Fixed point indices and fixed words at infinity of selfmaps of graphs

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What is Fixed Point Theory?

Let X be a space, and $f : X \to X$ a self-map.

- $x \in X$ is a fixed point of $f \iff f(x) = x$.
- $\mathrm{Fix} f := \{x \in X | f(x) = x\}$: the set of all fixed points of f.

Fixed Point Theory studies the nature of $\mathrm{Fix} f$ in relation to the space X and the map f, such as:

- Existence: is $\mathrm{Fix} f \neq \emptyset$?
- Number of fixed points $\#\mathrm{Fix} f$
- Behavior under homotopy: how $\mathrm{Fix} f$ changes when f changes continuously?
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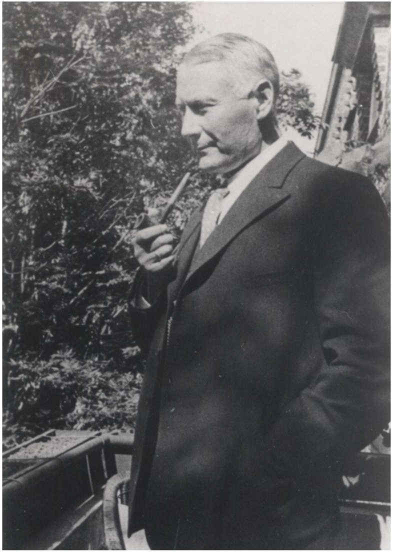
Nielsen fixed point theory



Jakob Nielsen (1890-1959)

Fixed point class

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition (path approach)

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\Longleftrightarrow$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

There is a subtle notion of empty fixed point class with $\mathrm{ind} = 0$. We omit the definition in this talk.

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Fixed point class

Let $p : \tilde{X} \to X$ be the universal covering of X, with deck group π identified with $\pi_1(X)$.

Definition (covering approach)

- For any lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ of f, the projection $p(\mathrm{Fix}\tilde{f})$ of its fixed point set is called a fixed point class of f.
- Two liftings $\tilde{f}$ and $\tilde{f}'$ of f are conjugate if there exists $\gamma \in \pi$ such that $\tilde{f}' = \gamma^{-1} \circ \tilde{f} \circ \gamma$.
- A fixed point class $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ carries a label by a conjugacy class of $\tilde{f}$.
- When $\mathrm{Fix}\tilde{f} = \emptyset$, we call $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ an empty fixed point class.

Empty fixed point classes have the same index 0 but may have different labels and hence be regarded as different. We would better think of them as hidden rather than nonexistent.

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Index: examples

For an isolated fixed point x_0 of $f : \mathbb{R}^n \to \mathbb{R}^n$, the index is defined:

$$\mathrm{ind}(f, x_0) := deg\varphi$$

where

$$\varphi : S^{n-1}_{x_0} \to S^{n-1}_{x_0}, \quad x \mapsto \frac{x - f(x)}{|x - f(x)|}.$$

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diff. map, x a isolated fixed point. Then

$$\mathrm{ind}(f, x) = sgn\,\det(I - Df_x) = (-1)^k.$$

If $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \lambda x$, then

$$\mathrm{ind}(f, 0) = \begin{cases} -1, & \lambda > 1, \\ 1, & \lambda < 1. \end{cases}$$

- If $n = 2$, f has a complex analytic expression $z \mapsto f(z)$, then $\mathrm{ind}(f, z_0) =$ multiplicity of the zero z_0 of the function $z - f(z)$.

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Definition

- A fixed point class $\mathbf{F}$ of f is essential if $\mathrm{ind}(f, \mathbf{F}) \neq 0$.
- Nielsen number $N(f) := \#\{essential\ fixed\ point\ classes\ of\ f\}$.
- Lefschetz number

$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

Lefschetz-Hopf Fixed Point Theorem

$$\sum_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathrm{ind}(f, \mathbf{F}) = L(f).$$

Nielsen number & Lefschetz number

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Example

For the identity $\mathrm{id} : X \to X$, the whole space X is the unique nonempty fixed point class and

$$\mathrm{ind}(X) = L(\mathrm{id}) = \chi(X),$$

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Nielsen Theory

Theorem (Nielsen)

The Nielsen number $N(f)$ is homotopy invariant.

