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FOURIER NONLINEAR GALERKIN APPROXIMATION FOR THE TWO DIMENSIONAL NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, for viscous incompressible Navier-Stokes equations with periodic boundary conditions, we prove the existence and uniqueness of the solution corresponding to its Fourier nonlinear Galerkin approximation. At the same time, we give its error estimates.

Key words Navier-Stokes equations, nonlinear Galerkin methods, Fourier methods

I. Introduction

Despite the considerable increase in the available computing power during the last years, the integration of Navier-Stokes equation (NSE) on large intervals of time, especially when Reynolds number is large and flow tends to turbulence, still remains a difficult problem whose solution is not at hand. Nonlinear Galerkin method (NGM) is numerical schemes for dissipative PDEs based on Approximate Inertial Manifolds (AIMs)[1], where its spatial discretization relies on a nonlinear manifold. Using this method, we can get higher accuracy under the same computational scale as that of classical Galerkin methods, thus it makes the study of long time behavior of the solution and the direct simulation of turbulence possible.

For nonslip boundary conditions, [2], [3] discuss NGMs in finite element and spectral cases. But they only give the existence of the approximate solutions, and their approximate forms are unsuitable for implementation. On the other hand, its error estimates are based on the comparison of NG solution and classical Galerkin one, this could not completely indicate the advantages of this method.

In this paper, for periodic boundary conditions, we give an easy implemented approximate form. Firstly, we prove its well-posedness, then directly give its error estimate between NG solution and true solution.

Setting $\Omega=(-\pi,\pi)^2$, the NSE, in velocity-pressure formulation, reads

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f && (\text{on } R\times\Omega)\\ &\nabla\cdot u=0 && (\text{on } R\times\Omega)\\ &\text{periodic boundary conditions} &&\\ &u(x,0)=a(x) && (\text{on } \Omega)\end{aligned}\right\}\tag{1.1}$$

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Here u and p stand for velocity and pressure, $\nu>0$ is kinetic viscosity, a the initial velocity field and f the volume force. Without loss of generality, we can assume $\nabla\cdot a=0$, $\int_\Omega a(x)dx=0$ and $\int_\Omega f(x,t)dx=0$ (See [4]).

We will work with Sobolev Spaces of periodic functions identifying Ω, $\forall m\in\mathcal{N}$

$$H^m(\Omega)=\left\{\phi:\phi=\sum_{k\in Z^2}c_k\exp[ik\cdot x],\ c_k=\overline{c_{-k}},\ \sum_{k\in Z^2}|k|^{2m}|c_k|^2<+\infty\right\}$$
$$\dot{H}^m(\Omega)=\{\phi:\phi\in H^m(\Omega),\ c_0=0\}$$

For any $\alpha\in R$, we define $H^\alpha(\Omega)$ by duality and interpolation as usual. We will use same notations for scalar and vector valued functions. Also, we introduce

$$H=\{\phi\in H^0(\Omega),\ \mathrm{div}\phi=0,\ \text{in weak sense}\}$$
$$V=\{\phi\in H^1(\Omega),\ \mathrm{div}\phi=0\},\ \dot{H}=H\cap\dot{H}^0(\Omega),\ \dot{V}=V\cap\dot{H}^1(\Omega)$$

Hereafter, we always use $(\cdot,\cdot)$ and $((\cdot,\cdot))=(\nabla\cdot,\nabla\cdot)$ to denote the inner products of L^2 and $H_0^1(\Omega)$. For any $\alpha\in R$, $\|\cdot\|_\alpha$ stands for standard H^α-norm, especially, $\|\cdot\|$ for that of L^2-norm. And denote $|\!|\!|\phi|\!|\!|=\sup_{t>0}\|\phi(t)\|$.

