi}) = \mathbb{E}[\mathbf{1}_{[0,\infty]}(y - w(x))) \mid \boldsymbol{\xi}].$$

As $w(x)$ is continuously distributed, the constraint (1f) can be written as

$$p(x) = \mathbb{E}_{\boldsymbol{\xi}}[C(x, \boldsymbol{\xi})] \le \epsilon,$$

and the sample average approximation constraint can be written as

$$\hat{p}_L(x) = \frac{1}{L}\sum_{l=1}^{L}[C(x, \xi(l))] \le \epsilon.$$

By the property of conditional expectation operator, we have that $C(x, \boldsymbol{\xi})$ is random lower

semi-continuous. Then by Theorem 7.48 and 7.51 in [7], $p(x)$ is continuous and $\hat{p}_L(x)$ converges to $p(x)$ w.p.1 uniformly on the feasible set of x.

Then we can obtain the convergence result of the optimal value by Theorem 5.5 in [7].

□

3.3. Sequential convex approximation algorithm

The problem (IC_L) is a non-convex optimization problem due to the bi-convex terms in (16). We thus consider using the sequential convex approximation approach. The basic idea is to decompose the problem into sub convex problems under the condition that a subset of variables is fixed.

For problem (IC_L), we first fix $\boldsymbol{z} = \boldsymbol{z}^n$ and update $\boldsymbol{u}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{h}, \boldsymbol{g}, \boldsymbol{q}$ by solving

$$
\begin{aligned}
&(SQ_1) \\
&\max_{u,b,s,h,g,q} \quad \frac{1}{L}\sum_{l=1}^{L}\Bigg(r_{m_\mathrm{b}}^{\mathrm{rf}} h_{m_\mathrm{b}} + \boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{B}_{m_\mathrm{b}} \boldsymbol{f}_{m_\mathrm{b}}(l) + \boldsymbol{\mu}_{m_\mathrm{b}}^\top \boldsymbol{u}_{m_\mathrm{b}} + r_{m_\mathrm{b}}^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_{m_\mathrm{b}} \\
&\qquad + \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)(r_m^{\mathrm{rf}} h_m + \boldsymbol{u}_m^\top \boldsymbol{B}_m \boldsymbol{f}_m(l) + \boldsymbol{\mu}_m^\top \boldsymbol{u}_m + r_m^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_m)\Bigg) && (17)\\
&\text{s.t.} \quad (1\mathrm{b}) - (1\mathrm{d}), (1\mathrm{g}) - (1\mathrm{j}), && (18)\\
&\quad \Phi^{-1}(z_l^n)\sqrt{\boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{\Sigma}_{\boldsymbol{m}_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)^2 \boldsymbol{u}_m^\top \boldsymbol{\Sigma}_{\boldsymbol{m}} \boldsymbol{u}_m} - r_{m_\mathrm{b}}^{\mathrm{rf}} h_{m_\mathrm{b}} - \boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{B}_{m_\mathrm{b}} \boldsymbol{f}_{m_\mathrm{b}}(l) \\
&\quad -\boldsymbol{\mu}_{m_\mathrm{b}}^\top \boldsymbol{u}_{m_\mathrm{b}} - r_{m_\mathrm{b}}^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_{m_\mathrm{b}} - \sum_{m\in\mathcal{M}_\mathrm{f}} e_m(l)(r_m^{\mathrm{rf}} h_m + \boldsymbol{u}_m^\top \boldsymbol{B}_m \boldsymbol{f}_m(l) + \boldsymbol{\mu}_m^\top \boldsymbol{u}_m \\
&\quad + r_m^{\mathrm{rf}} \mathbf{1}^\top \boldsymbol{u}_m) + y(l) \le 0, l = 1, ..., L, && (19)
\end{aligned}
$$

and then fix $\boldsymbol{u} = \boldsymbol{u}^n, \boldsymbol{b} = \boldsymbol{b}^n, \boldsymbol{s} = \boldsymbol{s}^n, \boldsymbol{h} = \boldsymbol{h}^n, \boldsymbol{g} = \boldsymbol{g}^n, \boldsymbol{q} = \boldsymbol{q}^n$ and update $\boldsymbol{z}$ by solving

$$
\begin{aligned}
(SQ_2) \quad &\max_{\boldsymbol{z}\in\mathbb{R}_+^L} \quad \sum_{l=1}^{L} \phi_l z_l && (20)\\
&\text{s.t.} \quad z_l \le \Phi\left(\frac{-y(l)+\mu(l)}{\sigma(l)}\right),\ l = 1, \cdots, L, && (21)\\
&\qquad \sum_{l=1}^{L} z_l \ge L(1-\epsilon), && (22)\\
&\qquad 0.5 \le z_l \le 1,\ l = 1, \ldots, L, && (23)
\end{aligned}
$$

where $\mu(l)$ and $\sigma(l)$ are calculated by (14) and (15). $\boldsymbol{\phi} = [\phi_1, \ldots, \phi_L]^\top$ is an artificial search direction for $\boldsymbol{z}$ as the objective function in (IC_L) does not contain $\boldsymbol{z}$.

We then solve (SQ_2) and (SQ_2) alternately in each iteration of the algorithm. The flowchart of the sequential convex approximation is given in Algorithm 1.

Proposition 2. *With a fixed $\boldsymbol{z}$ feasible to (SQ_2), the problem (SQ_1) is a convex programming problem; with fixed $\boldsymbol{u}, \boldsymbol{b}, \boldsymbol{s}, \boldsymbol{h}, \boldsymbol{g}, \boldsymbol{q}$ found by (SQ_1), (SQ_2) is a linear program.*

Algorithm 1 Sequential convex approximation

Require:

Choose an initial point $\boldsymbol{z}^0$ of $\boldsymbol{z}$ feasible to (22)-(23). Set $n = 1$.

