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Graphs determined by their generalized characteristic polynomials ☆

Wei Wang *, Feng Li, Hongliang Lu, Zongben Xu

School of Science, Xi'an Jiaotong University, Xi'an 710049, PR China

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ABSTRACT

For a given graph G with $(0, 1)$-adjacency matrix A_G, *the generalized characteristic polynomial* of G is defined to be $\phi_G = \phi_G(\lambda, t) = det(\lambda I - (A_G - tD_G))$, where I is the identity matrix and D_G is the diagonal degree matrix of G. In this paper, we are mainly concerned with the problem of characterizing a given graph G by its generalized characteristic polynomial ϕ_G. We show that graphs with the same generalized characteristic polynomials have the same degree sequence, based on which, a unified approach is proposed to show that some families of graphs are characterized by ϕ_G. We also provide a method for constructing graphs with the same generalized characteristic polynomial, by using GM-switching.

1. Introduction

Given a graph $G = (V, E)$ with adjacency matrix A_G, let D_G be the diagonal matrix with the (i, i)th entry being equal to the degree of the ith vertex. In [4], Cvetković et al. introduced a bivariate polynomial, denoted by $\phi_G(\lambda, t) = det(\lambda I_n - (A_G - tD_G))$ (or ϕ_G or simply ϕ is no confusion arises), which will be referred to as *the generalized characteristic polynomial* of G in the paper.

The polynomial $\phi_G(\lambda, t)$ generalizes some well known characteristic polynomials of graph G, e.g. the characteristic polynomial of graph G is $\phi_G(\lambda, 0)$; the characteristic polynomial of *the Laplacian matrix* $D_G - A_G$ of graph G is $(-1)^{|V|}\phi_G(-\lambda, 1)$; the characteristic polynomial of the *sign-less Laplacian matrix* $D_G + A_G$ of graph G is $\phi_G(\lambda, -1)$; the characteristic polynomial of the *normalized Laplacian matrix* $I - D_G^{-1/2} A_G D_G^{-1/2}$ of graph G is $(-1)^{|V|}\phi_G(0, -\lambda + 1)/det(D_G)$.

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* Corresponding author.
E-mail address: wang_weiw@163.com (W. Wang).

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In this paper, we are mainly concerned with the problem of characterizing graphs by their generalized characteristic polynomials. The motivations for us to study this problem are twofold:

First, our original interests come from the problem of spectral characterization of graphs – an old problem in spectral graph theory, which is far from resolved. The problem goes back to more than 50 years and originates from chemistry; recently, it has received a lot of attention from researchers. Most of the existing work has concentrated on showing some specific (new) families of graphs to be determined by the spectrum with respect to various matrices (e.g. the adjacency matrix, Laplacian matrix and the sign-less Laplacian matrix); see [3,10,11,13,15,19,21]. We refer the reader to the excellent surveys [6,7] for a background and related results on this topic.

The problem of spectral characterization of graphs clearly depends on the spectrum concerned. However, it turns out that characterizing graphs by a single kind of spectrum is generally a very hard problem and proving a given graph to be determined by its spectrum is usually quite complicated and involved. So it would be interesting to consider a mild modification of this problem – characterizing graphs by the spectrum with respect to several matrices associated with the given graph, simultaneously (see also [20]).

Actually, our problem of characterizing a graph by its generalized characteristic polynomial is equivalent to the spectral characterization of a graph with respect to the family of matrices $A_G - tD_G$ $(t \in \mathbf{R})$, simultaneously. Note that the spectrum of $A_G - tD_G$ $(t \in \mathbf{R})$ includes the spectrum of all the conventional matrices, e.g. the spectrum of the adjacency matrix, the spectrum of Laplacian matrix, the spectrum of the sign-less Laplacian matrix and the spectrum of the normalized Laplacian matrix, etc.

On the other hand, the generalized characteristic polynomial $\phi_G(\lambda, t)$ has also an amazing combinatorial interpretation as being equivalent to the Bartholdi zeta function. In [1], Bartholdi introduced a zeta function, known as the Bartholdi zeta function, which generalizes the well known Ihara–Selberg zeta function (see [12]) of a graph G. In particular, the reciprocal of the Bartholdi zeta function of a graph G is given as follows:

$$Z_G(u, t)^{-1} = \left(1 - (1-u)^2 t^2\right)^{|E|-|V|} det\left[I - tA_G + (1-u)(D_G - (1-u)I)t^2\right].$$

It is not difficult to show that $\phi_G(\lambda, t)$ determines the reciprocal of the Bartholdi zeta function and vice versa (see also [14]). Thus, it would be interesting to know graphs (or family of graphs) that are determined by their Bartholdi zeta functions.

In this paper, we first investigate some invariants of graphs with the same generalized characteristic polynomial, by using linear algebraic tools, and in particular, we show that the degree sequence of a graph G is determined by ϕ_G. Based on these properties, a unified approach is proposed to show that some families of graphs are characterized by ϕ_G.

