A discrete filled function algorithm for approximate global solutions of max-cut problems

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Abstract

A discrete filled function algorithm is proposed for approximate global solutions of max-cut problems. A new discrete filled function is defined for max-cut problems and the properties of the filled function are studied. Unlike general filled function methods, using the characteristic of max-cut problems, the parameters in proposed filled function need not be adjusted. This greatly increases the efficiency of the filled function method. By combining a procedure that randomly generates initial points for minimization of the filled function, the proposed algorithm can greatly reduce the calculation cost and be applied to large scale max-cut problems. Numerical results on different sizes and densities test problems indicate that the proposed algorithm is efficient and stable to get approximate global solutions of max-cut problems.

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1. Introduction

Filled function methods introduced by Ge in 1990 are a class of global optimization methods. Consider the following optimization problem:

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad x \in \Omega \subset \mathbb{R}^n,
\end{align*}
\]

where \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is a real valued continuous function. Let \( x_1^* \) be a local minimizer of problem (1.1), the basic idea of the filled function method is to construct an auxiliary function, called filled function, at point \( x_1^* \), that can be further minimized to get a point, say \( \bar{x} \), in a basin (see Definition 5 below) of \( f(x) \) lower than the basin containing \( x_1^* \) of \( f(x) \) when \( x_1^* \) is not a global minimizer. Then the minimization of \( f(x) \) is restarted from the point \( \bar{x} \). Repeat the process until a global minimizer of \( f(x) \) is obtained.

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The first filled function proposed in [13] has the form
\[ P(x; x^*, r, \rho) = \frac{1}{r + f(x)} \exp \left( -\frac{\|x - x^*\|_2^2}{\rho^2} \right), \] (1.2)
where \( x^* \) is a minimizer of the function \( f, r \) and \( \rho \) are two adjustable parameters. An unfavorable property of the filled function (1.2) is that the existence of term \( \exp(-\|x - x^*\|_2^2/\rho^2) \) makes changes in \( P(x; x^*; r, \rho) \) and \( \nabla P(x; x^*; r, \rho) \) indistinguishable when \( \|x - x^*\|_2^2 \) is large. Liu in [21] proposed the following filled function:
\[ L(x; x^*; a) = \frac{1}{\ln[1 + f(x) - f(x^*)]} - a\|x - x^*\|_2^2 \]
to improve the property of filled function (1.2), where \( a \) is a parameter. Xu et al. in [31] proposed a class of filled functions to overcome the disadvantage
\[ U(x; x^*; A, \gamma) = -\eta(f(x) - f(x^*)) - A\psi(\|x - x^*\|_2^2), \]
where the functions \( \eta(\cdot), \psi(\cdot) \) have some desired properties and \( A, \gamma \) are parameters. Recently, Zhang et al. in [32] proposed a new two-parameter filled function
\[ p(x; x^*; \rho, \mu) = \begin{cases} 
\|f(x) - f(x^*)\|_2^2 - \rho\|x - x^*\|_2^2 & \text{when } f(x) \geq f(x^*), \\
\|f(x^*) - f(x)\|_2^2 + \rho\|x - x^*\|_2^2 & \text{when } f(x) \leq f(x^*). 
\end{cases} \] (1.3)

Ge and Huang [14] employed the filled function method to solve nonlinear integer programming problems. Ng et al. [23] dealt with discrete optimization problems using the filled function (1.3). More results on filled function methods for discrete optimization problems can be found in Zhu [33], He et al. [18], Shang and Zhang [26] and Gu and Wu [17]. However, to the best of our knowledge, there is very few attempts for max-cut problems or other combinatorial optimization problems by using the filled function methods.

Max-cut problems are a kind of discrete optimization problems. Given a graph \( G(V; E) \), the problem is to find a partition of the node set \( S_1 \subset V \) and \( S_2 = V \setminus S_1 \), such that the sum of the weights on the edges connecting the two parts is maximized, where \( V \) and \( E \) are the sets of nodes and edges in the graph. Let \( W = (w_{ij})_{n \times n} \) be the symmetric weighted adjacency matrix of a given graph \( G(V, E) \), with \( w_{ij} \neq 0 \), for \( (i, j) \in E \) and \( w_{ij} = 0 \) for \( (i, j) \notin E \), the max-cut problem can be expressed as the following discrete quadratic optimization problem:
\[
(MC): \quad \min \quad f(x) = x^T W x \\
\text{s.t.} \quad x \in \mathbb{S},
\]
where \( \mathbb{S} = \{-1, 1\}^n \) and \( x = (x_1, \ldots, x_n)^T \). \( x \in \{-1, 1\}^n \) means that \( x_i \) takes either \(-1\) or \( 1 \) for all \( i = 1, 2, \ldots, n \).

The max-cut problem has long been known to be NP-complete [20], even for any un-weighted graphs [12], and has applications in circuit layout design and statistical physics [3]. Approximate algorithms, such as \( \rho \)-approximation algorithm [16], heuristic algorithms [9] and continuous algorithms [27,30] have been proposed to solve max-cut problems. Based on the following semi-definite programming (SDP) relaxation,
\[
(SDP): \quad \max \quad L \cdot X \\
\text{s.t.} \quad \text{diag}(X) = e, \\
X \succeq 0,
\]
Goemans and Williamson in [16] proposed a 0.878-approximation randomized algorithm for nonnegative weighted graphs, where \( e \in \mathbb{R}^n \) is the column vector with all ones and \( L = [\text{Diag}(We) - W]/4, \text{Diag}(W e) \) is the diagonal matrix with elements of the vector \( We \) being diagonal entries and \( \text{diag}(X) = (X_{11}, \ldots, X_{nn})^T, X_{ii}, i = 1, 2, \ldots, n \) are the diagonal entries of the matrix \( X \). Bertsimas-Ye in [6] and Zwick in [34] also proposed random algorithms for nonnegative weighted graphs with the same performance ratio.

Burer et al. in [9] proposed a rank-2 relaxation to max-cut problems and developed a continuous optimization heuristic to solve max-cut problems. Festa et al. in [11] designed, implemented and tested several pure and hybrid heuristics, such
as greedy randomized adaptive search procedure (GRASP), variable neighborhood search (VNS) and a path-relinking (PR) intensification heuristic for max-cut problems and obtained some satisfactory results on some well-known G-set graphs. Alperin and Nowak in [2] presented a smoothing heuristic for max-cut problems. The heuristic is based on a parametric optimization problem defined as a convex combination between the original problem and its Lagrangian relaxation. Starting from the Lagrangian relaxation, a path following method is proposed to obtain good solutions while gradually transforming the relaxed problem into the original problem formulated with an exact penalty function.

Xu et al. in [30,27] proposed continuous optimization methods to solve max-cut problems, in which max-cut problems were relaxed into a nonlinear continuous optimization problem with convex constraints:

$$\max \quad x^T L x$$

s.t. \( B(x_i, \sqrt{n}) \leq 0, \quad i = 1, \ldots, n, \quad \|x\|^2 \leq n, \)

where \( B(x_i, \sqrt{n}) \) is a \( B \)-function (see [30]). An obvious advantage of the continuous method is that it greatly reduces the CPU-time via without using linear search in each iteration. However, the solution obtained by the continuous methods cannot be guaranteed as a global solution, which motivates us to study global optimization method to deal with max-cut problems.

The main contribution of this paper is to propose a discrete filled function method to get approximate global solutions of max-cut problems. A discrete filled function is applied to solve max-cut problems for different size and dense graphs. Alperin and Nowak in [2] presented a smoothing heuristic for max-cut problems. The heuristic is based on a greedy randomized adaptive search procedure (GRASP), variable neighborhood search (VNS) and a path-relinking algorithm. By adding a subroutine for randomly generating initial points to minimizing the filled function, the proposed algorithm can greatly reduce the calculation cost, and can be applied to large scale max-cut problems.

Numerical experiments and comparisons on some randomly generated max-cut problems and on some max-cut problems.

The remainder of the paper is arranged as follows. In Section 2, some definitions and preliminaries are recalled. The new discrete filled function for problem (MC) is presented in Section 3, and the properties of the filled function are calculated in each iteration, and the parameters of the filled function need not to be adjusted in the process of implementing the algorithm. By adding a subroutine for randomly generating initial points to minimizing the filled function, the proposed algorithm can greatly reduce the calculation cost, and can be applied to large scale max-cut problems. Numerical experiments and comparisons on some randomly generated max-cut problems and on some well-known large scale max-cut problems are made to show the efficiency of the proposed algorithm.

The main contribution of this paper is to propose a discrete filled function method to get approximate global solutions of max-cut problems. A discrete filled function is applied to solve max-cut problems for different size and dense graphs concluding negative weight. Only at most \((n - K)(K \in \mathbb{Z}, 1 \leq K < n)\) inner products of two \(n\)-dimensional vectors are calculated in each iteration, and the parameters of the filled function need not to be adjusted in the process of implementing the algorithm. By adding a subroutine for randomly generating initial points to minimizing the filled function, the proposed algorithm can greatly reduce the calculation cost, and can be applied to large scale max-cut problems. Numerical experiments and comparisons on some randomly generated max-cut problems and on some well-known large scale max-cut problems are made to show the efficiency of the proposed algorithm.

For simplicity, we call a (strict) discrete path as a (strict) path. From the definition of the path and the property of a path-wise connected domain, it is clear that any two distinct points \(x, y \in S\) are (strict) path-wise connected, and hence \(S\) is a (strict) path-wise connected domain.

For convenience, in this section, we will give some definitions and lemmas without proofs.