Ensure:

while $n \leq n_{max}$ and $||\boldsymbol{z}^{n-1} - \boldsymbol{z}^n|| \geq \epsilon$ **do**

- Solve problem (SQ_1) with $\boldsymbol{z}^{n-1}$; let $\boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{s}^n, \boldsymbol{h}^n, \boldsymbol{g}^n, \boldsymbol{q}^n$ be an optimal solution of (SQ_1), let $\boldsymbol{\kappa}^n$ be the optimal Lagrangian multiplier (dual variable) of constraint (19), and let $\boldsymbol{v}^n$ be the optimal value of (SQ_1).
- Solve problem (SQ_2) with $\boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{s}^n, \boldsymbol{h}^n, \boldsymbol{g}^n, \boldsymbol{q}^n$, $\boldsymbol{\kappa}^n$, where

$$\phi_l = \kappa_l^n \cdot (\Phi^{-1})'(z_l^n) \sqrt{\boldsymbol{u}_{m_\mathrm{b}}^\top \Sigma_{m_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m \in \mathcal{M}_\mathrm{f}} e_m(l)^2 \boldsymbol{u}_m^\top \boldsymbol{\Sigma_m} \boldsymbol{u}_m},$$

  and $\mu(l)$ and $\sigma(l)$ are updated by (14) and (15) with $\boldsymbol{u}^n$.
  let $\tilde{\boldsymbol{z}}$ denote an optimal solution of (SQ_2).
- $\boldsymbol{z}^{n+1} \leftarrow \boldsymbol{z}^n + \tau(\tilde{\boldsymbol{z}} - \boldsymbol{z}^n)$, $n \leftarrow n + 1$. Here, $\tau \in (0, 1)$ is the step length.

end while

Output: $\boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{s}^n, \boldsymbol{h}^n, \boldsymbol{g}^n, \boldsymbol{q}^n$, $\boldsymbol{v}^n$

Proof. The only key point is to show the convexity of the term

$$\sqrt{\boldsymbol{u}_{m_\mathrm{b}}^\top \boldsymbol{\Sigma}_{m_\mathrm{b}} \boldsymbol{u}_{m_\mathrm{b}} + \sum_{m \in \mathcal{M}_\mathrm{f}} e_m(l)^2 \boldsymbol{u}_m^\top \boldsymbol{\Sigma_m} \boldsymbol{u}_m},$$

and the positiveness of $\Phi^{-1}(z_l^n)$. As the covariance matrix of the residuals from the multi-factor model is diagonal, the square root term can be reformulated as

$$\sqrt{\sum_{i=1}^{N_{m_b}} (\sigma_{i,m_\mathrm{b}} u_{i,m_\mathrm{b}})^2 + \sum_{m \in \mathcal{M}_\mathrm{f}} \sum_{i=1}^{N_m} (e_m(l) \sigma_{i,m} u_{i,m})^2},$$

which is a composition of the outer-level L_2-norm function, which is increasing and convex, and the inner-level linear terms.

The inverse distribution function $\Phi^{-1}(x)$ is nonnegative on $(0.5, 1)$. The last constraint in (SQ_2) guarantees that the value of z_l at each iteration step is larger than or equal to 0.5.

□

When these sub-problems are all convex, the objective function is continuous, and the feasible set is closed, the sequence convex approximation algorithm converges monotonically to a partial optimum point of (IC_L) (Theorem 4.7 [10]). When the objective function is a differentiable and biconvex function, (x, z) is a partial optimum point if and only if (x, z) is a stationary point (Corollary 4.3 [10]). Thus, the proposed algorithm converges to a stationary point of (IC_L).

4. Numerical experiments

In this section, we carry out a series of numerical tests of the proposed international portfolio selection model in practical international investment. Section 4.1 introduces the setting of the international portfolio selection model. In Section 4.2, we use the historical data in USA/Hong Kong/European markets to estimate the parameters of the factor model and the copula model. We then apply the model to the out-of-sample investment simulation by a rolling forward approach. We then show the statistics and cumulative curves of the out-of-sample portfolio return series using two different copulas in Sections 4.3 and 4.4. To better illustrate the superior performance of the proposed international portfolio selection model, we compare the performance with chance constrained portfolio selection model in a single market in Section 4.5.

4.1. Test setting

We, in this subsection, introduce the choice of the benchmark, stock pool, market factors, and exchange rates. We then introduce the basic setting of the investment procedure in the out-of-sample period.

4.1.1. Stock pool

We assume that the international portfolio selection model is run by an international investment fund based in the USA. The three targeting investment markets are the domestic USA market (base market), the European market, and the Hong Kong market.

We choose 55 stocks from Dow Jones Average Index in the USA market, 27 stocks from Euro Stoxx 50 Index in the European market, and 45 stocks from Hang Seng Index in the Hong Kong market to constitute the stock pool (Table 2). We select these stocks as they have complete historical data between 2006-2021. We collect these stocks' daily closed prices (in local currency) from November 01, 2006, to August 31, 2021, and compute their daily return rates starting from November 02, 2006.

