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To cite this article: Xinwei Zha et al 2020 Laser Phys. 30 035201

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Generalized monogamy linear entropy relations for multi-qubit pure states

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Received 8 January 2020
Accepted for publication 24 January 2020
Published 14 February 2020

Abstract
Monogamy of entanglement is an interesting property discovered recently in the context of multi-qubit entanglement. Here, we present a new kind of monogamous relations for pure quantum states of $N$ qubits. By comparing the entanglement monogamy equality proposed by Coffman, Kundu, and Wootters, we prove that there exists a strict triangle relation for the three-qubit system. Since the entropy defines the entanglement of a pure state of a pair of quantum systems; therefore, using linear entropy, we conjecture that there is a polygon relation of many-qubit systems. We also formalize the general triangle relation for the three-qubit system.

Keywords: monogamy of entanglement, $N$ qubits, linear entropy and entanglement monogamy

1. Introduction
Quantum entanglement is an essential feature of quantum mechanics, which distinguishes the quantum from the classical world [1–3]. One of the fundamental differences between quantum entanglement and classical correlations, a key property of entanglement is that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. In multipartite quantum systems, there can be several inequivalent types of entanglement among the subsystems, and the amount of entanglement with different types might not be directly comparable to each other. Coffman, Kundu, and Wootters (CKW) conceptualized the monogamy of entanglement (MOE) for a three-qubit system [4] whose bipartite quantum entanglement was measured already using concurrence [5] and have been proven recently for N-qubit [6]. Since, the formalization of the concept of MOE by CKW, many scientists have contributed to generalize MOE for multi-qubit system [7, 8] and tripartite quantum states [9] along with complementarity between tripartite quantum correlation and bipartite Bell-inequality violation [10]. The monogamy relation of entanglement is a way to characterize the different types of entanglement distribution. The monogamy relations give rise to the structures of entanglement in the multipartite setting. Recently, Zhu et al [11] present a monogamous relations based on concurrence and concurrence of assistance.

In this paper, we present a new kind of monogamous relations for pure quantum states of $N$ qubits. We derive monogamy relations for many-qubit systems using linear entropy. Furthermore, we formalize the general triangle relation for the three-qubit system. We also give some monogamy relations for four-, five- and six-qubit pure states, including the GHZ and W states.

2. Generalized monogamy relations for many-qubit systems
It is well known that the total quantum correlation of qubit $k$ with the remaining qubits $R_k$ can be characterized by the linear entropy [12]
\[ \tau_{k(R_k)} = 2 \left( 1 - tr \left( \rho_k^2 \right) \right). \] (1)
We can also define that the total quantum correlation of qubits \( ml \) with the remaining qubits \( R_{ml} \) can be characterized by the linear entropy using \([12]\)

\[
\tau_{ml}(R_{ml}) = 2 \left( 1 - tr \left( \rho_{ml}^2 \right) \right).
\]

(2)

Similarly, one can characterize the total quantum correlation of qubits \( stp \) with the remaining qubits \( R_{stp} \) using the linear entropy as

\[
\tau_{stp}(R_{stp}) = 2 \left( 1 - tr \left( \rho_{stp}^2 \right) \right).
\]

(3)

Following these, for an N-qubits pure state, we can obtain that the linear entropy is contributed by different levels of quantum correlations, i.e.,

\[
\tau_k(R_k) = 2 \left( 1 - tr \left( \rho_{k}^2 \right) \right),
\]

\[
\tau_{il}(R_{il}) = 2 \left( 1 - tr \left( \rho_{i}^2 \right) \right),
\]

\[
\tau_{ijkl}(R_{ijkl}) = 2 \left( 1 - tr \left( \rho_{ijkl}^2 \right) \right).
\]

(4A)

From \([13]\), we get

\[
Tr_{ij}\rho_{ij}^2 = \frac{1}{4} + \frac{1}{2}F_{ij}, \quad i = 1, 2, \ldots , n
\]

\[
Tr_{ik}\rho_{ik}^2 = \frac{1}{4} + \frac{1}{2} \left( F_{ik} + F_{ik} + F_{ik} + F_{ik} \right), \quad i = 1, 2, \ldots , n
\]

\[
Tr_{ijkl}\rho_{ijkl}^2 = \frac{1}{16} + \frac{1}{16} \left( F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} \right),
\]

where

\[
F_{ij} = \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2
\]

\[
= \sum_{\alpha=0,1,2,3} \langle \psi | \alpha_{ij} | \psi \rangle^2.
\]

\[
F_{ij} = \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2
\]

\[
+ \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2 + \langle \psi | \sigma_{ij} | \psi \rangle^2
\]

\[
= \sum_{\alpha=0,1,2,3} \langle \psi | \alpha_{ij} \alpha_{ij} | \psi \rangle^2.
\]

