Fixed point indices and fixed words at infinity of selfmaps of graphs

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Fixed point class: path approach

Let X be a connected compact polyhedron, and $f : X \to X$ a self-map. The fixed point set splits into a disjoint union of **fixed point classes**

$$\mathrm{Fix} f := \{x \in X | f(x) = x\} = \bigsqcup_{\mathbf{F} \in \mathrm{Fpc}(f)} \mathbf{F}$$

Definition (path approach)

Two fixed points $x, x' \in \mathrm{Fix}(f)$ are in the same fixed point class $\iff$ there is a path c (called a Nielsen path) from x to x' such that $c \simeq f \circ c$ rel endpoints.

The index of a fixed point class $\mathbf{F}$ is the sum

$$\mathrm{ind}(\mathbf{F}) := \mathrm{ind}(f, \mathbf{F}) := \sum_{x \in \mathbf{F}} \mathrm{ind}(f, x) \in \mathbb{Z}.$$

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For an isolated fixed point x_0 of $f : \mathbb{R}^n \to \mathbb{R}^n$, the index is defined:

$$\mathrm{ind}(f, x_0) := deg\varphi$$

where

$$\varphi : S^{n-1}_{x_0} \to S^{n-1}_{x_0}, \quad x \mapsto \frac{x - f(x)}{|x - f(x)|}.$$

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diff. map, x a isolated fixed point. Then

$$\mathrm{ind}(f, x) = sgn\det(I - Df_x) = (-1)^k.$$

If $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \lambda x$, then

$$\mathrm{ind}(f, 0) = \begin{cases} -1, & \lambda > 1, \\ 1, & \lambda < 1. \end{cases}$$

- If $n = 2$, f has a complex analytic expression $z \mapsto f(z)$, then $\mathrm{ind}(f, z_0) =$ multiplicity of the zero z_0 of the function $z - f(z)$.

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Fixed point class: covering approach

Let $p : \tilde{X} \to X$ be the universal covering of X, with deck group π identified with $\pi_1(X)$.

Definition (covering approach)

- For any lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ of f, the projection $p(\mathrm{Fix}\tilde{f})$ of its fixed point set is called a fixed point class of f.
- Two liftings $\tilde{f}$ and $\tilde{f}'$ of f are conjugate if there exists $\gamma \in \pi$ such that $\tilde{f}' = \gamma^{-1} \circ \tilde{f} \circ \gamma$
- A fixed point class $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ carries a label by a conjugacy class of $\tilde{f}$.
- When $\mathrm{Fix}\tilde{f} = \emptyset$, we call $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ an empty fixed point class.

Empty fixed point classes have the same index 0 but may have different labels and hence be regarded as different. We would better think of them as hidden rather than nonexistent.

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Reidemeister set

Let a reference lifting $\tilde{f}$ of f be chosen. Then every lifting of f can be unique written as $\beta \circ \tilde{f}$ for some $\beta \in \pi$.

- Each lifting $\tilde{f}$ induces an endomorphism $\tilde{f}_\pi : \pi \to \pi$ defined by
$$\tilde{f} \circ \gamma = \tilde{f}_\pi(\gamma) \circ \tilde{f}, \quad \gamma \in \pi.$$
- Two liftings $\beta \circ \tilde{f}$ and $\beta' \circ \tilde{f}$ are conjugate if and only if $\beta, \beta' \in \pi$ are $\tilde{f}_\pi$-conjugate, i.e., there exists $\gamma \in \pi$ such that
$$\beta' = \gamma\beta\tilde{f}_\pi(\gamma^{-1}).$$
- $\tilde{f}_\pi$-conjugacy class $[\beta]_{\tilde{f}_\pi} := \{\gamma\beta\tilde{f}_\pi(\gamma^{-1}) | \gamma \in \pi\}$ is said to be the coordinate for the fixed point class $p(\mathrm{Fix}(\beta \circ \tilde{f}))$.
- $\mathcal{R}(\tilde{f}_\pi) := \{[\beta]_{\tilde{f}_\pi} | \beta \in \pi\}$: Reidemeister set of $\tilde{f}_\pi$.