Define projection operator $P:H^0(\Omega)\to H$,

$$Pf=\sum_{k\in Z^2,k\neq 0}\left(I-\frac{k\cdot k^T}{|k|^2}\right)f_k\exp[ik\cdot x]+f_0,\ \forall f=\sum_{k\in Z^2}f_k\exp[ik\cdot x]\in H^0(\Omega)$$

And note that P commutes with derivations. At the same time, we still need to define the Stokes operator $A:\forall\phi\in D(A)=\{\phi\in\dot{H},\ \Delta\phi\in\dot{H}\}$, $A\phi=-\Delta\phi$. A can be extended as a positive defined and selfadjoint operator on $\dot{H}$. Thus we can define the powers of A, A^α. Indeed, $D(A^\alpha)=\{\phi\in\dot{H}^{2\alpha}(\Omega),\mathrm{div}\phi=0\}$ is a closed subspace of $\dot{H}^{2\alpha}(\Omega)$, and $\|A^\alpha\cdot\|$ is a kind of equivalent norms of it. It is well-known that $-\nu A$ can generate an analytic semigroup on $\dot{H}$, denoted by $\{\exp[-\nu At]\}_{t\geq 0}$, and there exist constants c, $\delta>0$ such that

$$\|A^\alpha\exp[-\nu At]\|\leqslant c\nu^{-\alpha}t^{-\alpha}\exp[-2\nu\delta t]\qquad(t>0,\ \alpha>0)\tag{1.2}$$

where δ depends only on A. From now on, we use c to represent a generic constant which has different meanings at different places.

Now, by projecting (1.1) onto H, we find u satisfying

$$\left.\begin{aligned}&\frac{du}{dt}+\nu Au+B(u,u)=Pf\\&u(0)=a\end{aligned}\right\}\tag{1.3}$$

and p satisfying $\Delta p=\nabla\cdot f-\nabla\cdot(u\cdot\nabla)u$, where $B(u,u)=P(u\cdot\nabla)u$.

In following analysis, we need some properties of $b(u;\ v,\ w)=\int_\Omega(u\cdot\nabla)v\cdot w dx$ [4],[5]

$$\left.\begin{aligned}&b(u;\ v,\ w)=-b(u;\ w,\ v) && (\forall u\in\dot{V},\ v,\ w\in V)\\&b(u;\ u,\ Au)=0 && (\forall u\in D(A))\\&b(w;\ u,\ Au)+b(u;\ w,\ Au)+b(u;\ u,\ Aw)=0 && (\forall u,\ w\in D(A))\end{aligned}\right\}\tag{1.4}$$

and for $s_i = m_i + a_i$, $m_i = [s_i] (i=1,2,3)$, if $\sum_{i=1}^{3} s_i > 1$ or $\sum_{i=1}^{3} s_i = 1$ but $s_1, s_2, s_3 \neq 1$,

$$\left.\begin{aligned} &|b(u; v, w)| \leqslant c\|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3} \\ &|b(u; v, w)| \leqslant c\|u\|_{m_1}^{1-a_1}\|u\|_{m_1+1}^{a_1}\|v\|_{m_2+1}^{1-a_2}\|v\|_{m_2+2}^{a_2}\|w\|_{m_3}^{1-a_3}\|w\|_{m_3+1}^{a_3} \end{aligned}\right\} \quad (1.5)$$

In addition, for the sake of convenience, we give the weak form of (1.3) here

$$\left.\begin{aligned} &(u_t, \phi) + \nu((u, \phi)) + b(u; u, \phi) = (Pf, \phi) \quad (\forall \phi \in \dot{V}) \\ &u(x, 0) = a(x) \qquad (\forall x \in \Omega) \end{aligned}\right\} \quad (1.6)$$

II. Fourier NG Approximation and Its Well-Posedness

For n, $N \in \mathscr{N}$, $n \ll N$, we introduce some finite dimensional subspaces

$$s_N = \left\{\phi : \phi = \sum_{-N \leqslant k_1, k_2 \leqslant N} \exp[ik \cdot x]\right\}$$

$$F_N = S_N \times S_N, \quad V_N = V \cap F_N, \quad \dot{V}_N = \dot{V} \cap F_N$$

Let P_N express the orthogonal projection operator from $L^2(\Omega)^2$ onto F_N, and denote $Q_N = I - P_N$. Now we state some basic properties of them

$$\|P_N\phi\|_m \leqslant cN^{m-\mu}\|\phi\|_\mu, \quad \|Q_N\phi\|_\mu \leqslant cN^{\mu-m}\|\phi\|_m \quad 0 \leqslant \mu \leqslant m, \ \phi \in H^m(\Omega) \quad (2.1)$$

