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A SIMPLE POSTPROCESS PROCEDURE FOR GALERKIN METHOD *

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Abstract: A kind simple postprocess procedure for classical Galerkin method for steady Navier-Stokes equations with stream function form was presented in this paper. The main ideal was to construct an approximate interactive rule between lower frequency components and higher frequency components by using the conception of Approximate Inertial Manifold (AIM) and a kind of new decomposition of the true solution. It is demonstrated in this paper that this kind of postprocess Galerkin method could derive a higher accuracy solution with lower computing efforts.

Key words: Navier-Stokes equations; approximate inertial manifold; non-singular solution
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Introduction

The concepts of Inertial Manifold (IM)[1] and Approximate Inertial Manifold (AIM)[2] for dissipative partial differential equations have attracted attention since 1988. These new concepts consider that there exist interactive rule or at least some approximate interactive rules between the higher frequency components and the lower frequency components of the solution. That is to say that we can compute or approximate compute the higher frequency components of the solution by using of the lower frequency components which is a finite dimensional vector in Sobolev space. Based upon this knowledge, a kind of new numerical scheme called nonlinear Galerkin method for dissipative PDEs is developed and studied by many authors (cf. [3～6] etc). One of the main advantages is that it presents higher convergence rate than classical Galerkin method. It is a numerical scheme for nonstationary dissipative PDEs. In present paper, we introduce a simple postprocess to the classical Galerkin method for steady Navier-Stokes with stream function form which is based on the virtue of AIM and can greatly improve the convergence rate by cheaper efforts. Our main ideal is to decompose the true solution according to its classical Galerkin approximation. Suppose $\boldsymbol{\psi}$ is the true solution and $\boldsymbol{\psi}_m$ is its classical Galerkin approximation, then we decompose the true solution as

$$\boldsymbol{\psi} = \psi + \boldsymbol{\psi}_m.$$

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Our main task is to get a reasonable approximation of ψ, denoted by Φ, such that we can use $\Phi + \psi_m$ to approximate ψ.

In fact, if the classical Galerkin solution ψ_m has the following convergence rates

$$\|\psi - \psi_m\|_1 \le \beta_1, \quad \|\psi - \psi_m\|_2 \le \beta_2,$$

we show that our corrected solution $\Phi + \psi_m$ has the following property

$$\|\Phi + \psi_m - \psi\|_2 \le c\beta_1^{1-\varepsilon}\beta_2^{1+\varepsilon} \qquad (\forall \varepsilon > 0).$$

In present paper, we will only consider the nonsingular case. For singular case, we will treat it later.

1 Preliminaries

Let us consider two-dimensional steady Navier-Stokes equations confined in a flat tori $\Omega = [-a, a] \times [-b, b]$ with periodic boundary conditions

$$\left.\begin{array}{ll} -\nu\Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \text{Periodic Boundary Conditions} & \text{on } \partial\Omega, \end{array}\right\} \tag{1}$$

where $u : \Omega \to R^2$ is the flow field, $p : \Omega \to R$ is the pressure, f represents the exterior force which drives the flow, $\nu > 0$ is the kinetic viscosity, Δ and ∇ denote the two-dimensional Laplace and gradient operator respectively.

To describe the solution both in detail and more easily, we introduce the following stream function and a nondimensional form of the equations. We first define the stream function ψ as

$$u_1 = \psi_y, \quad u_2 = -\psi_x,$$

where u_1 and u_2 represent the x component and y component of the velocity u respectively. Then the Navier-Stokes equations with above stream function form read

$$\nu\Delta^2\psi + J(\psi, \Delta\psi) = -\operatorname{rot} f,$$

here J is a bilinear form defined as

$$J(g, h) = \frac{\partial g}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial h}{\partial x}.$$

Obviously

$$J(g, h) = -J(h, g).$$

Now we introduce the following nondimensional transform:

$$(x, y) \mapsto \frac{b}{\pi}(x', y'), \quad \psi \mapsto \frac{\gamma b^4}{\nu\pi^4}\psi',$$

where $\gamma = \frac{\pi}{b}|\nabla f|_{-1}$ which stands for the strength of the exterior force. Then, after dropping the primes, we obtain

$$Gr^{-1}\Delta^2\psi + J(\psi, \Delta\psi) = Gr^{-1}F, \tag{2}$$

where Gr is the Grashoff number defined as

$$Gr = \frac{\gamma b^4}{\nu^2\pi^4},$$

and $F = -\operatorname{rot} f \,/\, |\nabla f|_{-1}$.

