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# Inverse indefinite Sturm–Liouville problems with three spectra

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ABSTRACT

We prove that the potential $q(x)$ of an indefinite Sturm–Liouville problem on the closed interval $[a,b]$ with the indefinite weight function $w(x)$ can be determined uniquely by three spectra, which are generated by the indefinite problem defined on $[a,b]$ and two right-definite problems defined on $[a,0]$ and $[0,b]$, where point 0 lies in $(a,b)$ and is the turning point of the weight function $w(x)$.



## 1. Introduction

Consider the indefinite Sturm–Liouville problem consisting of the equation

$$-y''(x)+q(x)y(x)=\lambda w(x)y(x), \quad x\in[a,b], \tag{1.1}$$

and the following self-adjoint boundary conditions

$$y'(a)-h_a y(a)=0, \tag{1.2}$$

$$y'(b)+h_b y(b)=0, \tag{1.3}$$

where $a<0<b$, the potential $q(x)\in L^1(a,b)$ is real-valued and the weight $w(x)$ is real-valued satisfying

$$xw(x)>0 \quad (x\neq 0), \qquad \big|w(x)\big|'\in AC(a,b). \tag{1.4}$$

Here, $h_a, h_b\in\mathbb{R}\cup\{+\infty\}$ ($h_{x_0}=+\infty$ is a shorthand notation for the Dirichlet boundary condition $y(x_0)=0$). It is well known [4] that the spectrum, denoted by $\sigma(q,w,h_a,h_b)$, of this problem consists of a countable infinity of simple eigenvalues which, except a finite numbers of non-real eigenvalues, lie in the real axis and are unbounded from both above and below.

Given $h_0\in\mathbb{R}\cup\{+\infty\}$, we consider the following interface condition at the turning point $x=0$ of weight function $w$

$$y'(0)+h_0 y(0)=0. \tag{1.5}$$

For the potential $q(x)$ and weight $w(x)$, two right-definite Sturm–Liouville problems on $[a,0]$ and $[0,b]$ are generated by

$$-y''(x)+q(x)y(x)=-\lambda\big|w(x)\big|y(x), \quad x\in[a,0), \tag{1.6}$$

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with boundary conditions (1.2) and (1.5), and

$$-y''(x)+q(x)y(x)=\lambda w(x)y(x), \quad x\in(0,b], \tag{1.7}$$

with boundary conditions (1.5) and (1.3), respectively. Clearly, the above two problems are self-adjoint in $L^2[0,a]$ and $L^2[0,b]$ respectively and their spectra, denote by $\sigma_L(q,w,h_a,h_0)$ and $\sigma_R(q,w,h_0,h_b)$, consist of a countable infinity of simple real eigenvalues, where $\sigma_L(q,w,h_a,h_0)$ is bounded from above and $\sigma_R(q,w,h_0,h_b)$ is bounded from below.

For classical inverse Sturm–Liouville problems, Gesztesy and Simon [3] and Pivovarchik [10] proved that, if the three spectra are pairwise disjoint, then the potential $q$ of a Sturm–Liouville problem can be uniquely determined by the spectra of the problems on three intervals $[0,1]$, $[0,a]$ and $[a,1]$ for some $a\in(0,1)$. Furthermore, [3] gave a counterexample to show that the pairwise disjoint of the spectra is necessary.

Our purpose of this paper is to extend the results of [3,10] to the above indefinite weight problems and improve the condition of pairwise disjoint in the left-definite case.

Consider another problem with $q(x)$ replaced by $\tilde{q}(x)$, where $\tilde{q}(x)\in L^1(a,b)$ is real-valued. For the remainder of this paper, we always assume that

$$\begin{cases}\sigma(q,w,h_a,h_b)=\sigma(\tilde{q},w,h_a,h_b),\\ \sigma_L(q,w,h_a,h_0)=\sigma_L(\tilde{q},w,h_a,h_0),\\ \sigma_R(q,w,h_0,h_b)=\sigma_R(\tilde{q},w,h_0,h_b).\end{cases} \tag{1.8}$$

It should be noted that the spectrum $\sigma(q,w,h_a,h_b)$ may contain a finite numbers of non-real eigenvalues, but the left-definite case.