**Definition 1.** A sequence \( \{x^{(i)}\}_{i=0}^u \) is called a discrete path in the set \( S \) between two points \( x^*, x^{**} \) in \( S \), where \( x^{(-1)} = x^*, x^{(u)} = x^{**}, x^{(i)} \in S \) for all \( i \), if \( x^{(i)} \neq x^{(j)} \), for all \( i \neq j \); and \( \|x^{(i)} - x^{(i-1)}\|_p = 2 \), for all \( i = 0, 1, 2, \ldots, u \). In addition, if \( \|x^{(i)} - x^*\|_p < \|x^{(i+1)} - x^*\|_p \), or \( \|x^{(i)} - x^*\|_p > \|x^{(i+1)} - x^*\|_p \) holds for all \( i \), then the sequence is called a strict discrete path in \( S \). If a (strict) discrete path exists for given points \( x^* \in S, x^{**} \in S \), then \( x^* \) and \( x^{**} \) are said to be (strict) path-wise connected in \( S \).

**Definition 2.** For any \( x \in S \), and any positive integer \( K, 1 \leq K \leq n \), the \( K \)-neighborhood of \( x \) under the \( L_p \)-norm is defined by

\[ N(x, K) = \{ y \in S : \|y - x\|_p \leq 2 \cdot K^{1/p} \}. \]
In particular, if \( K = 1 \), we write the 1-neighborhood of \( x \), \( N(x, 1) \), as \( N(x) \). The boundary of the \( K \)-neighborhood \( N(x, K) \) is defined by \( \partial N(x, K) = \{ y \in \mathbb{S} : ||y - x||_p = 2 \cdot K^{1/p} \} \). It is known that \( |\partial N(x, K)| = C_K^n \), and each point in \( \partial N(x, K) \) differs from point \( x \) in only \( K \) elements. The following lemma is obvious from Definition 2.

**Lemma 2.1.** For every \( K \in \mathbb{Z}, 1 \leq K \leq n \), and any \( x \in \mathbb{S} \), the number of points in \( N(x, K) \) is \( 1 + \sum_{i=1}^{K} C_{n}^i \). Especially, when \( K = 1 \), \( |N(x)| = n + 1 \).

**Definition 3.** A point \( x^* \in \mathbb{S} \) is called a local discrete minimizer of the function \( f \) over \( \mathbb{S} \), if \( f(x^*) \leq f(x) \), for all \( x \in N(x^*) \). Furthermore, if \( f(x^*) < f(x) \), for all \( x \in \mathbb{S} \), then \( x^* \) is called a global discrete minimizer of \( f \) over \( \mathbb{S} \).

Let \( e_i \in \mathbb{R}^n \) be the \( i \)th unit directional vector, and define a set of directions \( \mathcal{D} = \{ d : d = \pm 2 \cdot e_i, i = 1, \ldots, n \} \), then for any \( x \in \mathbb{S} \), there exists a direction \( d \in \mathcal{D} \), such that \( x + d \in N(x) \subset \mathbb{S} \).

**Definition 4.** For any \( x \in \mathbb{S} \), a direction \( d \in \mathcal{D} \) is called a descent direction of the function \( f \) at \( x \) over \( \mathbb{S} \), if \( x + d \in \mathbb{S} \) and \( f(x + d) < f(x) \). In addition, we call \( d^* \in \mathcal{D}_x \) the discrete steepest descent direction of the function \( f \) at \( x \) over \( \mathbb{S} \), if \( f(x + d^*) \leq f(x + d) \) for all \( d \in \mathcal{D}_x \), where \( \mathcal{D}_x = \{ d \in \mathcal{D} : x + d \in \mathbb{S}, f(x + d) < f(x) \} \).

**Definition 5.** Let \( x^* \) be a local minimizer of the function \( f \) over \( \mathbb{S} \), \( B^* \) is said to be a discrete basin of the function \( f \) at \( x^* \), if \( B^* \subset \mathbb{S} \) is a path-wise connected domain which contains \( x^* \) and in which the discrete steepest descent trajectory from any point in \( B^* \) converges to \( x^* \), but outside which the discrete steepest descent trajectory from any point in \( \mathbb{S}\setminus B^* \) does not converge to \( x^* \).

**Definition 6 (Xu et al. [31]).** Let \( x^* \) and \( x^{**} \) be two distinct local minimizers of the function \( f \) over \( \mathbb{S} \), the discrete basin \( B^{**} \) of \( f \) at \( x^{**} \) is said to be lower (or higher) than the discrete basin \( B^* \) of \( f \) at \( x^* \), if \( f(x^{**}) < f(x^*) \) (or \( f(x^{**}) > f(x^*) \)).

**Definition 7 (Ng et al. [23]).** Let \( x^* \) be a discrete minimizer of the function \( f \) over \( \mathbb{S} \), and \( B^* \) be the discrete basin of \( f \) at \( x^* \). A function \( H : \mathbb{S} \rightarrow \mathbb{R} \) is said to be a discrete filled function of \( f \) at \( x^* \), if it satisfies the following conditions:

1. \( x^* \) is a strict discrete local maximizer of \( H \) over \( \mathbb{S} \).
2. \( H \) has no discrete local minimizers in \( B^* \) and in any discrete basin of \( f \) higher than \( B^* \).
3. If \( f \) has a discrete basin, \( B^{**} \) say, lower than \( B^* \), then there is a point \( x^{**} \in B^{**} \) that minimizes \( H \) on the path \( \{x^*, \ldots, x^{**}, \ldots, x^*\} \) in \( \mathbb{S} \), where \( x^{**}, x^* \) are minimizers of \( f \) in \( B^{**} \) and \( B^* \), respectively.

3. A discrete filled function and its properties

Let \( x^* \) be a local minimizer of the function \( f \) in problem (MC); we modify the filled function (1.3) and define a function with simpler expression as

\[
H_p(x; x^*; z, \beta) = \begin{cases} \alpha[f(x) - f(x^*)] - \beta\|x - x^*\|_p, & f(x) > f(x^*), \\ [f(x^*) - f(x)] - \beta\|x - x^*\|_p, & f(x) \leq f(x^*), \end{cases} \tag{3.1} \]

where \( p(1 \leq p < \infty) \) is a constant, and \( z, \beta \) are two adjustable parameters. For simplicity, in the reminder of this paper, we only consider the case of \( p = 1 \), and denote \( H_1(x; x^*; z, \beta) \) as \( H(x; x^*; z, \beta) \), but it is not hard to extend all results of this paper to the case of \( 1 < p < \infty \). Since the specialities of the quadratic function \( f \) and the set \( \mathbb{S} \) in max-cut problems, we can simplify the proofs to show that \( H(x; x^*; z, \beta) \) is a filled function of \( f \) at \( x^* \), and assign a more efficient algorithm for global minimization of max-cut problems. The remainder of this section will be used to show that the function \( H(x; x^*; z, \beta) \) is a discrete filled function of the function \( f \) at point \( x^* \). The following lemma plays an important role in the analysis of this section.

**Lemma 3.1.** For any given \( x \in \mathbb{S} \) and for any \( y \in N(x) \), we have

\[
|f(y) - f(x)| \leq 4qM, \tag{3.2} \]
where \( q = \max\{q_i : i = 1, \ldots, n\} \leq n \) and \( M = \max\{\|w_i\|_1 : i = 1, \ldots, n\} \) are the \( i \)th column of the weighted adjacency matrix \( W \) of a given graph, and \( q_i (i = 1, \ldots, n) \) are the number of nonzero elements in vector \( w_i \) \((i = 1, \ldots, n)\).

**Proof.** Let \( x^l \in N(x) \) \((l = 1, 2, \ldots, n)\), where index \( l \) expresses that only the \( l \)th element of \( x^l \) differs from the \( l \)th element of \( x \), i.e.,
\[
x_i^l = \begin{cases} x_i, & i = 1, 2, \ldots, n, \; i \neq l, \\ -x_l, & i = l.
\end{cases}
\]

Since \( w_{lj} = 0 \), for \( i = 1, \ldots, n \), we have
\[
|f(x^l) - f(x)| = |(x^l)^T W x^l - x^T W x| = 4\sum_{i=1}^{n} x_i w_{lij} x_l - 2 \sum_{i=1}^{n} x_i w_{li} x_l - \sum_{i,j} x_i w_{ij} x_j \leq 4q M, \quad l = 1, 2, \ldots, n.
\]
(3.3)

This completes the proof. \[\square\]

**Theorem 3.2.** Let \( x^* \) be a local minimizer of the function \( f \) on \( \mathbb{S} \). If \( \beta > 0 \), and \( 0 \leq \alpha < \beta/(2q M) \), then the function \( H(x; x^*; \alpha, \beta) \) satisfies the first condition of Definition 7, i.e., \( x^* \) is a strict local maximizer of \( H(x; x^*; \alpha, \beta) \) on \( \mathbb{S} \). Furthermore, if \( x^* \) is a global minimizer of \( f \) on \( \mathbb{S} \), then \( H(x; x^*; \alpha, \beta) < 0 \) for all \( x \in \mathbb{S} \setminus \{x^*\} \).

**Proof.** Assume that \( x^* \) is a local minimizer of \( f \) on \( \mathbb{S} \). Since \( f(x) \geq f(x^*) \) and \( \|x - x^*\|_1 = 2 \) for any \( x \in N(x^*) \setminus \{x^*\} \).

It follows from Lemma 3.1 that when \( 0 \leq \alpha < \beta/(2q M) \), we have
\[
H(x; x^*; \alpha, \beta) = \alpha[f(x) - f(x^*)] - \beta\|x - x^*\|_1 \leq 4q M - 2\beta < 0 = H(x^*; x^*; \alpha, \beta).
\]

Hence, \( x^* \) is a strict local maximizer of \( H(x; x^*; \alpha, \beta) \) on \( \mathbb{S} \).