Because of $n \ll N$, we can decompose $\dot{V}_N$ as

$$\dot{V}_N = \dot{V}_n \oplus \dot{V}_N^n, \text{ where } \dot{V}_n = P_n \dot{V}_N, \quad \dot{V}_N^n = (I_N - P_n)\dot{V}_N = Q_{Nn}\dot{V}_N \quad (2.2)$$

Fourier NGM aims to search finite dimensional approximation of (1.7) in the form of

$$u_N = v^n + w_N^n, \quad v^n \in \dot{V}_n, \quad w_N^n \in \dot{V}_N^n$$

governed by

$$\left.\begin{aligned} &(v_t^n, \phi) + \nu((v^n, \phi)) + b(v^n + w_N^n; v^n + w_N^n, \phi) = (Pf, \phi) && (\forall \phi \in \dot{V}_n) \\ &\nu((w_N^n, \chi)) + b(v^n; v^n, \chi) = Pf, \phi) && (\forall \chi \in \dot{V}_N^n) \\ &v^n(x, 0) = P_n a(x) && (\forall x \in \Omega) \end{aligned}\right\} \quad (2.3)$$

Using semigroup $\{\exp[-\nu At]\}_{t \geqslant 0}$, we have for the abstract form of (2.3)

$$\left.\begin{aligned} &v^n(t) = \exp[-\nu At]P_n a - \int_0^t \exp[-\nu A(t-s)]P_n\{B(v^n + w_N^n, v^n + w_N^n) - Pf\}ds \\ &w_N^n(t) = \nu^{-1}A^{-1}Q_{Nn}\{Pf - B(v^n, v^n)\} \end{aligned}\right\} \quad (2.4)$$

Remark 2.1 If $N > 2n$, it is easy to verify $B(v^n, v^n) \in \dot{V}_{2n}$. So $w^n{}_N = \eta + \xi$, here $\eta \in \dot{V}_{2n}$, $\xi = \nu^{-1}A^{-1}Q_{Nn}Pf \in \dot{V}^c{}_{2n}$ and $\dot{V}^c{}_{2n}$ is the complement of $\dot{V}_{2n}$ in $\dot{V}$. Thus, no matter what value N takes, including $N = +\infty$, (2.3) and (2.4) are always finite dimensional systems. Its computational complexity is dominated by n, especially for autonomous systems.

In the rest we denote by $u = v + w$ the solution of (2.3), and $x = y + z$ the solution of (1.6), where $y = P_n x$, $z = Q_n x$. Now we state a useful inequality[8] as

Lemma 2.1 Suppose T, α and β be positive constants, $0 < \theta < 1$. Then for continuous function $f : [0, T] \to [0, +\infty)$ satisfying

$$f(t)\leqslant\alpha+\beta\int_0^t(t-s)^{-\theta}f(s)ds \qquad (0\leqslant t\leqslant T)$$

we have

$$f(t)\leqslant c\alpha\exp\{c\beta^{1/(1-\theta)}t\} \qquad (0\leqslant t\leqslant T)$$

where c depends only on θ. Especially, if $\alpha=0$, we have $f(t)=0$.

Theorem 2.1 Assume $a\in D(A^{1/2})$, $f\in C([0,\infty),\ \dot{H})$ and $|||f|||\leqslant C_f$ where $C_f>0$ is a constant. Then there exists some $n_0\in\mathcal{N}$, when $n_0\leqslant n\ll N\leqslant+\infty$, (2.3) has a unique solution $u=v+w$, and there is a constant $C_0>0$ independent of t, n, N and u such that $|||A^{1/2}u|||<C_0$.