With the above notations, we will prove

**Theorem 1.** *Let $a<0<b$, $h_a,h_b,h_0\in R\cup\{+\infty\}$ and $w(x)$ satisfy* (1.4)*. If* (1.8) *holds and the three sets in* (1.8) *are pairwise disjoint, then $q=\tilde{q}$ a.e. on $[a,b]$.*

**Remarks 1.** This result extends the results of [3,10] to the indefinite weight problems with one turning point. Indeed, we need only know two sets in (1.8) are disjoint, because the intersection of any two sets of them must be in the third set, see Proposition 2.2 in Section 2.

**Theorem 2.** *Let $a<0<b$ and $w(x)$ satisfy* (1.4)*. Let $q(x),\tilde{q}(x)\geqslant 0$ a.e. on $[a,b]$ and $h_a,h_b\in[0,+\infty]$, $h_0\in R\cup\{\infty\}$. If* (1.8) *holds and*

$$\mu_{-0}(h_0)\notin\sigma_R(q,w,h_0,h_b), \qquad \mu_{+0}(h_0)\notin\sigma_L(q,w,h_a,h_0), \tag{1.9}$$

*then $q=\tilde{q}$ a.e. on $[a,b]$. Here, $\mu_{+0}(h_0)$ is the least eigenvalue in $\sigma_R(q,w,h_0,h_b)$, $\mu_{-0}(h_0)$ is the greatest one in $\sigma_L(q,w,h_a,h_0)$.*

**Remarks 2.** Under the assumption about $q$ and $h_a$, $h_b$ in Theorem 2, the problem consisting of (1.1)–(1.3) is left-definite, and the condition of pairwise disjoint of three spectra is replaced by (1.9), which is easier checked. It needs only $\mu_{\pm 0}(h_0)$ are not in the intersection of $\sigma_R(q,w,h_0,h_b)$ and $\sigma_L(q,w,h_a,h_0)$. Actually, one needs only $\mu_{-0}(h_0)\notin\sigma_R(q,w,h_0,h_b)$ for $h_0\leqslant 0$ and $\mu_{+0}(h_0)\notin\sigma_L(q,w,h_a,h_0)$ for $h_0\geqslant 0$.

**Corollary 3.** *Under the hypotheses of Theorem* 2*, if $h_0=0$ or $h_0=\infty$ and* (1.8) *holds, then $q=\tilde{q}$ a.e. on $[a,b]$.*

In the case of Corollary 3, it is known [1,12] that the problem consisting of (1.6), and (1.2) and (1.5) is negative definite and the problem consisting of (1.7), and (1.3) and (1.5) is positive definite. So the disjoint condition (1.9) always holds.

This article is organized as follows. In Section 2, we will give the proof of Theorem 1. In Section 3 we will analyze the distribution of the eigenvalues of left-definite Sturm–Liouville problem defined on $[a,b]$, and other corresponding right-definite problems defined on $[a,0]$ and $[0,b]$. Then, we will prove Theorem 2. The methods used in this work rely on forward asymptotics of the $m$-functions, which will be collected in Appendix A.

## 2. Proof of Theorem 1

In this section, we will prove Theorem 1. The technique which we use to obtain Theorem 1 is an adaptation of the method discussed by F. Gesztesy and B. Simon in [3].

Apply to (1.1), (1.6) and (1.7) the Liouville transformation

$$t=t(x)=\int_0^x\sqrt{\left|w(\tau)\right|}\,\mathrm{d}\tau, \quad x\in[a,b], \qquad u(t)=\left|w(x)\right|^{1/4}y(x).$$

The transformed equations are

$$-u''(t) + Q(t)u(t) = \lambda\,\mathrm{sign}(t)u(t), \quad t \in [c, d],$$
$$-u''(t) + Q(t)u(t) = -\lambda u(t), \quad t \in [c, 0],$$
$$-u''(t) + Q(t)u(t) = \lambda u(t), \quad t \in [0, d],$$

respectively, where $c = t(a) = \int_0^a \sqrt{|w(\tau)|}\,\mathrm{d}\tau < 0$, $d = t(b) = \int_0^b \sqrt{|w(\tau)|}\,\mathrm{d}\tau > 0$ and

$$Q(t) = \left|w(x)\right|^{-\frac{1}{4}} \frac{d^2}{dt^2}\left|w(x)\right|^{\frac{1}{4}} + q(x)/\left|w(x)\right|. \tag{2.1}$$