Table 2: Stock pool

USA	AAPL	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE
	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM
	MRK	MSFT	NKE	PFE	PG	TRV	UNH	V	VZ
	WMT	XOM	ALEX	CAL	CHRW	CSX	EXPD	FDX	JBLU
	LSTR	LUV	NSC	R	UNP	UPS	AEP	AES	CNP
	D	DUK	ED	EIX	EXC	FE	FPL	NI	PCG
	PEG	SO	WMB						
Europe	ABI	AD	AIR	AIRP	ASML	AXAF	BNPP	DANO	ENGIE
	ENI	ESLX	INGA	ISP	LVMH	OREP	PERP	PHG	PRTP
	SAF	SAN	SASY	SCHN	SGEF	TTEF	VIV		
Hong Kong	00002	00003	00005	00006	00011	00012	00016	00027	00066
	00101	00175	00267	00291	00386	00388	00669	00688	00700
	00762	00823	00857	00868	00883	00939	00941	01038	01044
	01093	01109	01177	01211	01398	02018	02313	02318	02319
	02331	02388	02628	02688					

We show the accumulative return rate process of Dow Jones Average index, Hang Seng index and EURO STOXX index from 2008.11 to 2021.08 in Figure 3 to reflect the changing environments of the three markets.

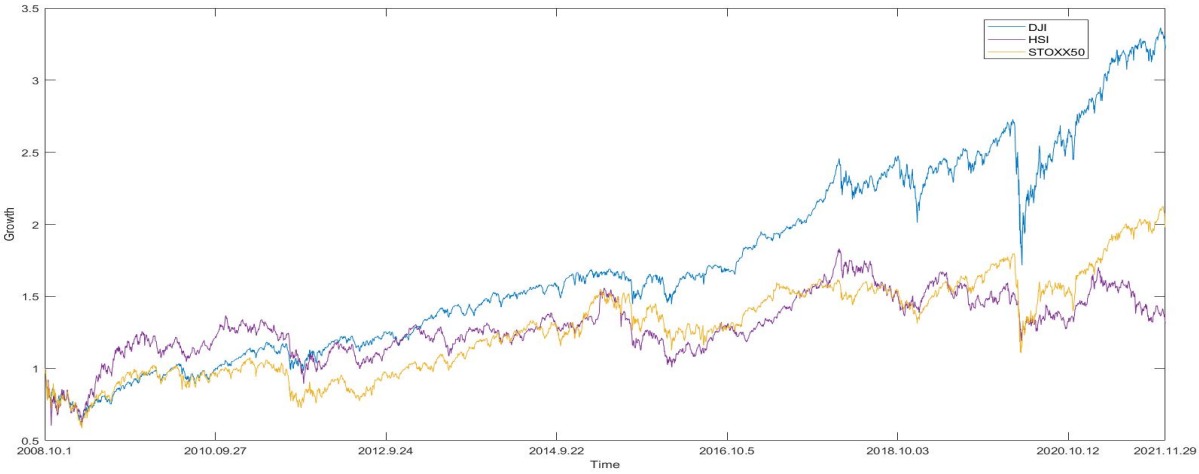


Figure 3: Accumulative return rate process (accounted in US dollars, discounted with exchange rate) of Dow Jones Average Index (USA market), Hang Seng index (Hong Kong market) and EURO STOXX (Europe market) from 2008.11 to 2021.08

4.1.2. Benchmark in chance constraint

The chance constraint of our model is to ensure that a lower bound of the international portfolio's daily return rate compared to the daily growth rate of S&P 500, with a large probability. We set the probability level as 90%. At each out-of-sample day, the random benchmark is 98% value of the S&P 500 investing wealth on the next day, which is the wealth we will have if we invest all wealth in S&P 500. To compute the observations of the benchmark, we fit a copula model of the return rate of S&P500 together with market factors and exchange rates and then generate some samples. We then use the samples of the return rate of S&P500 together with the wealth on the previous day to compute the observations/samples of the benchmark.

4.1.3. Market factors

We consider the basic three Fama-French factors for each market, i.e., Mkt-RF, SMB, and HML. We select Fama-French U.S. 3 Factors, Fama-French European 3 Factors and Fama-French Asia Pacific ex Japan 3 Factors for the three markets, respectively. We collect the daily historical data of these factors from November 02, 2006 to August 31, 2021 [2].

Table 3 presents some statistics of factors and corresponding Dickey-Fuller test results. The null hypothesis of the DF test is that a unit root is present in the time series model. In the table, the value of 'Decision' equal to 1 means we reject the unit-root null hypothesis, while 0 indicates a failure to reject the unit-root null. From the table, we can see that P-values are

[2]The historical data are downloaded from `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

near 0 and DF test statistics are less than 5% critical value, which means we can reject the null hypothesis with 95% confidence. Thus, we can believe that all factors series are stable.

Table 3: Basic statistics and Dickey-Fuller test of daily Fama-French factors in the three markets from November 02, 2006, to August 31, 2021

	USA			Asia			Europe		
	Mkt	SMB	HML	Mkt	SMB	HML	Mkt	SMB	HML
Mean	0.046	0.004	-0.013	0.028	-0.010	0.009	0.025	0.003	-0.01
Std.	1.30	0.62	0.81	1.14	0.56	0.55	1.30	0.53	0.52
Skewness	-0.35	0.18	0.49	-0.58	-1.20	0.20	-0.30	-0.80	0.25
Kurtosis	11.80	4.86	8.15	9.60	12.41	3.74	9.39	8.68	6.05
DF test statistic	-15.05	-45.91	-10.55	-22.51	-33.18	-19.70	-23.79	-10.62	-26.22
5% critical value	-2.86	-2.86	-2.86	-2.86	-2.86	-2.86	-2.86	-2.86	-2.86
P-value	0	0	0	0	0	0	0	0	0
Decision	1	1	1	1	1	1	1	1	1

4.1.4. Exchange rates

We choose USD as base currency. All currency exchanges are executed with respect to the base currency. Hence, we collect the daily exchange rate between HKD and USD as well as ERU and USD from November 01,2006 to August 31,2021.

The Dickey-Fuller test results show that the historical daily exchange rate series is unstable. Thus, we take the first-order difference to the exchange rate data. The Dickey-Fuller test results show that the growth rate (GR) series of the exchange rates is stable. Table 4 presents the statistics and results of the tests.