As it can be expected, we are able to give some general results for graphs determined by ϕ_G, which are not available for any single kind of spectrum. For example, we show that the graph G obtained from a graph Γ by adding some isolated vertices is still determined by ϕ_G, provided that Γ is determined by ϕ_Γ. It follows immediately that the disjoint union of the cycles and some isolated vertices $G := C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_s} \cup mK_1$ is determined by ϕ_G. We remark that however, this is in general not true for a single kind of spectrum such as the adjacency spectrum, the Laplacian spectrum, or the sign-less Laplacian spectrum.

Finally, we also provide a method for constructing graphs with the same generalized characteristic polynomial.

The rest of the paper is organized as follows: In Section 2, we give some properties of graphs with the same generalized characteristic polynomial. In Section 3, we present several methods to show that some family of graphs G to be determined by ϕ_G. In Section 4, we give a method for constructing graphs with the same ϕ-invariant. Conclusions are given in Section 5.

2. Some invariants of ϕ-cospectral graphs

In this section, we give some invariants of graphs with the same generalized characteristic polynomial. We start by fixing some notations.

Throughout the paper, we consider only *simple* graphs, i.e., undirected graphs without loops or multiple edges. For a given graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$, *the adjacency matrix* $A_G = (a_{ij})$ is an n by n matrix, where $a_{ij} = 1$ if i and j are adjacent; and $a_{ij} = 0$ otherwise. *The characteristic polynomial* of G is $P_G(\lambda) = det(\lambda I_n - A_G)$, and the multiset of eigenvalues of A_G is called the *spectrum* (or A-spectrum) of graph G.

Besides the adjacency matrix, there are other well known matrices associated with graph G. For example, *the Laplacian matrix* $L = D_G - A_G$, *the sign-less Laplacian matrix* $Q = A_G + D_G$ and *the normalized Laplacian matrix* $\tilde{L} = I - D_G^{-1/2} A_G D_G^{-1/2}$. The multiset of eigenvalues of the corresponding matrices is referred to as *the L-spectrum*, *the Q-spectrum* and the *$\tilde{L}$-spectrum*, respectively. In this paper, we are particularly interested in the spectrum with respect to a family of matrices $A_G - tD_G$ ($t \in \mathbf{R}$), which will be referred to as the *ϕ-spectrum* in the sequel.

Two graphs G and H are *cospectral* if they share the same spectrum. A graph G is said to be *determined by the spectrum* (DS for short) if any graph H that is cospectral with G is necessarily isomorphic to G. Of course, the spectrum concerned should be specified. So we have "determined by the spectrum of A (A-DS)", "determined by the spectrum of L (L-DS)", "determined by the spectrum of Q (Q-DS)", and "determined by the spectrum of $A_G - tD_G$ for any $t \in \mathbf{R}$ (ϕ-DS)", etc. Moreover, it is not difficult to verify that if two graphs are ϕ-cospectral, then they are A-cospectral, L-cospectral, Q-cospectral and $\tilde{L}$-cospectral, simultaneously (the $\tilde{L}$-cospectrality is less obvious, which is a simple consequence of the following Theorem 2.1).

It is clear that two graphs are cospectral with respect to the ϕ-spectrum iff they have the same generalized characteristic polynomial, and a graph G is characterized by ϕ_G iff G is ϕ-DS. For the ease of presentation, we will use the term "G is characterized by ϕ_G" and "G is ϕ-DS" interchangeably.

The following theorem shows that ϕ-cospectral graphs share the same degree sequence. This is usually quite useful in proving the ϕ-DS property of graphs.

Theorem 2.1. *If $\phi_G = \phi_H$, then graphs G and H have the same degree sequence.*

To prove Theorem 2.1, we need several lemmas below. The proof of the first lemma can be found in any linear algebra text book, and is omitted.

Lemma 2.2. *Let U and V be two n by n matrices. Then we have $det(\lambda I_n + UV) = det(\lambda I_n + VU)$.*

Lemma 2.3. *Let u_i and v_i be n-dimensional column vectors. Let D be an n by n diagonal matrix. Then, we have*

$$det\left(I_n + t\sum_{i=1}^{n} Du_i v_i^T\right) = \sum_{k=0}^{n} t^k \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \begin{vmatrix} u_{i_1}^T Dv_{i_1} & u_{i_1}^T Dv_{i_2} & \cdots & u_{i_1}^T Dv_{i_k} \\ u_{i_2}^T Dv_{i_1} & u_{i_2}^T Dv_{i_2} & \cdots & u_{i_2}^T Dv_{i_k} \\ \cdots & \cdots & \ddots & \cdots \\ u_{i_k}^T Dv_{i_1} & u_{i_k}^T Dv_{i_2} & \cdots & u_{i_k}^T Dv_{i_k} \end{vmatrix}.$$

