Transient energy density distribution of a rod under high-frequency excitation

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Abstract

Energy finite element analysis (EFEA) is an efficient way to solve high-frequency structural dynamics response problems. Up to now, EFEA has been used to deal with time-independent vibration problems. However, it is still necessary to understand the time dependent details of energy density distribution of a structure. To study the transient response of a rod under high-frequency sinusoidal excitation, the transient energy density governing equation for a rod is presented. The governing equation is solved, and the solution is verified using an analytical method. Example application to a rod is presented to illustrate the feasibility.

1. Introduction

Structural dynamics problems have been of growing interest to scientists and engineers for a long time. A great number of numerical studies about structural dynamics responses, both at steady state and at transient state, have been performed [1,2] using finite element methods (FEM). However, FEM is not necessarily highly accurate at higher frequencies, at least not without huge computing resources, an energy-based approach could be useful in these areas.

Statistical energy analysis (SEA) and energy finite element analysis are two energy-based approaches used to solve high frequency dynamics problems. SEA has been used for the analysis of steady state [3] and transient responses [4,5]. However, in SEA, the local modeling details designers are usually concerned with are ignored because SEA is based on the division of sub-structures. In comparison, the energy finite element method was developed by Nefske and Sung [6], Wohlever and Bernhard [7], Bouthier and Bernhard [8] to predict the high-frequency average response of built-up structural acoustic systems consisting of subsystems such as rods, beams, plates, and acoustical enclosures [9]. In this method, governing differential equations are derived in terms of energy density variables and a finite element approach is employed to solve them numerically. The ability to prescribe local model details is one of the main benefits of EFEA. Previous research concerning EFEA has paid more attention to vibrational responses at steady state; however, most mechanical faults in machinery reveal themselves through transient events in vibration signals. In order to expand the application of EFEA to vibrational responses at transient state, this disparity is considered.

Simple structural members, such as rods, are studied first when a new analytical method is developed. In this study, a transient energy density equation for longitudinal vibration of a rod with sinusoidal excitation is solved using the Fourier Transform method. The energy density distributions at various times and locations were obtained. This study aims to explore this method for solving high frequency transient vibrational responses for a rod, as it may lead to the development of an effective method for finding the high frequency transient response of complex structures.
2. The transient energy density equation of a rod

There are two steps in EFEA: deriving governing differential equations in terms of energy density variables and employing a finite element approach for solving those equations numerically. The energy density governing equation is derived from the motion equation using the principle of energy balance.

The equation of motion for a rod with sinusoidal excitation can be written as follows:

\[ E_s \frac{\partial^2 u(x,t)}{\partial x^2} - \rho \frac{\partial^2 u(x,t)}{\partial t^2} = F \delta(x-x_0) \exp(i\omega t) \]  

(1)

where \( u(x,t) \) is the longitudinal displacement, \( t \) is time, \( x \) is the cartesian coordinate, \( x_0 \) is the coordinate of the point force excited, \( F \) is the amplitude of excitation, \( S \) is the cross sectional area of the rod, \( \rho \) is the mass density, \( E_s \) is the complex elastic module, and \( \eta \) is the damping loss factor.

Substituting Eq. (2) into Eqs. (4) and (5), and because of Eq. (3), the time averaged power and the time averaged energy density are, respectively:

\[ q = -\frac{1}{2} E_s \left( \frac{\partial u(x,t)}{\partial x} \right)^2 \left( \frac{\partial u(x,t)}{\partial t} \right)^2 + \frac{1}{2} \rho S \left( \frac{\partial u}{\partial t} \right)^2 \]  

(4)

\[ e = \frac{1}{2} \rho u^2 + \frac{1}{2} E_s \left( \frac{\partial u}{\partial x} \right)^2 \]  

(5)

where the first term is the potential energy density and the second term is kinetic energy density.