Similarly

\[
F_{ijkl} = \sum_{\alpha_1=0,1,2,3} \sum_{\alpha_2=0,1,2,3} \sum_{\alpha_3=0,1,2,3} \sum_{\alpha_4=0,1,2,3} \langle \psi | \alpha_1 \alpha_2 \alpha_3 \alpha_4 | \psi \rangle^2
\]

\[
F_{ijkl} = \sum_{\alpha_1=0,1,2,3} \sum_{\alpha_2=0,1,2,3} \sum_{\alpha_3=0,1,2,3} \sum_{\alpha_4=0,1,2,3} \langle \psi | \alpha_1 \alpha_2 \alpha_3 \alpha_4 | \psi \rangle^2
\]

(5)

(6)

On the other hand, we have \([14]\)

\[
Tr_{ij}\rho_{ij} = \frac{1}{4} + \frac{1}{4} \left( -F_{ij} - F_{ij} + F_{ij} \right), \quad i = 12, 13, \ldots , n
\]

\[
Tr_{ik}\rho_{ik} = \frac{1}{4} + \frac{1}{4} \left( -F_{ik} - F_{ik} + F_{ik} + F_{ik} - F_{ik} \right)
\]

\[
Tr_{ijkl}\rho_{ijkl} = \frac{1}{16} + \frac{1}{16} \left( -F_{ijkl} - F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} + F_{ijkl} \right),
\]

(7)

where

\[
\tilde{\rho}_{ij} = \sigma_0^{\otimes 2} \rho_{ij}^{\otimes 2} \tilde{\rho}_{ik} = \sigma_0^{\otimes 3} \rho_{ik}^{\otimes 3} \tilde{\rho}_{ijkl} = \sigma_0^{\otimes 4} \rho_{ijkl}^{\otimes 4}
\]

It is obvious that such invariants satisfy

\[
F_i \geq 0, F_{ij} \geq 0, F_{ik} \geq 0, F_{ijkl} \geq 0.
\]

From equations (4)–(7), we can obtain

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl}.
\]

(8)

Since, \( tr\rho_{ijkl} \rho_{ijkl} \geq 0 \), we have

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl} \geq 0.
\]

(9)

This equation is just of that \([11]\).

Also, we can obtain

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl} \geq 0.
\]

(10)

As \( tr\rho_{ijkl} \rho_{ijkl} \geq 0 \) we have

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl} \geq 0.
\]

(11)

That is

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl} \geq 0.
\]

(12)

Therefore, we have

\[
\tau_{1(R_1)} + \tau_{2(R_2)} \geq 2r_p\rho_{ijkl} \geq 0.
\]

(13)

3. The monogamy relations for three-, five- and six-qubit systems

For three-qubit pure states,

\[
|\psi_{ABC}^\alpha = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + a_4|100\rangle + a_5|101\rangle + a_6|110\rangle + a_7|111\rangle.
\]

(14)

We have \( \tau_{A(B)} = \tau_{A(B)C} \), \( \tau_{B(C)} = \tau_{B(C)} \), \( \tau_{AB(A)C} = \tau_{AB(A)} = \tau_{C(A)} = \tau_{C(A)} \).

Using equation (9), we obtain a triangle relation

\[
\tau_{A(B)} + \tau_{B(C)} - \tau_{C(A)} \geq 0.
\]

(15)

For GHZ state, \( |\psi_{ABC}^{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \), we have \( \tau_{A(B)} = \tau_{B(C)} = \tau_{C(A)} = 1 \), we have \( \tau_{A(B)} + \tau_{B(C)} - \tau_{C(A)} > 0 \);

For W state, \( |\psi_{ABC}^{W} = \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle + |110\rangle) \), we have \( \tau_{A(B)} = \tau_{B(C)} = \tau_{C(A)} = \frac{1}{4} \), we have \( \tau_{A(B)} + \tau_{B(C)} - \tau_{C(A)} > 0 \).