Proof. Define $U := [Du_1, Du_2, \ldots, Du_n]$ and $V := [v_1, v_2, \ldots, v_n]$. Let $M := \sum_{i=1}^{n} Du_i v_i^T = UV^T$ and $N := U^T V$. Then it follows from Lemma 2.2 that

$$\begin{aligned} det(I_n + tM) &= det\left(I_n + tUV^T\right) \\ &= det\left(I_n + tV^T U\right) \\ &= det\left[\left(I_n + tV^T U\right)^T\right] \\ &= det\left(I_n + tU^T V\right) \end{aligned}$$

$$= det(I_n + tN)$$
$$= t^n det\left(\frac{1}{t}I_n + N\right).$$

The coefficient of t^k in $det(I_n + tM)$ equals the coefficients of $\left(\frac{1}{t}\right)^{n-k}$ in $det\left(\frac{1}{t}I_n + N\right)$, which in turn equals the sum of all principal minors of N of order k. Then the lemma follows by noticing that $\begin{vmatrix} u_{i_1}^T Dv_{i_1} & u_{i_1}^T Dv_{i_2} & \cdots & u_{i_1}^T Dv_{i_k} \\ u_{i_2}^T Dv_{i_1} & u_{i_2}^T Dv_{i_2} & \cdots & u_{i_2}^T Dv_{i_k} \\ \cdots & \cdots & \ddots & \cdots \\ u_{i_k}^T Dv_{i_1} & u_{i_k}^T Dv_{i_2} & \cdots & u_{i_k}^T Dv_{i_k} \end{vmatrix}$ is actually the principal minor of N with row and column indices being $i_1, i_2, \ldots, i_k$, respectively. □

Lemma 2.4 [8, p. 186]. *Let ξ_i be the normalized eigenvectors of the adjacency matrix A_G of graph G with associated eigenvalues λ_i ($i = 1, 2, \ldots, n$). Then $(\lambda I_n - A_G)^{-1} = \sum_{i=1}^{n} \frac{\xi_i \xi_i^T}{\lambda - \lambda_i}$.*

Proof of Theorem 2.1. Suppose that $det(\lambda I_n - (A_G - tD_G)) = det(\lambda I_n - (A_H - tD_H))$ for any λ and t. Then it is clear that graphs G and H are A-cospectral. Let λ_i be the common eigenvalues of their adjacency matrices. Let ξ_i and η_i be the normalized eigenvectors of G and H associated with λ_i, respectively. Then we have

$$\begin{aligned} det(\lambda I_n - (A_G - tD_G)) &= det(\lambda I_n - A_G) det\left(I_n + tD_G(\lambda I_n - A_G)^{-1}\right) \\ &= det(\lambda I_n - A_G) det\left(I_n + \sum_{i=1}^{n} \frac{tD_G \xi_i \xi_i^T}{\lambda - \lambda_i}\right). \end{aligned} \tag{1}$$

Similarly, we have

$$det(\lambda I_n - (A_H - tD_H)) = det(\lambda I_n - A_H) det\left(I_n + \sum_{i=1}^{n} \frac{tD_H \eta_i \eta_i^T}{\lambda - \lambda_i}\right). \tag{2}$$

Note that $det(\lambda I_n - A_G) = det(\lambda I_n - A_H)$, it follows from Eqs. (1) and (2) that

$$det\left(I_n + \sum_{i=1}^{n} \frac{tD_G \xi_i \xi_i^T}{\lambda - \lambda_i}\right) = det\left(I_n + \sum_{i=1}^{n} \frac{tD_H \eta_i \eta_i^T}{\lambda - \lambda_i}\right). \tag{3}$$

Comparing the coefficients of t^k in both sides of Eq. (3) gives that

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{\begin{vmatrix} \xi_{i_1}^T D_G \xi_{i_1} & \xi_{i_1}^T D_G \xi_{i_2} & \cdots & \xi_{i_1}^T D_G \xi_{i_k} \\ \xi_{i_2}^T D_G \xi_{i_1} & \xi_{i_2}^T D_G \xi_{i_2} & \cdots & \xi_{i_2}^T D_G \xi_{i_k} \\ \cdots & \cdots & \ddots & \cdots \\ \xi_{i_k}^T D_G \xi_{i_1} & \xi_{i_k}^T D_G \xi_{i_2} & \cdots & \xi_{i_k}^T D_G \xi_{i_k} \end{vmatrix}}{(\lambda - \lambda_{i_1})(\lambda - \lambda_{i_2}) \cdots (\lambda - \lambda_{i_k})} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{\begin{vmatrix} \eta_{i_1}^T D_H \eta_{i_1} & \eta_{i_1}^T D_H \eta_{i_2} & \cdots & \eta_{i_1}^T D_H \eta_{i_k} \\ \eta_{i_2}^T D_H \eta_{i_1} & \eta_{i_2}^T D_H \eta_{i_2} & \cdots & \eta_{i_2}^T D_H \eta_{i_k} \\ \cdots & \cdots & \ddots & \cdots \\ \eta_{i_k}^T D_H \eta_{i_1} & \eta_{i_k}^T D_H \eta_{i_2} & \cdots & \eta_{i_k}^T D_H \eta_{i_k} \end{vmatrix}}{(\lambda - \lambda_{i_1})(\lambda - \lambda_{i_2}) \cdots (\lambda - \lambda_{i_k})}, \tag{4}$$

for $k = 1, 2, \ldots, n$ and where we have used Lemma 2.3.

