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A SMALL EDDY CORRECTION METHOD FOR NONLINEAR DISSIPATIVE EVOLUTIONARY EQUATIONS*

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Abstract. Considering the interaction between the large and small eddy components of solutions and using the idea of the Newton iteration, a small eddy correction method is proposed for approximating and numerically solving nonlinear dissipative PDEs of parabolic type, in particular the Navier–Stokes equations (NSE). We assume that the large eddy approximation to the solution is known. Formally applying the Newton iterative procedure to the small eddy equation, we then generate approximate systems. It is shown that the first two steps in fact lead to the standard Galerkin method (SGM) and the so-called optimum nonlinear Galerkin method (ONG), and therefore the small eddy correction method is actually a certain generalization of SGM and ONG. The boundedness and convergence analysis are presented in the framework of the two-dimensional NSE. The results show that the small eddy correction method can greatly improve the accuracy of SGM approximate solutions.

Key words. dissipative equations, spectral methods, Newton iteration, multilevel method

AMS subject classifications. 65N30, 76D05

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1. Introduction.

Considering an evolutionary dissipative nonlinear PDE system of parabolic type, for example, the Navier–Stokes equation (NSE), despite the considerable increase in the available computing power during the past few years, numerically solving this kind of system, especially the integration of evolutionary NSE on large time intervals under physically realistic situations, still remains a difficult problem whose solution is not close at hand. We thereby intend to solve dissipative evolution PDEs in dynamically nontrivial situations, i.e., when the long-term behavior is not merely convergent to a steady state. In this case, the solution to be simulated remains time dependent, and as time goes to infinity, it converges to a set, an attractor, which can be a complicated set (a fractal). Studying the complicated structure of this set, which to some extent is reflected by the long-term behavior of the solution, is of great importance to understanding the nature of turbulent phenomena. That is one of the main reasons why people are so interested in the long-term behavior of the solution and the construction of more accurate and effective numerical schemes.

We present this work in the context of the functional form of the two-dimensional NSE in divergence-free Hilbert space H defined on an open bounded domain $\Omega \subset \mathcal{R}^2$ with smooth boundary $\partial\Omega$,

$$\begin{cases} \dfrac{du}{dt} + \nu A u + B(u,u) = f, \\ u(0) = a. \end{cases} \tag{1.1}$$

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Here u stands for the velocity field, f is the external force which drives the flow, $\nu > 0$ is the viscosity, and a is the initial velocity field. A and $B(\cdot, \cdot)$ are the Stokes operator and bilinear operator whose detailed definitions will be given in section 2.

For a given positive integer $m \in \mathcal{N}$, let us denote by H_m and P_m a finite-dimensional subspace of H and an associated orthogonal projection from H onto H_m, that is, $H_m = P_m H$. Then the SGM approximate system of (1.1) reads

$$\begin{cases} \dfrac{du_m}{dt} + \nu A u_m + P_m B(u_m, u_m) = P_m f, \\ u_m(0) = P_m a. \end{cases} \tag{1.2}$$

It follows from [6] that, for sufficiently large $t_0 > 0$,

$$|u(t) - u_m(t)|_{\mathcal{L}^2(\Omega)} \le c(t) L_m \lambda_{m+1}^{-1} \quad \forall t \ge t_0, \tag{1.3}$$

where $\lambda_{m+1} > 0$ is the $(m+1)$th eigenvalue of the Stokes operator A, which tends to $+\infty$ when m tends to infinity and

$$L_m \sim \left(1 + \log \frac{\lambda_m}{\lambda_1}\right)^{\frac{1}{2}}. \tag{1.4}$$

On the other hand, if we denote $Q_m = I - P_m$ and

$$u = p + q \quad \text{with} \quad p = P_m u, \ q = Q_m u,$$

we can rewrite (1.1) in the following coupled system:

$$\frac{dp}{dt} + \nu A p + P_m B(p + q, p + q) = P_m f, \tag{1.5}$$

$$\frac{dq}{dt} + \nu A q + Q_m B(p + q, p + q) = Q_m f. \tag{1.6}$$

Set

$$e = u - u_m, \quad e_p = P_m e = p - u_m, \quad e_q = Q_m e = q.$$

Simple calculation shows that

$$|e_p(t)|_{\mathcal{L}^2(\Omega)} \le c e^{ct} \sup_{0 \le s \le t} |e_q(s)|_{\mathcal{L}^2(\Omega)}. \tag{1.7}$$

That is to say that the large eddy error and thus the total error of the standard Galerkin method (SGM) can be controlled by the small eddy error. In other words, to improve the accuracy of the SGM approximation, we have only to improve the approximation of the small eddy components, which is approximated by 0 in the case of SGM. This is the basic idea of this work.