Due to the non-stationariness of the exchange rate series, we fit the copula model with the growth rate of the exchange rate rather than the exchange rates themselves.

Table 4: Basic statistics of and Dickey-Fuller test of historical exchange rate series and their growth rate (GR) series from November 02, 2006 to August 31, 2021

	Mean	Std.	Skewness	Kurtosis	DF test statistic	5% critical value	P-value	Decision
HKD/USD	0.13	0.00	-1.03	-0.22	-2.81	-2.86	0.06	0
EUR/USD	1.26	0.13	0.30	-0.82	-1.83	-2.86	0.36	0
GR(HKD/USD)	0.00	0.00	0.49	9.26	-13.88	-2.86	0	1
GR(EUR/USD)	0.00	0.01	0.11	2.70	-61.56	-2.86	0	1

4.1.5. Transaction settings

We have the following settings for transactions in the international investment process.

- The initial wealth at the beginning of the first out-of-sample investment period is 1,000 US dollars in cash;
- The transaction cost is proportional to transaction wealth with both buying cost rate c_b and selling cost rate c_s to be 0.001 in all three markets;
- The risk-free rate is $r_m^{rf} = 0$ in all markets;

- The maximal holding proportion of a single stock is $\lambda = 0.3$ for all markets;
- The upper bound of the maximal cash transfer into a foreign market in one period is 30% of total wealth. The upper bound of the cash amount maximal transfer out from a foreign market is 30% of total wealth (account in local currency);
- The forex transfer cost is proportional to transaction wealth with a cost rate of 0.01. All forex transfers can be realized in real-time;
- The maximal holding limitation of a single stock is 30% of current wealth.

4.1.6. Out-of-sample test procedure: a rolling forward way

We divide all historical data into two parts: the in-of-sample period is from 2006.11.02 to 2008.11.03., and the out-of-sample period is from 2006.11.03 to 2021.8.31.

We carry out the out-of-sample test in a rolling forward way. We first use the data in the in-sample period to determine the optimal portfolio of the tested model. We invest with the optimal portfolio for the first out-of-sample day and compute the portfolio return with the actual return data on that out-of-sample day. Then, we update the in-sample period by adding the new day in the out-of-sample period and removing the first day in the in-sample period. We then estimate the distribution of the factors and exchange rates by using the data in the updated in-sample period, re-solve the resulting portfolio selection problem, determine the optimal portfolio, and compute its return rates on the second day of the out-of-sample period. We carry out the out-of-sample test by rolling forward until the end of the out-of-sample period; this would provide us with a return series with 3309 out-of-sample daily return rates.

4.2. Copula fitting

At each out-of-sample day, we fit a copula model with the historical data of market factors, the growth rate of the exchange rates, and the S&P500 index in the in-sample period. We always fit the marginal distribution as a Student's t distribution. We use two kinds of copulas, the Gaussian copula and the Student's t copula, to fit the dependence structure of the selected random variables, respectively. We take the t copula as the instance to illustrate the fitting procedure in what follows. The fitting procedure for the Gaussian copula is likewise. To illustrate the fitting procedure, we show the fitting results with all data, including both the in-sample and out-of-sample periods.

Step 1 Fitting marginal distribution. By Tables 3 and 4, we can find that the historical data of market factors, exchange rates, and the benchmark are high kurtosis and heavy-tailed. Thus we fit their marginal distribution by a Student's t distribution, which can better capture the leptokurtosis nature of the data than the Gaussian distribution. We plot some of the fitted marginal distribution in Figure 4 and the estimated distribution parameters in Table 5. From the figure we can observe that the Student's t distribution can better capture the distribution characteristics of the data.

Step 2 Generating pseudo-observations. We generate some pseudo-observations $\widehat{U}_t$ lie in $(0, 1)$, from the original data $X_{i,j}$, $i = 1, \ldots, d$, $j = 1, \ldots, n$, of the market factors, exchange rates and benchmark, by using the c.d.f. function F_j, $j = 1, \ldots, n$, of marginal distributions fitted in step 1. d is the sample size. $n = 12$ is the dimensional of the copula function.

$$\widehat{U}_i = (\widehat{U}_{i,1}, ..., \widehat{U}_{i,n})^\top := (F_1(X_{i,1}), ..., F_n(X_{i,n}))^\top, \ i = 1, ..., d.$$