If \( x^* \) is a global minimizer of \( f \) on \( \mathbb{S} \), then for any \( x \in \mathbb{S} \setminus \{x^*\} \), there exists a integer \( K \) \((1 \leq K \leq n)\), such that \( x \in \partial N(x^*; K) \) that implies \( \|x - x^*\|_1 = 2 K \). Hence, there must exist at least a path \( \{x_j\}_{j=0}^{K} \) from \( x^* \) to \( x \) satisfying \( x^* = x_0, x = x_K \). Since
\[
|f(x) - f(x^*)| = |f(x_K) - f(x_{K-1}) + f(x_{K-1}) - f(x_{K-2}) + \cdots + f(x_1) - f(x_0)| \leq \sum_{i=1}^{K} |f(x_i) - f(x_{i-1})|,
\]
and \( x_i \in N(x_{i-1}) \). It follows from Lemma 4.1 that
\[
|f(x) - f(x^*)| \leq \sum_{i=1}^{K} 4q M = 4K q M.
\]

Thus,
\[
H(x; x^*; \alpha, \beta) = \alpha[f(x) - f(x^*)] - \beta\|x - x^*\|_1 \leq 4K q M - 2K \beta < 0 = H(x^*; x^*; \alpha, \beta),
\]
when \( \beta > 0 \), and \( 0 \leq \alpha < \beta/(2q M) \). \[\square\]
Lemma 3.3. Let \( x^* \) be a local minimizer of the function \( f \) on \( S, x \in S \) and \( f(\overline{x}) \geq f(x^*) \). If \( \beta > 0 \) and \( 0 \leq \alpha \leq \beta/(2qM) \), and if there exists direction \( \overline{d} \in \mathcal{D} \), such that \( \overline{x} + \overline{d} \in S, f(\overline{x} + \overline{d}) \geq f(x^*) \) and \( \|\overline{x} + \overline{d} - x^*\|_1 > \|\overline{x} - x^*\|_1 \), then we have
\[
H(\overline{x} + \overline{d}; x^*; \alpha, \beta) < f(\overline{x}) - f(x^*) < H(x^*; x^*; \alpha, \beta) = 0.
\] (3.4)

Proof. Denote \( \mathcal{D}_0 = \{\overline{d} \in \mathcal{D} : \overline{x} + \overline{d} \in S, f(\overline{x} + \overline{d}) \geq f(x^*), \|\overline{x} + \overline{d} - x^*\|_1 > \|\overline{x} - x^*\|_1 \} \). If \( \mathcal{D}_0 \) is nonempty, then \( f(\overline{x} + \overline{d}) \leq f(\overline{x}) \), then \( H(\overline{x} + \overline{d}; x^*; \alpha, \beta) < H(\overline{x}; x^*; \alpha, \beta) \) when \( \beta > 0 \). If \( f(\overline{x} + \overline{d}) \geq f(\overline{x}) \), it follows from Lemma 3.1 that the right-hand side of (3.5) is not greater than \( 4\alpha qM - 2\beta \), i.e.,
\[
H(\overline{x} + \overline{d}; x^*; \alpha, \beta) \leq 4\alpha qM - 2\beta.
\]
Hence, when \( 0 \leq \alpha \leq \beta/(2qM) \),
\[
H(\overline{x} + \overline{d}; x^*; \alpha, \beta) \leq H(\overline{x}; x^*; \alpha, \beta)
\]
holds. The second inequality in (3.4) directly follows from Theorem 3.2. \( \square \)

Lemma 3.3 indicates that for any point \( \overline{x} \in S \), especially, when \( \overline{x} \) lies either in the basin \( B^* \) or basin higher than \( B^* \), if \( \mathcal{D}_0 \) is nonempty, then \( \overline{x} \) is not a local minimizer of the function \( H \). That is, we have the following result.

Theorem 3.4. Let \( x^* \) be a local minimizer of the function \( f \) on \( S \), and \( B^* \) be the discrete basin of \( f \) at \( x^* \). If \( \beta > 0, 0 \leq \alpha \leq \beta/(2qM) \), then \( H(x; x^*; \alpha, \beta) \) has no any local minimizer in \( B^* \) and in any basin higher than \( B^* \).

Proof. Let \( U^* \) denote the union of all basins of \( f \) higher than \( B^* \), it is clear from Definition 6 that \( f(x) > f(x^*) \) holds for any point \( x \in U^* \). Note that \( -x^* \not\in U^* \) via \( f(-x^*) = f(x^*) \) and \( -x^* \not\in B^* \) via the fact that \( x^* \) oneself is a minimizer of \( f \). Hence for any \( \overline{x} \in B^* \) or \( \overline{x} \in U^* \), it follows that \( \overline{x} \neq -x^* \). Thus, there exists at least a direction \( \overline{d} \in \mathcal{D} \), such that \( \overline{x} + \overline{d} \in S \) and \( \|\overline{x} + \overline{d} - x^*\|_1 > \|\overline{x} - x^*\|_1 \). If \( f(\overline{x} + \overline{d}) < f(x^*) \), then we can obtain a point, say \( x^{**} \), satisfying \( f(x^{**}) < f(x^*) \) by minimizing the function \( f \) from \( \overline{x} \). It means \( \overline{x} \) is in a basin of \( f \) lower than the basin \( B^* \) that contradicts \( \overline{x} \in B^* \) or \( \overline{x} \in U^* \). Thus the set \( \mathcal{D}_0 \) is nonempty. From Lemma 3.3, \( \overline{x} \) is not a local minimizer of the function \( H(x; x^*; \alpha, \beta) \). \( \square \)

Theorem 3.4 indicates that if \( \beta > 0 \) and \( 0 \leq \alpha \leq \beta/(2qM) \), then \( H(x; x^*; \alpha, \beta) \) satisfies the second condition of Definition 7. From Lemma 3.3 and Theorem 3.4, we can obtain the following conclusions. When \( \beta > 0 \) and \( 0 \leq \alpha \leq \beta/(2qM) \),

(i) for any \( x \in B^*, x \neq x^* \), if \( d \) satisfies \( \|x + d - x^*\|_1 > \|x - x^*\|_1 \) and \( f(x + d) \geq f(x^*) \), then \( d \) is a descent direction of \( H \) at \( x \),

(ii) for any \( x \in U^* \), if \( d \) satisfies \( \|x + d - x^*\|_1 > \|x - x^*\|_1 \) and \( f(x + d) \geq f(x^*) \), then \( d \) is a descent direction of \( H \) at \( x \).

Theorem 3.5. Let \( f(x^*) > f(x^{**}) \), and \( B^*, B^{**} \) be basins of the function \( f \) at \( x^* \), \( x^{**} \) in \( S \), respectively. Assume that \( \overline{x} \in B^{**} \), and \( \overline{x} + \overline{d} \in B^{**} \), where \( \overline{d} \) is a descent direction of \( f \) at \( \overline{x} \) in \( S \) and satisfies
\[
f(\overline{x} + \overline{d}) < f(x^*) \leq f(\overline{x}).
\] (3.6)

Suppose that there is a strict path \( \{x^{(i)}\}_{i=-1}^u \) in \( S \) between \( x^* (=x^{(-1)}) \) and \( \overline{x}(=x^{(u)}) \), such that
\[
f(x^{(i)}) \geq f(x^*),
\] (3.7)
for all $i = 0, 1, \ldots, u$. If $0 \leq \alpha < \beta/(2qM)$, and $0 < \beta < \beta_0$, then $\overline{x}$ is a minimizer of the function $H(x; x^*; \alpha, \beta)$ on the path $\{x^{(i)}\}_{i=-1}^{u+1}$ in $\Sigma$, where $x^* = (x^{(-1)}), \overline{x} = x^{(u+1)}, x^{**} = x^{(u+v)}$ and

$$\beta_0 = \frac{2qM[f(x^*) - f(\overline{x} + \overline{d})]}{f(x^*) - f(x^*) + 4qM} > 0. \quad (3.8)$$

**Proof.** It is sufficient to show $H(x; x^*; \alpha, \beta) < H(x^{(u+1)}; x^*; \alpha, \beta) = H(\overline{x} + \overline{d}; x^*; \alpha, \beta)$, and $H(x; x^*; \alpha, \beta) < H(x^{(u+1)}; x^*; \alpha, \beta)$ hold on the path. Since $x^{(0)}$ and $x^*$ are adjacent points in the (strict) path $\{x^{(i)}\}_{i=-1}^{u+1}, x^{(0)} \in B^*$ and $H(x^{(0)}; x^*; \alpha, \beta) = 0$ holds from Theorem 3.2 when $\beta > 0, 0 \leq \alpha < \beta/(2qM)$.

Since $\{x^{(i)}\}_{i=-1}^{u+1}$ is a strict path starting from $x^*$, assume that $\|x^{(i-1)} - x^*\|_1 < \|x^{(i)} - x^*\|_1$ holds for all $i = 0, 1, \ldots, u$. In view of (3.7) and Lemma 3.3, for all the points $\{x^{(i)}\}, i = -1, 0, \ldots, u$ on the strict path, we have

$$H(x^{(i)}; x^*; \alpha, \beta) < H(x^{(i-1)}; x^*; \alpha, \beta), \quad i = 1, \ldots, u.$$  

Especially,

$$H(\overline{x}; x^*; \alpha, \beta) < H(x^{(u+1)}; x^*; \alpha, \beta) < H(x^{(u+1)}; x^*; \alpha, \beta), \quad j = 2, \ldots, u + 1. \quad (3.9)$$

On the other hand, since $\overline{d}$ satisfies (3.6),

$$H(\overline{x} + \overline{d}; x^*; \alpha, \beta) - H(\overline{x}; x^*; \alpha, \beta) = f(x^*) - f(\overline{x} + \overline{d}) - \alpha[f(x) - f(x^*)] - \beta(\|\overline{x} + \overline{d} - x^*\|_1 - \|x - x^*\|_1)$$

$$> f(x^*) - f(\overline{x} + \overline{d}) - \frac{\beta}{2qM}[f(\overline{x}) - f(x^*)] - 2\beta \quad (3.10)$$

holds when $0 \leq \alpha < \beta/(2qM)$.