For a large number of high-frequency vibration designs, a shorter wavelength brings the local maximum and minimum values very close to each other in space and can be estimated using their average. From the above relationship, we can determine that the average power \( q \) is proportional to a derivative of the average energy density \( \bar{e} \):

\[ \bar{q} = -\frac{C_i^2}{\eta \omega} \frac{d \bar{e}}{dx} \]  

(7)

The formula is the relationship of energy transfer in the rod and reflects its characteristics.

According to the principle of energy balance on a differential rod element, the time change rate of energy within the control volume must be equal to the net power entering the volume minus the power dissipated within the volume.

\[ \frac{d \bar{e}}{dt} = (\pi_{in} - \nabla \cdot \bar{T}) - \pi_{diss} \]  

(8)

where \( \bar{e} \) is the energy density, \( \pi_{in} \) is the power density imposed on the control volume in unit time, \( \pi_{diss} \) is the dissipated power density within the differential control volume, which is the dissipated energy in unit volume in unit time, \( \bar{T} \) is energy intensity and is defined by the energy passed in unit area in unit time, the positive direction is pointing to the exterior surface, and \( \nabla \cdot \bar{T} \) indicates a divergence operation. The formula of energy balance holds true for any elastic medium at not only steady state, but also transient state.

For one dimensional structural vibration, such as the longitudinal vibration of a rod, power is written as \( q = \bar{q} = \bar{T} \cdot \bar{n} = \mid \bar{T} \mid \), and \( \nabla \cdot \bar{T} = \bar{q}/\bar{r} \) where \( \cdot \) indicates dot product. According to Eq. (7), \( \nabla \cdot \bar{T} = (C_i^2/\eta \omega) (d^2 \bar{e}/dx^2) \). The energy consumption of the structure due to loss factors during a period is written \( \pi_{diss} = \eta \omega \bar{e} \). Eq. (8) is written as

\[ \frac{d \bar{e}}{dt} = (\frac{C_i^2}{\eta \omega}) \frac{d^2 \bar{e}}{dx^2} + \eta \omega \bar{e} = \pi_{in} \]  

(9)

Eq. (9) is the transient energy density governing equation for a rod.
3. The solution with an initial condition

At the initial stage, the response of a linear structure to loads that vary harmonically with time is both the superimposed transient response and steady-state response. Considering that the length of the rod is very long, the movement of the rod far from the border – after the initial disturbance for a relatively short time – will be studied. At this time, the boundary conditions have not affected the movement of the rod, so the boundary conditions can be ignored.

The governing equation and the initial conditions are:

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial t^2} - \frac{C_t^2}{\eta \omega} \frac{\partial \varphi}{\partial x} + \eta \omega \varphi &= \pi_n \\
\varphi|_{t=0} &= \phi(x)
\end{align*}
\]  

(10)

Regarding \( t \) as a parameter, the following homogeneous equation will be solved first:

\[
\begin{align*}
\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \frac{C_t^2}{\eta \omega} \frac{\partial \tilde{\varphi}}{\partial x} + \eta \omega \tilde{\varphi} &= 0 \\
\tilde{\varphi}|_{t=0} &= \tilde{\phi}(\lambda)
\end{align*}
\]  

(11)

The Fourier transform on \( x \) for both sides of Eq. (11) is performed, then:

\[
\begin{align*}
\hat{\varphi}(\lambda, t) &= \phi(\lambda) \exp \left( - \left( \eta \omega + \frac{C_t^2}{\eta \omega} \lambda^2 \right) t \right)
\end{align*}
\]  

(14)

The inverse Fourier Transform of function \( \exp(-\left(\frac{C_t^2}{\eta \omega}\lambda^2\right)t) \) is

\[
F^{-1} \left[ \exp \left( - \left( \frac{C_t^2}{\eta \omega} \lambda^2 \right) t \right) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( - \frac{C_t^2}{\eta \omega} \lambda^2 t \right) \exp(i\lambda x) d\lambda = \frac{\sqrt{\eta \omega}}{2C_t \sqrt{\pi t}} \exp \left( - \frac{\eta \omega x^2}{4C_t^2 t} \right)
\]  

(15)