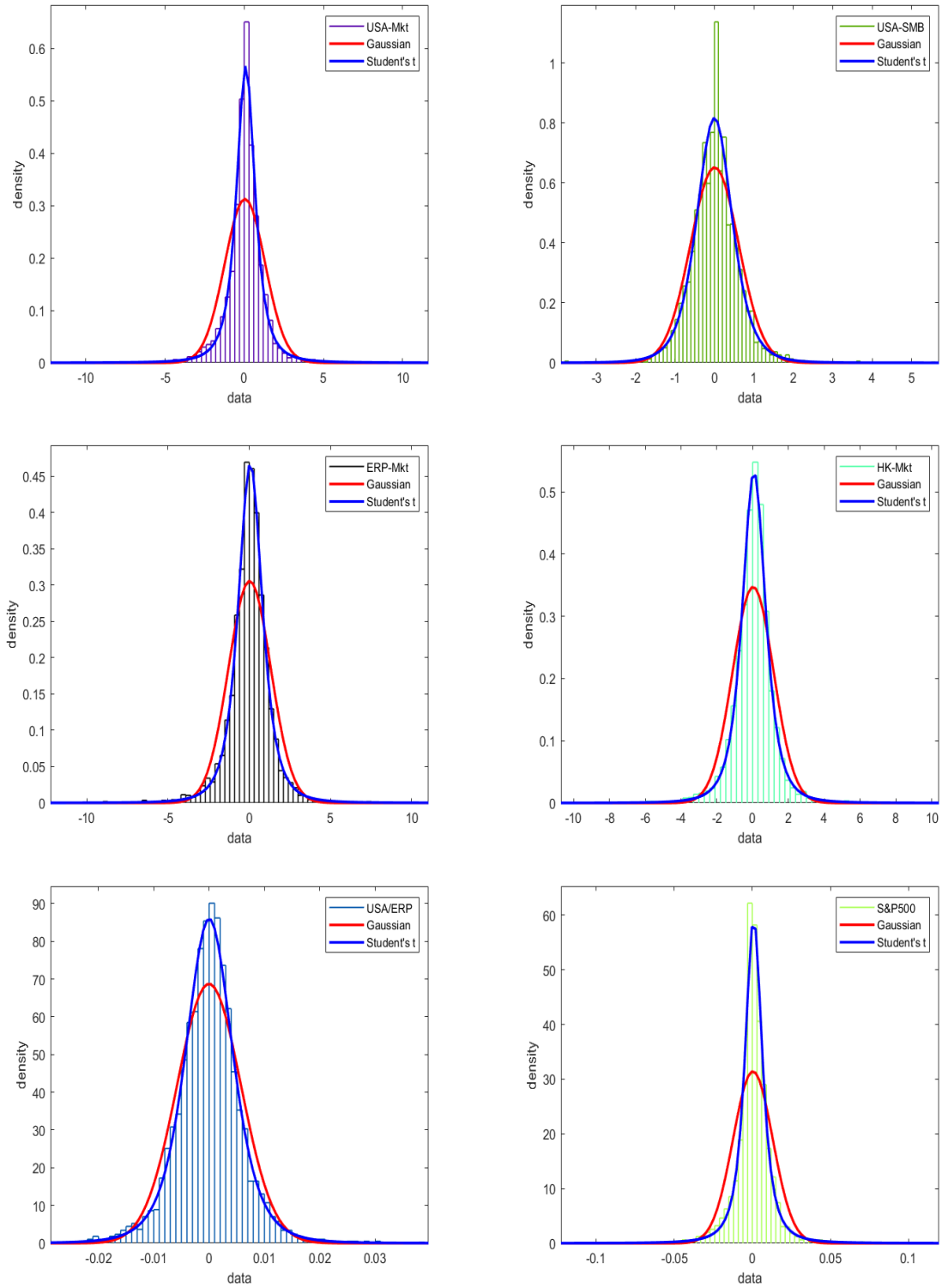


Figure 4: Density functions of some marginal distributions (with Gaussian and Student's t fitting)

Table 5: Estimated parameters of some marginal Student's t distributions

	USA			European			Hong Kong			growth rate of exchange rate		benchmark
	Mkt	SMB	HML	Mkt	SMB	HML	Mkt	SMB	HML	EUR/USD	HKD/USD	S&P500
Location	0.098	-0.001	-0.034	0.067	0.014	-0.021	0.076	0.008	0.002	2.6E-05	-2.4E-10	0.001
Scale	0.627	0.463	0.400	0.782	0.353	0.358	0.687	0.365	0.403	0.004	1.0E-06	0.006
Degree of Freedom	2.132	4.676	2.160	2.834	3.393	3.527	2.857	3.476	4.125	4.453	0.298	2.029

Step 3 Maximizing the likelihood function. We show the detailed procedure of fitting t copula. The fitting for the Gaussian copula is similar. By using the explicit formula of the density function (9), we can write the log-likelihood function of t copula $c^T_{R,\nu}$ with d samples as:

$$\ln L(R,\nu;\widehat{U}_1,...,\widehat{U}_d) = \sum_{i=1}^{d} \ln t_{R,\nu}\left(T_\nu^{-1}(\widehat{U}_{i,1}),\cdots,T_\nu^{-1}(\widehat{U}_{i,n})\right) - \sum_{i=1}^{d}\sum_{j=1}^{n} t_\nu\left(T_\nu^{-1}(\widehat{U}_{i,j})\right). \quad (24)$$

Here $t_{R,\nu}$ is the joint density of n-dimensional t distribution $t_n(\nu,0,R)$ with mean value 0, degree of freedom ν and linear correlation matrix R. t_ν is the density of univariate Student's t distribution $t_1(\nu,0,1)$ with degrees of freedom ν and mean value 0.

Solving the above maximum likelihood estimation problem

$$\{\widehat{R},\widehat{\nu}\} = \arg\max_{R,\nu} \ln L(R,\nu;\widehat{U}_1,...,\widehat{U}_d)$$

yields the optimal parameter of the Student's t copula function.

4.3. Out-of-sample performance

By using the rolling forward approach introduced in Section 4.1.6, we carry out the out-of-sample test for the international portfolio selection models with Gaussian copula and Student's t copula, respectively. This rolling forward procedure provides us with two return series with 3309 out-of-sample daily return rates. We show the wealth process of the Gaussian copula model in the out-of-sample period in Figure 5, as well as the optimal portfolio allocation in different markets. Figure 6 shows two more detailed figures.

The corresponding wealth process and portfolio allocation process of the Student's t copula model are shown in Figure 7 and Figure 8.

4.3.1. Portfolio process of the Gaussian copula model

From Figure 3, Figure 5 and Figure 6, we have the following observations:

- The international investment model with Gaussian copula (denoted as 'Gaussian model') mainly invests in the USA and Hong Kong markets, and does not invest in European market all the time.
- The Gaussian copula model initially invests in the USA market, with a small portion in stocks and a large portion in risk-free asset. When the growth rate of the Hang Seng Index surged in 2009, the model started to transfer money from the USA market to the Hong Kong market at the end of 2009. After that, the proportions invested in Hong Kong and USA markets stabilized at 0.8 and 0.2, respectively, with slight fluctuations.

