Asymmetric acoustic transmission in graded beam

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ABSTRACT

We demonstrate the dynamic effective material parameters and vibration performance of a graded beam. The structure of the beam was composed of several unit cells with different fill factors. The dispersion relations and energy band structures of each unit cell were calculated using the finite element method (FEM). The dynamic effective material parameters in each unit cell of the graded beam were determined by the dispersion relations and energy band structures. Longitudinal wave propagation was investigated using a numerical method and FEM. The results show that the graded beam allows asymmetric acoustic transmission over a wide range of frequencies.

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1. Introduction

In the last few decades, phononic crystals have attracted much attention owing to their unique acoustic properties [1–9]. In particular, their effective material parameters can be negative [1,2], anisotropic [3], or dependent on the frequencies of vibration [4,5], resulting in some unusual phenomena, such as low-frequency forbidden bands [5,6] and negative refraction [7]. These exotic effects have been intensely studied and many intriguing devices have been designed, such as the invisible cloak [3], superlens [7], and acoustic shield [8]. With recent developments in graded phononic crystals, the innovative properties of this structure promise a wide variety of applications, such as a plane lens and acoustic absorber. By introducing the concept of a gradient-index, Lin et al. [9] designed a graded phononic crystal to control the propagation of acoustic waves. The gradient refractive index profile in a gradient-index phononic crystal is obtained by adjusting the material parameters [9], fill factor [9,10] and lattice constants [11]. Researchers have developed graded phononic crystals and investigated the focusing of the Lamb waves [12] and Rayleigh waves [13], both numerically and experimentally. Using graded phononic crystals, Yu [14] designed a broadband acoustic absorber.

In parallel, by using a nonlinear material [15,16], an asymmetric structure [17,18], and a graded structure [19–23], the asymmetric acoustic transmission effect has been realized. Asymmetric acoustic transmission is important in many applications, such as acoustic rectifying devices or sound diodes. Researchers have proposed two-dimensional prism models consisting of phononic crystals [19,20] or acoustic metamaterials [21] for unidirectional transmission. Prism structures can be regarded as spatial gradient structures. Chen et al. [22] proposed graded grating phononic crystal slabs that support broad bi-directional asymmetric Lamb wave transmission. Zhu et al. [23] designed a four-body composite structure, which consists of two reflectors and two metasurfaces. The metasurfaces are fabricated by using graded grooves. By changing the thickness of a graded plate, Krylov et al. [24,25] investigated the “acoustic black holes” for flexural waves. However, most of the aforementioned works focus on the properties of wave propagation in graded phononic crystals or acoustic metamaterials. Besides, it is necessary to investigate the relationship between the properties of wave propagation and effective material parameters of graded phononic crystals.

In this work, a gradient fill factor of each unit cell is introduced into the phononic crystals to obtain graded beam. The dispersion relations and energy band structures of each unit cell are calculated using the finite element method (FEM). Then, the dynamic effective material parameters in each unit cell of the graded beam are determined using the dispersion relations and energy band structures. The vibration performance of the graded beam is then investigated by using a numerical method and FEM. The paper is organized as follow. In Section 2, the model of the structure is presented and explained. And the calculation methods of dispersion relations and energy band structures based on FEM are described. The setup of numerical model of effective materials parameters is presented. In Section 3, the vibration performance of
the graded beam is investigated based on the FEM and numerical method. Finally, conclusions are given in Section 4.

2. Effective parameters of a graded beam

The model studied is shown in Fig. 1. A graded beam is composed of several unit cells with length $a$ and radius $r$. Each unit cell is composed of beam-A (white regions in Fig. 1) with length $a_1 = (1 - F) a$ and beam-B (gray regions in Fig. 1) with length $a_2 = Fa$. Beam-A and beam-B are made of polymethylmethacrylate (PMMA) and steel, respectively. The material properties of PMMA are $\rho_{\text{PMMA}} = 1142 \text{ kg m}^{-3}$, $E_{\text{PMMA}} = 2 \text{ GPa}$, and $\sigma_{\text{PMMA}} = 0.389$; and those for steel are $\rho_{\text{steel}} = 7782 \text{ kg m}^{-3}$, $E_{\text{steel}} = 210.6 \text{ GPa}$, and $\sigma_{\text{steel}} = 0.3$. By adjusting the fill factor $F$ of beam-B, the gradient material properties of the beam can be changed.