The boundary conditions (1.2), (1.3) and (1.5) are transformed to

$$u'(c) - H_c u(c) = 0, \qquad u'(d) + H_d u(d) = 0, \qquad u'(0) + H_0 u(0) = 0,$$

respectively, where $H_{t_0} = |w(x_0)|^{-\frac{1}{2}} h_{x_0} - |w(x_0)|^{\frac{1}{4}} [\frac{d}{dt}|w(x)|^{\frac{1}{4}}]_{x=x_0}$ ($x_0 = a, 0, b$ are corresponding to $t_0 = c, 0, d$). It is clear that $\sigma(Q, \mathrm{sign}(x), H_c, H_d) = \sigma(q, w, h_a, h_b)$, $\sigma_L(Q, -1, H_c, H_0) = \sigma_L(q, w, h_a, h_0)$, $\sigma_R(Q, 1, H_0, H_d) = \sigma_R(q, w, h_0, h_b)$. This shows that the Liouville transformation does not change the spectra of the above three Sturm–Liouville problems. By (2.1), $q(x)$ and $Q(t)$ determine uniquely each other, when $w(x)$ is given. So we need only prove that Theorem 1 holds in the case of $w(x) = \mathrm{sign}(x)$.

In order to prove Theorem 1, we need the following lemma on asymptotics, poles and residues determining a meromorphic Herglotz function, see [3, Theorem 2.3].

**Lemma 2.1.** *(See [3].) Let $f_1(z)$ and $f_2(z)$ be two meromorphic Herglotz functions with identical sets of poles and residues, respectively. If*

$$f_1(ix) - f_2(ix) \to 0, \quad \text{as } x \to \infty,$$

*then $f_1 = f_2$.*

**Proof of Theorem 1.** Let $v_-(x,\lambda)$ be the solution of

$$-y''(x) + q(x)y(x) = -\lambda y(x), \quad x \in [a, 0],$$

which satisfies the initial conditions

$$v_-(a) = 1, \qquad v_-'(a) = h_a,$$

for $h_a \neq +\infty$ or

$$v_-(a) = 0, \qquad v_-'(a) = 1$$

for $h_a = +\infty$. Similarly, $v_+(x,\lambda)$ is the solution of

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, b],$$

satisfies the initial conditions

$$v_+(b) = 1, \qquad v_+'(b) = -h_b, \tag{2.2}$$

for $h_b \neq +\infty$ or

$$v_+(b) = 0, \qquad v_+'(b) = -1,$$

for $h_b = +\infty$. It is known [11, p. 11] that $v_\pm(x,\lambda)$ and $v_\pm'(x,\lambda)$ are entire functions of $\lambda$ of order $\frac{1}{2}$ for any fixed $x$.

Define the Weyl $m$-functions [3]

$$m_-(\lambda) = -\frac{v_-'(0,\lambda)}{v_-(0,\lambda)}, \qquad m_+(\lambda) = \frac{v_+'(0,\lambda)}{v_+(0,\lambda)}, \tag{2.3}$$

both $m_-(\lambda)$ and $m_+(\lambda)$ are the Herglotz functions, that is, analytic functions in the upper half-plane $\mathbb{C}^+$, with positive imaginary part.

Let

$$W(\lambda) = \begin{vmatrix} v_-(0,\lambda) & v_+(0,\lambda) \\ v_-'(0,\lambda) & v_+'(0,\lambda) \end{vmatrix} = v_-(0,\lambda)v_+'(0,\lambda) - v_-'(0,\lambda)v_+(0,\lambda). \tag{2.4}$$

Then $W(\lambda)$ is an entire function. The zeros of $W(\lambda)$ are precisely the points of $\sigma(q, w, h_a, h_b)$. In fact, if $\lambda^* \in \sigma(q, w, h_a, h_b)$, then there is a constant $k \neq 0$ such that $v_+(x,\lambda^*) = k v_-(x,\lambda^*)$ $(x \in [a, b])$. Consequently, $W(\lambda^*) = 0$ by (2.4). Conversely, if