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Let X be a bouquet of n circles with one 0-cell and n 1-cells $a_1, \ldots, a_n$, and $f : X \to X$ a cellular map. Then

$$\pi := \pi_1(X) = \langle a_1, a_2, \ldots, a_n | - \rangle \cong F_n.$$

Let $\mathbf{F}$ denote the fixed point class labeled by $\tilde{f} : \tilde{X} \to \tilde{X}$, and $\varphi = \tilde{f}_\pi$ be the induced injective endomorphism of $\tilde{f}$, that is,

$$\tilde{f} \circ \gamma = \varphi(\gamma) \circ \tilde{f}, \quad \gamma \in \pi.$$

Then the coordinate of $\mathbf{F}$ is the φ-conjugacy class $[1]_\varphi$.

Proposition (Reidemeister trace formula)

The π-generalized Lefschetz number

$$L_\pi(f) := \sum_{[\beta]_\varphi \in \mathcal{R}(\varphi)} \mathrm{ind}(f, [\beta]_\varphi) \cdot [\beta]_\varphi = [1]_\varphi - \sum_{j=1}^{n} [\frac{\partial \varphi(a_j)}{\partial a_j}]_\varphi \in \mathbb{Z}\mathcal{R}(\tilde{f}_\pi).$$

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Fixed subgroups: definitions

For any group G, denote the set of endomorphisms of G by $\mathrm{End}(G)$.

Definition

For an endomorphism $\phi \in \mathrm{End}(G)$, the fixed subgroup of ϕ is

$$\mathrm{Fix}\phi := \{g \in G | \phi(g) = g\}.$$

For a family $\mathcal{B}$ of endomorphisms of G (i.e., $\mathcal{B} \subseteq \mathrm{End}(G)$), the fixed subgroup of $\mathcal{B}$ is

$$\mathrm{Fix}\mathcal{B} := \{g \in G | \phi(g) = g, \forall \phi \in \mathcal{B}\} = \bigcap_{\phi \in \mathcal{B}} \mathrm{Fix}\phi.$$

Compactification of free group

- F: **free group** of rank n. $\phi : F \to F$ injective endomorphism.
  $\Lambda = \{g_1, \dots, g_n\}$: a basis (i.e, a free generating set) of F.
- $F =$ the set of reduced words in the letters $g_i^{\pm 1}$.
  $\partial F =$ the set of infinite reduced words $W = w_1 w_2 \cdots w_i \cdots$, i.e., $w_i \in \Lambda^{\pm} = \{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$ and $w_i \neq w_{i+1}^{-1}$.
  $\bar{F} := F \sqcup \partial F$. $W_i := w_1 \cdots w_i$.
- $|W|$: word length of $W \in F$ with respect to Λ.
  $W \wedge V$:=the longest common initial segment of W and V.
- The initial segment metric $d_{i.s} : \bar{F} \times \bar{F} \to \mathbb{R}_{\geq 0}$ is defined:

$$d_{i.s}(W, V) = \frac{1}{1 + |W \wedge V|}, \quad W \neq V.$$

  With this metric, $\bar{F}$ is compact (compactification as a hyper. group in the sense of Gromov), and F is dense in $\bar{F}$.
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Compactification of free group

- F: **free group** of rank n. $\phi : F \to F$ injective endomorphism.
  $\Lambda = \{g_1, \ldots, g_n\}$: a basis (i.e, a free generating set) of F.
- F = the set of reduced words in the letters $g_i^{\pm 1}$.
  ∂F = the set of infinite reduced words $W = w_1 w_2 \cdots w_i \cdots$,
  i.e., $w_i \in \Lambda^{\pm} = \{g_1^{\pm 1}, \ldots, g_n^{\pm 1}\}$ and $w_i \neq w_{i+1}^{-1}$.
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Extended map on the boundary

- A sequence of reduced words $V_p \in \bar{F}$ converges to an infinite word $W \in \partial F \iff \lim_{p \to +\infty} |W \wedge V_p| = +\infty$.
- The natural actions of F and $\mathrm{Aut}(F)$ on F extend continuously to $\bar{F}$: a left multiply $W : F \to F$ by a word $W \in F$ and an automorphism $f : F \to F$ extend uniquely to homeomorphisms $W : \partial F \to \partial F$ and $\bar{f} : \partial F \to \partial F$, respectively.
- Any f.g. subgroup $F' < F$ is quasi-convex, and hence an inclusion induces a natural embedding $\partial F' \hookrightarrow \partial F$. For an injective endomorphism $\phi : F \to F$, since $F \cong \phi(F) < F$, we have $\partial F \cong \partial(\phi(F)) \hookrightarrow \partial F$. Therefore

Lemma

Let $\phi : F \to F$ be an injective endomorphism of F. Then ϕ can be extended to a continuous injective map $\bar{\phi} : \partial F \to \partial F$.