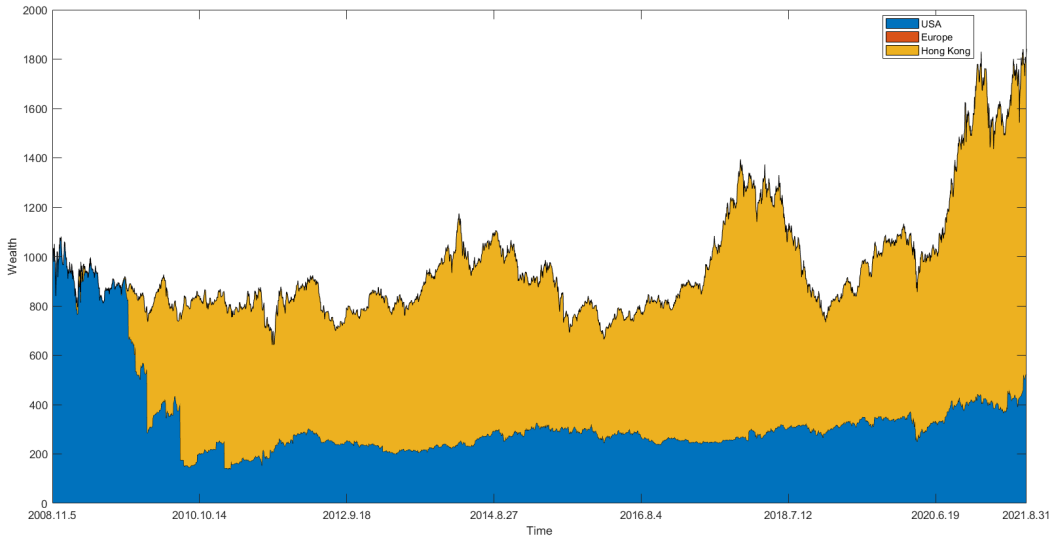
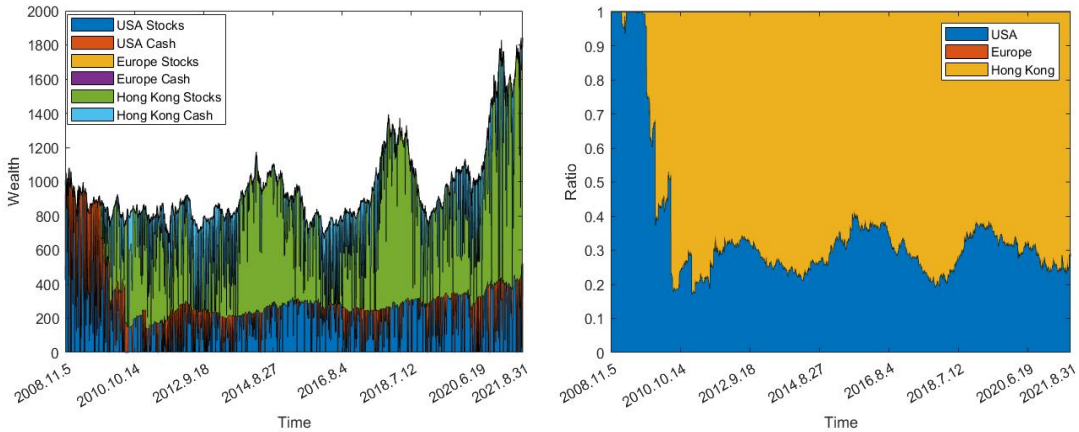


Figure 5: Out-of-sample wealth process and portfolio composition of the Gaussian copula model



(a) Detailed portfolio composition process of the Gaussian copula model

(b) The ratio of portfolio composition process of the Gaussian copula model

Figure 6: Detailed out-of-sample portfolio composition process of the Gaussian copula model

- The accumulative wealth of the Gaussian model reaches its peaks in 2015, 2018, and 2021 and then falls back, which is in line with the Hang Seng Index.
- The Gaussian model is not sensitive to market decline. It cannot withdraw investment in time when the Hong Kong market falls, and still mainly invests funds in Hong Kong market.