Proof Obviously, the second equation of (2.3) can define a mapping

$$\Phi: v\in\dot{V}_n\to w=\Phi(v)\in\dot{V}_N^* \tag{2.5}$$

In fact, the graph of Φ is a sort of AIMs of Galerkin form. We will use Schauder's fixed point theory to demonstrate the existence of the solution of (2.3). First of all, we must introduce a bounded closed convex subset as usual

$$K=\{\phi\in C([0,\infty),\dot{V}_n),\ |||A^{1/2}\phi|||\leqslant\rho\}\subset C([0,\infty),\dot{V}_n) \tag{2.6}$$

Here $\rho>0$ is a certain constant. Now consider the map Ψ: $K\to C([0,\infty),\ \dot{V}_n)$ defined by

$$\left.\begin{aligned}&\text{Given } \lambda\in K,\ \text{find } (v,w)\in C([0,\infty),\ \dot{V}_n)\times C([0,\infty),\ \dot{V}_N^*)\ \text{such that}\\ &(v_t,\phi)+\nu((v,\phi))+b(v+w;\ v+w,\phi)=(Pf,\phi) \qquad (\forall\phi\in\dot{V}_n)\\ &\nu((w,\chi))+b(\lambda;\ \lambda,\chi)=(Pf,\chi) \qquad (\forall\chi\in\dot{V}_N^*)\\ &v(x,0)=P_na(x) \qquad (\forall x\in\Omega)\end{aligned}\right\} \tag{2.7}$$

From the second equation of (2.7) and $|||A^{1/2}\lambda|||\leqslant\rho$, we know

$$\|A^{3/4}w\|\leqslant\nu^{-1}n^{-1/4}(n^{-1/2}C_f+\rho^2) \qquad (\forall t\geqslant0) \tag{2.8}$$

Taking $\phi=Av$ in the first equation of (2.7) and noticing (1.4)—(1.5) and (2.8), we can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{1/2}v\|^2+\nu\|Av\|^2&=(Pf,Av)-b(w;\ w,Av)-b(v;\ w,Av)-b(w;\ v,Av)\\ &\leqslant C_f\|Av\|+cn^{-1}\|A^{3/4}w\|^2\|Av\|+cn^{-1/2}\|A^{3/4}w\|\|Av\|^2\\ &\leqslant\nu^{-1}C_f^2+\frac{\nu}{4}\|Av\|^2+cn^{-2}\nu^{-1}\|A^{3/4}w\|^4+cn^{-1/2}\|A^{3/4}w\|\|Av\|^2\\ &\leqslant\nu^{-1}C_f^2+\frac{\nu}{4}\|Av\|^2+cn^{-3}\nu^{-5}(C_f+\rho^2)^4+cn^{-3/4}\nu^{-1}(C_f+\rho^2)\|Av\|^2\end{aligned}$$

If we take $n_0\geqslant c\nu^{-8/3}(C_f+\rho^2)^{\frac{4}{3}}$ (more precisely, n_0 can be estimated to $n_0\geqslant c\nu^{-(2+\varepsilon)}$ for any $\varepsilon>0$ small enough), when $n\geqslant n_0$ it yields

$$\frac{d}{dt}\|A^{1/2}v\|^2+\nu\|Av\|^2\leqslant2\nu^{-1}C_f^2+c\nu^3 \qquad (\forall t\geqslant0)$$

Noticing $\nu\|Av\|^2\geqslant\lambda_1c\nu\|A^{1/2}v\|^2\triangleq\nu_1\|A^{1/2}v\|^2$, in which $\lambda_1>0$ is a constant, the above expression can turn to be a differential inequality about $\|A^{1/2}v\|^2$. Then integrate it on [0, t].

denoting $\|A^{1/2}a\|^2+2\nu^{-2}C_f^2+c\nu^2$ by M_0^2, we have

$$\|A^{1/2}v(t)\|\leqslant M_0 \quad (\forall t\geqslant 0) \tag{2.9}$$

Obviously, if we take $\rho<M_0$, Ψ is a mapping from K into K. On the other hand, because of (2.8) and (2.9), we have

$$|||A^{1/2}(v+w)|||\leqslant c\nu^3C_f(C_f+M_0^2)^{-2}+M_0+c\nu\triangleq C_0 \tag{2.10}$$

Since $\dot{V}_n$ is finite dimensional and $\dot{V}_n\subset C^\infty(\Omega)^2$. It is easy to verify that $C^1([0,\infty),\dot{V}_n)$ compactly embeds in $C([0,\infty),\ \dot{V}_n)$. Thus Ψ also is completely continuous. By Schauder's fixed point theory, we know that Ψ has a fixed point in K, that is, there is at least a solution of (2.3) in K.