Indeed, many authors have already applied this idea to developing new methods and techniques. One of these new methods arises in connection with the concept of the approximate inertial manifold (AIM) (see [7]) and is called the nonlinear Galerkin method (NGM). For example, Marion and Temam proposed the first and the most frequently discussed NGM in [15]:

$$\begin{cases} \dfrac{dp_m}{dt} + \nu A p_m + P_m B(p_m + q_m, p_m + q_m) = P_m f, \\ q_m = \Phi(p_m), \end{cases} \tag{1.8}$$

where $\Phi : H_m \to Q_m H$ provides an approach to approximating the small eddy components, which are determined by solving the following steady Stokes problem in $Q_m H$:

$$\nu A q_m + Q_m B(p_m, p_m) = Q_m f.$$

It is shown that

$$|u(t) - (p_m(t) + \Phi(p_m(t)))|_{\mathcal{L}^2(\Omega)} \leq c(t) L_m \lambda_{m+1}^{-\frac{3}{2}} \quad \forall t \geq t_0. \tag{1.9}$$

That is, $p_m + \Phi(p_m)$ is a much better approximation to u than the SGM approximation u_m. From that point on, many authors investigated this version of the NGM (see [1], [5], [6], [14], [16], [17], [18], etc.) and its applications (see [3], [19], etc.). To improve the effectiveness of numerical schemes, Garcia-Archilla, Novo, and Titi proposed a kind of postprocessing procedure to the Galerkin method (PPGM) in [8] based on the same mapping Φ in (1.8). For any prescribed time T, they use $\Phi(u_m(T))$ to approximate $q(T)$ and show that

$$|u(T) - (u_m(T) + \Phi(u_m(T)))|_{\mathcal{L}^2(\Omega)} \leq c(T) L_m^4 \lambda_{m+1}^{-\frac{3}{2}} \quad \forall T \geq t_0. \tag{1.10}$$

Equations (1.9) and (1.10) indicate that both the NGM (1.8) and PPGM can greatly improve the accuracy of the SGM approximation and be more effective than the SGM. But there are some problems which have already been discussed in [12]. For example, they used a steady Stokes problem to approximate the small eddy equation (1.6) and overlooked the self evolution of the small eddy components. This is only valid when the small eddy part of the solution varies very slowly.

To overcome this defect of NGM (1.8), some people have already given certain modifications (see [9], [10], [11], [13], [20]). One of these modifications is the so-called optimal nonlinear Galerkin method (ONG) (see [10], [11]):

$$\frac{d\bar{p}_m}{dt} + \nu A \bar{p}_m + P_m B(\bar{p}_m + \bar{q}_m, \bar{p}_m + \bar{q}_m) = P_m f, \tag{1.11}$$

$$\frac{d\bar{q}_m}{dt} + \nu A \bar{q}_m + Q_m B(\bar{p}_m, \bar{q}_m) + Q_m B(\bar{q}_m, \bar{p}_m) = Q_m[f - B(\bar{p}_m, \bar{p}_m)], \tag{1.12}$$

where the small eddy equation in (1.8) is replaced by a generalized unsteady Stokes problem (1.12), and therefore the self evolution of the small eddy components is involved. Furthermore, it is shown in this paper that

$$|u(t) - (\bar{p}_m + \bar{q}_m)|_{\mathcal{L}^2(\Omega)} \leq c(t) L_m^4 \lambda_{m+1}^{-2} \quad \forall t \geq t_0. \tag{1.13}$$

To get a reliable long-term simulation of the NSE, we still have to consider the problem of error accumulation. Unfortunately, almost all numerical methods for the NSE in general (including the SGM, NGM, PPGM, and ONG) will lead to an exponentially increasing error, which makes the long-term simulation almost meaningless. A possible remedy is to improve the accuracy of the approximate solution without too much computational cost. Meanwhile, we have to mention that the rapid increase of available computing power in the past few years as well as the rapid development of computer networks and parallel computing techniques, which can integrate many CPUs in a system, make the large scale computation possible. If computing facilities are not a problem, we want to know whether it is possible to get a reliable simulation in a large time interval.

From the point of view of our basic idea mentioned above (the small eddy correction), the NGM, PPGM, and ONG schemes all do a small eddy correction once, and their analysis shows that they can improve the convergence rate of the SGM without too much computational cost. It seems that one cannot improve the convergence rate any more if the correction is performed only once. Then it is very natural for us to think about certain iteration methods for improving the convergence rate further. And it is also very natural for us to think about the Newton iteration because of its fast convergence speed. Based upon these considerations, we propose a small eddy correction method in this paper (see the later definition (3.1)–(3.3)). Moreover, the analysis shows that

$$|u(t) - u^l(t)|_{\mathcal{L}^2(\Omega)} \leq c(t) L_m^2 (L_m \lambda_{m+1}^{-1})^{2^l}, \tag{1.14}$$

where u^l is the small eddy correction approximation and $l \in \mathcal{N}$ is the number of iteration steps used.