Thus the solution for the homogeneous equation is

\[
\varphi(x, t) = \frac{\sqrt{\eta \omega}}{2C_t \sqrt{\pi t}} \exp(-\eta \omega t) \int_{-\infty}^{\infty} \phi(\xi) \exp \left( - \frac{\eta \omega (x-\xi)^2}{4C_t^2 t} \right) d\xi
\]  

(16)

for the following non-homogeneous equation:

\[
\begin{align*}
\frac{\partial^2 \varphi}{\partial t^2} - \frac{C_t^2}{\eta \omega} \frac{\partial \varphi}{\partial x} + \eta \omega \varphi &= \pi_n \\
\varphi|_{t=0} &= 0
\end{align*}
\]  

(17)

Its solution can be written as

\[
\varphi(x, t) = \int_0^t w(x, t; \tau) d\tau
\]  

(18)

where \( w(x, t; \tau) \) is the solution of the following questions:

\[
\begin{align*}
\frac{\partial w}{\partial t} - \frac{C_t^2}{\eta \omega} \frac{\partial w}{\partial x} + \eta \omega w &= 0 \\
\frac{\partial w}{\partial t}|_{t=\tau} &= \pi_n(x, \tau)
\end{align*}
\]  

(19)

Using Eq. (16), the solution of problem (17) is interpreted as

\[
\varphi(x, t) = \frac{\sqrt{\eta \omega}}{2C_t \sqrt{\pi t}} \int_0^t \int_{-\infty}^{\infty} \pi_n(x, \tau) \exp \left( - \frac{\eta \omega (x-\xi)^2}{4C_t^2 (t-\tau)} \right) \exp(-\eta \omega (t-\tau)) d\xi d\tau
\]  

(20)

By the superposition principle, the solution of the original problem is the superposition of Eqs. (16) and (20), which can be written as

\[
\varphi(x, t) = \frac{\sqrt{\eta \omega}}{2C_t \sqrt{\pi t}} \int_{-\infty}^{\infty} \pi_n(x, \tau) \exp \left( - \frac{\eta \omega (x-\xi)^2}{4C_t^2 \tau} \right) \exp(-\eta \omega (t-\tau)) d\xi d\tau
\]  

(21)

4. Verification of the solution

To prove that Eq. (21) is the solution of problem (10), the following validation is performed. First, we need to prove Eq. (16) is the solution of problem (11). Because \( \phi(t \to 0) \) is positive and limited, then \( |\phi(x)| \leq M \).

From Eq. (16), we know that

\[
|\Xi(x,t)| \leq M \frac{1}{\sqrt{\pi}} \exp(-\eta \omega t) \int_{-\infty}^{\infty} \frac{\sqrt{\eta \omega}}{2C_l \sqrt{t}} \exp\left(-\frac{\eta \omega(x-\xi)^2}{4C_l^2 t}\right) d\xi
\]

(22)

Let \( \zeta = \sqrt{\eta \omega}(\xi-x)/2C_l \sqrt{t} \), due to \( \int_{-\infty}^{\infty} \exp(-\zeta^2) d\zeta = \sqrt{\pi} \), then

\[
M \frac{1}{\sqrt{\pi}} \exp(-\eta \omega t) \int_{-\infty}^{\infty} \frac{\sqrt{\eta \omega}}{2C_l \sqrt{t}} \exp\left(-\frac{\eta \omega(x-\xi)^2}{4C_l^2 t}\right) d\xi = M \exp(-\eta \omega t)
\]

(23)

Because \( t > 0 \), \( M \exp(-\eta \omega t) \) is a constant value, which shows that Eq. (16) is convergent, and the resulting function \( \Xi(x,t) \) is a bounded function and less than the border of the initial value.