Now, equation (2) is defined in following rectangle:

$$\Omega = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right] \times [-\pi, \pi], \ \alpha = \frac{b}{a}.$$

Of course, there are many other ways to get a nondimensional form. Our nondimensionalization is charactered by an exterior force of order $O(Gr^{-1})$.

If we denote by V_2 the following periodical sobolev space

$$V_2 = \left\{ \psi = \sum_{k \in Z^2, k \neq 0} c_k e^{i(k_1 \alpha x + k_2 y)}, c_k = \overline{c_{-k}}, \sum_{k \in Z^2, k \neq 0} | c_k |^2 (k_1^4 \alpha^4 + k_2^4) < +\infty \right\},$$

and $(\cdot, \cdot)$ the L^2 inner product, $\langle \cdot, \cdot \rangle$ the dual pair and $\| \cdot \|_l$ the sobolev norm in $H^l(\Omega)$, especially, $\| \cdot \|$ the L^2 norm, we can get the weak form of (2) as

$$\left.\begin{array}{l} \text{find } \psi \in V_2, \text{such that} \\ a(\psi, v) + j(\psi, \Delta\psi, v) = Gr^{-1}\langle F, v \rangle \quad (\forall v \in V_2), \end{array}\right\} \tag{3}$$

where

$$a(\psi, v) = Gr^{-1}(\Delta\psi, \Delta v),$$

$$j(\psi, \Delta\psi, v) = (J(v, \psi), \Delta\psi) = (\psi_y v_x, \Delta\psi) - (\psi_x v_y, \Delta\psi).$$

From the definition of the trilinear form j, we can easily derive following properties of j

$$\left.\begin{array}{ll} j(\psi, \Delta\psi, v) = -j(v, \Delta\psi, \psi), & \text{particular } j(v, \Delta\psi, v) = 0, \\ j(\psi, \Delta\psi, v) = -j(\psi, v, \Delta\psi), & \text{particular } j(\psi, \Delta v, \Delta v) = 0. \end{array}\right\} \tag{4}$$

and its estimation (see [7])

$$j(u, \Delta u, w) \leq c_1 \| u \|_{s_1+1} \| v \|_{s_2+2} \| w \|_{s_3+1}, \tag{5}$$

where $s_1 + s_2 + s_3 \geq 1$, $(s_1, s_2, s_3) \neq (0,0,1), (0,1,0), (1,0,0)$, $c_1 > 0$ is a constant depending on Ω.

Let us denote by P_m the L^2-orthogonal projection in V_2 and $V_{2,m}$ the projection of V_2 under P_m, that is

$$V_{2,m} = \left\{ \psi = \sum_{|k| < m, k \neq 0} c_k e^{i(k_1 \alpha x + k_2 y)}, c_k = \overline{c_{-k}} \right\}.$$

Then the Galerkin approximate solutions (3) satisfies

$$\left.\begin{array}{l} \text{find } \psi_m \in V_{2,m} \text{ such that} \\ a(\psi_m, v) + j(\psi_m, \Delta\psi_m, v) = Gr^{-1}\langle F, v \rangle \quad (\forall v \in V_{2,m}). \end{array}\right\} \tag{6}$$

And we further suppose that the errors between ψ_m and ψ are

$$\| \eta \|_1 \leq \beta_1(Gr, m), \ \| \eta \|_2 \leq \beta_2(Gr, m), \text{ where } \eta = \psi - \psi_m. \tag{7}$$

2 Decomposition of the True Solution

From the view point of L^2-projection, (7) are the optimal error estimates. One can not improve the error estimates of $\| \psi - \psi_m \|_l, l = 1,2$ further more because of the restriction of approximation theory, that is, they have the same order as the truncated error $\|(I - P_m)\psi\|_l$, $l = 1, 2$. Of course, to improve the estimates of (7), one may approximate $(I - P_m)\psi$ by using some kind of postprocess techniques. Unfortunately, once we get a suitable approximation of $(I - P_m)\psi$, the distance between $P_m\psi$ and ψ_m will become the dominant factor of the total error and will restrict the convergence rate of the postprocess solution. If we consider this problem under other projection which maybe different from L^2-projection, the result may also be quite different. Intuitively, we decompose the solution of (3) as

$$\psi = \mathrm{\psi} + \psi_m, \tag{8}$$

where ψ_m is the classical Galerkin approximation of ψ. We identify ψ_m the lower frequency component of ψ and $\mathrm{\psi}$ the higher frequency component. Of course, according to L^2-projection, $\mathrm{\psi}$ has higher frequency components as well as lower frequency components. To shed light on the decomposition, we use the following formal "orthogonal projection": $Q_m : V_2 \to V_{2,m}$

$$\left.\begin{aligned}
&\forall \Phi \in V_2,\ \text{find } Q_m\Phi \in V_{2,m} \text{ such that}\\
&a(\Phi - Q_m\Phi, v) + j(\Phi - Q_m\Phi, \Delta\psi_m, v) + j(\psi_m, \Delta(\Phi - Q_m\Phi), v) +\\
&\quad j(\Phi - Q_m\Phi, \Delta(\Phi - Q_m\Phi), v) = 0 \qquad (\forall v \in V_{2,m}).
\end{aligned}\right\} \tag{9}$$