Multiplying on both sides of Eq. (4) by $(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$ and comparing the coefficients of λ^{n-k} gives that

$$\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\begin{vmatrix}\xi_{i_1}^TD_G\xi_{i_1} & \xi_{i_1}^TD_G\xi_{i_2} & \cdots & \xi_{i_1}^TD_G\xi_{i_k}\\ \xi_{i_2}^TD_G\xi_{i_1} & \xi_{i_2}^TD_G\xi_{i_2} & \cdots & \xi_{i_2}^TD_G\xi_{i_k}\\ \cdots & \cdots & \ddots & \cdots\\ \xi_{i_k}^TD_G\xi_{i_1} & \xi_{i_k}^TD_G\xi_{i_2} & \cdots & \xi_{i_k}^TD_G\xi_{i_k}\end{vmatrix} = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\begin{vmatrix}\eta_{i_1}^TD_H\eta_{i_1} & \eta_{i_1}^TD_H\eta_{i_2} & \cdots & \eta_{i_1}^TD_H\eta_{i_k}\\ \eta_{i_2}^TD_H\eta_{i_1} & \eta_{i_2}^TD_H\eta_{i_2} & \cdots & \eta_{i_2}^TD_H\eta_{i_k}\\ \cdots & \cdots & \ddots & \cdots\\ \eta_{i_k}^TD_H\eta_{i_1} & \eta_{i_k}^TD_H\eta_{i_2} & \cdots & \eta_{i_k}^TD_H\eta_{i_k}\end{vmatrix} \quad (5)$$

for $k=1,2,\ldots,n$.
Define

$$P_1 := [\xi_1,\xi_2,\ldots,\xi_n],\quad P_2 := [\eta_1,\eta_2,\ldots,\eta_n]. \quad (6)$$

Then P_1 and P_2 are orthogonal matrices. Let

$$M_G := \left(\xi_i^TD_G\xi_j\right)_{n\times n} = P_1^TD_GP_1 \quad\text{and}\quad M_H := \left(\eta_i^TD_H\eta_j\right)_{n\times n} = P_2^TD_HP_2.$$

Then by Eq. (5), we have $det(\lambda I-M_G)=det(\lambda I-M_H)$, i.e., Matrices M_G and M_H are similar (since both of them are symmetric matrices). Therefore, there exists an orthogonal matrix Q such that $Q^TM_GQ = M_H$. That is, $Q^TP_1^TD_GP_1Q = P_2^TD_HP_2$ or equivalently, $\left(P_1QP_2^T\right)^T D_G\left(P_1QP_2^T\right) = D_H$. It follows from $\left(P_1QP_2^T\right)^T = \left(P_1QP_2^T\right)^{-1}$ that D_G and D_H are similar. Moreover, note that both of them are diagonal matrices, and hence, the set of diagonal entries of D_G and that of D_H must be equal. This completes the proof. □

By the lemma above, we can assume without loss of generality that G and H are indexed such that $D_G=D_H$. The following lemma shows that by appropriately choosing the orthonormal eigenvectors ξ_i of the adjacency matrix A_G of graph G, $\xi_i^TD_G\xi_i$ $(i=1,2,\ldots,n)$ are invariants for ϕ-cospectral graphs.

Lemma 2.5. *Suppose that $\phi_G=\phi_H$ and $D_G=D_H$. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the common eigenvalues of the adjacency matrices of G and H. Then there exist normalized eigenvectors ξ_i (resp. η_i) of matrices A_G (resp. A_H) associated with λ_i such that*

$$\xi_i^TD_G\xi_i = \eta_i^TD_G\eta_i, \quad \textit{for } i=1,2,\ldots,n. \quad (7)$$

Proof. Let ξ_i (resp. η_i) be any normalized eigenvectors of the adjacency matrix A_G (resp. A_H) of graph G (resp. H) associated with λ_i, for $i=1,2,\ldots,n$.

Then according to the proof of Theorem 2.1, Eq. (4) holds. Let $k=1$ in Eq. (4), we have

$$\sum_{i=1}^{n}\frac{\xi_i^TD_G\xi_i}{\lambda-\lambda_i} = \sum_{i=1}^{n}\frac{\eta_i^TD_G\eta_i}{\lambda-\lambda_i}. \quad (8)$$

If $\lambda_1,\lambda_2,\ldots,\lambda_n$ are all distinct, then by Eq. (8), the lemma clearly holds. Next, we show that if graph G (resp. H) has multiple eigenvalues, we can always choose the corresponding eigenvectors of A_G (resp. A_H) appropriately such that the lemma holds.

Assume without loss of generality that λ_1 is a multiple eigenvalue of graph G with multiplicity $m_1\geqslant 2$. Then it follows from Eq. (8) that

$$\sum_{j=1}^{m_1}\xi_j^TD_G\xi_j = \sum_{j=1}^{m_1}\eta_j^TD_G\eta_j. \quad (9)$$

Note that the eigenspace $V_{\lambda_1} = \{x \in \mathbf{R}^n | A_G x = \lambda_1 x\}$ of A_G corresponding to λ_1 is m_1-dimensional. Similarly, the eigenspace $W_{\lambda_1} = \{x \in \mathbf{R}^n | A_H x = \lambda_1 x\}$ of A_H corresponding to λ_1 is also m_1-dimensional. We have to choose orthonormal eigenvectors ξ_j (resp. η_j), $j = 1, 2, \ldots, m$, from V_{λ_1} (resp. W_{λ_1}) to meet the requirements that $\xi_j^T D_G \xi_j = \eta_j^T D_G \eta_j$ for $j = 1, 2, \ldots, m$. We show that this is always possible.