To obtain the effective material parameters of the graded beam, each unit cell is treated as a cell of a periodic beam. Thus, the effective density and elastic modulus of each unit cell can be calculated by using its energy band structure [26]. The dispersion relation and energy band structure of each unit cell are calculated by using the FEM. The periodic boundary conditions are considered on two sides of a unit cell. The dispersion relations and eigenmodes are obtained by varying the Bloch wave vector in the first Brillouin zone and by solving a spectral problem. The eigen- vectors represent the modal displacement fields. The mechanical energy can be calculated from the deformation of the structure.

Fig. 2(a) shows the dispersion relations for a longitudinal wave in eleven periodic beams with the various fill factors. As the fill factor increases, the dispersion bands of the periodic beams first fall and then rise along the $\Gamma X$ orientation. We found that the fill factor of the lowest dispersion curve is equal to 0.5. In fact, when the fill factor tends to 0.5, the proportion of two materials in the periodic beam will approach the same value, and this change causes a stronger dispersion. Conversely, when the fill factor tends to 0 or 1, the dispersion curves will rise and approach the dispersion curve of the beam of the dominant material. Fig. 2 (b) shows the relationships between the wave vector and mechanical energy of each unit cell with various fill factors. It should be noted that the energy is not only related to the wave vector, but also to the vibration amplitude. Thus, the vibration amplitude should be normalized to eliminate the effect of the amplitude on the energy band structure. The proportion of steel increases with the fill factor. In addition, the mechanical energy density of steel is higher than that of PMMA when their vibration amplitudes are equal. Thus, the energy bands of the periodic beams rise along the $\Gamma X$ orientation as the fill factor increases.

We consider each unit beam to be effectively equivalent to a homogeneous beam with effective density, $\rho_{\text{eff}}$, and elastic modulus, $E_{\text{eff}}$. This homogeneous beam has the same size as the unit beam, as shown in Fig. 3. Meanwhile, the dispersion relations and energy band structures of the homogeneous beam are the same as those of the unit beam.

For the longitudinal waves propagating in the homogeneous beam, the equation of motion is given by

$$E_{\text{eff}}\frac{\partial^2 u}{\partial x^2} + \rho_{\text{eff}}\frac{\partial^2 u}{\partial t^2} = 0,$$  \hspace{1cm} (1)

where $u$ denotes the displacement of a structure and $S$ is the cross-sectional area. The harmonic solution of Eq. (1) is

$$u(x, t) = A \sin(kx - \omega t),$$  \hspace{1cm} (2)

where $A$ denotes the vibration amplitude of a structure and $k$ and $\omega$ are the wave vector and angular frequency, respectively. Substituting Eq. (2) into Eq. (1), the dispersion relation can be expressed as

$$\rho_{\text{eff}}\omega^2 = E_{\text{eff}}k^2$$  \hspace{1cm} (3)

The effective density and elastic modulus cannot be uniquely determined by dispersion relations, as shown in Eq. (3). According to the Ref. [27], the accompanying energy equation of Eq. (1) is easily obtained

$$\frac{\partial}{\partial x} \left( \frac{1}{2}\rho_{\text{eff}} S \left( \frac{\partial u}{\partial x} \right)^2 \right) + \frac{1}{2} E_{\text{eff}} S \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial}{\partial t} \left( -E_{\text{eff}} S \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right) = 0$$  \hspace{1cm} (4)

The first term, which takes derivative with respect to $t$, represents the energy density and the second term, which takes derivative with respect to $x$, represents the energy flux. Substituting Eq. (2) into the energy density term in Eq. (4), the energy density, $w$, is approximately given by

$$w = \frac{1}{2} \rho_{\text{eff}} S a^2 \sin^2(kx - \omega t) + \frac{1}{2} E_{\text{eff}} S k^2 a^2 \cos^2(kx - \omega t)$$  \hspace{1cm} (5)