To this extend, by taking $\Phi = \psi$, we have

$$\psi_m - Q_m\psi = 0,\ \mathrm{\psi} = (I - Q_m)\psi.$$

Obviously, the error between true solution and the classical Galerkin solution only comes from the "truncated" error and $\mathrm{\psi}$ is "orthogonal" with $V_{2,m}$, that is

$$a(\mathrm{\psi}, v) + j(\mathrm{\psi}, \Delta\psi_m, v) + j(\psi_m, \Delta\mathrm{\psi}, v) + j(\mathrm{\psi}, \Delta\mathrm{\psi}, v) = 0 \ (\forall v \in V_{2,m}). \tag{10}$$

As for whether (9) can define a projection Q_m from V_2 onto $V_{2,m}$ or not, it is meaningless here. The only useful result is (10).

Now we can rewrite equation (3) according to (8) as

$$\begin{aligned}
&a(\mathrm{\psi}, v) + j(\mathrm{\psi}, \Delta\psi_m, v) + j(\psi_m, \Delta\mathrm{\psi}, v) + j(\mathrm{\psi}, \Delta\mathrm{\psi}, v) +\\
&\quad a(\psi_m, v) + j(\psi_m, \Delta\psi_m, v) = Gr^{-1}\langle F, v\rangle \qquad (\forall v \in V_2).
\end{aligned} \tag{11}$$

3 Postprocess Procedure

As is well-known (e. g. [1], [2]), the theory of Approximate Inertial Manifold (AIM) asserts that there at least exists an approximate interactive rule between higher frequency components and lower frequency components of the solution. And this kind of rule can determine so called AIM which attracts all orbits into a thin neighborhood of it. In our steady case, the equilibrium point, that is the solution of (3), will be inside this thin neighborhood. In other words, the higher frequency components can be approximately expressed by the lower frequency components.

To construct this kind of interactive rule, we need to form a mapping from lower frequency components to higher frequency components, that is a mapping $\boldsymbol{\Phi}$ from $V_{2,m}$ into V_2 in our context. For this purpose, we recall the equaiton (3). It defines a mapping $\mathbf{F}$ from V_2 to V_2^* by

$$\left.\begin{aligned}&\forall\psi \in V_2\text{, find } \mathbf{F}\,(\psi) \in V_2^*\text{ such that}\\ &\langle \mathbf{F}\,(\psi)\,, v\rangle = a(\psi, v) + j(\psi,\Delta\psi, v) - Gr^{-1}\langle F, v\rangle \qquad (\forall v \in V_2)\,.\end{aligned}\right\} \qquad (12)$$

Then (3) is equivalent to

$$\mathbf{F}\,(\psi) = 0. \qquad (13)$$

Similarly, the Galerkin equation (6) can define a mapping $\mathbf{F}_m$ from $V_{2,m}$ into $V_{2,m}^*$ like (12), and (6) can be equivalent to

$$\mathbf{F}_m(\psi_m) = 0. \qquad (14)$$

The Frechet derivatives of $\mathbf{F}$ and $\mathbf{F}_m$ at ψ and ψ_m are denoted by $D\mathbf{F}\,(\psi)$ and $D\mathbf{F}_m(\psi_m)$ separately which are mappings from V_2 to V_2^* and $V_{2,m}$ to V_2^* respectively

$$\left.\begin{aligned}&\langle D\mathbf{F}\,(\psi)\,w\,, v\rangle = a(w\,, v) + j(\psi,\Delta w\,, v) + j(w\,,\Delta\psi, v) \;\; (\forall w\,, v \in V_2)\,,\\ &\langle D\mathbf{F}_m(\psi_m)\,w\,, v)\rangle = a(w\,, v) + j(\psi_m\,,\Delta w\,, v) + j(w\,,\Delta\psi_m\,, v)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\forall w\,, v \in V_{2,m})\,,\end{aligned}\right\} \qquad (15)$$