(16)
\[ \tau_A(R_a) = \tau_B(A(BCD)), \quad \tau_B(R_b) = \tau_B(B(ACDE)), \quad \tau_C(R_c) = \tau_C(ABD), \quad \tau_D(R_d) = \tau_D(A(BDCE)), \]
\[ \tau_{AB}(R_{ab}) = \tau_B(A(BCDE)), \quad \tau_{AC}(R_{ac}) = \tau_C(ABD), \quad \tau_{AD}(R_{ad}) = \tau_D(A(BDCE)), \quad \tau_{BC}(R_{bc}) = \tau_B(C(DAEC)), \]
\[ \tau_{BCD}(R_{bcd}) = \tau_{B(BCD)}, \]  
Using equation (13), we obtain the triangle relation as
\[ 3(\tau_{BCD} + \tau_{BD}(ACDE) + \tau_{CD}(AB)) - (\tau_{AB}(ACDE) + \tau_{AC}(BDCE) + \tau_{AD}(BACE) + \tau_{BCD}(ACDE)) 
+ \tau_{BD}(ACE) + \tau_{CE}(ABD) + \tau_{DE}(ABC) \geq 0. \]

For GHZ state, \( |\varphi\rangle_{ABCD} = \frac{1}{\sqrt{2}} (|00000\rangle + |1111\rangle) \), we have
\[ \tau_{AB}(ACDE) = \tau_{AC}(BDCE) = \tau_{AD}(BACE) = \tau_{BCD}(ACDE) = \tau_D(ABCD) = 1, \]
\[ \tau_{BCD}(ACDE) = \tau_{B(ACDE)} = \tau_{D(ABCD)} = 1. \]

4. Conclusion

We found a general linear entropy relation for pure quantum states of \( N \)-qubit. We also proved that in general all many-qubit systems, there exist strict monogamy laws for quantum correlations. Moreover, we may also establish monogamy equality for three-qubit pure states, which is analogous to CKW equality. Finally, based on the results presented, we conjecture that there is relation many-qubit systems. We believe that these results can add a new family to such equalities and can play a vital role in quantum communication, computing, information process and information disturbance tradeoff phenomenon in composite systems.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (61705182), National Science Foundation of Shannxi Province (2017JQ6024).

Appendix A. The monogamy relations for five- and six-qubit systems

For five-qubit systems
\[ \tau_A(R_a) = \tau_B(A(BCD)), \quad \tau_B(R_b) = \tau_B(B(ACDE)), \quad \tau_C(R_c) = \tau_C(ABD), \quad \tau_D(R_d) = \tau_D(A(BDCE)), \]
\[ \tau_{AB}(R_{ab}) = \tau_B(A(BCDE)), \quad \tau_{AC}(R_{ac}) = \tau_C(ABD), \quad \tau_{AD}(R_{ad}) = \tau_D(A(BDCE)), \quad \tau_{BC}(R_{bc}) = \tau_B(C(DAEC)), \]
\[ \tau_{BCD}(R_{bcd}) = \tau_B(C(DAEC)), \]  
Using equation (13), we obtain the triangle relation as
\[ 3(\tau_{BCD} + \tau_{BD}(ACDE) + \tau_{CD}(AB)) - (\tau_{AB}(ACDE) + \tau_{AC}(BDCE) + \tau_{AD}(BACE) + \tau_{BCD}(ACDE)) 
+ \tau_{BD}(ACE) + \tau_{CE}(ABD) + \tau_{DE}(ABC) \geq 0. \]

For GHZ state, \( |\varphi\rangle_{ABCD} = \frac{1}{\sqrt{2}} (|00000\rangle + |1111\rangle) \), we have
\[ \tau_{AB}(ACDE) = \tau_{AC}(BDCE) = \tau_{AD}(BACE) = \tau_{BCD}(ACDE) = \tau_D(ABCD) = 1, \]
\[ \tau_{BCD}(ACDE) = \tau_{B(ACDE)} = \tau_{D(ABCD)} = 1. \]
\[ \tau_{AB(CDE)} = \tau_{AC(BDE)} = \tau_{AD(BCD)} = \tau_{AE(BCD)} = \tau_{BC(ABD)} = \tau_{BD(ACE)} = \tau_{CD(ABE)} = \tau_{CE(ABD)} = \tau_{DE(ABC)} = 1. \]