Let $m = \min_j \left\{\eta_j^T D_G \eta_j\right\}$ and $M = \max_j \left\{\eta_j^T D_G \eta_j\right\}$. Since every $\xi_j^T D_G \xi_j$ (resp. $\eta_j^T D_G \eta_j$) is positive, there must exists a term on the left hand side of Eq. (9), say $\xi_1^T D_G \xi_1$, which satisfies $m \leqslant \xi_1^T D_G \xi_1 \leqslant M$. Next, we show that there exists a unit vector $\tilde{\eta}_1 \in W_{\lambda_1}$ with $\xi_1^T D_G \xi_1 = \tilde{\eta}_1^T D_G \tilde{\eta}_1$.

Let $\tilde{\eta}_1 = (\eta_1, \ldots, \eta_{m_1})\, x$, where $x = (x_1, x_2, \ldots, x_{m_1})^T$. Then $\tilde{\eta}_1^T D_G \tilde{\eta}_1 = x^T B x$, where $B = \begin{bmatrix} \eta_1^T D_G \eta_1 & \cdots & \eta_1^T D_G \xi_{m_1} \\ \cdots & \cdots & \cdots \\ \eta_{m_1}^T D_G \eta_1 & \cdots & \eta_{m_1}^T D_G \eta_{m_1} \end{bmatrix}$ is a positive-definite matrix.

Let $\eta_k^T D_G \eta_k = m = \min_j \left\{\eta_j^T D_G \eta_j\right\}$ and $\eta_l^T D_G \eta_l = M = \max_j \left\{\eta_j^T D_G \eta_j\right\}$. Then the quadratic form $f(x) = x^T B x$ attains values m and M at $x = e_k$ and $x = e_l$, respectively, where e_k and e_l are the kth and the lth vectors of the standard orthonormal basis of $\mathbf{R}^{m_1}$. Note that $f(x)$ is continuous on the unit sphere $||x||_2 = 1$ (here and below $||\cdot||_2$ is the Euclidean norm), and $\xi_1^T D_G \xi_1$ lies between m and M. There must exist an $\tilde{x}$ with $||\tilde{x}||_2^2 = \tilde{x}_1^2 + \cdots + \tilde{x}_{m_1}^2 = ||\tilde{\eta}_1||_2^2 = 1$ such that $\tilde{x}^T B \tilde{x} = \xi_1^T D_G \xi_1$, i.e., $\xi_1^T D_G \xi_1 = \tilde{\eta}_1^T D_G \tilde{\eta}_1$. Still use the same notation η_1 to denote the $\tilde{\eta}_1$ that we found.

Now choose orthonormal eigenvectors, still denoted by $\xi_2, \ldots, \xi_{m_1}$, from $V_{\lambda_1} \cap (span\ \xi_1)^{\perp}$, choose orthonormal eigenvectors, still denoted by $\eta_2, \ldots, \eta_{m_1}$ from $W_{\lambda_1} \cap (span\ \eta_1)^{\perp}$. Then eliminating the identical terms $\xi_1^T D_G \xi_1 = \eta_1^T D_G \eta_1$ from both sides of Eq. (9), we get that

$$\sum_{j=2}^{m_1} \xi_j^T D_G \xi_j = \sum_{j=2}^{m_1} \eta_j^T D_G \eta_j. \tag{10}$$

Using the same arguments as above, we can find $\tilde{\eta}_2 \in W_{\lambda_1} \cap (span\ \eta_1)^{\perp}$ such that $\tilde{\eta}_2{}^T D_G \tilde{\eta}_2 = \xi_2^T D_G \xi_2, \ldots$, continuing this process, we will find ξ_i and η_i satisfying the requirements of the lemma for $i = 1, 2, \ldots, m_1$.

Similar arguments can be applied to all the other multiple eigenvalues, and this completes the proof. □

3. Methods for finding ϕ-DS graphs

Based on the previous analysis, in this section, we provide a unified approach to show that some family of graphs are determined by the ϕ-spectrum. Our main observation is that if two graphs G and H are ϕ-cospectral, then they have the same degree sequence and we can choose the normalized eigenvectors ξ_i (resp. η_i) of the adjacency matrix A_G (resp. A_H) of graph G (resp. H) such that Eq. (7) holds. This turns out to be useful in showing a graph to be ϕ-DS.

First, we give a method for constructing large ϕ-DS graphs from smaller ϕ-DS graphs. Given k disjoint graphs G_i, $i = 1, 2, \ldots, k$, $V(G_i) \cap V(G_j) = \phi$ for $i \neq j$. The *sum* (or *disjoint union*) of graphs G_i is a graph G with vertex set $V(G) = \cup_{i=1}^k V(G_i)$ and edge set $E(G) = \cup_{i=1}^k E(G_i)$. When all G_i's are identical, we use kG_1 to denote the sum of k copies of graph G_1.

A natural question is: Is the disjoint union of DS graphs still DS? Generally, we cannot expect an affirmative answer. The following theorem shows that a ϕ-DS graph is still ϕ-DS after adding some isolated vertices.