If we set

$$\begin{aligned}\mathbf{L}\,(w\,, v) &= \langle D\mathbf{F}\,(\psi)\,w\,, v\rangle,\\ \mathbf{L}_m(w\,, v) &= \langle D\mathbf{F}_m(\psi_m)\,w\,, v\rangle,\end{aligned}$$

we have

$$\mathbf{L}_m(w\,, v) = \mathbf{L}\,(w\,, v) - j(\psi,\Delta w\,, v) - j(w\,,\Delta\psi, v)\,. \qquad (16)$$

If ψ is a nonsingular solution of (13), then $D\mathbf{F}\,(\psi)$ is an isomorphism from V_2 onto V_2^*, therefore there exists a constant $\alpha_0 > 0$ such that

$$\left.\begin{aligned}&\inf_{w\in V_2}\sup_{v\in V_2}\frac{\mathbf{L}\,(w\,, v)}{\|w\|_2\,\|v\|_2} \geq \alpha_0\,,\\ &\inf_{v\in V_2}\sup_{w\in V_2}\frac{\mathbf{L}\,(w\,, v)}{\|w\|_2\,\|v\|_2} \geq \alpha_0.\end{aligned}\right\} \qquad (17)$$

Next lemma describes the conditions which ensure one point $\tilde{\psi}$ close to nonsingular solution ψ is also a nonsingular solution, for the proof, readers can see [8].

Lemma **3.1** Assume $\tilde{V}_2 \subset V_2$ is a finite dimensional subspace and $\tilde{\mathbf{F}}$ is a smooth mapping from $\tilde{V}_2$ into V_2^*. Let Φ be a nonsingular point of $\mathbf{F}\,(\Phi)$ and denote

$$\sigma(\Phi) = \|D\mathbf{F}\,(\Phi)^{-1}\|_{\mathbf{L}(V_2^*, V_2)}\,,$$

$$\mu(\tilde{\Phi}) = \|D\mathbf{F}\,(\Phi) - D\mathbf{F}\,(\Phi)\|_{\mathbf{L}(V_2, V_2^*)}\,.$$

If $\tilde{\Phi}$ is so close to Φ such that

$$\sigma(\Phi)\,\mu(\tilde{\Phi}) < 1\,, \qquad (18)$$

$D\tilde{\mathbf{F}}\,(\tilde{\Phi})$ is an isomorphism from $\tilde{V}_2$ onto $\tilde{V}_2^*$. Hence $\tilde{\Phi}$ is a nonsingular point of $\tilde{\mathbf{F}}$.

Thanks to the above lemma, we can easily get the following corollary.

Corollary **3.1** Assume that ψ is a nonsingular solution of (3). If m is chosen large enough such that

$$\beta_2(Gr, m) < \frac{\alpha_0}{2c_1}, \tag{19}$$

the classical Galerkin approximate solution ψ_m is also a nonsingular point of $\mathbf{F}_m$.

Proof (17) shows that

$$\sigma(\psi) = \| D\mathbf{F}^{-1}(\psi) \|_{\mathbf{L}(V_2^*, V_2)} \leq \alpha_0^{-1}. \tag{20}$$

On the other hand, if we substitute $\widetilde{\mathbf{F}}$ and $\widetilde{\Phi}$ in Lemma 3.1 with $\mathbf{F}_m$ and Φ_m, and $\widetilde{V}_2$ with $V_{2,m}$, due to (16) and (5), we have

$$\begin{aligned}\mu(\psi_m) &= \| D\mathbf{F}(\psi) - D\mathbf{F}_m(\psi_m) \|_{\mathbf{L}(V_2, V_2^*)} = \\ &\sup_{w, v \in V_2} \frac{\langle (D\mathbf{F}(\psi) - D\mathbf{F}_m(\psi_m))w, v\rangle}{\|w\|_2 \|v\|_2} = \\ &\sup_{w, v \in V_2} \frac{j(\psi, \Delta w, v) + j(w, \Delta\psi, v)}{\|w\|_2 \|v\|_2} \leq \\ &2c_1 \|\psi\|_2 \leq 2c_1 \beta_2(Gr, m).\end{aligned}$$

Now we can obtain the corollary by using (18). □

From (15), we know

$$\begin{aligned}\mathbf{L}_m(w, v) &\geq \mathbf{L}(w, v) - |j(\psi, \Delta w, v)| - |j(w, \Delta\psi, v)| \geq \\ &\mathbf{L}(w, v) - 2c_1\beta_2(Gr, m) \|w\|_2 \|v\|_2.\end{aligned}$$