Corollary (Nielsen)

If $g \simeq f$, then g has at least $N(f)$ fixed points. That is,

$$M[f] := \min\{\#\mathrm{Fix}(g) | g \simeq f\} \geq N(f).$$

Theorem (Jiang, 1980)

Let X be a compact, connected polyhedron and let $f : X \to X$ be a map. If X does not have local separating points and X is not a surface (with or without boundary) then

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Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Compactification of free group

- F: **free group** of rank n. $\phi : F \to F$ injective endomorphism.
  $\Lambda = \{g_1, \ldots, g_n\}$: a basis (i.e, a free generating set) of F.
- F = the set of reduced words in the letters $g_i^{\pm 1}$.
  ∂F = the set of infinite reduced words $W = w_1 w_2 \cdots w_i \cdots$,
  i.e., $w_i \in \Lambda^{\pm} = \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$ and $w_i \neq w_{i+1}^{-1}$.
  $\bar{F} := F \sqcup \partial F$. $W_i := w_1 \cdots w_i$.
- $|W|$: word length of $W \in F$ with respect to Λ.
  $W \wedge V$:=the longest common initial segment of W and V.
- The initial segment metric $d_{i.s} : \bar{F} \times \bar{F} \to \mathbb{R}_{\geq 0}$ is defined:

$$d_{i.s}(W, V) = \frac{1}{1 + |W \wedge V|}, \quad W \neq V.$$

  With this metric, $\bar{F}$ is compact (compactification as a hyper. group in the sense of Gromov), and F is dense in $\bar{F}$.
  ∂F: a compact space homeo. to a Cantor set when $n \geq 2$.

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- F = the set of reduced words in the letters $g_i^{\pm 1}$. ∂F = the set of infinite reduced words $W = w_1 w_2 \cdots w_i \cdots$, i.e., $w_i \in \Lambda^{\pm} = \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$ and $w_i \neq w_{i+1}^{-1}$. $\bar{F} := F \sqcup \partial F$. $W_i := w_1 \cdots w_i$.
- $|W|$: word length of $W \in F$ with respect to Λ. $W \wedge V$:=the longest common initial segment of W and V.
- The initial segment metric $d_{i.s} : \bar{F} \times \bar{F} \to \mathbb{R}_{\geq 0}$ is defined:

$$d_{i.s}(W, V) = \frac{1}{1 + |W \wedge V|}, \quad W \neq V.$$

With this metric, $\bar{F}$ is compact (compactification as a hyper. group in the sense of Gromov), and F is dense in $\bar{F}$. ∂F: a compact space homeo. to a Cantor set when $n \geq 2$.

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Extended map on the boundary

- A sequence of reduced words $V_p \in \bar{F}$ converges to an infinite word $W \in \partial F \iff \lim_{p\to+\infty} |W \wedge V_p| = +\infty$.
- The natural actions of F and $\mathrm{Aut}(F)$ on F extend continuously to $\bar{F}$: a left multiply $W : F \to F$ by a word $W \in F$ and an automorphism $f : F \to F$ extend uniquely to homeomorphisms $W : \partial F \to \partial F$ and $\bar{f} : \partial F \to \partial F$, respectively.
- Any f.g. subgroup $F' < F$ is quasi-convex, and hence an inclusion induces a natural embedding $\partial F' \hookrightarrow \partial F$. For an injective endomorphism $\phi : F \to F$, since $F \cong \phi(F) < F$, we have $\partial F \cong \partial(\phi(F)) \hookrightarrow \partial F$. Therefore

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Attracting fixed words at infinity

Let $\phi : F \to F$ be an injective endomorphism, and $W = w_1 \cdots w_i \cdots$ be a **fixed infinite reduced word** of ϕ.