Let (v_1,w_1), (v_2,w_2) are a couple of solutions of (2.3) and denote $v=v_1-v_2$, $w=w_1-w_2$. From (2.4), we know

$$v(t)=-\int_0^t\exp[-\nu A(t-s)]P_n\{B(v+w,v_1+w_1)+B(v_2+w_2,v+w)\}ds$$

By the properties (1.2) and (1.5) and using the second equation of (2.4), finally we get

$$\|v(t)\|\leqslant C_1\int_0^t(t-s)^{-3/4}\exp[-2\nu\dot{\delta}(t-s)]\|v(s)\|ds \tag{2.11}$$

where C_1 is a constant depending only on M_0, C_f, a and ν. Set $g(t)=e^{2\nu\delta t}\|v(t)\|$, we have

$$g(t)\leqslant C_1\int_0^t(t-s)^{-3/4}g(s)ds \tag{2.12}$$

Now by using Lemma 2.1, we can immediately get $g(t)=0$. Of cause, $\|v(t)\|=0$ for $t\geqslant 0$. Noticing (2.5) again, we can claim that $(v_1,w_1)=(v_2,w_2)$, that is, (2.3) has a unique solution under the sense of L^2.

III. Error Estimates

First of all, we give some priori estimates

Lemma 3.1[6] Assume $a\in D(A)$. If

i) $f=0$, there exists a constant $k_0(a)$, such that

$$\|Ax(t)\|\leqslant k_0(a)\|Aa\|\exp[-2\nu_1\delta t] \quad (t\geqslant 0) \tag{3.1}$$

ii) $f\in C([0,\ +\infty),\ \dot{H}^1(\Omega))$, there exists a constant $L(t)$ such that

$$\|Ax(t)\|\leqslant L(t) \quad (t\geqslant 0) \tag{3.2}$$

Thanks to [6] and [7], we can get following estimates immediately.

Lemma 3.2 Suppose $a\in D(A)$. If

i) $f=0$, there is a constant $k_1(a)$ such that

$$\|A^{1/2}x_t(t)\|\leqslant k_1(a)\exp[-\nu_1 t] \quad (t\geqslant 1) \tag{3.3}$$

ii) $f\in C^1([0,+\infty),\ \dot{H}^0(\Omega))$, there is a constant depending on t, which we still denote by $L(t)$, such that

$$\|A^{1/2}x_t(t)\|\leqslant L(t) \quad (t\geqslant 1) \tag{3.4}$$

In the rest, we always denote $\sigma=\min\{1/4, \delta\}$. We first consider the error estimate of homogenous equations

Theorem 3.1 Assume that $a\in D(A)$, $f=0$. Then there is some $n_0\in\mathscr{N}$ such that for any $\varepsilon\in(0, 1)$, when $n\geqslant n_0$, there exists a constant $C_\varepsilon>0$ related on ε such that

$$\|A^{1/2}(v-y)(t)\|\leqslant C_\varepsilon n^{\varepsilon-1}(n^{-2}+N^{-1})\exp[-\sigma\nu_1 t] \qquad (t\geqslant 1) \tag{3.5}$$

Proof Taking $\phi=Av$, $\chi=Aw$ in (2.3) and adding them up, by using (1.4), we have

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}v\|^2+\nu\|A(v+w)\|^2=b(v; w, Aw)+b(w; v, Aw)=-b(w; w, Av) \tag{3.6}$$

From Theorem 2.1, we know for some $n_1\in\mathscr{N}$, when $n\geqslant n_1$, we have $\|A^{1/2}(v+w)\|\leqslant C_0$. Thus by using (1.5) and Agmon's inequality, we have

$$|b(w; w, Av)|\leqslant c\|w\|\|A^{1/2}w\|\|Av\|^{1/2}\|A^2v\|^{1/2}\leqslant cC_0n^{-1}\|A(v+w)\|^2$$