Assume m is large enough such that (19) holds, then we can easily derive

$$\left.\begin{aligned}\inf_{w \in V_2} \sup_{v \in V_2} \frac{\mathbf{L}_m(w, v)}{\|w\|_2 \|v\|_2} \geq \frac{\alpha_0}{2}, \\ \inf_{v \in V_2} \sup_{w \in V_2} \frac{\mathbf{L}_m(w, v)}{\|w\|_2 \|v\|_2} \geq \frac{\alpha_0}{2}.\end{aligned}\right\} \tag{21}$$

Now, we can define a mapping from $V_{2,m}$ into V_2 as

$$\left.\begin{aligned}&\forall w \in V_{2,m}, \text{find } \Phi(w) \in V_2 \text{ such that} \\ &\mathbf{L}_m(\Phi(w), v) + \langle \mathbf{F}(w), v\rangle = 0 \qquad (\forall v \in V_2).\end{aligned}\right\} \tag{22}$$

Obviously, substituting ψ_m with w and omitting higher order item $j(\psi, \Delta\psi, v)$ in (11), one can obtain (22) directly. In other words, (22) is the linear part of (11). (22) provides us a method to make some correction to the classical Galerkin approximation ψ_m.

Then, we will show that a simple postprocess procedure of ψ_m based on the approximate interactive rule (22) can greatly improve the convergence rate of classical Galerkin approximate solution. From the view ponit of (8), the classical Galerkin solution ψ_m is exactly the lower frequency part of the true solution ψ respecting to decomposition (8) and the error between ψ and ψ_m only comes from the “truncate ”error ψ. If a reasonable correction of ψ, denoted by Φ, can be derived from ψ_m, $\Phi + \psi_m$ may be a much better approximation of ψ than ψ_m. This is the fundamental of our postprocess procedure.

Noticing (22), for any $w \in V_{2,m}$, we can get $\Phi(w) \in V_2$. We will use $\phi = \Phi(\psi_m)$ to complete our correction. Now we can write the postprocess scheme as

$$\left.\begin{aligned} &\text{find } \psi_m \in V_{2,m},\ \phi \in V_2 \text{ such that} \\ &a(\psi_m, v) + j(\psi_m, \Delta\psi_m, v) = Gr^{-1}\langle F, v\rangle \qquad (\forall v \in V_{2,m}), \\ &\phi = \Phi(\psi_m). \end{aligned}\right\} \tag{23}$$

Because (22) is a linear system of Φ, we only need to solve a finite dimensional nonlinear equation in (23). The following theorem shows that $\phi + \psi_m$ will approximate ψ much better than ψ_m.

Theorem 3.1 Suppose that ψ and ψ_m are the solutions of (3) and (6) respectively. Furthermore, ψ is a nonsingular solution of (3). If we choose m large enough such that (19) is valid, we have

$$\| \phi + \psi_m - \psi \|_2 \le \frac{2c_1 k(\Omega)}{\alpha_0} \beta_1^{1-\varepsilon}(Gr, m)\, \beta_2^{1+\varepsilon}(Gr, m) \qquad (\forall \varepsilon > 0), \tag{24}$$

where $k(\Omega) > 0$ is a constant depending on Ω.

Proof Taking $w = \psi_m$ in (22) and subtracting (22) from (11), it yields

$$\mathbf{L}_m(\psi - \Phi(\psi_m), v) + j(\psi, \Delta\psi, v) = 0 \qquad (\forall v \in V_2).$$

By using (21) and take $(s_1, s_2, s_3) = (\varepsilon, 0, 1)$ in (5) for any $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have

$$\frac{\alpha_0}{2}\|\psi - \Phi(\psi_m)\|_2 \le \sup_{v \in V_2} \frac{\mathbf{L}_m(\psi - \Phi(\psi_m), v)}{\|v\|_2} =$$

$$\sup_{v \in V_2} \frac{-j(\psi, \Delta\psi, v)}{\|v\|_2} \le c_1 \|\psi\|_{1+\varepsilon}\beta_2(Gr, m).$$

Noticing the following interpolation inequality in Sobolev space[9]:

$$\|w\|_\varepsilon \le k(\Omega)\|w\|_1^\varepsilon \|w\|^{1-\varepsilon} \qquad \left(\forall \varepsilon \in \left(0, \frac{1}{2}\right), w \in H^1(\Omega)\right),$$

where $k(\Omega) > 0$ is a constant only depending on Ω. Thus

$$\|\psi\|_{1+\varepsilon} \le k(\Omega)\|\psi\|_1^{1-\varepsilon}\|\psi\|_2^{\varepsilon}. \tag{25}$$

Now we can derive the result from (25). $\square$

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