Considering the average values of $\cos^2(kx\cdot o\cdot t)$ and $\sin^2(kx\cdot o\cdot t)$ in one period are both equal to one half, we have

$$w = \frac{1}{4} \rho_{\text{eff}} S a^2 + \frac{1}{4} E_{\text{eff}} S k^2 a^2$$  \hspace{1cm} (6)

Eq. (6) shows that the energy at each point is equal to the sum of the kinetic and potential energy at that point. The total energy, $W$, of the longitudinal vibration of the beam is calculated using

$$W = \int_0^L w \, dx$$  \hspace{1cm} (7)

In a mechanical structure without damping, the maximum kinetic energy, $T_{\text{max}}$, the maximum potential energy, $U_{\text{max}}$, and the total energy are equal:

$$W = T_{\text{max}} = U_{\text{max}}$$  \hspace{1cm} (8)

Substituting Eq. (3) and Eq. (7) into Eq. (8), we obtain the maximum kinetic and potential energies of a homogeneous beam.

$$T_{\text{max}} = W = \frac{1}{2} \rho_{\text{eff}} S a^2$$

$$U_{\text{max}} = W = \frac{1}{2} E_{\text{eff}} S k^2 a^2$$  \hspace{1cm} (9)

Eq. (9) shows that the maximum kinetic and potential energies are functions where the dependent variables are the effective material parameters, $\rho_{\text{eff}}$ and $E_{\text{eff}}$. Thus, the effective material parameters can be expressed as

<table>
<thead>
<tr>
<th>$F$</th>
<th>$T_{\text{max}}$</th>
<th>$U_{\text{max}}$</th>
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</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
</tr>
<tr>
<td>$0.1$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
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<tr>
<td>$0.2$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
</tr>
<tr>
<td>$0.8$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
</tr>
<tr>
<td>$0.9$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{2} \rho_{\text{eff}} S a^2$</td>
<td>$\frac{1}{2} E_{\text{eff}} S k^2 a^2$</td>
</tr>
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Fig. 1. Configuration of the graded beam.
As mentioned above, $A$ of a unit beam has been normalized, and thus, a constant. In the low frequency limit, $\rho_{\text{eff}}(0)$ and $E_{\text{eff}}(0)$ are equal to the static effective density and elastic modulus, respectively. Thus, $A$ can be obtained

$$A = \lim_{\omega \to 0} \sqrt{\frac{2W}{\rho_{\text{se}} a^2}},$$

(11)

where $\rho_{\text{se}}$ is the static effective density, which is given in Ref. [28] as

$$\begin{align*}
\rho_{\text{eff}} &= \frac{2T_{\text{max}}}{ Sa^2 A a} \\
E_{\text{eff}} &= \frac{2U_{\text{max}}}{Sk A a} \\
\end{align*}$$

(10)

Fig. 4(a) and (b) show the effective density and elastic modulus, respectively. The black lines are the static effective material parameters, and they are also the dynamic effective material parameters in the long-wave limit in Fig. 4. The gray regions show the changing ranges of dynamic effective material parameters with the frequency from 0 to 5000 Hz. It can be observed that when the fill factor increases from 0 to 1, the effective density and elastic modulus increase. The gray regions first increase in width, and then become narrow. The increase of fill factor causes the decrease in the amount of PMMA. Thus, both the effective density and elastic modulus increase. In addition, the dispersion effect is strengthened when the fill factor increases from 0 to 0.5 and the range of the dynamic effective material parameter increases. The dispersion effect is diminished when the fill factor increases from 0.5 to 1. Thus, the range of the dynamic effective material parameter decrease. By comparing Fig. 4(a) and (b), we find that the dynamic effective density is larger than the static effective density, however, the dynamic effective elastic modulus is smaller than the static effective elastic modulus.