Definition

1. W is an attracting fixed word of ϕ if
$$\lim_{i \to +\infty} |W \wedge \phi(W_i)| - i = +\infty.$$
2. W is an attracting fixed point of ϕ if $\exists$ a neighborhood $\mathcal{U}$ of $W \in \bar{F}$ s.t.
$$W' \in \mathcal{U} \Longrightarrow \lim_{p \to +\infty} \phi^p(W') = W.$$

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The number $a(\phi)$

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- *Let W be an attracting fixed word of ϕ. Then $\exists\ i_0$ s.t* $|W \wedge W'| \geq i_0 \Longrightarrow |W \wedge \phi(W')| > |W \wedge W'|$.
- *W is an attracting fixed word $\Longleftrightarrow$ W is an attracting fixed point $\Longrightarrow$ $W \notin \partial(\mathrm{Fix}\phi)$.*

Definition

- Two fixed infinite words $W, W' \in \partial F$ of ϕ are equivalent if $\exists$ a fixed word $U \in \mathrm{Fix}(\phi)$ s.t. $W' = UW$.
- Let $\mathscr{A}(\phi)$ be the set of equivalence classes of attracting fixed words of ϕ, and $a(\phi)$ the cardinality of $\mathscr{A}(\phi)$.

Remark: Let $\mathcal{A}(\phi)$ be the set of attracting fixed words of ϕ. Then $\mathscr{A}(\phi) = \mathrm{Fix}(\phi) \backslash \mathcal{A}(\phi)$, the set of orbits of $\mathrm{Fix}(\phi)$ acting on $\mathcal{A}(\phi)$.

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An example

Let $F = \langle g \rangle \cong \mathbb{Z}$. Then any endomorphism $\phi : F \to F$ has the form $\phi(g) = g^k$. The boundary ∂F consists of two points: $gg \cdots g \cdots$ and $g^{-1}g^{-1} \cdots g^{-1} \cdots$. We have

k	$\phi(g)$	$\mathrm{Fix}(\phi)$	$\mathrm{rkFix}(\phi)$	$a(\phi)$
0	1	$\{1\}$	0	N/A
1	g	$\mathbb{Z}$	1	0
> 1	g^k	$\{1\}$	0	2
< 0	g^k	$\{1\}$	0	0

For the identity *id*, each element in F is fixed. It is obvious that the two infinite words are both fixed, but are not attracting.

Theorem (Gaboriau-Jaeger-Levitt-Lustig, 1998)

Let ϕ be an **automorphism** *of a free group F_n. Then*

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Rank of a fixed point class

Let $f : X \to X$ be a π_1-injective selfmap of a connected finite graph, and $\mathbf{F}$ be a fixed point class of f. For a fixed point $x \in \mathbf{F}$, let $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ be the induced endomorphism.

Definition

Define

$$\mathrm{rk}(f, \mathbf{F}) := \mathrm{rkFix}(f_\pi), \quad a(f, \mathbf{F}) := a(f_\pi),$$

and the improved characteristic of $\mathbf{F}$ to be

$$\mathrm{ichr}(f, \mathbf{F}) := 1 - \mathrm{rk}(f, \mathbf{F}) - a(f, \mathbf{F}).$$

The definition above is independent of the choice of $x \in F$.

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Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1), a(f_0, \mathbf{F}_0) = a(f_1, \mathbf{F}_1).$$

Hence the index $\mathrm{ind}(\mathbf{F})$ and the improved characteristic $\mathrm{ichr}(\mathbf{F})$ are homotopy invariants.

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$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1), a(f_0, \mathbf{F}_0) = a(f_1, \mathbf{F}_1).$$

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Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

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Mutation invariance

Among selfmaps of compact polyhedra, homotopy and commutation generates an equivalence relation:

Definition

A sequence $\{f_i : X_i \to X_i | i = 0, \cdots, k\}$ of self-maps is a mutation if for each i, either

1. $X_{i+1} = X_i$ and $f_{i+1} \simeq f_i$, or
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A mutation sets up a one-one correspondence between fixed point classes of the end maps.