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Attracting fixed words at infinity

Let $\phi : F \to F$ be an injective endomorphism, and $W = w_1 \cdots w_i \cdots$ be a **fixed infinite reduced word** of ϕ.

Definition

1. W is an attracting fixed word of ϕ if
$$\lim_{i \to +\infty} |W \wedge \phi(W_i)| - i = +\infty.$$

2. W is an attracting fixed point of ϕ if $\exists$ a neighborhood $\mathcal{U}$ of $W \in \bar{F}$ s.t.
$$W' \in \mathcal{U} \Longrightarrow \lim_{p \to +\infty} \phi^p(W') = W.$$

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The number $a(\phi)$

The two definitions above are equivalent.

Proposition

W is an attracting fixed word $\Longleftrightarrow$ W is an attracting fixed point $\Longrightarrow$ $W \notin \partial(\mathrm{Fix}\phi)$.

Definition

- Two fixed infinite words $W, W' \in \partial F$ of ϕ are equivalent if $\exists$ a fixed word $U \in \mathrm{Fix}(\phi)$ s.t. $W' = UW$.
- Let $\mathscr{A}(\phi)$ be the set of equivalence classes of attracting fixed words of ϕ, and $a(\phi)$ the cardinality of $\mathscr{A}(\phi)$.

Remark: Let $\mathcal{A}(\phi)$ be the set of attracting fixed words of ϕ. Then $\mathscr{A}(\phi) = \mathrm{Fix}(\phi)\backslash\mathcal{A}(\phi)$, the set of orbits of $\mathrm{Fix}(\phi)$ acting on $\mathcal{A}(\phi)$.

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An example

Let $F = \langle g \rangle \cong \mathbb{Z}$. Then any endomorphism $\phi : F \to F$ has the form $\phi(g) = g^k$. The boundary ∂F consists of two points: $gg \cdots g \cdots$ and $g^{-1}g^{-1} \cdots g^{-1} \cdots$. We have

k	$\phi(g)$	$\mathrm{Fix}(\phi)$	$\mathrm{rkFix}(\phi)$	$a(\phi)$
0	1	$\{1\}$	0	N/A
1	g	$\mathbb{Z}$	1	0
> 1	g^k	$\{1\}$	0	2
< 0	g^k	$\{1\}$	0	0

For the identity *id*, each element in F is fixed. It is obvious that the two infinite words are both fixed, but are not attracting.

Theorem (Gaboriau-Jaeger-Levitt-Lustig, 1998)

Let ϕ be an **automorphism** *of a free group F_n. Then*

$$\mathrm{rkFix}(\phi) + a(\phi)/2 \leq n.$$

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Improved characteristic $\mathrm{ichr}(\mathbf{F})$

Let $f : X \to X$ be a π_1-injective selfmap of a connected finite graph, $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ a fixed point class of f labeled by $\tilde{f} : \tilde{X} \to \tilde{X}$, and $\tilde{f}_\pi : \pi \to \pi$ the induced endomorphism by $\tilde{f}$.

Definition

Define

$$\mathrm{rk}(f, \mathbf{F}) := \mathrm{rk}(\mathrm{Fix}\tilde{f}_\pi), \quad a(f, \mathbf{F}) := a(\tilde{f}_\pi),$$

the characteristic of $\mathbf{F}$ to be

$$\mathrm{chr}(f, \mathbf{F}) := 1 - \mathrm{rk}(\mathrm{Fix}\tilde{f}_\pi),$$

and the improved characteristic of $\mathbf{F}$ to be

$$\mathrm{ichr}(f, \mathbf{F}) := 1 - \mathrm{rk}(\mathrm{Fix}\tilde{f}_\pi) - a(\tilde{f}_\pi).$$

For brevity, we may write $\mathrm{rk}(\mathbf{F}), a(\mathbf{F}), \mathrm{chr}(\mathbf{F})$ and $\mathrm{ichr}(\mathbf{F})$ if no confusion exists for the selfmap f.