If we take $n_0>\max\{n_1, c\nu^{-1}C_0^{-1}\}$, when $n\geqslant n_0$, we obtain

$$\frac{d}{dt}\|A^{1/2}v\|^2+4\nu_1\|A^{1/2}v\|^2\leqslant 0 \qquad (t\geqslant 0) \tag{3.7}$$

Therefore

$$\left.\begin{aligned}&\|A^{1/2}v\|\leqslant\|A^{1/2}a\|\exp[-2\sigma\nu_1 t] &&(t\geqslant 0)\\ &\|A^{1/2}w\|\leqslant c\|A^{1/2}a\|\exp[-4\sigma\nu_1 t] &&(t\geqslant 0)\end{aligned}\right\} \tag{3.8}$$

Now we set up to deal with the error estimates. Firstly, we give some symbols

$$e=x-u, \quad e_v=y-v, \quad e_w=z-w$$

Obviously, $e=e_v+e_w$. On the other hand, notice that e_w has following decomposition

$$e_w=P_Ne_w+Q_Ne_w=P_Ne_w+Q_Nz \tag{3.9}$$

Then we have for $\forall\chi\in\dot{V}_N^*$,

$$\nu((P_Ne_w, \chi))=-(z_t,\chi)-b(e_v; y,\chi)-b(v; e_v,\chi)-b(y; z, \chi)-b(z; x,\chi)$$

Now by using Lemmas 3.1, 3.2 and estimate (3.8), we have following estimates for above terms

$$\begin{aligned}
|(z_t,\chi)|&\leqslant\|z_t\|\|\chi\|\leqslant k_1n^{-2}\exp[-2\sigma\nu_1 t]\|A^{1/2}\chi\|\\
|b(e_v; y, \chi)|&\leqslant c\|A^{1/2}e_v\|\|A^{1/2}y\|\|A^{1/4}\chi\|\\
&\leqslant k_0\|Aa\|n^{-1/2}\exp[-2\sigma\nu_1 t]\|A^{1/2}e_v\|\|A^{1/2}\chi\|\\
|b(v; e_v,\chi)|&\leqslant c\|A^{1/2}v\|\|A^{1/2}e_v\|\|A^{1/4}\chi\|\\
&\leqslant c\|Aa\|n^{-1/2}\exp[-2\sigma\nu_1 t]\|A^{1/2}e_v\|\|A^{1/2}\chi\|\\
|b(y; z, \chi)|&\leqslant c\|Ay\|\|A^{1/2}z\|\|\chi\|\leqslant k_0^2\|Aa\|^2n^{-2}\exp[-2\sigma\nu_1 t]\|A^{1/2}\chi\|\\
|b(z; x, \chi)|&\leqslant c\|A^{1/4}z\|\|Ax\|\|\chi\|\leqslant k_0^2\|Aa\|^2n^{-5/2}\exp[-2\sigma\nu_1 t]\|A^{1/2}\chi\|
\end{aligned}$$

Thus

$$\|A^{1/2}P_Ne_w\|\leqslant M_1n^{-2}\exp[-2\sigma\nu_1 t]+M_2n^{-1/2}\exp[-2\sigma\nu_1 t]\|A^{1/2}e_v\|$$

where $M_1=c(k_1+k_0^2\|Aa\|^2)$, $M_2=(k_0^2\|Aa\|+1)\|Aa\|$. Therefore

$$\|A^{1/2}e_w\|\leqslant M_1(n^{-2}+N^{-1})\exp[-2\sigma\nu_1 t]+M_2 n^{-1/2}\exp[-2\sigma\nu_1 t]\|A^{1/2}e_v\| \quad (3.10)$$

Moreover, by using the formal solutions of y and v, we obtain

$$\begin{aligned}A^{1/2}e_v=&-\int_1^t A^{1/2}\exp[-\nu A(t-s)]P_n\{B(e_v,y)+B(v,e_v)+B(e_v,z)+B(v,e_w)\\&+B(z,e_v)+B(e_w,v)+B(e_w,z)+B(z,e_w)\}ds\\=&-\int_1^t A^{1-\epsilon/3}\exp[-\nu A(t-s)]\Big\{\sum_{i=1}^{8}I_i\Big\}ds\end{aligned}$$