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Bounds for graphs & surfaces

Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite **graph** *or a connected compact hyperbolic* **surface**, *and $f : X \to X$ is a* **selfmap**. *Then*

(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f;

(B) *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X),$$

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Theorem (Z., 2012)

Suppose M is a compact orientable **Seifert** *3-manifold with hyperbolic orbifold, and $f : M \to M$ is a homeomorphism. Then*

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$$\mathcal{B} = \begin{cases} 4(3 - \mathrm{rk}\pi_1(M)) & M \text{ is a closed surface } F \times S^1 \\ 4(2 - \mathrm{rk}\pi_1(M)) & \text{others} \end{cases}.$$

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Main Results

Theorem (Z.-Zhao, 2020)

Let X be a connected finite graph and $f : X \to X$ be a π_1-injective selfmap. Then for every fixed point class $\mathbf{F}$ of f, we have

$$\mathrm{ind}(\mathbf{F}) \leq \mathrm{ichr}(\mathbf{F}).$$

When the Euler characteristic $\chi(X) \geq 0$, the equality $\mathrm{ind}(\mathbf{F}) = \mathrm{ichr}(\mathbf{F})$ holds immediately.

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If X is a connected finite graph with $\chi(X) = -1$ and $f : X \to X$ is a π_1-injective selfmap, then for every essential fixed point class $\mathbf{F}$ of f, we have

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$\mathrm{ind} = \mathrm{ichr}$: empty fixed point class

An example supporting $\mathrm{ind} = \mathrm{ichr}$ for empty fixed point classes.

Example

Let $f : (R_2, *) \to (R_2, *), a \mapsto b, b \mapsto a$ be a selfmap of the graph R_2 with one vertex $*$ and two edges a, b.

Fix a universal covering $q : \tilde{R}_2 \to R_2$ with a given point $\tilde{*} \in q^{-1}(*)$, and a lifting $\tilde{a} : (I, 0, 1) \to (\tilde{R}_2, \tilde{*}, \tilde{a}(1))$ of the loop a. Then $\mathrm{Fix}\tilde{f} = \emptyset$, namely, the fixed point class $\mathbf{F}_a$ is empty.

The f-route a induces an injective endomorphism

$$f_a : \pi_1(R_2, *) \to \pi_1(R_2, *), \quad a \mapsto aba^{-1}, \quad b \mapsto a^{-1},$$

with $\mathrm{Fix}(f_a) = \langle aba^{-1}b^{-1} \rangle \cong \mathbb{Z}$.

By computing, we have $|f_a(W_i)| \leq i + 2$,, it implies that f_a has no attracting fixed words. So $a(f_a) = 0$ and hence

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Example (Jiang, 1984)

Let $f : (R_2, *) \to (R_2, *)$ be a selfmap of the graph R_2 with one vertex $*$ and two edges a, b, such that the induced endomorphism of $\pi_1(R_2, *)$ is given by $f_*(a) = a^{-1}$ and $f_*(b) = a^{-1}b^2$.
[Jiang1984] showed that f has two nonempty fixed point classes and both of them have indices zero.
Note that $\mathrm{Fix}(f_*) = \{1\}$ and f_* has an attracting fixed word

$$f_*^{\infty}(b^{-1}) = b^{-2}ab^{-4}ab^{-2}a \cdots .$$

Hence, $a(f_*) = 1$ and the nonempty fixed point class consisting of $\{*\}$ has

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As corollaries, we have

Corollary (Z.-Zhao, 2020)

Suppose X is a connected finite graph but not a tree, and $f : X \to X$ is a π_1-injective selfmap. Then

$$\sum_{\mathbf{F} \in \mathrm{Fpc}(f)} \max\{0, \ \ \mathrm{rk}(\mathbf{F}) + a(\mathbf{F})/2 - 1\} \leq -\chi(X).$$

Theorem (Z.-Zhao, 2020)

Let ϕ be any **injective endomorphism** *of a free group F_n. Then*

$$\mathrm{rkFix}(\phi) + a(\phi)/2 \leq n.$$

Gaboriau-Jaeger-Levitt-Lustig proved the inequality above for automorphisms of F_n, by using groups acting on $\mathbb{R}$-trees. Our proof for general case is based on Theorem J-W-Z and Bestvina-Handel's train track maps.