It can be verified that the right-hand side of (3.10) is nonnegative when

$$\beta \leq \frac{2qM[f(x^*) - f(\overline{x} + \overline{d})]}{f(x^*) - f(x^*) + 4qM} = \beta_0,$$

that is, when $0 < \beta < \beta_0$, we have

$$H(\overline{x}; x^*; \alpha, \beta) < H(\overline{x} + \overline{d}; x^*; \alpha, \beta). \quad (3.11)$$

Therefore, $\overline{x}$ minimizes $H(x; x^*; \alpha, \beta)$ on the path $\{x^{(i)}\}_{i=-1}^{u+1}$ in $\Sigma$ between $x^*$ and $x^{(u+1)} = \overline{x} + \overline{d}$. If $\overline{x} + \overline{d} = x^{**}$, then the conclusion of the theorem holds. If $\overline{x} + \overline{d} \neq x^{**}$, since $\overline{x} + \overline{d} \in B^*$, then there exists a steepest descent trajectory $\{x^{(i)}\}_{i=u+1}^{u+v}$ of $f$ in $\Sigma$ from $\overline{x} + \overline{d}$ to $x^{**} = x^{(u+v)}$. Hence $\overline{x}$ is a minimizer of $H(x; x^*; \alpha, \beta)$ on the path $\{x^{(i)}\}_{i=-1}^{u+v}$. \(\square\)

In view of Theorem 3.5, when $\overline{x} + \overline{d} \neq x^{**}$, the path $\{x^{(a+j)}\}_{j=1}^{v}$ is a steepest descent trajectory of the function $f$ from $x^{(u+1)}$, and hence

$$f(x^{(u+1)}) > f(x^{(u+2)}) > \ldots > f(x^{(a+j)}) = f(x^{**}), \quad (3.12)$$

$$x^{(a+j+1)} \in N(x^{(a+j)}), \quad j = 1, 2, \ldots, v - 1. \quad (3.13)$$

Let

$$\beta(j) = f(x^{(a+j)}) - f(x^{(a+j+1)})$$

$$= 4 \sum_{i=1}^{n} x^{(a+j)}_{i} w_{i} x^{(a+j)}_{i} > 0, \quad j = 1, \ldots, v - 1,$$

where $x^{(a+j)}_{i}$ is the $i$th element of the vector $x^{(a+j)}$, and $l_j$ is defined by

$$l_j = \arg \min \left\{ -4 \sum_{i=1}^{n} x^{(a+j)}_{i} w_{i} x^{(a+j)}_{i} : l = 1, \ldots, n \right\}.$$
Theorem 3.6. Let the conditions of Theorem 3.5 be satisfied and \( \bar{x} + \bar{a} \neq x^* \). Assume that the path \( \{x^{(a+j)}\}_{j=1}^v \) satisfies \( \|x^{(a+j+1)} - x^*\|_1 > \|x^{(a+j)} - x^*\|_1 \) for all \( j \in \{1, \ldots, v-1\} \). If

\[
0 < \beta < \min \left\{ \frac{\beta(j)}{2} : j = 1, \ldots, v-1 \right\}
\]

(3.14)

then \( \bar{x} \) is the global minimizer of the function \( H \) on the path \( \{x^{(i)}\}_{i=1}^v \).

Proof. Combining inequality (3.9), it is sufficient to show

\[
H(\bar{x}; x^*; \alpha, \beta) < H(x^{(u+j)}; x^*; \alpha, \beta), \quad j = 1, \ldots, v.
\]

Since \( \|x^{(a+j+1)} - x^*\|_1 > \|x^{(a+j)} - x^*\|_1, \|x^{(a+j+1)} - x^*\|_1 - \|x^{(a+j)} - x^*\|_1 = 2, j \in \{1, \ldots, v-1\} \) from (3.13). It follows from the definition of the function \( H \) and (3.12) that

\[
H(x^{(a+j)}; x^*; \alpha, \beta) - H(x^{(a+j+1)}; x^*; \alpha, \beta) = f(x^{(a+j+1)}) - f(x^{(a+j)}) + 2\beta
\]

(3.15)

holds for all \( j \in \{1, 2, \ldots, v-1\} \). When \( \beta \) is chosen to satisfy inequality (3.14), the value of the right-hand side of (3.15) is

\[
-\beta(j) + 2\beta < 0, \quad j = 1, \ldots, v-1.
\]

Thus

\[
H(x^{(u+j)}; x^*; \alpha, \beta) < H(x^{(u+j+1)}; x^*; \alpha, \beta), \quad j = 1, \ldots, v-1.
\]

Combining (3.11), it follows that

\[
H(\bar{x}; x^*; \alpha, \beta) < H(x^{(u+j)}; x^*; \alpha, \beta), \quad j = 1, \ldots, v.
\]

This completes the proof. \( \Box \)

Theorems 3.5 and 3.6 indicate that the function \( H(x; x^*; \alpha, \beta) \) satisfies the third condition of Definition 7. From the analysis above, we can conclude that when \( 0 < \beta < \beta_0 \) and \( 0 \leq \alpha < \beta/(2qM) \), \( H(x; x^*; \alpha, \beta) \) is a discrete filled function of the function \( f \) at point \( x^* \) on \( S \).

4. The algorithm

The parameters \( \alpha \) and \( \beta \) play important role in filled function algorithm for global optimization. A typical approach to select \( \alpha \), \( \beta \) in the literatures is to first give appropriate initial estimations of \( \alpha \), \( \beta \), and then adjust them step by step in the process of implementing an algorithm. The shortcoming of the approach is that the algorithm needs to restart after \( \alpha \), \( \beta \) are adjusted, which will increase lots of loads of computation. In the algorithm proposed in this paper, \( \alpha \), \( \beta \) can be given independent of variable \( x \in S \) and need not to be adjusted in the implementation of the algorithm.

4.1. The estimation of \( \alpha \), \( \beta \)

From Theorems 3.2 and 3.5, it is not difficult to find that when \( \alpha \), \( \beta \) satisfy

\[
0 < \beta < \beta_0 = \frac{2qM [f(x^*) - f(\bar{x} + \bar{a})]}{f(\bar{x}) - f(x^*) + 4qM},
\]

(4.1)

\[
0 \leq \alpha < \frac{\beta}{2qM},
\]

\( H(x; x^*; \alpha, \beta) \) is a desired filled function of \( f \) at \( x^* \) on \( S \), where \( x^*, \bar{x}, \) and \( \bar{x} + \bar{a} \) satisfy (3.6).
Since \( f(\bar{x}) - f(\bar{x} + \bar{d}) \leq 4qM \), we have
\[
0 < f(\bar{x}) - f(x^*) + 4qM \leq 8qM - \left[ f(x^*) - f(\bar{x} + \bar{d}) \right].
\]
It follows that
\[
\beta_0 = \frac{2qM[f(x^*) - f(\bar{x} + \bar{d})]}{f(\bar{x}) - f(x^*) + 4qM} \geq \frac{2qM[f(x^*) - f(\bar{x} + \bar{d})]}{8qM - \left[ f(x^*) - f(\bar{x} + \bar{d}) \right]}.
\]  (4.2)
For a given matrix \( W \), let \( \varepsilon \) denote the precision of calculation in implementing the algorithm, it is clear that \( 0 < \varepsilon \leq 1 \) and, from (3.6),
\[
f(x^*) - f(\bar{x} + \bar{d}) \geq \varepsilon > 0.
\]  (4.3)
Let
\[
g(t) = \frac{2qMt}{8qM - t}, \quad t \in [0, 4qM],
\]
then \( g(t) \) is a monotone increasing function with respect to \( t \) in the interval \([0, 4qM]\). Thus, from (4.2) and (4.3),
\[
\beta_0 \geq g(f(x^*) - f(\bar{x} + \bar{d})) \geq g(\varepsilon) = g(0) = 0.
\]
If the value of \( \beta \) is taken as
\[
\beta = \beta(W) = \frac{1}{2} g(\varepsilon) = \frac{qM \varepsilon}{8qM - \varepsilon},
\]  (4.4)
then \( 0 < \beta < \beta_0 \) holds for any \( \bar{x} \in \mathbb{S} \), where \( \beta(W) \) means that the value of \( \beta \) only depends on the given matrix \( W \) and is independent of variable \( \bar{x} \in \mathbb{S} \). After the value of \( \beta \) is chosen, any value of \( \varepsilon \) satisfying (4.1) can be selected, for instance,
\[
\varepsilon = \varepsilon(W) = \frac{\beta(W)}{4qM} = \frac{\varepsilon}{4(8qM - \varepsilon)}.
\]  (4.5)
It is clear that \( \varepsilon \) satisfies \( 0 \leq \varepsilon < \beta/(2qM) = \beta(W)/(2qM) \) for any \( \bar{x} \in \mathbb{S} \).

4.2. The statement of the algorithm

In this subsection, we will state the filled function algorithm and the detail of implementing the algorithm will be given in the following subsections. Once a local minimizer, \( x^*_k \) say, of the function \( f \) is obtained, initial points from which the minimization of the filled function \( H(x; x^*_k; \varepsilon(W), \beta(W)) \) is started are randomly generated in the neighborhood \( N(x^*_k) \).