3. The vibration performance of the graded beam

We have assumed that the graded beam is effectively equivalent to a functionally graded material beam with density, $\rho(x)$ and elastic modulus, $E(x)$. The dashed lines in Fig. 4 shows the material parameters as a function of position in the beam. The density and elastic modulus can be expressed as

$$\begin{align*}
\rho(x) &= \rho_A \frac{x}{11a} \rho_b + \rho_b \frac{11a - x}{11a} \\
E(x) &= \frac{11a E_A E_b}{E_B + (11a - x)} \\
\end{align*}$$

(13)

When a longitudinal wave propagates in the functionally graded material beam, the vibration equation can be written as

$$\frac{d}{dx} \left( E(x) \frac{du(x)}{dx} \right) + \omega^2 \rho(x) u(x) = 0,$$

(14)

where $u(x)$ is the longitudinal particle displacement of the functionally graded material beam. Expanding the bracketed expression in Eq. (14), we obtain
\[ u' + \frac{E}{E}u' + k^2u = 0, \]

where \( k = \sqrt{\omega \rho(x)/E(x)} \). The solution of Eq. (15) in the usual geometrical-acoustic representation is sought using

\[ u(x) = A(x)e^{ibx}, \]

where \( A(x) \) and \( B(x) \) are the slowly varying amplitude and phase of the wave, respectively. Substituting Eq. (16) into Eq. (15), setting the real part to zero and discarding terms with second and higher derivatives of the slowly varying functions \( A(x) \) and \( B(x) \), we obtain

\[ dB = \sqrt{k^2} dx \]

We are only concerned with the positive value of the square root on the right-hand side of Eq. (17). The solution of this equation corresponding to the wave propagation in the positive direction along \( x \)-axis is

\[ B = \int k dx \]

By substituting Eq. (16) into Eq. (15) and setting the imaginary part equal to zero, the transport equation is obtained.

\[ 2A'B' + AB'' + \frac{E}{E}AB' = 0 \]  

(19)

The transport equation describes evolution of the amplitude, \( A(x) \), while the wave propagates in the functionally graded material beam. Substituting Eq. (18) into the transport equation, Eq. (19), we obtain \( A(x) \) as follows:

\[ A(x) = \frac{C}{\sqrt{\omega \rho(x)/E(x)}}. \]  

(20)

where \( C \) is an arbitrary constant. The amplitude is dependent on the frequency, density, and elastic modulus. The transmission coefficients of this functionally graded material beam are easily deduced.

\[
\begin{align*}
T_{LN} = A_L/A_L = \left( \frac{E_L\rho_L}{E_R\rho_R} \right)^{1/4} \\
T_{RN} = A_R/A_R = \left( \frac{E_R\rho_R}{E_L\rho_L} \right)^{1/4}
\end{align*}
\]

(21)

where \( T_{LN} \) and \( T_{RN} \) are the transmission coefficients of the incident waves emanating from the left and right sides of the sample, respectively. \( A_L \) and \( A_R \) are the amplitudes at the left and right ends of the functionally graded material beam, respectively. \( \rho_L, \rho_R, E_L \) and \( E_R \) are the material parameters of the left and right ends of the functionally graded material beam, respectively. Eq. (21) has a simple physical significance: the transmission coefficients are dependent on the material parameters of each end of the structure. Thus, the transmission coefficients can be adjusted by changing the material parameters at each end of the functionally graded material beam.

In order to investigate the transmission properties of the graded beam by manipulating longitudinal waves, we calculated the transmission coefficients of the graded beam by using the FEM. In this simulation model, the free boundary condition is considered at all edges of the structures. The displacement excitation sources, with a frequency sweeping from 0 to 5000 Hz, are applied on one side of the graded beam. The transmitted displacement values are detected and recorded on the other side of the graded beam and the transmission coefficient is defined as

\[ T = \frac{U'_0}{U'_1}. \]  

(22)

Fig. 4. The dynamic effective (a) density and (b) elastic modulus of the graded beam.

Fig. 5. The transmission coefficients of the incident waves emanating from (a) left and (b) right sides of the sample.
where $U_r$ and $U_l$ are the amplitudes of the transmitted and incident displacement, respectively. By varying the excitation frequency of the incident waves, the transmission coefficient is obtained.