Using (1.2), we can get

$$\|A^{1/2}e_v\|\leqslant c\int_1^t (t-s)^{\epsilon/3-1}\exp[-2\sigma\nu_1(t-s)]\Big\{\sum_{i=1}^{8}\|I_i\|\Big\}ds \quad (3.11)$$

In the following, we give the estimates of $\|I_i\|$

$$\begin{aligned}
\|I_1\|&=\|A^{\epsilon/3-1/2}P_nB(e_v,y)\|\leqslant\|Ay\|\|A^{1/2}e_v\|\leqslant k_0\|Aa\|\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_v\|\\
\|I_2\|&=\|A^{\epsilon/3-1/2}P_nB(v,e_v)\|\leqslant\|A^{1/2}v\|\|A^{1/2}e_v\|\leqslant\|Aa\|\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_v\|\\
\|I_3\|&=\|A^{\epsilon/3-1/2}P_nB(e_v,z)\|\leqslant\|Az\|\|A^{1/2}e_v\|\leqslant k_0\|Aa\|\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_v\|\\
\|I_4\|&=\|A^{\epsilon/3-1/2}P_nB(v,e_w)\|\leqslant n^{\epsilon}\|A^{1/2}v\|\|e_w\|\leqslant\|Aa\|n^{\epsilon-1}\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_w\|\\
\|I_5\|&=\|A^{\epsilon/3-1/2}P_nB(z,e_v)\|\leqslant\|Az\|\|A^{1/2}e_v\|\leqslant k_0\|Aa\|\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_v\|\\
\|I_6\|&=\|A^{\epsilon/3-1/2}P_nB(e_w,v)\|\leqslant n^{\epsilon}\|A^{1/2}v\|\|e_w\|\leqslant\|Aa\|n^{\epsilon-1}\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_w\|\\
\|I_7\|&=\|A^{\epsilon/3-1/2}P_nB(e_w,z)\|\leqslant n^{\epsilon}\|A^{1/2}z\|\|e_w\|\leqslant k_0\|Aa\|n^{\epsilon-1}\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_w\|\\
\|I_8\|&=\|A^{\epsilon/3-1/2}P_nB(w,e_w)\|\leqslant n^{\epsilon}\|A^{1/2}w\|\|e_w\|\leqslant\|Aa\|n^{\epsilon-1}\exp[-2\sigma\nu_1(t)]\|A^{1/2}e_w\|
\end{aligned}$$

Together with (3.10), we know that

$$\sum_{i=1}^{8}\|I_i\|\leqslant M_3\exp[-2\sigma\nu_1 s]\|A^{1/2}e_v\|+M_4\exp[-4\sigma\nu_1 s]n^{\epsilon-1}(n^{-2}+N^{-1}) \quad (3.12)$$

Here $M_3=(k_0+1)(M_2+1)\|Aa\|$, $M_4=(k_0+1)M_1\|Aa\|$. Combine (3.12) and (3.11), it holds.

$$\begin{aligned}\|A^{1/2}e_v\|\leqslant& M_3\int_1^t (t-s)^{\epsilon/3-1}\exp[-2\sigma\nu_1(t-s)]\|A^{1/2}e_v\|ds\\&+M_4e^{-2\sigma\nu t}n^{\epsilon-1}(n^{-2}+N^{-1})\int_1^t (t-s)^{\epsilon/3-1}\exp[-2\sigma\nu_1 s]ds \quad (3.13)\end{aligned}$$

Set $g(t)=\exp[2\sigma\nu_1 t]\|A^{1/2}e_v\|$, then we can rewrite (3.13) as

$$g(t)\leqslant M_3\int_0^t (t-s)^{\epsilon/3-1}g(s)ds+M_4n^{\epsilon-1}(n^{-2}+N^{-1})\int_0^t (t-s)^{\epsilon/3-1}\exp[-2\sigma\nu_1 s]ds \quad (3.14)$$