Algorithm (DFFA): The Discrete Filled Function Algorithm

**Step 0.** (Initialization)

(1) Input the matrix \( W \), an initial point \( x_0 \), the number \( n_1 (\leq n) \) and set \( k := 0 \).

(2) Calculate \( \beta = \beta(W) \) and \( \varepsilon = \varepsilon(W) \).

**Step 1.** Minimize \( f \) from \( x_k \) using a local minimization method and obtain a local minimizer \( x^*_k \) of \( f \) on \( \mathbb{S} \), set \( I = 1, \bar{N} = \{x^*_k\} \).

**Step 2.** Randomly generate an initial point \( x^*_i \) in the set \( N(x^*_k) \backslash \bar{N} \).

**Step 3.** Minimize the filled function \( H(x; x^*_k; \varepsilon(W), \beta(W)) \) starting from the point \( x^*_i \). If a point \( \bar{x} \) is obtained, such that either \( x^*_i \) satisfies \( f(\bar{x}) < f(x^*_k) \) or \( \bar{x} \) lies in a basin of \( f \) lower than the current basin, then set \( x_{k+1} = \bar{x}, k = k + 1 \) and goto Step 1.

**Step 4.** If \( I < n_1 \), set \( \bar{N} = \bar{N} \cup \{x^*_k\}, I = I + 1 \), goto Step 2.

**Step 5.** (Termination) If \( I = n_1 \), then return \( x^*_k \) and stop.
Remark 1. The index $i$ of $x_i^*$ in step 2 means that only the $i$th element of $x_i^*$ differs from that of $x_k^*$. Also, the index $i$ is distinct with the counter $I$. In fact, by the discussion of the Section 4.3, the index $i$ is a function of the counter $I$.

Remark 2. The termination condition $I = n_1$ indicates that we do not get a better solution than the current $x_k^*$ after arbitrary $n_1$ points out of $n$ points in $N(x_k^*)$ have been used as initial points to minimize $H(x; x_k^*; z(W), b(W))$.

4.3. Minimizing $f$ using local search

Given a point $x = (x_1, \ldots, x_n)^T \in S$, 1-neighborhood search is used to find a point $x_i^* \in N(x)$ satisfying $f(x_i^*) = \min\{f(x) : x \in N(x)\}$. If $x_i^* \neq x$, then replacing $x$ with $x_i^*$ and repeating the process until a point $x_i^*$ satisfying $f(x_i^*) = \min\{f(x) : x \in N(x_i^*)\}$ is found, which indicates that $x_i^*$ is a local minimizer of $f$.

Let $x^p = (x_1^p, \ldots, x_p^p, \ldots, x_n^p)^T \in N(x)\setminus\{x\}$ with $x_p^p = x_i$, for all $i \neq p$ and $x_p^p = -x_p$, i.e., only the $p$th element of $x^p$ differs from the $p$th element of $x$. Let $\delta(p)$ denote the difference between $f(x^p)$ and $f(x)$,

$$
\delta(p) = f(x^p) - f(x) = (x^p)^T W x^p - (x)^T W x
$$

(4.6)

here we use the fact $w_{jj} = 0, i = 1, \ldots, n$. Calculate

$$
\delta(i_*) = \min\{\delta(p) : p = 1, 2, \ldots, n\}.
$$

(4.7)

If $\delta(i_*) \geq 0$, then $x$ is a local minimizer of $f$. Otherwise $x_i^*$ satisfies $f(x_i^*) < f(x^p)$, for all $p = 1, 2, \ldots, n$, where $x^p \in N(x)\setminus\{x\}, p = 1, 2, \ldots, n$.

If $\delta(i_*) < 0$ 1-neighborhood local search can be continued at the point $x_i^*$. Assume that the index $i_*$ satisfies (4.7), that is

$$
x_i^* = (x_1^i, \ldots, x_{i_*}^i, \ldots, x_n^i)^T = (x_1, \ldots, -x_{i_*}, \ldots, x_n)^T.
$$

Let $x^{p,i_*}$ be the point in $N(x_i^*)$ from which only $p$th element differs from the $p$th element of $x_i^*$ for $p = 1, \ldots, n$, $p \neq i_*$, and we denote $x^{p,i_*} = (x_1, \ldots, -x_p, \ldots, -x_{i_*}, \ldots, x_n)^T$. Then

$$
\delta(i_*, p) = f(x^{p,i_*}) - f(x_i^*) = (x^{p,i_*})^T W (x^{p,i_*}) - (x_i^*)^T W (x_i^*)
$$

$$
= -4 \sum_{i=1}^n x_i^i w_{ip} x_p^{i_*}
$$

$$
= \delta(p) - 8 x_{i_*}^i w_{ip} x_p^{i_*}
$$

$$
= \delta(p) + 8 x_{i_*} w_{ip} x_p \quad (p \neq i_*),
$$

$$
\delta(i_*, i_*) = -\delta(i_*),
$$

(4.8)

where the $\delta(p)$ is given in (4.6). If $\delta(i_*, p) \geq 0$, for all $p = 1, 2, \ldots, n$, then $x_i^*$ is a local minimizer of $f$. Otherwise, we calculate $\delta(i_*, p_*) = \min\{\delta(i_*, p) : p = 1, 2, \ldots, n, p \neq i_*\}$, set $x_i^* = x^{p_*,i_*}, i_* = p_*$ and repeat the process above until a local minimizer of $f$ is found. We summary the 1-neighborhood local search in the following Algorithm (A1).
Algorithm (A1): 1-neighborhood local search

Step 1. Given an initial point \( x^0 \in \mathcal{S} \), calculate \( f_0 = f(x^0) \).

Step 2. Calculate

\[
\delta(p) = -4 \sum_{i=1}^{n} x^0_i w_{ip} x^0_p, \quad p = 1, 2, \ldots, n.
\]

Step 3. Calculate

\[
\delta(i_s) = \min \{ \delta(p) : p = 1, 2, \ldots, n \}.
\]

Step 4. If \( \delta(i_s) \geq 0 \), then set \( x^* = x^0 \), and return \( x^* \) as a local minimizer of \( f \), stop.

Step 5. Set \( f_0 = f_0 + \delta(i_s) \), and calculate

\[
\delta(p) = \begin{cases} 
\delta(p) + 8x^0_i w_{ip} x^0_p, & p \neq i_s, \\
-\delta(i_s), & p = i_s.
\end{cases}
\]

set \( x^0_{i_s} = -x^0_{i_s} \), goto Step 3.

4.4. Generating initial points to minimize \( H \)

For any minimizer \( x^*_k \) of \( f \), a point \( x^i_k \in N(x^*_k)\backslash\{x^*_k\} \) is called a good point in \( N(x^*_k) \), if a point \( x \) in a basin of \( f \) lower than the current basin can be obtained by minimizing the filled function \( \mathcal{H}(x; x^*_k; \alpha(W), \beta(W)) \) starting from \( x^i_k \). Let \( \mathcal{S}_s \) denote the set of all minimizers of the function \( f \) on \( \mathcal{S} \). It is clear that if \( x^*_k = \arg \max \{ f(x^*): x^* \in \mathcal{S}_s \} \), then there may exist more good points in \( N(x^*_k) \). On the contrary, if \( x^*_k = \arg \min \{ f(x^*): x^* \in \mathcal{S}_s \} \), then there does not exist any good point in \( N(x^*_k) \). Thus the number of good points in \( N(x^*_k) \) can reflect completely the performance of the minimizer \( x^*_k \), that is, the number of good points exists, the better performance the minimizer \( x^*_k \) has.

However, it is still challenging for efficiently validating whether a point in \( N(x^*_k) \) is a good point. A typical and direct approach as presented in the literatures to find an initial point for minimizing \( H \) is to test one by one each point in \( N(x^*_k)\backslash\{x^*_k\} \). Since there are \( n \) elements in \( N(x^*_k)\backslash\{x^*_k\} \), it may not be advisable for the approach when \( n \) is large. Especially, for max-cut problems, it is unlikely to confirm all points in \( N(x^*_k) \) being good points or exclude all points in \( N(x^*_k) \) not being good points. The approach proposed in this paper is to randomly generate \( n_1 (\leq n) \) points from \( N(x^*_k)\backslash\{x^*_k\} \) as initial points to minimize \( H \), where \( n_1 \) is called sample size. The probability of each point in \( N(x^*_k)\backslash\{x^*_k\} \) sampled is \( n_1/n \), and \( n_1 \) does not need to be chosen too close to \( n \) via \( \mathcal{S} \) is a connect domain. The numerical results in Section 5 (see Table 5) indicate that after the random generating subroutine is added into the filled function algorithm, the better performance of the proposed algorithm is displayed and the costs of computation and CPU-time are greatly reduced.

Now, we state the random process of generating initial points. For given \( n \), assume that we need to draw out randomly \( n_1 \) points from \( n \) points in \( N(x^*_k)\backslash\{x^*_k\} \). Set \( I_1 = [n/n_1] \), where \([a]\) denotes the integer part of the real number \( a \). If \( n/n_1 \) is an integer, then we can partition the set \( N = \{1, \ldots, n\} \) into \( n_1 \) disjoint subsets and each subset has \( I_1 \) integers, that is \( N_1 = \{1, \ldots, I_1\}, N_2 = \{I_1 + 1, \ldots, 2I_1\}, \ldots, N_{n_1} = \{(n_1 - 1)I_1 + 1, \ldots, n_1 I_1\} \). Let \( c \) be a random number in \((0, 1]\), \( i = (I - 1) \cdot I_1 + [c \cdot I_1] + 1 \in N_I, I = 1, 2, \ldots, n_1 \), and we take \( x^i_k \) as an initial point to minimize \( H(x; x^*; \alpha, \beta) \), where the index \( i \) means the \( i \)th element of \( x^i_k \) differs from the \( i \)th element of \( x^*_k \). The random number \( c \in (0, 1) \) can be generated from the uniform distribution \((0, 1)\).