In this section we will demonstrate that the function expressed by (16) will satisfy (11)a, because \( t > 0 \). Regarding \( \zeta \) as a parameter, the function under the integral sign,

\[
\frac{\sqrt{\eta \omega}}{2C_l \sqrt{\pi}} \exp(-\eta \omega t) \exp\left(-\frac{\eta \omega(x-\xi)^2}{4C_l^2 t}\right)
\]

(24)

For \( x \) and \( t \), it satisfies (11)a. Then, we will prove that the derivative in Eq. (11) can be obtained by the derivative under the integral sign in (16). As the integral limit is infinite, in order to ensure the derivation under the integral sign is possible, the integral obtained by the derivative under the integral sign is consistent with convergence. Let us consider the partial derivative on \( x \), the integral that derivative on the function under the integral sign to \( x \) is expressed by

\[
\frac{\sqrt{\eta \omega}}{2C_l \sqrt{\pi}} \exp(-\eta \omega t) \int_{-\infty}^{\infty} \frac{-\eta \omega(x-\xi)}{2C_l^2 t} \phi(x) \exp\left(-\frac{\eta \omega(x-\xi)^2}{4C_l^2 t}\right) d\xi
\]

(25)

Eq. (25) is always uniform convergence within the framework of \( t \geq t_0 > 0 \) (where \( t_0 \) is any positive number). So the following formula is established when \( t > 0 \):

\[
\frac{\partial \Xi(x,t)}{\partial x} = \frac{\sqrt{\eta \omega}}{2C_l \sqrt{\pi}} \exp(-\eta \omega t) \int_{-\infty}^{\infty} \frac{-\eta \omega(x-\xi)}{2C_l^2 t} \phi(x) \exp\left(-\frac{\eta \omega(x-\xi)^2}{4C_l^2 t}\right) d\xi
\]

(26)

Therefore the derivation for \( x \) can be obtained by the derivative under the integral sign. Similarly, we can show that the other partial derivatives in Eq. (11)a also can be obtained by the derivative under the integral sign so that the function expressed by (16) satisfies (11)a.

The rest of the work is proving that the function identified by Eq. (16) satisfies the initial conditions in Eq. (11)b, i.e., to prove that for any \( x_0 \), \( e(x,t) \to \phi(x_0) \) when \( t \to 0 \) and \( x \to x_0 \). Therefore, we need to prove that \( \delta > 0 \) can be found for any value \( \varepsilon \) satisfied \( \varepsilon > 0 \). The following formula is established when \( |x-x_0| \leq \delta, \ 0 \leq t \leq \delta \)

\[
|e(x,t)-\phi(x_0)| \leq \varepsilon
\]

(27)

Let

\[
\zeta = \frac{\sqrt{\eta \omega}(x-x_0)/2C_l \sqrt{t}}{2C_l \sqrt{\pi} \exp(-\eta \omega t)}
\]

(28)

\[
\phi(x_0) \text{ can be expressed by}
\]

\[
\phi(x_0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \phi(x) \exp(-\xi^2) d\xi
\]

(29)

then

\[
\Xi(x,t)-\phi(x_0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[ \exp(-\eta \omega t) \phi(x) \right] \exp(-\xi^2) d\xi
\]

(30)

For the any value \( \varepsilon \) satisfied \( \varepsilon > 0 \), taking \( N > 0 \), and large enough, then:

\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\xi^2) d\xi \leq \frac{\varepsilon}{6M}, \ \frac{1}{\sqrt{\pi}} \int_{-N}^{-N} \exp(-\xi^2) d\xi \leq \frac{\varepsilon}{6M}
\]

(31)
Let $N$ be invariable. We can find $\delta > 0$ because of the continuity of $\phi(x)$. The following formula can be established if $|x-x_0| \leq \delta$ and $0 < t \leq \delta$,

$$\left| \exp(-\eta\omega t)\phi\left(x+\frac{2C_1\sqrt{\zeta}}{\sqrt{\eta\omega}}\right)-\phi(x_0) \right| \leq \frac{e}{3} \cdots (-N \leq \zeta \leq N) \quad (32)$$