It is easy to verify $\int_0^t (t-s)^{\epsilon/3-1}\exp[-2\sigma\nu_1 s]\,ds\leqslant\frac{3}{\varepsilon}+\frac{1}{2\sigma\nu_1}\triangleq\frac{\tilde{C}_\epsilon}{M_4}$. Due to Lemma 2.1, we get

$$\|A^{1/2}e_v\|\leqslant\tilde{C}_\epsilon n^{\epsilon-1}(n^{-2}+N^{-1})\exp\{M_3^{3/\epsilon}-2\sigma\nu_1\}t \qquad (t\geqslant1) \quad (3.15)$$

Here we obtained the above inequality for $t\geqslant1$. If we carry out the above procedure for

$t \geqslant t_0 > 1$, thanks to the definition of M_3 and Lemma 3.1, we know that there must exist some $t_0 > 1$ such that $M_3^{3/\varepsilon} \leqslant \sigma\nu_1$. So we can claim that there exists some constant $\bar{C}_\varepsilon > 0$ depending only on a, f, ν, σ and ε such that

$$\|A^{1/2}e_v\| \leqslant \tilde{C}_\varepsilon \bar{C}_\varepsilon n^{\varepsilon-1}(n^{-2}+N^{-1})\exp\{-\sigma\nu_1 t\}, \qquad (t \geqslant 1)$$

If we denote $C_\varepsilon = \tilde{C}_\varepsilon \bar{C}_\varepsilon$, then we can get (3.5).

Corollary 3.1 Under the same conditions of Theorem 3.1, we have

$$\|A^{1/2}(z-w)\| \leqslant c(n^{-2}+N^{-1})\exp[-\sigma\nu_1 t]$$

For inhomogeneous case, we can get following estimate by using the same method.

Theorem 3.2 Suppose $a \in D(A)$ and $f \in C([0,\infty), \dot{H}^1(\Omega)) \cap C^1([0,\infty), \dot{H}^0(\Omega))$. For any $\varepsilon \in (0,1)$, there is some constant independent of n and N, which we still denote by $L(t)$, such that

$$\left.\begin{aligned} &\|A^{1/2}(y-v)\| \leqslant L(t)n^{\varepsilon-1}(n^{-2}+N^{-1}) \qquad (t \geqslant 1) \\ &\|A^{1/2}(z-w)\| \leqslant L(t)(n^{-2}+N^{-1}) \qquad (t \geqslant 1) \end{aligned}\right\} \tag{3.16}$$

Remark 3.1 From Theorems 3.1, 3.2 and Corollary 3.1, we find that the accuracy of low frequency part is better than that of high ones. In the other words, the final accuracy is determined by that of high frequency part. How to improve the accuracy of the high frequency part, is the key problem of the construction of effective algorithms.

Remark 3.2 See E. Weinan[6], the author got following estimates for Fourier-Galerkin method

$$\left.\begin{aligned} &\|A^{1/2}(x-u_g)(t)\| \leqslant CN^{-1}\exp\{-\sigma t\} \quad (t \geqslant 0 \text{ homogeneous case}) \\ &\|A^{1/2}(x-u_g)(t)\| \leqslant C(t)N^{-1} \quad (t \geqslant 0 \text{ inhomogeneous case}) \end{aligned}\right\} \tag{3.17}$$

where u_g is the Fourier-Galerkin approximation of (1.6) in $\dot{V}_N$. (3.17) indicates that the approximate order is $O(N^{-1})$. And we need to solve a nonlinear evolutionary equation in the frequency area $1-N$. But from Theroems 3.1, 3.2 and Corollary 3.1, we know that, to obtain the same error order, we just need to take $n = O(N^{1/2})$ for Fourier NGM. At the same time, noticing the form of (2.3), in fact, we only need to solve a stationary Stokes problem in large frequency area $(n+1)-N$. Indeed, from Remark 2.1 (especially for autonomous systems), we only need to solve it in $(n+1)-2n$. We only solve a nonlinear evolutionary equation in the relative small frequency range $1-n$. It is well-known that the computational complexity for numerically simulating nonlinear evolutionary equations, mainly comes from time integration and nonlinearity. As indicated above, the Fourier NGM is helpful to decrease computational complexity. In other words, under the same computational scale, Fourier NGM can get much higher precision. Thus it makes the direct simulation of turbulence possible under current computing ability. On the other hand, this method can get much better approximation for low frequency part (see (3.5) and the first expression of (3.16)).

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