If \( n/n_1 \) is a fraction, then set \( n_0 = n - n_1 I_1 \). Since \( n_0 < n_1 \), we can also partition the set \( N = \{1, \ldots, n\} \) into \( n_1 \) disjoint subsets of \( N \), where each subset of the first \( n_1 - n_0 \) subsets has \( I_1 \) integers and each subset of last \( n_0 \) subsets has \( I_1 + 1 \) integers. Then we also take \( x^i_k \) as an initial point to minimize \( H(x; x^*; \alpha, \beta) \), where the index \( i \) has the same meaning as above and for \( I = 1, 2, \ldots, n_1 - n_0 \),

\[
i = (I - 1) \cdot I_1 + [c \cdot I_1] + 1,
\]
or for $I = n_1 - n_0 + 1, \ldots, n_1 - n_0 + j, \ldots, n_1$,

$$
i = (n_1 - n_0)I_1 + (I_1 + 1)(I - (n_1 - n_0) - 1) + [(I_1 + 1) \cdot c] + 1
$$

$$
= (I_1 + 1)(I - 1) - (n_1 - n_0) + [(I_1 + 1) \cdot c] + 1,
$$

$c \in (0, 1)$ is a random number generated from $\mathcal{U}(0, 1)$.

### 4.5. Minimizing the filled function $H$

In this subsection, we will describe the method of minimizing $H$ in detail. Let $x_k^* = (x_k^*_{i1}, \ldots, x_k^*_{km}, \ldots, x_k^*_{kn})^T \in \mathbb{S}$ denote the current local minimizer of $f$ in problem (MC) and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in \partial N(x_k^*, K)$, $1 \leq K \leq n$. Denote $\mathcal{X}_K = \{y \in \mathbb{S} : y \in N(\bar{x}), \|y - x_k^*\|_1 > \|\bar{x} - x_k^*\|_1\}$, then we have the following result.

#### Lemma 4.1. There only exist $(n - K)$ elements in set $\mathcal{X}_K$.

**Proof.** Without loss of generality, assume that $\bar{x} = -x_{ks}$ satisfies

$$
\bar{x}_s = -x_{ks}^*, \quad s = 1, 2, \ldots, K,
$$

$$
\bar{x}_s = x_{ks}^*, \quad s = K + 1, K + 2, \ldots, n.
$$

For any $y \in \mathcal{X}_K$, since $y \in N(\bar{x})$, either there exists an index $s \in \{1, \ldots, K\}$ or $s \in \{K + 1, \ldots, n\}$, such that $y_s = -\bar{x}_s$. If $s \in \{1, \ldots, K\}$, then there are only $K - 1$ different elements between $y$ and $x_k^*$. It follows that $\|y - x_k^*\|_1 < \|\bar{x} - x_k^*\|_1$ that contradicts $y \in \mathcal{X}_K$. Hence $s \in \{K + 1, \ldots, n\}$, and each $s \in \{K + 1, \ldots, n\}$ corresponds to an element in $\mathcal{X}_K$. That shows that there are only $n - K$ elements in $\mathcal{X}_K$, i.e., $|\mathcal{X}_K| = n - K$.

Let $x_k^j = (x_{k1}^j, \ldots, x_{km}^j, \ldots, x_{kn}^j)^T$ be a point that is randomly generated in $N(x_k^*) \setminus \{x_k^*\}$, where the superscript $i$ expresses that the sign of only the $i$th element of $x_k^j$ differs from the sign of the $i$th element of $x_k^*$. Since $x_k^j \in \partial N(x_k^*, 1)$, $|\mathcal{X}_1| = n - 1$, where $\mathcal{X}_1 = \{x \in \mathbb{S} : x \in N(x_k^*), \|x - x_k^*\|_1 > \|\bar{x} - x_k^*\|_1\}$. Denote $\mathcal{X}_1 = \{x_{k1}^{i1}, \ldots, x_{km}^{ij}, \ldots, x_{kn}^{in}\}$, where $x_k^{ij} = (x_{k1}^{ij}, \ldots, x_{km}^{ij}, \ldots, x_{kn}^{ij})^T \in \mathcal{X}_1$ and $x_k^{jm} = x_k^{km} (m = 1, 2, \ldots, n, m \neq j)$, $x_k^{ij} = -x_k^{ij}$.

Now, we present the process of minimizing the filled function $H$ starting from an initial point $x_k^j \in N(x_k^*)$. Assume that $f(x_k^j) \geq f(x_k^*)$ holds for all points $x_k^j \in \mathcal{X}_1$ ($j = 1, \ldots, n, j \neq i$) (if there exists $x_k^j \in \mathcal{X}_1$ such that $f(x_k^j) < f(x_k^*)$, then $x_k^j$ is in a basin lower than the current basin containing $x_k^*$). Since $f(x_k^j) \geq f(x_k^*)$, $\|x_k^{ij} - x_k^*\|_1 > \|x_k^j - x_k^*\|_1$, and $\alpha(W)$, $\beta(W)$ are calculated by (4.4), (4.5), respectively, it follows from Lemma 3.3 that

$$
\delta_H(j) = H(x_k^j; x_k^*; \alpha(W), \beta(W)) - H(x_k^j; x_k^*; \alpha(W), \beta(W)) < 0, \quad j = 1, \ldots, n, \quad j \neq i.
$$

On the other hand,

$$
\delta_H(j) = \alpha(W)[f(x_k^j) - f(x_k^*)] - \beta(W)[\|x_k^{ij} - x_k^*\|_1 - \|x_k^j - x_k^*\|_1]
$$

$$
= -4\alpha(W) \sum_{m=1}^n x_{km}^{ij} w_{mj} x_{kj}^j - 2\beta(W)
$$

$$
= \alpha(W) \delta_f(j) - 2\beta(W), \quad j = 1, \ldots, n, \quad j \neq i,
$$

where $\delta_f(j) = f(x_k^{ij}) - f(x_k^*)$. Thus we can find an index $j$ to minimize $\delta_H(j)$, that is equivalent to minimize $\delta_f(j)$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Let $j_0 = \arg \min_j \{\delta_f(j) : j \in \{1, 2, \ldots, n\} \setminus \{i\}\}$
then for all $x^{ij}_k \in \mathcal{X}_1$, we have

$$H(x^{ij}_k; x^*_n; \alpha(W), \beta(W)) \leq H(x^{ij}_k; x^*_n; \alpha(W), \beta(W)),$$

and $x^{ij+}_k$ is used as the next iterate point for minimizing $H(x; x^*_k; \alpha(W), \beta(W))$.

After $x^{ij+}_k \in \partial N(x^*_n, 2)$ is found, we consider the points in set $\mathcal{X}_2 = \{ x \in \mathbb{S} : x \in N(x^{ij+}_k), \| x - x^*_n \|_1 > \| x^{ij+}_k - x^*_n \|_1 \}$ and $|\mathcal{X}_2| = n - 2$. There are two possible cases.

1. If there exists a point $x \in \mathcal{X}_2$, such that $f(x) < f(x^*_n)$, then $x$ is a point in a basin of $f$ lower than the current basin and we can re-minimize $f$ using Algorithm (A1) from $x$.
2. If for all $x \in \mathcal{X}_2$, $f(x) \geq f(x^*_n)$, then we can find a point $\tilde{x} \in \mathcal{X}_2$ satisfying $H(\tilde{x}; x^*_n; \alpha(W), \beta(W)) \leq H(x^{ij+}_k; x^*_n; \alpha(W), \beta(W))$.

Let

$$\mathbf{Algorithm (A2): Minimizing the Filled Function H}$$

**Step 1.** Let $x^i_k \in N(x^*_n) \setminus N$ be a randomly generated point, set $\tilde{x}^i_k = x^i_k$. Calculate

$$\delta_0 = f(\tilde{x}^i_k) - f(x^*_n) = -4 \sum_{m=1}^{\infty} x^*_{km} w_{mi} x^*_{ki},$$

set $K = 1$ and $\bar{I} = \{ i \}$.

**Step 2.** If $K = [\frac{n}{2}]$, goto step 2 of Algorithm (SA). Otherwise, for all $x^i_k \in \mathcal{X}_K = \{ x \in N(\tilde{x}^i_k) : \| x - x^*_n \|_1 > \| \tilde{x}^i_k - x^*_n \|_1 \}$.

Calculate

$$\delta_f(j) = f(x^{ij+}_k) - f(\tilde{x}^i_k) = -4[ (x^i_{km})^T w_j ] \tilde{x}^i_{kj}, \quad j = 1, \ldots, n, \quad j \notin \bar{I},$$

$$j_0 = \arg \min \{ \delta_f(j) : j \in \{ 1, 2, \ldots, n \} \setminus \bar{I} \},$$

$$\delta_K(j_0) = f(x^{ij+}_k) - f(x^*_n) = \delta_{K-1}(j_0) + \delta_f(j_0).$$

Here, $\delta_0(j_0) = 0$.

**Step 3.** If $\delta_K(j_0) = f(x^{ij+}_k) - f(x^*_n) < 0$, then goto Algorithm (A1) to minimize $f$ starting from $x^{ij+}_k$.

**Step 4.** Set

$$\tilde{x}^i_k = x^{ij+}_k, \quad K = K + 1, \quad \bar{I} = \bar{I} \cup \{ j_0 \},$$

goto step 2.