$W(\lambda^*) = 0$, then there is a constant $k$ such that $v_+(0,\lambda^*) = kv_-(0,\lambda^*)$ and $v'_+(0,\lambda^*) = kv'_-(0,\lambda^*)$ by (2.4). We assert that $k \neq 0$. If not so, $v_+(x,\lambda) \equiv 0$, by uniqueness theorem of solution, it is in contradiction with (2.2). Hence

$$\varphi(x) = \begin{cases} kv_-(x,\lambda^*), & x \in [a,0], \\ v_+(x,\lambda^*), & x \in [0,b], \end{cases}$$

is the eigenfunction corresponding to eigenvalue $\lambda^* \in \sigma(q, \operatorname{sign}(x), h_a, h_b)$.

Now define a meromorphic function

$$g(\lambda) = \begin{cases} \frac{W(\lambda)}{v_-(0,\lambda)v_+(0,\lambda)}, & h_0 = \infty, \\ \frac{W(\lambda)}{[v'_-(0,\lambda)+h_0 v_-(0,\lambda)][v'_+(0,\lambda)+h_0 v_+(0,\lambda)]}, & h_0 \in \mathbb{R}. \end{cases}$$

It is clear that the set of poles of $g(\lambda)$ is precisely $\sigma_L(q, \operatorname{sign}(x), h_a, h_0) \cup \sigma_R(q, \operatorname{sign}(x), h_0, h_b)$. It should be noted that

$$W(\lambda) = v_-(0,\lambda)\big[v'_+(0,\lambda) + h_0 v_+(0,\lambda)\big] - \big[v'_-(0,\lambda) + h_0 v_-(0,\lambda)\big]v_+(0,\lambda).$$

So,

$$g(\lambda) = \begin{cases} \frac{v'_+(0,\lambda)}{v_+(0,\lambda)} - \frac{v'_-(0,\lambda)}{v_-(0,\lambda)}, & h_0 = \infty, \\ -\frac{v_+(0,\lambda)}{v'_+(0,\lambda)+h_0 v_+(0,\lambda)} + \frac{v_-(0,\lambda)}{v'_-(0,\lambda)+h_0 v_-(0,\lambda)}, & h_0 \in \mathbb{R}, \end{cases}$$
$$= M_+(\lambda) + M_-(\lambda),$$

where

$$M_+(\lambda) = \begin{cases} m_+(\lambda), & h_0 = \infty, \\ \frac{1}{-h_0 - m_+(\lambda)}, & h_0 \in \mathbb{R}, \end{cases} \qquad M_-(\lambda) = \begin{cases} m_-(\lambda), & h_0 = \infty, \\ \frac{1}{h_0 - m_-(\lambda)}, & h_0 \in \mathbb{R}. \end{cases} \tag{2.5}$$

It is easy to see that both $M_+(\lambda)$ and $M_-(\lambda)$ are Herglotz functions. Define $\tilde{m}_+(\lambda)$, $\tilde{m}_-(\lambda)$, $\tilde{M}_+(\lambda)$, $\tilde{M}_-(\lambda)$ and $\tilde{g}(\lambda)$ in an analogous manner with $q(x)$ replaced by $\tilde{q}(x)$.

Let

$$F(\lambda) = g(\lambda)/\tilde{g}(\lambda).$$

Then $F$ is an entire function, since $g$ has the same zeros and poles as $\tilde{g}$, by hypothesis (1.8). For any $\varepsilon > 0$, using Theorems A.1 and A.2 for $h_0 = \infty$; and Theorems A.3 and A.4 for $h_0 \in \mathbb{R}$, respectively, we infer that

$$F(\lambda) = g(\lambda)/\tilde{g}(\lambda) = 1 + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

holds in both sectors of $\varepsilon \leqslant \arg\lambda \leqslant \pi - \varepsilon$ and $\pi + \varepsilon \leqslant \arg\lambda \leqslant 2\pi - \varepsilon$. By Liouville's theorem, we have