Then, we have
\[
3(\tau_{A(BCDE)} + \tau_{B(ACDE)} + \tau_{C(ABDE)} + \tau_{D(ABCE)} + \tau_{E(ABCD)})
\]
\[
> \left( \tau_{AB(CDE)} + \tau_{AC(BDE)} + \tau_{AD(BCD)} + \tau_{AE(BCD)} + \tau_{BC(ABD)} + \tau_{BD(ACE)} + \tau_{BE(ACD)} + \tau_{CD(ABE)} + \tau_{CE(ABD)} + \tau_{DE(ABC)} \right).
\]

For W state, \(|\varphi\rangle_{ABCDE} = \frac{1}{\sqrt{5}} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle).\]
\[
\tau_{A(BCDE)} = \tau_{B(ACDE)} = \tau_{C(ABDE)} = \tau_{D(ABCE)} = \tau_{E(ABCD)} = \frac{16}{25}.
\]

Then, we have
\[
3(\tau_{A(BCDE)} + \tau_{B(ACDE)} + \tau_{C(ABDE)} + \tau_{D(ABCE)} + \tau_{E(ABCD)})
\]
\[
= \left( \tau_{AB(CDE)} + \tau_{AC(BDE)} + \tau_{AD(BCD)} + \tau_{AE(BCD)} + \tau_{BC(ABD)} + \tau_{BD(ACE)} + \tau_{BE(ACD)} + \tau_{CD(ABE)} + \tau_{CE(ABD)} + \tau_{DE(ABC)} \right).
\]

For maximally five-qubit entangled states [14]
\[
\tau_{A(BCDE)} = \tau_{B(ACDE)} = \tau_{C(ABDE)} = \tau_{D(ABCE)} = \tau_{E(ABCD)} = 1
\]
\[
\tau_{AB(CDE)} = \tau_{AC(BDE)} = \tau_{AD(BCD)} = \tau_{AE(BCD)} = \tau_{BC(ABD)} = \tau_{BD(ACE)} = \tau_{BE(ACD)} = \tau_{CD(ABE)} = \tau_{CE(ABD)} = \tau_{DE(ABC)} = \frac{3}{2}.
\]

Then, we have
\[
3(\tau_{A(BCDE)} + \tau_{B(ACDE)} + \tau_{C(ABDE)} + \tau_{D(ABCE)} + \tau_{E(ABCD)})
\]
\[
= \left( \tau_{AB(CDE)} + \tau_{AC(BDE)} + \tau_{AD(BCD)} + \tau_{AE(BCD)} + \tau_{BC(ABD)} + \tau_{BD(ACE)} + \tau_{BE(ACD)} + \tau_{CD(ABE)} + \tau_{CE(ABD)} + \tau_{DE(ABC)} \right).
\]

For six -qubit systems

For six-qubit state \(|\varphi\rangle_{ABCDEF}, using equation (13), we obtain relation
\[
10 \left( \tau_{AB(CDEF)} + \tau_{B(ACDEF)} + \tau_{C(ABDEF)} + \tau_{D(ABCFE)} + \tau_{E(ABDFC)} + \tau_{F(ABCD)} \right)
\]
\[
+ 6 \left( \tau_{AB(DEF)} + \tau_{ABD(CEF)} + \tau_{ABE(CDF)} + \tau_{ABC(CDE)} + \tau_{ACD(BDE)} + \tau_{ACE(BDF)} + \tau_{ACF(BDE)} + \tau_{ADF(BCF)} + \tau_{AEF(BCD)} + \tau_{BEF(ACD)} + \tau_{BDF(ACE)} \right)
\]
\[
- 7 \left( \tau_{AB(DEF)} + \tau_{AC(BDE)} + \tau_{AD(BCF)} + \tau_{AE(BCD)} + \tau_{AF(BCDE)} + \tau_{BDF(ACE)} + \tau_{BE(ACD)} + \tau_{BDF(ACE)} + \tau_{BEF(ACD)} + \tau_{BDF(ACE)} + \tau_{BDF(ACE)} \right) \geq 0.
\]

(A.2)