Fig. 5(a) and (b) show the transmission coefficients for incident waves emanating from the left and right sides, respectively. The solid lines are the transmission coefficients, $T_{LF}$ and $T_{RF}$, of the FEM simulations and the dashed lines are the results of the analytical solutions. The transmission coefficients $T_{LF}$ and $T_{RF}$ are significantly different along the entire frequency range. There are four peaks and troughs in the $T_{RF}$ curve in the frequency range 0–5000 Hz, as shown in Fig. 5(a). The troughs fall to very small values close to 0.1, which means that the transmitted displacement value is only 10% of the excitation amplitude. Alternatively, three peaks and troughs appear in Fig. 5(b) in the frequency range of 0–5000 Hz. The troughs fall to values of ~5: which means that the wave amplitude is magnified by a factor of five when the longitudinal wave propagates from the right side to the left side of the graded beam. By comparing solid and dashed lines in Fig. 5, we find that all troughs of $T_{RF}$ and $T_{LF}$ curves are close to the results of the analytical solution.

The large difference between $T_{RF}$ and $T_{LF}$ clearly indicates that the asymmetric acoustic transmission phenomenon exists in this structure. In order to quantify the phenomenon of asymmetric transmission, we have introduced the asymmetric transmission ratio defined as $\eta = T_{RF}/T_{LF}$. Fig. 6 shows the asymmetric transmission ratio of the system. We find that the value of the asymmetric transmission ratio remains greater than 1 in the four bands below 5000 Hz, such as 530–1830 Hz, 1900–3240 Hz, 3290–4600 Hz, and 4640–5000 Hz, shown as the light gray regions. These ranges account for 86.2% of the frequency range. The asymmetric transmission ratio reaches a value of up to 10 in the ranges of 900–1560 Hz, 2120–3070 Hz, 3470–4480 Hz, and 4790–5000 Hz, shown by the dark gray regions. These ranges account for 56.6% of the frequency range. There are three peaks, at 1100 Hz, 2600 Hz, and 4090 Hz, where the asymmetric transmission ratio is greater than 1000. Their locations approximately correspond to the peaks in Fig. 5(b). The four ranges where the asymmetric transmission ratio is less than 1 are 0–520 Hz, 1840–1890 Hz, 3250–3280 Hz, and 4610–4630 Hz. These ranges correspond to the peaks in Fig. 5(a).

To understand the asymmetric transmission effect further, we simulated and measured the transmitted displacement field distribution in our system. As shown in Fig. 7(a) and (b), the displacement excitation sources with a frequency of 3500 Hz are applied on the left and right sides of the graded beam, respectively. The longitudinal wave propagates through the structure in the forward direction at a frequency of 4250 Hz and the amplitude of the longitudinal wave becomes larger. Additionally, two waves with different wavelengths are found in Fig. 7(a). Since the material parameters of the graded beam increase in the forward direction (as shown in Fig. 4), the wavelength, $\lambda_1$, is smaller than $\lambda_2$.

On the contrary, in the case where the longitudinal wave propagates in the backward direction, the amplitude of the longitudinal wave becomes smaller. Moreover, there exist two waves with different wavelengths in Fig. 7(b): by comparing the wavelengths of the two waves, we find that the wavelength, $\lambda_4$, is larger than $\lambda_3$.

4. Conclusions

In conclusion, we have demonstrated the dynamic effective material parameters and vibration performance of a graded beam. The dynamic effective material parameters in each unit cell of the graded beam were determined by using the dispersion relations and energy band structures via the FEM. The numerical results showed that the effective density and elastic modulus of the graded beam are functions where the dependent variables are the frequency and fill factors. Longitudinal wave propagation was investigated by using a numerical method and the FEM. The results show that the graded beam allows asymmetric acoustic transmission over a wide range of frequencies and the displacement ratio depends on the material parameter ratio of each end of the graded beam. The graded beam described in this work may be useful in applications such as acoustic diodes, acoustic rectifiers, amplified vibration, and vibration isolation.

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