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Homotopy invariance

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

Theorem (Homotopy invariance)

Under the correspondence via a homotopy H,

$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1), a(f_0, \mathbf{F}_0) = a(f_1, \mathbf{F}_1).$$

Hence the index $\mathrm{ind}(\mathbf{F})$ and the improved characteristic $\mathrm{ichr}(\mathbf{F})$ are homotopy invariants.

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$$\mathrm{ind}(f_0, \mathbf{F}_0) = \mathrm{ind}(f_1, \mathbf{F}_1), \mathrm{rk}(f_0, \mathbf{F}_0) = \mathrm{rk}(f_1, \mathbf{F}_1), a(f_0, \mathbf{F}_0) = a(f_1, \mathbf{F}_1).$$

Hence the index $\mathrm{ind}(\mathbf{F})$ and the improved characteristic $\mathrm{ichr}(\mathbf{F})$ are homotopy invariants.

A homotopy $H = \{h_t\} : f_0 \simeq f_1 : X \to X$ gives rise to a natural one-one correspondence

$$H : \mathbf{F}_0 \mapsto \mathbf{F}_1$$

from the fixed point classes of f_0 to the fixed point classes of f_1.

Remark. A homotopy may create or remove fixed point classes. The correspondence is one-one only when empty fixed point classes are taken into account.

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Commutation invariance

Suppose $\phi : X \to Y$ and $\psi : Y \to X$ are maps. Then $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are said to differ by a commutation. The map ϕ sets up a natural one-one correspondence

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from the fixed point classes of $\psi \circ \phi$ to the fixed point classes of $\phi \circ \psi$.

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Theorem (Jiang-Wang-Z., 2011)

Suppose X is either a connected finite **graph** *or a connected compact hyperbolic* **surface**, *and $f : X \to X$ is a* **selfmap**. *Then*

(A) $\mathrm{ind}(\mathbf{F}) \leq \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of f;*

(B) *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geq 2\chi(X),$$

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Main Results

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A criterion for a fixed point

The first consequence of the main Theorem is

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Let $f : X \to X$ be a π_1-injective selfmap of a connected finite graph, and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f to the universal covering $\tilde{X}$. If

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then $\tilde{f}$ has a fixed point.

Proof.

Suppose $\tilde{f}$ has no fixed point, i.e., the fixed point class $\mathbf{F} = p(\mathrm{Fix}\tilde{f})$ is empty. Then $\mathrm{ichr}(\mathbf{F}) = \mathrm{ind}(\mathbf{F}) = 0$. So

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Fixed words in free groups

The second consequence:

Corollary (Z.-Zhao, 2020)

Suppose X is a connected finite graph but not a tree, and $f : X \to X$ is a π_1-injective selfmap. Then

$$\sum_{\mathbf{F}\in \mathrm{Fpc}(f)} \max\{0, \quad \mathrm{rk}(\mathbf{F}) + a(\mathbf{F})/2 - 1\} \leq -\chi(X).$$

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Let ϕ be any **injective endomorphism** *of a free group F_n. Then*

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Gaboriau-Jaeger-Levitt-Lustig proved the inequality above for automorphisms of F_n, by using groups acting on $\mathbb{R}$-trees. Our proof for general case is based on Theorem J-W-Z and Bestvina-Handel's train track maps.

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Let $\mathrm{Trace}(\phi^{\mathrm{ab}})$ be the trace of a matrix of ϕ^{ab}. For any $c \in F_n$, let $i_c : F_n \to F_n, \ g \mapsto cgc^{-1}$ be the inner automorphism induced by c.

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An example for $\mathrm{ind} = \mathrm{ichr}$

Example

Let $f : (R_2, *) \to (R_2, *), a \mapsto b, b \mapsto a$ be a selfmap of the graph R_2 with one vertex $*$ and two edges a, b.

Fix a universal covering $q : \tilde{R}_2 \to R_2$ with a given point $\tilde{*} \in q^{-1}(*)$, and a lifting $\tilde{a} : (I, 0, 1) \to (\tilde{R}_2, \tilde{*}, \tilde{a}(1))$ of the loop a. Then $\mathrm{Fix}\tilde{f} = \emptyset$, namely, the fixed point class $\mathbf{F}_a$ is empty.

The f-route a induces an injective endomorphism

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By computing, we have $|f_a(W_i)| \leq i + 2$,, it implies that f_a has no attracting fixed words. So $a(f_a) = 0$ and hence

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Thanks ! 谢 谢!