For GHZ state, \(|\varphi\rangle_{ABCDEF} = \frac{1}{\sqrt{2}} (|000000\rangle + |111111\rangle), we have
\[
\tau_{A(BCDEF)} = \tau_{B(ACDEF)} = \tau_{C(ABDEF)} = \tau_{D(ABCEF)} = \tau_{E(ABCF)} = \tau_{F(ABCD)} = 1,
\]
\[
\tau_{ABC(DEF)} = \tau_{ABD(CEF)} = \tau_{ABE(CDF)} = \tau_{ABF(CDE)} = \tau_{ACD(BEF)} = \tau_{ACE(BDF)} = \tau_{ACF(BDE)}
\]
\[
= \tau_{ADE(BCF)} = \tau_{ADF(BCE)} = \tau_{AEF(BCD)} = 1,
\]
\[
\tau_{ABC(DEF)} = \tau_{AC(BDE)} = \tau_{AD(BCF)} = \tau_{AE(BCD)} = \tau_{BD(ACE)} = \tau_{BE(ACD)} = \tau_{BEF(ACD)}
\]
\[
= \tau_{BC(DEF)} = \tau_{CD(ABE)} = \tau_{CE(ABF)} = \tau_{CF(ABD)} = \tau_{DE(ABC)} = \tau_{DF(ABCE)} = \tau_{EF(ABCD)} = 1.
\]
We have
\[ 10 \left( \tau_{ABCDEF} + \tau_{B(ACDEF)} + \tau_{C(ABDEF)} + \tau_{D(ABCEF)} + \tau_{E(ABCFD)} + \tau_{F(ABCDF)} \right) 
+ 6 \left( \tau_{ABC(DEF)} + \tau_{AB(DCEF)} + \tau_{A(BCDF)} + \tau_{AC(BDEF)} + \tau_{AD(BCFE)} + \tau_{AE(BCFD)} + \tau_{AF(BCDE)} + \tau_{ACF(BDE)} + \tau_{ACF(BCE)} + \tau_{ACF(BCD)} \right) 
- 7 \left( \tau_{AB(CDEF)} + \tau_{AC(BDEF)} + \tau_{AD(BCEF)} + \tau_{AE(BCDF)} + \tau_{AF(BCDE)} + \tau_{AEF(BCD)} + \tau_{AFC(BDE)} + \tau_{AFC(BCE)} + \tau_{AFC(BCD)} \right) 
+ \tau_{BF(ACDE)} + \tau_{CDF(ABEF)} + \tau_{C(ABEDEF)} + \tau_{D(ABCEFD)} + \tau_{E(ABCFDE)} \right) \geq 0. \]

For W state, \( |\psi\rangle_{ABCD} = \frac{1}{\sqrt{5}} (|000000\rangle + |000101\rangle + |001000\rangle + |010000\rangle + |100000\rangle), \)

\[ \tau_{ABCDEF} = \tau_{B(ACDEF)} = \tau_{C(ABDEF)} = \tau_{D(ABCEF)} = \tau_{E(ABCFD)} = \tau_{F(ABCDF)} = \frac{5}{9} \]

\[ \tau_{ABC(DEF)} = \tau_{ABD(CF)} = \tau_{ABE(CDF)} = \tau_{ACD(BE)} = \tau_{ACF(BDE)} = \tau_{AD(BCFE)} = \tau_{AD(BCFD)} = \tau_{AD(BCDE)} = \tau_{A(BCDEF)} = \frac{1}{9} \]

\[ \tau_{AB(CDEF)} = \tau_{AC(BDEF)} = \tau_{AD(BCEF)} = \tau_{AE(BCDF)} = \tau_{AF(BCDE)} = \tau_{BC(ADEF)} = \tau_{BD(ACDE)} = \tau_{BE(ACDF)} = \tau_{CD(ABFE)} = \frac{3}{5} \]

\[ 10 \left( \tau_{A(BCDEF)} + \tau_{B(ACDEF)} + \tau_{C(ABDEF)} + \tau_{D(ABCEF)} + \tau_{E(ABCFD)} + \tau_{F(ABCDF)} \right) + 
6 \left( \tau_{ABC(DEF)} + \tau_{ABD(CF)} + \tau_{ABE(CDF)} + \tau_{ACD(BE)} + \tau_{ACF(BDE)} + \tau_{AD(BCFE)} + \tau_{AD(BCFD)} + \tau_{AD(BCDE)} + \tau_{A(BCDEF)} \right) 
- 7 \left( \tau_{AB(CDEF)} + \tau_{AC(BDEF)} + \tau_{AD(BCEF)} + \tau_{AE(BCDF)} + \tau_{AF(BCDE)} + \tau_{AEF(BCD)} + \tau_{AFC(BDE)} + \tau_{AFC(BCE)} + \tau_{AFC(BCD)} \right) 
+ \tau_{BF(ACDE)} + \tau_{CDF(ABEF)} + \tau_{C(ABEDEF)} + \tau_{D(ABCEFD)} + \tau_{E(ABCFDE)} \right) \geq 0. \]