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For an injective endomorphism $\phi : F_n \to F_n$ of a free group F_n, it induces an endomorphism ϕ^{ab} of the abelianization of F_n,

$$\phi^{\mathrm{ab}} : \mathbb{Z}^n \to \mathbb{Z}^n.$$

Let $\mathrm{Trace}(\phi^{\mathrm{ab}})$ be the trace of a matrix of ϕ^{ab}. For any $c \in F_n$, let $i_c : F_n \to F_n,\ g \mapsto cgc^{-1}$ be the inner automorphism induced by c.

Theorem (Z.-Zhao, 2020)

Let ϕ be an injective endomorphism of F_n. Then $\exists\ c \in F_n$ s.t

$$\mathrm{rkFix}(i_c \circ \phi) = a(i_c \circ \phi) = 0$$

if the trace $\mathrm{Trace}(\phi^{\mathrm{ab}}) < 1$; and

$$\mathrm{rkFix}(i_c \circ \phi) + a(i_c \circ \phi) > 1$$

if $n \leq 2$ and $\mathrm{Trace}(\phi^{\mathrm{ab}}) > 1$.

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For an injective endomorphism $\phi : F_n \to F_n$ of a free group F_n, it induces an endomorphism ϕ^{ab} of the abelianization of F_n,

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Let $\mathrm{Trace}(\phi^{\mathrm{ab}})$ be the trace of a matrix of ϕ^{ab}. For any $c \in F_n$, let $i_c : F_n \to F_n, \ g \mapsto cgc^{-1}$ be the inner automorphism induced by c.

Theorem (Z.-Zhao, 2020)

Let ϕ be an injective endomorphism of F_n. Then $\exists$ $c \in F_n$ s.t

$$\mathrm{rkFix}(i_c \circ \phi) = a(i_c \circ \phi) = 0$$

if the trace $\mathrm{Trace}(\phi^{\mathrm{ab}}) < 1$*; and*

$$\mathrm{rkFix}(i_c \circ \phi) + a(i_c \circ \phi) > 1$$

if $n \leq 2$ *and* $\mathrm{Trace}(\phi^{\mathrm{ab}}) > 1$.

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Theorem (Bestvina-Handel, Dicks-Ventura, 1990s)

Let $f : X \to X$ be a π_1-injective map of a connected graph (not a tree) X. Then f has the same homotopy type as a graph selfmap $\beta : Z \to Z$, where Z is a connected graph without vertices of valence 1 and all fixed points of β are vertices, and there is a β-invariant proper subgraph Z_0, containing all vertices of Z. The map $\beta : (Z, Z_0) \to (Z, Z_0)$ of the pair is of one of the following types.
Type 1 : β *sends Z into Z_0.*
Type 2 : β *cyclically permutes the edges in $Z \backslash Z_0$.*
Type 3 : β *expands edges of $Z \backslash Z_0$ by a factor $\lambda > 1$ w.r.t a non-negative metric L supported on $Z \backslash Z_0$, and has the properties (a)-(c) below.*
(a) *For every oriented edge e in $Z \backslash Z_0$, $D\beta(e)$ lies in $Z \backslash Z_0$.*
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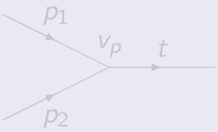

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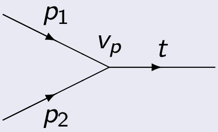

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Using Equation 0.1&0.2, the Main Result

$$\operatorname{ind}(\mathbf{F}) \leq \operatorname{ichr}(\mathbf{F})$$

can be proved by working inductively.

ichr for RTT

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Using Equation 0.1&0.2, the Main Result

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ichr for RTT

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Proof of Equation 0.2, I: lifting to universal covering

Fix a universal covering $q : \tilde{Z} \to Z$ of Z. Pick $\tilde{v}_0 \in q^{-1}(v_0)$ and a lifting $\tilde{\beta} : \tilde{Z} \to \tilde{Z}$ of β with $\tilde{\beta}(\tilde{v}_0) = \tilde{v}_0$, then $\mathbf{F} = q(\mathrm{Fix}\tilde{\beta})$, and the lifting $\tilde{\beta}$ induces an injective endomorphism $\beta_\pi : \pi \to \pi$ defined by

$$\tilde{\beta} \circ \gamma = \beta_\pi(\gamma) \circ \tilde{\beta}, \quad \forall \gamma \in \pi.$$