$$F(\lambda) \equiv 1,$$

which therefore concludes

$$g(\lambda) = \tilde{g}(\lambda).$$

Now from (2.5), we see that the poles of $M_+(\lambda)$ and $M_-(\lambda)$ are precisely the points of $\sigma_L(q, \operatorname{sign}(x), h_a, h_0)$ and $\sigma_R(q, \operatorname{sign}(x), h_0, h_b)$, respectively. Note that $\sigma_L(q, \operatorname{sign}(x), h_a, h_0)$ and $\sigma_R(q, \operatorname{sign}(x), h_0, h_b)$ are disjoint. We have

$$\operatorname{res} M_+\big(\lambda^*\big) = \operatorname{res} g\big(\lambda^*\big),$$

for all $\lambda^* \in \sigma_R(q, \operatorname{sign}(x), h_0, h_b)$, which means

$$\operatorname{res} M_+\big(\lambda^*\big) = \operatorname{res} \tilde{M}_+\big(\lambda^*\big),$$

for all $\lambda^* \in \sigma_R(q, \operatorname{sign}(x), h_0, h_b)$. This, together with Lemma 2.1 and Theorem A.3, gives

$$M_+(\lambda) = \tilde{M}_+(\lambda).$$

Therefore $q(x) = \tilde{q}(x)$ on $[0,b]$ by Borg theorem [2,8,9]. Similarly, we can show $q(x) = \tilde{q}(x)$ on $[a,0]$. Thus, completes the proof of Theorem 1. □

**Proposition 2.2.** *Under the hypotheses of Theorem* 2, *the intersection of any two sets in* (1.8) *must be in the third set.*

**Proof.** If $\lambda^* \in \sigma_L(q, w, h_a, h_0) \cap \sigma_R(q, w, h_0, h_b)$, then both $v_-(x, \lambda^*)$ and $v_+(0, \lambda^*)$ satisfy the same interface condition (1.5). This gives $W(\lambda^*) = 0$ and $\lambda^* \in \sigma(q, w, h_a, h_b)$.

If $\lambda^* \in \sigma_L(q, w, h_a, h_0) \cap \sigma(q, w, h_a, h_b)$, then $v_-(x, \lambda^*)$ satisfies all boundary conditions (1.2), (1.5), and (1.3), since $v_-(x, \lambda^*)$ is the eigenfunction corresponding eigenvalue $\lambda^*$ on $[a, 0]$ and $[a, b]$. This concludes that $v_-(x, \lambda^*)$ is the eigenfunction corresponding eigenvalue $\lambda^*$ on $[0, b]$. So $\lambda^* \in \sigma_R(q, w, h_0, h_b)$.

The case of $\lambda^* \in \sigma_L(q, w, h_a, h_0) \cap \sigma(q, w, h_a, h_b)$ is similar to the above case. This finishes the proof. □

Proposition 2.2 shows that the condition of pairwise disjoint is equivalent to

$$\sigma_L(q, w, h_a, h_0) \cap \sigma_R(q, w, h_0, h_b) = \phi.$$

## 3. Left-definite case

In this section, we shall prove Theorem 2. For the remainder of this section, we always assume that $q(x) \geqslant 0$ a.e. on $[a, b]$ and $h_a, h_b \geqslant 0$. It is known [5,12] that the problem (1.1)–(1.3) is left-definite. Let $\sigma(q, w, h_a, h_b) = \{\lambda_n \mid n \in \mathbb{Z}^* := \mathbb{Z} \cup \{-0\}\}$. Then all of the eigenvalues are real and can be formed as

$$-\infty \leftarrow \cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty.$$

Let $\sigma_L(q, w, h_a, h_0) = \{\mu_{-n}(h_0) \mid n \in \mathbb{N}^* := \mathbb{N} \cup \{0\}\}$ and $\sigma_R(q, w, h_0, h_b) = \{\mu_n(h_0) \mid n \in \mathbb{N}^*\}$. There are inequalities among $\lambda_{\pm n}$ and $\mu_{\pm n}(0)$ and $\mu_{\pm n}(+\infty)$ as the following lemma.

**Lemma 3.1.** *(See [12].)*

$$\mu_{-n-1}(+\infty) < \lambda_{-n} < \mu_{-n}(0) < \mu_{-n}(+\infty),$$

*with* $\mu_{-0}(0) < 0$ *and*

$$\mu_n(\infty) < \mu_n(0) < \lambda_n < \mu_{n+1}(\infty),$$

*with* $\mu_0(0) > 0$.