For maximally six-qubit entangled states [14, 16]

\[ \tau_{A(BCDEF)} = \tau_{B(ACDEF)} = \tau_{C(ABDEF)} = \tau_{D(ABCEF)} = \tau_{E(ABCFD)} = \tau_{F(ABCDF)} = 1, \]

\[ \tau_{ABC(DEF)} = \tau_{ABD(CF)} = \tau_{ABE(CDF)} = \tau_{ACD(BE)} = \tau_{ACF(BDE)} = \tau_{AD(BCFE)} = \tau_{AD(BCFD)} = \tau_{AD(BCDE)} = \tau_{A(BCDEF)} = \frac{14}{8} \]

\[ \tau_{AB(CDEF)} = \tau_{AC(BDEF)} = \tau_{AD(BCEF)} = \tau_{AE(BCDF)} = \tau_{AF(BCDE)} = \tau_{BC(ADEF)} = \tau_{BD(ACDE)} = \tau_{BE(ACDF)} = \tau_{CD(ABFE)} = \frac{3}{2} \]

\[ 10 \left( \tau_{A(BCDEF)} + \tau_{B(ACDEF)} + \tau_{C(ABDEF)} + \tau_{D(ABCEF)} + \tau_{E(ABCFD)} + \tau_{F(ABCDF)} \right) + 
6 \left( \tau_{ABC(DEF)} + \tau_{ABD(CF)} + \tau_{ABE(CDF)} + \tau_{ACD(BE)} + \tau_{ACF(BDE)} + \tau_{AD(BCFE)} + \tau_{AD(BCFD)} + \tau_{A(BCDEF)} \right) 
- 7 \left( \tau_{AB(CDEF)} + \tau_{AC(BDEF)} + \tau_{AD(BCEF)} + \tau_{AE(BCDF)} + \tau_{AF(BCDE)} + \tau_{BC(ADEF)} + \tau_{BD(ACDE)} + \tau_{BE(ACDF)} \right) 
+ \tau_{BF(ACDE)} + \tau_{CDF(ABEF)} + \tau_{C(ABEDEF)} + \tau_{D(ABCEFD)} + \tau_{E(ABCFDE)} \right) \geq 0. \]
Appendix B

Table B1. Summary of the triangular relationship for the multi-qubit states.

<table>
<thead>
<tr>
<th>N</th>
<th>State</th>
<th>Triangle relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Pure</td>
<td>$\tau_{A(BC)} + \tau_{B(AC)} - \tau_{C(AB)} \geq 0$</td>
</tr>
<tr>
<td></td>
<td>GHZ</td>
<td>$\tau_{A(BC)} + \tau_{B(AC)} - \tau_{C(AB)} &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>$\tau_{A(BC)} + \tau_{B(AC)} - \tau_{C(AB)} &gt; 0$</td>
</tr>
<tr>
<td>4</td>
<td>Pure</td>
<td>$\tau_{A(BCD)} + \tau_{B(ACD)} + \tau_{C(ABD)} \geq \tau_{A(BCD)} + \tau_{AC(BD)} + \tau_{BC(AD)}$</td>
</tr>
<tr>
<td></td>
<td>GHZ</td>
<td>$\tau_{A(BCD)} + \tau_{B(ACD)} + \tau_{C(ABD)} &gt; \tau_{A(BCD)} + \tau_{AC(BD)} + \tau_{BC(AD)}$</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>$\tau_{A(BCD)} + \tau_{B(ACD)} + \tau_{C(ABD)} &gt; \tau_{A(BCD)} + \tau_{AC(BD)} + \tau_{BC(AD)}$</td>
</tr>
<tr>
<td>5</td>
<td>Pure</td>
<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD})$</td>
</tr>
<tr>
<td></td>
<td>GHZ</td>
<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD}) &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>W</td>
<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD}) &gt; 0$</td>
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<tr>
<td>Maximal</td>
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<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD}) &gt; 0$</td>
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<td>Pure</td>
<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD})$</td>
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<td>$\tau \sum (\tau_{ABCDEF} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD} + \tau_{ACDE} + \tau_{ABCD}) &gt; 0$</td>
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</table>

References

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