**Remark 3.** By the symmetry structure of the set $\mathbb{S}$, for any integer $K$ with $[n/2] < K \leq n$, if $x \in \partial N(x^*_n, K)$, then $-x \in \partial N(x^*_n, n - K)$ and $f(-x) = f(x)$. Thus if we cannot find a point $\tilde{x}$ in a basin of $f$ lower than the current basin along a steepest descent path of $H(x; x^*_n; \alpha(W), \beta(W))$ from point $x^i_k \in N(x^*_n)$ to some point in $\partial N(x^*_n, \{ n/2 \})$, then it is not necessary to continue the search in $\partial N(x^*_n, \{ n/2 \})$. Hence, when $K = [n/2]$ in step 2, we need to restart the minimization of filled function $H(x; x^*_n; \alpha(W), \beta(W))$ by generating another initial point.

**Remark 4.** In order to obtain the next initial iterate point $x^{ij+}_k$ from previous initial point $\tilde{x}^i_k$ for minimizing $H$, we need to calculate equality (4.9) and (4.10), that is, only the need to calculate the inner product of two $n$-dimensional vectors at most $(n - |\bar{I}|)$ times, instead of calculating directly the value $f(x^{ij}_k)$ ($j \in \{ 1, 2, \ldots, n \} \setminus \bar{I}$).
5. Numerical results

In this section, some experimented results are reported on the following two classes of test max-cut problems:

- Randomly generated graphs \((20 \leq n \leq 200)\).
- 12 G-set graphs \((n \geq 800)\).

When \(n_1 = n\), in Algorithm DFFA, it means that in case necessary all points in \(N(x^*_k) \backslash \{x^*_k\}\) will be generated as initial points to minimize the filled function \(H(x; x^*_k; \alpha(W), \beta(W))\). We call the algorithm as Algorithm (DA).

The first class of test problems are generated by the matlab code

\[
c = \text{floor}(w \times \text{abs} (\text{full}(\text{spRANDSYM}(n, a)))) ,
\]

\[
W = c - \text{diag}(\text{diag}(c)),
\]

where the parameter \(a\) reveals the density of nonzero entries in matrix \(W\) and \(w\) reflects the weighted values of matrix \(W\).

When \(20 \leq n \leq 100\), we randomly generate 18 problems with \(w = 1\), denoted by S1, \ldots, S18 and call them as S-set problems, and only the Algorithm (DA) is implemented for all S-set problems. When \(120 \leq n \leq 200\), we randomly generate eight problems with \(w\) as a random integer in \([1,50]\), denoted by L1, \ldots, L8 and call them as L-set problems. The 12 G-set graphs are G1, G2, G3, G11, G12, G13, G14, G15, G16, G43, G44 and G45 that were created by using a graph generator, rudy, written by Pro. Rinaldi. We only implement Algorithm (DFFA) with \(n_1 = \lfloor n/2\rfloor\) for the 12 G-set problems. All the results presented in this section for the proposed algorithm are implemented in a 1.6 GHz Pentium IV personal computer with 256 Mb of RAM.

The elements \(w_{ij}\) of matrix \(W\) are integers for all test problems. Thus we take \(\varepsilon = 1\) in Eqs. (4.4) and (4.5). By (4.4), the value \(\beta\) decreases and tends to \(1/8\) with the increasing of the product \(qM\). Hence, a simple choice for the value of \(\beta\) is \(\beta = 1/8\) that satisfies \(\beta < \beta_0\) for all test problems. Thus, we only need to calculate the value of \(\alpha\) by (4.5) for different test problems. These values are given in Tables 1–3. It can be seen from Tables 1–3 that with the increasing of the dimension or the density of matrix \(W\), \(\alpha\) is decreasing via the value \(q\) and \(M\) increasing.

<table>
<thead>
<tr>
<th>Problem</th>
<th>(n)</th>
<th>(a)</th>
<th>(\alpha(W))</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>20</td>
<td>0.09</td>
<td>0.0052</td>
</tr>
<tr>
<td>S2</td>
<td></td>
<td>0.30</td>
<td>0.0013</td>
</tr>
<tr>
<td>S3</td>
<td></td>
<td>0.60</td>
<td>8.9446e−4</td>
</tr>
<tr>
<td>S4</td>
<td>30</td>
<td>0.09</td>
<td>0.0070</td>
</tr>
<tr>
<td>S5</td>
<td></td>
<td>0.30</td>
<td>1.3055e−3</td>
</tr>
<tr>
<td>S6</td>
<td></td>
<td>0.60</td>
<td>8.1300e−4</td>
</tr>
<tr>
<td>S7</td>
<td>40</td>
<td>0.09</td>
<td>0.0052</td>
</tr>
<tr>
<td>S8</td>
<td></td>
<td>0.30</td>
<td>8.9446e−4</td>
</tr>
<tr>
<td>S9</td>
<td></td>
<td>0.60</td>
<td>2.1786e−4</td>
</tr>
<tr>
<td>S10</td>
<td>60</td>
<td>0.09</td>
<td>1.7422e−3</td>
</tr>
<tr>
<td>S11</td>
<td></td>
<td>0.30</td>
<td>3.3444e−4</td>
</tr>
<tr>
<td>S12</td>
<td></td>
<td>0.60</td>
<td>2.0564e−4</td>
</tr>
<tr>
<td>S13</td>
<td>80</td>
<td>0.09</td>
<td>1.3155e−3</td>
</tr>
<tr>
<td>S14</td>
<td></td>
<td>0.30</td>
<td>2.4814e−4</td>
</tr>
<tr>
<td>S15</td>
<td></td>
<td>0.60</td>
<td>1.3024e−4</td>
</tr>
<tr>
<td>S16</td>
<td>100</td>
<td>0.09</td>
<td>8.6956e−4</td>
</tr>
<tr>
<td>S17</td>
<td></td>
<td>0.30</td>
<td>1.6717e−4</td>
</tr>
<tr>
<td>S18</td>
<td></td>
<td>0.60</td>
<td>6.2570e−5</td>
</tr>
</tbody>
</table>
Table 2
The values of parameter $\alpha$ for randomly generated the 8 L-set problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>$\alpha$</th>
<th>$w$</th>
<th>$\alpha(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>120</td>
<td>0.30</td>
<td>47</td>
<td>3.5568e$-7$</td>
</tr>
<tr>
<td>L2</td>
<td>0.60</td>
<td>7</td>
<td></td>
<td>1.2084e$-6$</td>
</tr>
<tr>
<td>L3</td>
<td>160</td>
<td>0.30</td>
<td>30</td>
<td>3.6837e$-7$</td>
</tr>
<tr>
<td>L4</td>
<td>0.60</td>
<td>14</td>
<td></td>
<td>3.0114e$-7$</td>
</tr>
<tr>
<td>L5</td>
<td>180</td>
<td>0.30</td>
<td>47</td>
<td>1.9473e$-7$</td>
</tr>
<tr>
<td>L6</td>
<td>0.60</td>
<td>9</td>
<td></td>
<td>4.0575e$-7$</td>
</tr>
<tr>
<td>L7</td>
<td>200</td>
<td>0.30</td>
<td>17</td>
<td>4.6358e$-7$</td>
</tr>
<tr>
<td>L8</td>
<td>0.60</td>
<td>46</td>
<td></td>
<td>5.5325e$-8$</td>
</tr>
</tbody>
</table>

Table 3
The values of parameter $\alpha$ for the 12 G-set problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>$n$</th>
<th>Density (%)</th>
<th>$\alpha(W)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>800</td>
<td>6.12</td>
<td>1.3334e$-5$</td>
</tr>
<tr>
<td>G2</td>
<td>1.4348e$-5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>G3</td>
<td>1.3127e$-5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>G11</td>
<td>800</td>
<td>0.63</td>
<td>0.0040</td>
</tr>
<tr>
<td>G12</td>
<td></td>
<td></td>
<td>0.0040</td>
</tr>
<tr>
<td>G13</td>
<td></td>
<td></td>
<td>0.0040</td>
</tr>
<tr>
<td>G14</td>
<td>800</td>
<td>1.58</td>
<td>3.5870e$-6$</td>
</tr>
<tr>
<td>G15</td>
<td></td>
<td></td>
<td>2.6710e$-6$</td>
</tr>
<tr>
<td>G16</td>
<td></td>
<td></td>
<td>4.1412e$-6$</td>
</tr>
<tr>
<td>G43</td>
<td>1000</td>
<td>2.10</td>
<td>5.5658e$-5$</td>
</tr>
<tr>
<td>G44</td>
<td></td>
<td></td>
<td>4.5658e$-5$</td>
</tr>
<tr>
<td>G45</td>
<td></td>
<td></td>
<td>4.5658e$-5$</td>
</tr>
</tbody>
</table>

For each of all $S$-set and $L$-set problems, we run Algorithms (DFFA) (or (DA)) and Continuation Algorithm [30] 10 times, respectively. The numerical results and comparisons are listed in Tables 4 and 5. In both the tables, the columns headed with $F^*$ and $F_C^*$ present the largest values to max-cut problems generated by Algorithms (DFFA) (or (DA)) and Continuation Algorithm in 10 tests, respectively. $N$ and $NH$ denote the number of local minimizers of $f$ which is found by Algorithm (DFFA) or (DA) and the number of mean times minimizing the filled function $H$, that is, the number of mean initial points generated for minimizing $H$ 10 times, respectively. The CPU-time, denoted by time(s), is the mean value of CPU-time in 10 tests. $F_0^*$ stands for the max-cut value associated with the first minimizer $x_0^*$ of $f$ obtained by Algorithm (A1) starting from an initial point $x_0$, where the initial point $x_0$ is generated randomly by the procedure $x_0 := \text{sign(unifrnd}(-1, 1, n, 1))$, where unifrnd$(-1, 1, n, 1)$ is a Matlab function which generates an $n$-dimensional uniformly distributed vector whose elements lie in $(-1, 1)$. $\text{sign}(x)$ is a sign function that takes 1 when $x \geq 0$ and -1 when $x < 0$.