Theorem 3.1. *Let Γ be a ϕ-DS graph. Let G be a graph obtained from Γ by adding m isolated vertices, i.e., $G = \Gamma \cup mK_1$. Then G is ϕ-DS.*

Proof. Suppose that H is a graph with $\phi_G = \phi_H$. We show that H is isomorphic to G.

By Theorem 2.1, G and H have the same degree sequence. Note that G contains at least m isolated vertices, it follows that H must have at least m isolated vertices. Therefore, we can write $H = H' \cup mK_1$ for some graph H'.

Moreover, it is easy to verify that

$$\phi_G(\lambda, t) = \phi_\Gamma(\lambda, t)\phi_{mK_1}(\lambda, t), \phi_H(\lambda, t) = \phi_{H'}(\lambda, t)\phi_{mK_1}(\lambda, t).$$

It follows that $\phi_\Gamma(\lambda, t) = \phi_{H'}(\lambda, t)$. Thus, Γ and H' are ϕ-cospectral. By the assumption that Γ is ϕ-DS, we get that H' is isomorphic to Γ and, hence H is isomorphic to G. □

Example 1. It is known (see e.g. [6]) that the sum of disjoint cycles $\Gamma := C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_s}$ is A-DS and hence ϕ-DS. It follows from Theorem 3.1 that $\Gamma \cup mK_1$ is ϕ-DS.

We remark that in general, Theorem 3.1 is *not* true for a single kind of spectrum. For example, as pointed out in [6], the graph $C_4 \cup K_1$ is not A-DS (since $C_4 \cup K_1$ and $K_{1,4}$ are A-cospectral but non-isomorphic), although C_4 is A-DS. The graph $C_3 \cup K_1$ is not Q-DS, which has a Q-cospectral mate $K_{1,3}$. Nevertheless, C_3 is Q-DS. The graph $\bar{C}_6 \cup K_1$ is not L-DS, which has a L-cospectral mate $\overline{2K_2 \cup K_1} \cup K_2$. However, $\bar{C}_6$ is L-DS, since C_6 is A-DS and hence L-DS (for regular graphs, A-DS, L-DS and Q-DS are all equivalent).

We mention in passing that a similar result was shown in [7], that is, if a graph G with the largest Laplacian eigenvalue being equal to the order n of G (which is equivalent to that the complement $\bar{G}$ of G is disconnected) is L-DS, then $G \cup mK_1$ is L-DS. The last example mentioned above provides a situation that this proposition is in general not true for an arbitrary L-DS graph G.

Next, we give a method for obtaining ϕ-DS graphs by using another kind of graph operation.

Let G be a graph with degree sequence (d_1^{n-1}, d_2), where the exponents denote the multiplicity and $d_1 \neq d_2$. We call such a graph a *bi-degree graph with a dominating vertex.* A graph G is said to be almost d-regular if its degree sequence is $(d^k, (d+1)^{n-k})$. The following theorem gives a method for determining whether a bi-degree graph with a dominating vertex is ϕ-DS.

Theorem 3.2. *Let G_i $(i = 1, 2, \ldots, s)$ be almost d-regular graphs. Let G be obtained from the disjoint union $\cup_{i=1}^{s} G_i$ by adding a new vertex v, which is connected to all the vertices of graph G_i with degree d. Suppose that the degree $\tilde{d}$ of v in G is not equal to $d+1$. If the disjoint union $\cup_{i=1}^{s} G_i$ is A-DS, then G is ϕ-DS.*

Proof. Let H be a graph with $\phi_H = \phi_G$. We show that H is isomorphic to G.

Assume without loss of generality that v is indexed as $n = 1 + \sum_{i=1}^{s} n_i$, where n_i is the order of graph G_i. Then the degree sequence of H is $((d+1)^{n-1}, \tilde{d})$. Assume that w is the vertex in H with degree $\tilde{d}$, which is also indexed as n.

Let λ_i $(i = 1, 2, \ldots, n)$ be the eigenvalues of A_G. By Lemma 2.5, there exist normalized eigenvectors ξ_i (resp. η_i) of graphs G (resp. H) associated with λ_i such that $\xi_i^T D_G \xi_i = \eta_i^T D_H \eta_i$ for each i. Note that $D_G = D_H = diag(d+1, d+1, \ldots, d+1, \tilde{d})$. It follows that

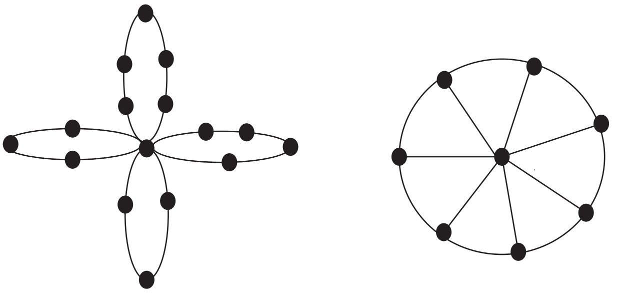
$$(d+1)\xi_i^T\xi_i + (\tilde{d} - d - 1)\left(\xi_i^T e_n\right)^2 = (d+1)\eta_i^T\eta_i + (\tilde{d} - d - 1)\left(\eta_i^T e_n\right)^2.$$

Since $\tilde{d} - d - 1 \neq 0$ and $\xi_i^T\xi_i = \eta_i^T\eta_i = 1$, we get that $\left(\xi_i^T e_n\right)^2 = \left(\eta_i^T e_n\right)^2$ for each i, where $e_n = (0, 0, \ldots, 1)^T$ is the nth vector of the standard orthonormal basis of $\mathbf{R}^n$.