For general $h_0 \in \mathbb{R}$, we have the distribution of the eigenvalues as the following lemma.

**Lemma 3.2.** *For any* $h_0 \in R$, *and* $n \in N^*$:

(i) $$\mu_{-n-1}(\infty) < \begin{matrix} \lambda_{-n} \\ \mu_{-n}(h_0) \end{matrix} < \mu_{-n}(\infty),$$

*with* $\mu_{-0}(0) < 0$ *and* $\mu_{-0}(\infty) = +\infty$, *and*

$$\lim_{h_0 \to +\infty} \mu_{-n}(h_0) = \mu_{-n-1}(\infty), \qquad \lim_{h_0 \to -\infty} \mu_{-n}(h_0) = \mu_{-n}(\infty).$$

*In particular, if* $h_0 \geqslant 0$, *then* $\lambda_{-n} < \mu_{-n}(h_0)$.

(ii) $$\mu_n(\infty) < \begin{matrix} \lambda_n \\ \mu_n(h_0) \end{matrix} < \mu_{n+1}(\infty),$$

*with* $\mu_0(0) > 0$ *and* $\mu_0(\infty) = -\infty$, *and*

$$\lim_{h_0 \to +\infty} \mu_n(h_0) = \mu_n(\infty), \qquad \lim_{h_0 \to -\infty} \mu_n(h_0) = \mu_{n+1}(\infty).$$

*In particular, if* $h_0 \leqslant 0$, *then* $\lambda_n > \mu_n(h_0)$.

For convenience and comprehension of reader, see Fig. 1.

Fig. 1 above shows the changing situation of each $\mu_{\pm n}(h_0)$ as $h_0$ through the real axis from $-\infty$ to $+\infty$. Here, $\mu_n^N$ ($\mu_n^D$, resp.) is a shorthand notation for $\mu_n(0)$ ($\mu_n(\infty)$, resp.).

**Proof of Lemma 3.2.** The above facts are obtained immediately by Lemma 3.1 and the dependence of eigenvalues on the boundary conditions, see [6,7].

It is easy to see that, in the left-definite case, for any $h_0$, there is at most one common eigenvalue $\mu_{+0}(h_0)$ or $\mu_{-0}(h_0)$ belonging to $\sigma_R(q, w, h_0, h_b)$ and $\sigma_L(q, w, h_a, h_0)$. □

**Proof of Theorem 2.** If (1.9) holds in the left-definite case, by Lemma 3.2, then $\sigma_R(q, w, h_0, h_b)$ and $\sigma_L(q, w, h_a, h_0)$ are disjoint. Based on condition (1.8), by Theorem 1 we infer $q = \tilde{q}$ a.e. on $[a, b]$. □

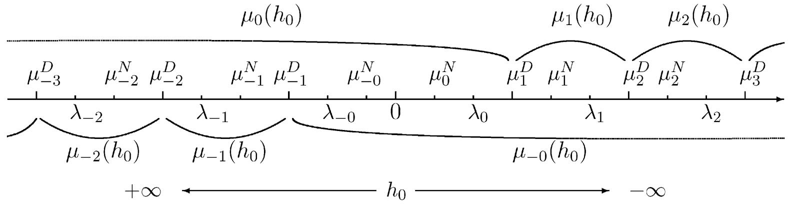


**Fig. 1.** Distribution of the eigenvalues.

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## Appendix A. Asymptotic behavior of *m*-functions

As discussing alone the right-definite problem associated with the equation

$$-y''(x) + q(x)y(x) = -\lambda y(x), \quad x \in [a, 0],$$

we can obtain the asymptotic behaviors of $m_-(\lambda)$ and $M_-(\lambda)$ from $m_+(\lambda)$ and $M_+(\lambda)$ by redefining the branch $\sqrt{-\lambda}$ (i.e. $\sqrt{\lambda}$). When the branch $\sqrt{\lambda}$ has been defined for the right-definite problem associated with the equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, b],$$

$\sqrt{-\lambda}$ must be considered in distinct cases of sign about real part of $\sqrt{\lambda}$.