Endow $\tilde{Z}$ with a metric d with each edge length 1. Then the map

$$j : \pi \to \tilde{Z}, \quad \gamma \mapsto \gamma(\tilde{v}_0)$$

is π-equivariant (i.e. $\alpha(j(\gamma)) = j(\alpha\gamma)$ for any $\alpha, \gamma \in \pi$), and gives a quasi-isometric embedding from the covering transformation group π to the covering space $\tilde{Z}$. This induces a π-equivariant homeomorphism $\bar{j} : \partial\pi \to \partial\tilde{Z}$ between $\partial\pi$ and the space $\partial\tilde{Z}$ of ends of $\tilde{Z}$. It follows that the extension of $\tilde{\beta}$ to $\partial\tilde{Z}$ agrees with the extension of β_π to $\partial\pi$.

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Proof of Equation 0.2, II: a bijective correspondence

Lemma

An attracting fixed word $W \in \partial\pi$ of β_π defines an attracting fixed point $\bar{j}(W) \in \partial\tilde{Z}$ of $\tilde{\beta}$, and the π-equivariant homeomorphism $\bar{j} : \partial\pi \to \partial\tilde{Z}$ induces a bijective correspondence

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between the set $\mathcal{A}(\beta_\pi)$ of attracting fixed words of β_π in $\partial\pi$ and the set $\mathcal{A}(\tilde{\beta})$ of attracting fixed points of $\tilde{\beta}$ in $\partial\tilde{Z}$.

Here a fixed end $\mathcal{E} \in \mathcal{A}(\tilde{\beta})$ represented by a ray $\tilde{\rho} = \tilde{e}_1 \cdots \tilde{e}_i \cdots \subset \tilde{Z}$ is an *attracting fixed point* of $\tilde{\beta}$, if there exists a number $N > 0$ such that for any point $\tilde{x} \in \tilde{Z}$, we have

$$d([\tilde{v}_0, \tilde{x}] \cap \tilde{\rho}) > N \Longrightarrow \lim_{k \to +\infty} d([\tilde{v}_0, \tilde{\beta}^k(\tilde{x})] \cap \tilde{\rho}) = +\infty,$$

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Proof of Equation 0.2, III: a key Lemma

Let $\beta : (Z, Z_0) \to (Z, Z_0)$ be an RTT of Type 3, $v_0 \in \mathbf{F} \subset \mathrm{V}(Z)$ a fixed point of β, and $\beta_\pi : \pi_1(Z, v_0) \to \pi_1(Z, v_0)$ the induced injective endomorphism of β. Then

Lemma (Z.-Zhao, 2020)

(1) *Every oriented edge $e \in \Delta(\mathbf{F})$ defines an equivalence class $\mathscr{W}_e$ of attracting fixed words of β_π, and $\mathscr{W}_e$ does not contain any attracting fixed word of $(\beta_0)_\pi : \pi_1(Z_0, v_0) \to \pi_1(Z_0, v_0)$.*
(2) *Suppose $e_i \in \Delta(\mathbf{F})$ $(i = 1, 2)$ are two distinct edges with initial points $e_i(0)$ two (possibly the same) fixed points in $\mathbf{F}$. Then $\mathscr{W}_{e_1}$ and $\mathscr{W}_{e_2}$ are equal if and only if there exists an indivisible Nielsen path $p = p_1\bar{p}_2$ as in $\mathrm{Type\ 3(c)}$ of Theorem BH such that $e_i = D\beta(p_i)$ are the initial edges of the β-legal paths p_i for $i = 1, 2$.*
(3) *For every equivalence class $\mathscr{W}$ of attracting fixed words of β_π not containing an attracting fixed word of $(\beta_0)_\pi$, there exists an oriented edge $e \in \Delta(\mathbf{F})$ such that $\mathscr{W} = \mathscr{W}_e$.*

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Thanks！　　谢　谢！