4.3.2. Portfolio process of the Student's t copula model

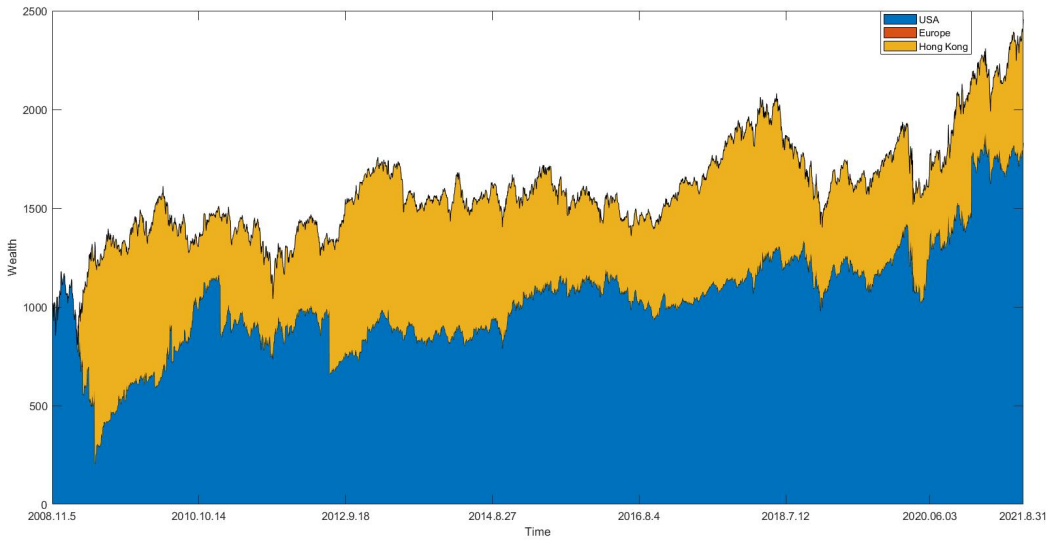
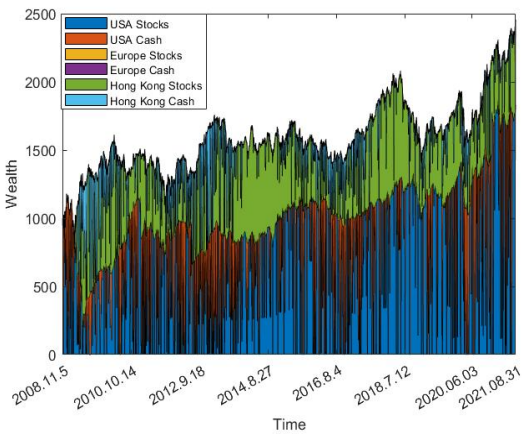
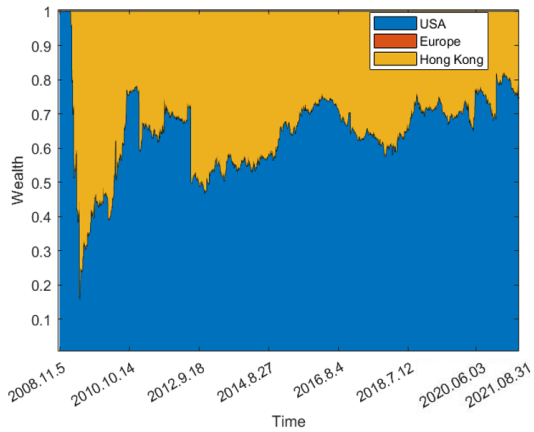


Figure 7: Out-of-sample portfolio composition process of the Student's t copula model



(a) Detailed portfolio composition process of the Student's t copula model



(b) The ratio of portfolio composition process of the Student's t copula model

Figure 8: Detailed out-of-sample portfolio composition process of the Student's t copula model

From Figure 3, Figure 7 and Figure 8, we have the following observations:

- The international investment model with Student's t copula (denoted as 't copula model') does not invest in European market all the time.
- The t copula model invests balancedly in the USA and Hong Kong markets. It started to transfer money from the USA market to the Hong Kong market at the beginning of 2009, similar to the Gaussian copula model. With the development of time, the proportion invested in the USA market rebounds to 60%-70%.
- The wealth accumulation of the t copula model is more stable than the Gaussian copula model, with only one major draw-down after 2018.
- From 2016 to 2018, the t copula model increased the proportion of investment in the Hong Kong market due to the prosperity of the Hong Kong market (can be observed from the Hang Seng Index). After the decline of the Hong Kong market at the end of 2018, the model retreats its investment back to the USA market. The proposition invested in the USA market rebounded to 70% after 2018.
- The t copula model is much more sensitive to market fluctuations than the Gaussian copula model. It responds quickly when the market turns to bull or bear. For instance, it transferred money to Hong Kong earlier in 2009 when the Hong Kong market surged. It retreated money to the USA when the Hong Kong market retreated, avoiding large swings in overall wealth accumulation. Aggregate wealth grows more steadily and accumulates more than the Gaussian model at the end of the period. This phenomenon happens whenever the environment of the Hong Kong market or the USA market changes.
- The t copula model always captures the market environment changes and reacts to market volatility much earlier than the Gaussian copula model. As a result, it can distribute wealth internationally more rationally, which leads to higher final wealth.

4.4. Analysis of the observations

4.4.1. The cruel truth: Europe is keeping in recession

By 2020, European Union's (EU) GDP has never surpassed its 2008 peak in these 13 years due to the impact of the European sovereign debt crisis, the refugee crisis in Europe, and the Russia-Ukraine conflict.

For instance, in 13 years, Germany's GDP has grown by only 13.14%, the highest of the EU's four central European countries. As for the UK and France, their GDP growth rates are 2.57% and 0.68%, respectively. Even more brutal, Italy's GDP fell by 12.08% in 13 years. At the same time, EU GDP grew by 5.23% in 13 years from 2008 to 2021. However, the global GDP grew by 48.32% in 13 years, nearly ten times the EU's. That is to say, the economic growth rate of the EU is far below the global average growth rate. Moreover, considering the cumulative inflation rate, Europe's economic growth rate is negative in net value. Overall, the European economy has been in recession in the out-of-sample period we studied.

Thus, it illustrates the reasonability and rationality of our international portfolio selection model, which chooses not to invest in the European market to avoid market recession. It also shows the importance of considering international portfolio management rather than a single market.

4.4.2. International transfer between USA and Hong Kong: keeping away from systemic risk in a single market

According to the discussion above, it is reasonable that our model only invests in USA and Hong Kong markets. Compared with the continued recession in European markets, the economies of the USA and Hong Kong recovered quickly from the great recession in 2008-2009. The proposed international portfolio selection model transferred money from the USA market to the Hong Kong market in late 2008 as the USA was hit hard by the financial crisis. Since a large number of high quality shares were issued in the Hong Kong market in 2009, there has been a surge in the Hong Kong market. As a result, our investment proportion in Hong Kong once rose to 85% in 2009. Then, as the US economy recovered and the stock market grew steadily, the cash began to flow back into the USA market. After that, the proportion of investment in the Hong Kong market increased with several rapid increases in the Hang Seng Index and then decreased with the decline after the peak. Notably, in 2015, the Hong Kong market was during a bear market impacted by the Chinese financial crisis. Thus, both the t and Gaussian copula models reduce the proportion of the Hong Kong market in its international portfolio. In 2019, Hong Kong was affected by social movements, and the annual economic data shrank by 1.2%, which also left a recession in the stock market. Correspondingly, the t copula model timely adjusted the allocation of wealth by reducing the proportion in Hong Kong.