Then

$$|e(x,t)-\phi(x_0)| = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\eta\omega t)\phi\left(x+\frac{2C_1\sqrt{\zeta}}{\sqrt{\eta\omega}}\right)-\phi(x_0) \exp(-\zeta^2) d\zeta \leq \frac{1}{\sqrt{\pi}} \int_{-N}^{N} \exp(-\eta\omega t)\phi\left(x+\frac{2C_1\sqrt{\zeta}}{\sqrt{\eta\omega}}\right)$$

$$-\phi(x_0)\exp(-\zeta^2) d\zeta + \frac{2M}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\zeta^2) d\zeta \leq \frac{e}{3} \frac{1}{\sqrt{\pi}} \int_{-N}^{N} \exp(-\zeta^2) d\zeta + \frac{4M}{6M} \leq \frac{e}{3} + \frac{2e}{3} = e \quad (33)$$

As a result, we have verified that the function expressed by integral (16) is indeed the bounded solution of Cauchy problem (11). Therefore, the function expressed by the integral (21) is indeed the bounded solution of Cauchy problem (10).

5. An example and discussions

Let us consider an example. A rod is made of aluminum with length $l=10$ m, Young’s modulus is $E=7.1\times10^9$ Pa, Poisson’s ratio is 0.30, mass density is 7800 kg m$^{-3}$, radius frequency $\omega=2\pi \times 2000$ rad s$^{-1}$, hysteresis damping factor $\eta=0.005$, and the input power is 200 W at one end of the rod.

Because the energy density everywhere on the rod is 0 at initial time, $\phi(\zeta)=0$.

From Eq. (21):

$$\tau(x,t) = \frac{\sqrt{\eta\omega}}{2C_1\sqrt{\pi}} \int_{0}^{t} \frac{200}{\sqrt{t-\tau}} \exp\left(-\frac{\eta\omega(x^2)}{4C_1^2(t-\tau)}\right) \exp(-\eta\omega(t-\tau)) d\tau \quad (34)$$

Let the sum of $N$ segment replace the integral in Eq. (34), then:

$$\tau(x,t) = \frac{\sqrt{\eta\omega}}{2C_1\sqrt{\pi}} \sum_{n=1}^{N} \frac{200}{\sqrt{t-(nt/N)}} \exp\left(-\frac{\eta\omega(x^2)}{4C_1^2(t-(nt/N))}\right) \exp\left(-\eta\omega\left(t-\frac{nt}{N}\right)\right) \frac{t}{N} \quad (35)$$

The rod is divided into 100 elements and 101 nodes. At time $t=0.2$, 0.4, 0.6, 0.8, and 1 s, the distribution of energy density on the rod is as shown in Fig. 1. Selecting six points: A, B, C, D, E, and F, uniformly located on the rod, point A is the excitation point. The energy density on a point selected against time is expressed in Fig. 2.

Fig. 1 shows that the energy density decreases with distance from the location of the load. These results demonstrate that vibration energy is transmitted from the load point to the other points. The results tend to agree with those found through steady-state analysis.

From Fig. 2, the energy density increases quickly from 0 to 0.06 s and decreases slowly from 0.06 s. Energy density is tending to a fixed value, which can be regarded as its steady response.

6. Conclusions

In this paper, we have solved and verified the transient energy density governing equation of a rod using an analytical method. The solution has been implemented into the high-frequency vibrational response of a rod. The transient energy density
density distribution of the rod was obtained. The result actually is reasonable and indicates that it is feasible to investigate the transient high-frequency vibrational response of a structure with EFEA. However, the work in this paper is still at its beginning. Future studies should explore a methodology to study complex structural vibration response analysis in the transient state, where the problem is comprised of both boundary and initial conditions in order to show that the capability of EFEA transient analysis in practice. When material and geometric nonlinearities are taken into account, the first term coefficient of Eq. (1) is no longer constant, and Eq. (1) becomes nonlinear. To solve the nonlinear differential equation, a possible method, it is necessary to suppose that the system is a weakly nonlinear system, and the nonlinear part can be separated from the linear part. After determining the linear solution, it must be substituted into the nonlinear equation to arrive at a reasonable correction term which is then used to obtain the approximate nonlinear dynamic solution. Thus the study in this paper could still be the basis for nonlinear system studies, but further many researches needs to be performed to explore the potential fully.

Appendix A. Supporting information

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.jsv.2011.02.028.

References