Let $\sqrt{\lambda} = \xi + i\eta$ with the argument in $[0, \pi)$. Then $\eta \geqslant 0$ and $\sqrt{-\lambda} = -\eta + i\xi$ which has the argument in $[\pi/2, 3\pi/2)$. As $|\lambda| \to \infty$, we have known the asymptotic formulas

$$v_+(x, \lambda) = \begin{cases} \cos[\sqrt{\lambda}(b - x)] + O(e^{\eta(b-x)}/\sqrt{\lambda}), & h_b \in \mathbb{R}, \\ \sin[\sqrt{\lambda}(b - x)]/\sqrt{\lambda} + O(e^{\eta(b-x)}/\lambda), & h_b = \infty, \end{cases}$$

$$v'_+(x, \lambda) = \begin{cases} \sqrt{\lambda}\sin[\sqrt{\lambda}(b - x)] + O(e^{\eta(b-x)}), & h_b \in \mathbb{R}, \\ -\cos[\sqrt{\lambda}(b - x)] + O(e^{\eta(b-x)}/\sqrt{\lambda}), & h_b = \infty, \end{cases}$$

and

$$v_-(x, \lambda) = \begin{cases} \cos[\sqrt{-\lambda}(x - a)] + O(e^{|\xi|(x-a)}/\sqrt{\lambda}), & h_a \in \mathbb{R}, \\ \sin[\sqrt{-\lambda}(x - a)]/\sqrt{-\lambda} + O(e^{|\xi|(x-a)}/\lambda), & h_a = \infty, \end{cases}$$

$$v'_-(x, \lambda) = \begin{cases} -\sqrt{-\lambda}\sin[\sqrt{-\lambda}(x - a)] + O(e^{|\xi|(x-a)}), & h_a \in \mathbb{R}, \\ \cos[\sqrt{-\lambda}(x - a)] + O(e^{|\xi|(x-a)}/\sqrt{\lambda}), & h_a = \infty. \end{cases}$$

And then, it is easy to obtain the asymptotic formulas of $m$-functions as the following theorems (see [9]).

**Theorem A.1.** *For any $\varepsilon > 0$, if $\varepsilon \leqslant \arg\lambda \leqslant 2\pi - \varepsilon$, then*

$$m_+(\lambda) = i\sqrt{\lambda}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right), \quad \text{as } \lambda \to \infty.$$

**Theorem A.2.** *For any $\varepsilon > 0$,*

(i) *if $\varepsilon \leqslant \arg\lambda \leqslant \pi - \varepsilon$, then, as $\lambda \to \infty$,*

$$m_-(\lambda) = -\sqrt{\lambda}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right);$$

(ii) *if $\pi + \varepsilon \leqslant \arg\lambda \leqslant 2\pi - \varepsilon$, then, as $\lambda \to \infty$,*

$$m_-(\lambda) = \sqrt{\lambda}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right).$$

**Proof.** Theorem A.1 is the result as a classical Sturm–Liouville problem [9]. It is a corollary of Theorem A.1 in the case (i) of Theorem A.2 by replacing $\sqrt{\lambda}$ with $\sqrt{-\lambda}$. For the case (ii) of Theorem A.2, we need take the branch $-\sqrt{-\lambda}$, such that its imaginary part is positive. □

**Theorem A.3.** *Fixed $h_0 \in R$. For any $\varepsilon > 0$, if $\varepsilon \leqslant \arg\lambda \leqslant 2\pi - \varepsilon$, then*

$$M_+(\lambda) = \frac{i}{\sqrt{\lambda}}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right), \quad \text{as } \lambda \to \infty.$$

**Theorem A.4.** *Fixed $h_0 \in R$. For any $\varepsilon > 0$,*

(i) *if $\varepsilon \leqslant \arg\lambda \leqslant \pi - \varepsilon$, then, as $\lambda \to \infty$,*

$$M_-(\lambda) = \frac{1}{\sqrt{\lambda}}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right);$$

(ii) *if $\pi + \varepsilon \leqslant \arg\lambda \leqslant 2\pi - \varepsilon$, then, as $\lambda \to \infty$,*

$$M_-(\lambda) = -\frac{1}{\sqrt{\lambda}}\left(1 + o\left(\frac{1}{\sqrt{\lambda}}\right)\right).$$

Theorems A.3 and A.4 are the immediate corollary of Theorems A.1 and A.2.

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