Choose the sign of η_i appropriately such that $\xi_i^T e_n = \eta_i^T e_n$ for each i. Let P_1 and P_2 be defined as in Eq. (6). Let $Q = P_1 P_2^T$. Then it follows that

$$Q^T A_G Q = A_H, \quad Q^T e_n = e_n. \tag{11}$$

By the second equality in Eq. (11), we get that Q is of the form $Q = diag(Q_1, 1)$, where Q_1 is an orthogonal matrix of order $n-1$. By the first equality in Eq. (11), we get $Q_1^T A_{G-v} Q_1 = A_{H-w}$, i.e., the



(A) (B)

Fig. 1. (A) The rose graph $R(6, 5, 4, 4)$ and (B) the wheel graph W_8.

two graphs $G - v$ and $H - w$ are A-cospectral. Since $G - v = \cup_{i=1}^{s} G_i$ is A-DS, $H - w$ is isomorphic to $G - v$. Thus, $H - w$ is the disjoint unions $\cup_{i=1}^{s} H_i$ of s graphs H_i, where H_i is isomorphic to G_i. Note that H_i is almost d-regular, and the degree sequence of H is $((d+1)^{n-1}, \tilde{d})$. It follows that w is adjacent to all vertex of degree d in H_i. Since v is also adjacent to all the vertex of degree d in G_i. Therefore, H is actually isomorphic to G. □

Let G_1 and G_2 be two disjoint graphs. *The product* of G_1 and G_2, denoted by $G_1 \times G_2$, is defined to be a graph G obtained from $G_1 + G_2$ by adding all the edges between $V(G_1)$ and $V(G_2)$. The following theorem shows that if G_i's are d_i-regular graphs and the disjoint union $\cup_{i=1}^{s} G_i$ is A-DS, then the graph obtained by adding a new vertex v connecting with all the vertices of G_i, is ϕ-DS.

Theorem 3.3. *Let G_i $(i = 1, 2, \ldots, s)$ be d-regular graphs. Let $G = (G_1 + G_2 + \cdots + G_s) \times v$. Assume that G is irregular. If the disjoint union $\cup_{i=1}^{s} G_i$ is A-DS, then G is ϕ-DS.*

The proof of the Theorem 3.3 is similar to that of Theorem 3.2, and is omitted. We remark that as mentioned in [6], if $s > 1$, Theorem 3.3 holds for L-spectrum. However, it is generally not true for L-spectrum when $s = 1$. Next, we give some examples to illustrate Theorems 3.2 and 3.3 (Fig. 1).

Example 2. *The wheel graph W_{n+1}* on $n + 1$ vertices is a graph obtained from the cycle C_n by adding a new vertex connecting with all the other vertices. It was shown in [21] that the wheel graph (except for W_7) is L-DS. Let $G_1 = C_n$. Note that C_n is A-DS, it follows from Theorem 3.3 that W_{n+1} is ϕ-DS.

Example 3. Let v_i be any vertex in the cycle C_{l_i}, $i = 1, 2, \ldots, p$. *The rose graph* , denoted by $R(l_1, l_2, \ldots, l_p)$, is a graph obtained from the cycles $C_{l_1}, C_{l_2}, \ldots, C_{l_p}$ by identifying v_i $(i = 1, 2, \ldots, p)$ as one vertex. When $p = 2$, the so-called ∞-graphs without triangles were shown in [17] to be L-DS; when $p = 3$, the rose graph with three petals were shown in [18] to be Q-DS.

Take $G_i = P_{l_i - 1}$. Note that $G_1 + G_2 + \cdots + G_p$ is A-DS (see e.g. [7]). It follows from Theorem 3.2 that the rose graph $R(l_1, l_2, \ldots, l_p)$ $(l_i \geqslant 3)$ is ϕ-DS.

4. Construction of graphs with the same ϕ-spectrum

In this section, we give a method for constructing non-regular and cospectral graphs with respect to the ϕ-spectrum, which is based on a method of Godsil and McKay [9]. Note that two graphs are cospectral with respect to the ϕ-spectrum means that they are A-cospectral, L-cospectral, Q-cospectral and $\tilde{L}$-cospectral, simultaneously. In [2], the author asked the question of how to construct pairs of non-regular graphs which are cospectral with respect to the adjacency spectrum, the Laplacian and the normalized Laplacian spectrum, simultaneously. The method in this section also gives an answer

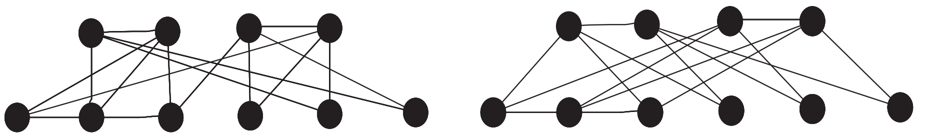


Fig. 2. A pair of non-regular, non-isomorphic ϕ-cospectral graphs.

to this question in a stronger sense. (As pointed out by the reviewer, our construction is essentially equivalent to GM*-switching in [5].)