In conclusion, the proposed international portfolio selection model with chance constraint and copula structure can well capture the dynamic dependence structure between different markets and dynamically adjust the international wealth allocation to avoid loss and systemic risk in one market. The t copula model performs even better in capturing the non-linear international dependence structure.

4.5. Comparison of international and single-market investment

We test the investment performances in three single markets respectively. For a single market model, we only consider the stocks in the market and fit a t copula for three factors in the market and the benchmark without considering the exchange rate. Other constraints are the same as the international investment model.

Figure 9 shows the out-of-sample wealth processes of three single market models, including USA, Europe, Hong Kong. As a comparison, we show the wealth process of the international portfolio selection model with t copula in the blue line.

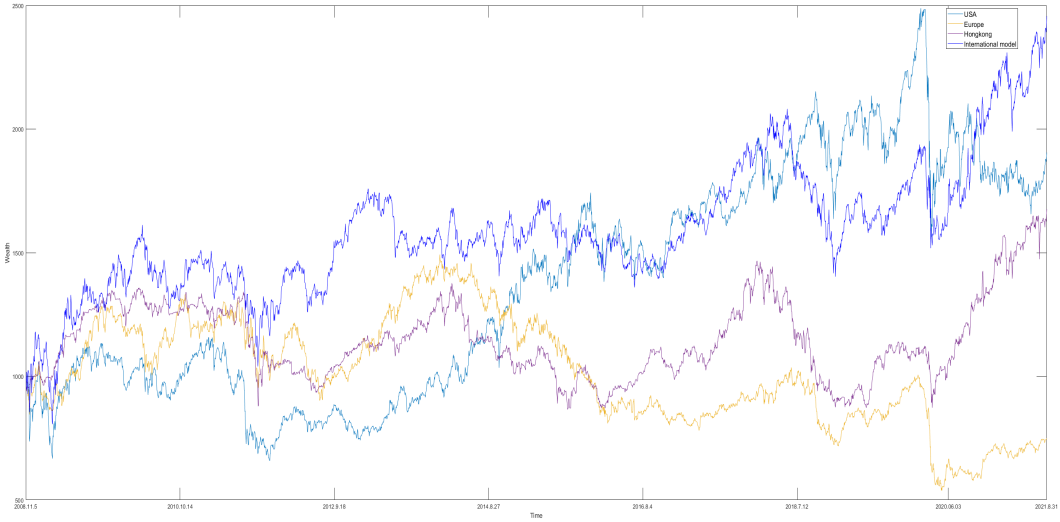


Figure 9: Wealth process of international and single market portfolio selection models with t copula

From Figure 9, we can find that the final wealth of the international portfolio selection model is higher than the investment in one single market. Table 6 shows some statistics of the out-of-sample return rate series of the four models. Here, 'Sharpe' is the Sharpe ratio, the ratio of mean to std; 'DSDV' is the downside semi-deviation; 'MDD' is the maximum drawdown; 'CR' is the Calmar ratio, the ratio of the annualized return to maximum drawdown.

Table 6: Some statistics of four models' out-of-sample daily return rate series

	Mean	Std	Max	Min	Sharpe	DSDV	MDD	CR
International model	3.7e-04	0.0137	0.1221	-0.1286	0.0267	0.0096	0.3548	2.2891
USA market model	3.2e-04	0.0158	0.1386	-0.1361	0.0203	0.0158	0.4307	1.8051
Europe market model	-4.7e-06	0.0128	0.1252	-0.1087	-3.7e-04	0.0093	0.6400	1.0243
Hong Kong market model	2.2e-04	0.0112	0.0848	-0.0811	0.0196	0.0081	0.4049	1.8776

From Table 6, we can find that the international portfolio selection model gets a higher average return rate and greater Sharpe ratio than the domestic portfolio selection model in any single market. Moreover, the maximum drawdown of the international investment model is the smallest, illustrating that international investment can efficiently resist market drawdown.

Investment in a single USA or Hong Kong market provides a positive average return rate, while the European market has a negative average return and the largest maximum drawdown. It illustrates again the long recession of European economics and why the international investment model does not invest in the Europe market.

We can find from Figure 9 that single USA market model has a significant drawdown around May 2020, which is coincident with the drop of the DJI index in Figure 3. However, the drawdown of international model at that time is much smaller than the USA market model. That is because the international portfolio selection model increases the investment proportion in the Hong Kong market, which suffers a lower drawdown. This phenomenon illustrates that the international portfolio selection model can efficiently reduce the systematic risk by diverse

investments in different markets and gain higher returns by international investment transfer.

5. Summary and future work

In this paper, we set up a stochastic optimization model with chance constraints for international investing management. We use a multi-factor model to capture the return rate of stocks and a copula model to characterize the relationships between factors in different markets and exchange rates. Then we solve the optimization problem by using the partial sampling algorithm and the sequential convex approximation algorithm. We apply the model with Gaussian or Student's t copula in practical numerical experiments. We find that t copula can better characterize the non-linear dependence structure in different markets. Meanwhile, the international portfolio selection model performs better than the chance-constrained portfolio selection model in a single market.

Our model can also be applied to bankrupt constraints, safety-first constraints, or Value-at-Risk constraints. The dynamic extension of the proposed international portfolio selection model is a promising topic. Meanwhile, the distributionally robust optimization counterpart of the proposed model is a promising research direction to deal with the ambiguity of the distributions in the proposed model.

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