In order to see how the value $F^*$ is close to the global optimal value of the max-cut problem, we calculate the ratio $S_U = (\sum F^* / \sum S_U)\%$, where $S_U$ is an upper bound of $F^*$, which is generated by solving problem (SDP) using the Matlab software package for solving semidefinite programming, SDPACK [1].

Table 6 gives the results and comparisons between the hybrid GRASP-VNS [11] (hybrid greedy randomized adaptive search procedure with variable neighborhood search) heuristic, denoted as gvnss, and the Algorithm (DFFA) on the 12 G-set large size test problems. In the table, the columns headed with gvnss present the approximate values to max-cut problems generated by the gvnss heuristic and these values are quoted from [11]. The upper bound $S_U$ for the 12 G-set
Table 4
The numerical comparisons of continuous method with Algorithm (DA) for the 18 S-set problems with running continuous method, Algorithm (DA) 10 times, respectively, for each problem

<table>
<thead>
<tr>
<th>Problem</th>
<th>( S_U )</th>
<th>Continuous</th>
<th>Algorithm (DA)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( F^*_C )</td>
<td>( F^*_0 )</td>
</tr>
<tr>
<td>S1</td>
<td>9</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>S2</td>
<td>27</td>
<td>27</td>
<td>25</td>
</tr>
<tr>
<td>S3</td>
<td>43</td>
<td>40</td>
<td>38</td>
</tr>
<tr>
<td>S4</td>
<td>14</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>S5</td>
<td>37</td>
<td>34</td>
<td>34</td>
</tr>
<tr>
<td>S6</td>
<td>74</td>
<td>69</td>
<td>66</td>
</tr>
<tr>
<td>S7</td>
<td>29</td>
<td>26</td>
<td>25</td>
</tr>
<tr>
<td>S8</td>
<td>72</td>
<td>66</td>
<td>61</td>
</tr>
<tr>
<td>S9</td>
<td>124</td>
<td>116</td>
<td>110</td>
</tr>
<tr>
<td>S10</td>
<td>55</td>
<td>46</td>
<td>49</td>
</tr>
<tr>
<td>S11</td>
<td>164</td>
<td>155</td>
<td>144</td>
</tr>
<tr>
<td>S12</td>
<td>311</td>
<td>293</td>
<td>289</td>
</tr>
<tr>
<td>S13</td>
<td>97</td>
<td>90</td>
<td>89</td>
</tr>
<tr>
<td>S14</td>
<td>268</td>
<td>249</td>
<td>239</td>
</tr>
<tr>
<td>S15</td>
<td>465</td>
<td>437</td>
<td>426</td>
</tr>
<tr>
<td>S16</td>
<td>153</td>
<td>142</td>
<td>139</td>
</tr>
<tr>
<td>S17</td>
<td>409</td>
<td>379</td>
<td>370</td>
</tr>
<tr>
<td>S18</td>
<td>762</td>
<td>715</td>
<td>706</td>
</tr>
<tr>
<td>Sum</td>
<td>3113</td>
<td>2905</td>
<td>3012</td>
</tr>
</tbody>
</table>

Bold value represents, for the same as test problem, the largest max-cut value obtained by algorithm DFFA, DA, CA or GRASP-VNS.

problems comes from [7]. The value \( F^* \) is obtained by running Algorithm (DFFA) only one time. The following observations can be made based on the results in Tables 4–6.

1. For different density graphs, the proposed filled algorithm is efficient for solving max-cut problems. Comparison with \( F^*_C \) and gvn, the value \( F^* \) is obviously greatly improved.
2. Although Algorithm (DFFA) uses less points in \( N(x^*_k)\setminus\{x^*_k\} \) than Algorithm (DA) as initial points to minimizing \( H(x; x^*_k; z(W); \beta(W)) \), the value \( F^* \) and the ratio \( S_U\% \) obtained by Algorithm (DFFA) are not less than that obtained by Algorithm (DA). Moreover, Algorithm (DFFA) has obtained a better solution than Algorithm (DA) for L4. Especially, Algorithm (DFFA) only needs to spend almost half of CPU-time of Algorithm (DA).
3. The ratio \( S_U\% \) reflects that the obtained solution by the proposed algorithm is very close to the global solution for all the test problems.
4. The value \( \rho = (F^* - F^*_0)/N \) listed in the final column of Table 6 indicates that the speed of improving from \( F^*_0 \) to \( F^* \) is fast for problems G1, G2, G3, G43, G44 and G45, which also reflects the filled function algorithm is promising for solving max-cut problems.
5. Although we have greatly reduced the computation cost by random generating initial points and avoiding to calculate the function value repeatedly, the CPU-time is still large. This is because \( N \), the number of the local minimizers of the function \( f \) and the times of minimizing the filled function \( H \) are large. That is, the larger the \( N \) and \( NH \) are, the more the CPU-time is.

6. Conclusions

A discrete filled function algorithm is proposed to find approximate global solutions for NP-hard max-cut problems. The algorithm is implemented via two phase cycle: in the first phase, the objective function \( f \) in problem (MC) is
Table 5
The numerical comparisons of continuous method with Algorithm (DA), (DFFA) for the 8 L-set problems with running continuous method, Algorithm (DA) and (DFFA) 10 times, respectively, for each problem

<table>
<thead>
<tr>
<th>Problem</th>
<th>$S_U$</th>
<th>Continuous</th>
<th>Algorithm (DA)/(DFFA)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$F^*_C$</td>
<td>Time (s)</td>
</tr>
<tr>
<td>L1</td>
<td>49738</td>
<td>46423</td>
<td>6.25</td>
</tr>
<tr>
<td>L2</td>
<td>11765</td>
<td>11122</td>
<td>6.86</td>
</tr>
<tr>
<td>L3</td>
<td>55241</td>
<td>51525</td>
<td>7.67</td>
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<tr>
<td>L4</td>
<td>43860</td>
<td>41690</td>
<td>8.77</td>
</tr>
<tr>
<td>L5</td>
<td>108190</td>
<td>102082</td>
<td>9.80</td>
</tr>
<tr>
<td>L6</td>
<td>33856</td>
<td>32078</td>
<td>9.72</td>
</tr>
<tr>
<td>L7</td>
<td>46810</td>
<td>44122</td>
<td>11.56</td>
</tr>
<tr>
<td>L8</td>
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<td>14.02</td>
</tr>
<tr>
<td>Sum</td>
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<td>545030</td>
<td></td>
</tr>
<tr>
<td>%SU</td>
<td>100</td>
<td>94.77</td>
<td></td>
</tr>
</tbody>
</table>

Bold value represents, for the same as test problem, the largest max-cut value obtained by algorithm DFFA, DA, CA or GRASP-VNS.

Table 6
The numerical comparisons of gvns heuristic with Algorithm (DFFA) for 12 G-set problems with only running Algorithm (DFFA) once time for each problem

<table>
<thead>
<tr>
<th>Problem</th>
<th>$S_U$</th>
<th>gvns</th>
<th>Algorithm (DFFA)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$F^<em>_0$ $N$ $NH$ $F^</em>$ Time (s) $\rho$</td>
</tr>
<tr>
<td>G1</td>
<td>12078</td>
<td>11475</td>
<td>11355 14 165 11557</td>
</tr>
<tr>
<td>G2</td>
<td>12084</td>
<td>11499</td>
<td>11280 21 196 11563</td>
</tr>
<tr>
<td>G3</td>
<td>12077</td>
<td>11507</td>
<td>11315 19 217 11549</td>
</tr>
<tr>
<td>G11</td>
<td>627</td>
<td>544</td>
<td>448 40 218 557</td>
</tr>
<tr>
<td>G12</td>
<td>621</td>
<td>542</td>
<td>418 41 227 550</td>
</tr>
<tr>
<td>G13</td>
<td>645</td>
<td>572</td>
<td>430 37 202 576</td>
</tr>
<tr>
<td>G14</td>
<td>3187</td>
<td>3009</td>
<td>2926 21 198 3029</td>
</tr>
<tr>
<td>G15</td>
<td>3169</td>
<td>3008</td>
<td>2891 28 214 3021</td>
</tr>
<tr>
<td>G16</td>
<td>3172</td>
<td>2985</td>
<td>2896 22 169 3025</td>
</tr>
<tr>
<td>G43</td>
<td>7027</td>
<td>6583</td>
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</tr>
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<td>G44</td>
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<td>6373 22 293 6578</td>
</tr>
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<td>7020</td>
<td>6553</td>
<td>6372 25 267 6573</td>
</tr>
<tr>
<td>Sum</td>
<td>68729</td>
<td>64834</td>
<td>65165</td>
</tr>
<tr>
<td>%SU</td>
<td>100</td>
<td>94.33</td>
<td>94.81</td>
</tr>
</tbody>
</table>

Bold value represents, for the same as test problem, the largest max-cut value obtained by algorithm DFFA, DA, CA or GRASP-VNS.
minimized using the 1-neighborhood local search to obtain a local minimizer, and then in the second phase, the discrete filled function from some neighbor points of the local optimizer is minimized and the two cycles are repeated until the stop conditions are satisfied. The properties of the proposed filled function are analyzed. The characteristics of max-cut problems are used to show that the parameter values in the filled function need not be adjusted. This greatly increases the efficiency of the proposed filled function method. Numerical results and comparisons are reported to indicate that the filled function algorithm is efficient for max-cut problems. For large scale graphs, the algorithm may spend more time to find a desired solution which is due to the number of the local minimizers of the function $f$ and the times of minimization of the filled function $H$ from different initial points in a neighbor of a local minimizer of $f$. Hence further works on the proposed algorithm are required to refine the algorithm in theory and implementation.

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References