Theorem 4.1 cf. [5]. *Let G_1 be a d-regular graph with adjacency matrix A_1, and G_2 be an arbitrary graph with adjacency matrix A_2. Suppose the order m (resp. k) of G_1 (resp. G_2) is even. Let C be an $m \times k$ (0,1)-matrix such that $Ce_m = \frac{k}{2}e_m$ and $C^T e_k = \frac{m}{2}e_k$, where e_m (resp. e_k) is all-one vector of order m (resp. k). Let the adjacency matrices of graphs G and H be given as follows:*

$$A_G = \begin{bmatrix} A_1 & C \\ C^T & A_2 \end{bmatrix}, \quad A_H = \begin{bmatrix} A_1 & J - C \\ J^T - C^T & A_2 \end{bmatrix}, \tag{12}$$

where J is an m by k all-one matrix. Then $\phi_G = \phi_H$.

Proof. The proof is similar to that in [5], and is omitted. □

As an illustration, we give a concrete example of ϕ-cospectral graphs, which is taken from Fig. 3 in [5].

Example 4. Let A_1 and A_2 be the adjacency matrices of graphs $2K_2$ and $P_3 \bigcup 3K_1$, respectively. Let $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$. Then, Eq. (14) gives a pair of non-regular, non-isomorphic ϕ-cospectral graphs (see Fig. 2).

5. Concluding remarks

The problem of characterizing graphs by their generalized characteristic polynomial (or equivalently, by their ϕ-spectrum) is considered in the paper. As we have shown, it is comparatively easier to characterize graphs using a family of spectrum simultaneously, rather than by a single kind of spectrum, and we have given several methods for finding graphs that are determined by the ϕ-spectrum. However, there are many problems needed to be further explored in the future:

1. Does there exist a simple characterization of graphs with the same ϕ-spectrum? Clearly, if there exists an orthogonal matrix Q such that $Q^T A_G Q = A_H$ and $Q^T D_G Q = D_H$, then $\phi_G = \phi_H$. Is the converse true? We would like to see a proof or a counterexample.
2. Can more invariants of ϕ-cospectral graphs be derived? If so, we could have good chance to find more ϕ-DS graphs.
3. Can the combinatorial interpretation of the generalized characteristic polynomial as an equivalence of Bartholdi zeta function be helpful in finding ϕ-DS graphs? This line of research might be an interesting future work.

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References

[1] L. Bartholdi, Counting paths in graphs, Enseign. Math. 45 (1999) 83–131.
[2] S. Butler, A note about cospectral graphs for the adjacency and normalized Laplacian matrices, Linear and Multilinear Algebra, doi:10.1080/03081080902722741.
[3] R. Boulet, Spectral characterizations of sun graphs and broken sun graphs, Discrete Math. Theor. Comput. Sci. 11 (2009) 149–160.
[4] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic Press, New York, 1982.
[5] W.H. Haemers, E. Spence, Enumeration of cospectral graphs, European J. Combin. 25 (2004) 199–211.
[6] E.R. van Dam, W.H. Haemers, Which graphs are determined by their spectrum?, Linear Algebra Appl. 373 (2003) 241–272.
[7] E.R. van Dam, W.H. Haemers, Developments on spectral characterizations of graphs, Discrete Math. 309 (2009) 576–586.
[8] C. Dodsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.
[9] C.D. Godsil, B.D. McKay, Constructing cospectral graphs, Aequationes Math. 25 (1982) 257–268.
[10] N. Ghareghani, G.R. Omodi, B. Tayfeh-Rezaie, Spectral characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. 420 (2007) 483–489.
[11] W.H. Haemers, X. Liu, Y. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.
[12] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[13] X. Liu, Y. Zhang, X. Gui, The multi-fan graphs are determined by their Laplacian spectrum, Discrete Math. 308 (2008) 4267–4271.
[14] H.K. Kim, J. Lee, A generalized characteristic polynomial of graphs having a semifree action, Discrete Math. 308 (2008) 555–564.
[15] G.R. Omodi, K. Tajbakhsh, The starlike trees are determined by their Laplacian spectrum, Linear Algebra Appl. 422 (2007) 654–658.
[16] X. Shen, Y. Hou, Y. Zhang, Graphs Z_n and some graphs related to Z_n are determined by their spectrum, Linear Algebra Appl. 404 (2005) 58–68.
[17] J.F. Wang, Q.X. Huang, F. Belardo, E.M. Li Marzi, On the spectral characterizations of ∞-graphs, Discrete Math. 310 (2010) 1845–1855.
[18] J.F Wang, Q.X. Huang, F. Belardo, Sign-less spectral characterizations of 3-rose graphs, Util. Math., in press.
[19] W. Wang, C.X. Xu, On the spectral characterization of T-shape trees, Linear Algebra Appl. 414 (2006) 492–501.
[20] W. Wang, C.X. Xu, A sufficient condition for a family of graphs being determined by their generalized spectra, European J. Combin. 27 (2006) 826–840.
[21] Y. Zhang, X. Liu, X. Yong, Which wheel graphs are determined by their Laplacian spectra, Comput. Math. Appl. 58 (2009) 1887–1890.