This paper is arranged as follows. In section 2, the detail of a two-dimensional NSE is given. In section 3, we introduce a small eddy correction method by formally applying the Newton iteration to the small eddy equation (1.6). Then we prove that this procedure is a bounded procedure and can generate a bounded approximate solution of (1.1). In section 4, we present some error estimates of the proposed small eddy correction method and show that it can lead to a very accurate approximation of the solution of the NSE. Finally in section 5, for a one-dimensional Burger's equation and a two-dimensional NSE, we give a full discrete form of the small eddy correction scheme and then present some numerical results to check the high accuracy and high effectiveness of this method.

2. Navier–Stokes equations. We consider the NSE on a smooth bounded domain $\Omega \in \mathcal{R}^2$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = F, & (x,t) \in \Omega \times (0, +\infty), \\ \nabla \cdot u = 0, & (x,t) \in \Omega \times [0, +\infty), \\ u|_{t=0} = a(x), & x \in \Omega, \\ \text{Dirichlet or periodic boundary conditions.} & \end{cases} \tag{2.1}$$

Here $u(x,t)$ and $p(x,t)$ stand for the velocity field and pressure, respectively; $F(x,t)$ the external force, which drives the flow; $a(x)$ the initial velocity, which satisfies $\nabla \cdot a = 0$; and $\nu > 0$ the kinetic viscosity.

Now, we define the Hilbert space H:

$$H = \{v \in (L^2(\Omega))^2 : \nabla \cdot v = 0,\ v \cdot n|_{\partial\Omega} = 0\}$$

in the case of the homogeneous Dirichlet boundary condition, where n denotes the unit outward normal vector to $\partial\Omega$, or

$$H = \left\{ v \in (L^2_{per}(\Omega))^2 : \nabla \cdot v = 0,\ \int_\Omega v dx = 0 \right\}$$

in the case of a periodic boundary condition. The space H is equipped with the usual L^2-inner product $(\cdot,\cdot)$ and norm $|\cdot| = (\cdot,\cdot)^{\frac{1}{2}}$ and is a closed subspace of $(L^2(\Omega))^2$.

We also define a Hilbert space

$$V = \{v \in (H_0^1(\Omega))^2 : \nabla \cdot v = 0\}$$

or

$$V = \left\{v \in (H_{per}^1(\Omega))^2 : \nabla \cdot v = 0, \int_\Omega v dx = 0\right\}$$

depending on the use of boundary conditions. Equipped with the usual H^1-inner product and norm, it is a Hilbert space. Let us denote by P the orthogonal projection from $(L^2(\Omega))^2$ onto H and project (2.1) onto H. Then we can get the functional form (1.1), where $A = -P\Delta$ is the Stokes operator which will be $-\Delta$ in the case of a periodical boundary condition, $B(u, u) = P[(u \cdot \nabla)u]$ and $f = PF$, which is assumed to be time independent or in $L^\infty(\mathcal{R}^+, H)$.

For the Stokes operator A, it is well known that it is a symmetric, positive definite, self-adjoint, and unbounded operator in H with compact inverse. Therefore, its eigenvalues and the associated eigenfunctions admit

$$A\phi_i = \lambda_i \phi_i, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty,$$

and the set of eigenfunctions $\{\phi_1, \phi_2, \ldots\}$ forms a unit orthonormal basis of H. For any $\alpha \in \mathcal{R}$, the space

$$D(A^\alpha) = \left\{v = \sum_{i=1}^\infty v_i \phi_i : \sum_{i=1}^\infty v_i^2 \lambda_i^{2\alpha} < +\infty\right\}$$

is a Hilbert space if it is equipped with the natural inner product and norm

$$(\cdot, \cdot)_{D(A^\alpha)} = (A^\alpha \cdot, A^\alpha \cdot), \qquad |\cdot|_{D(A^\alpha)} = |A^\alpha \cdot|.$$

It is known that $D(A^{\frac{1}{2}}) = V$, and $\{\phi_1, \phi_2, \ldots\}$ is also an orthonormal basis of $D(A^\alpha)$. Here and later on, we denote by $|\cdot|_s$ the $(H^s(\Omega))^2$ norm, and by $|\cdot|_\infty$ the $(L^\infty(\Omega))^2$ norm. Especially, we use $|\cdot|$ to denote $|\cdot|_0$. For the spatial periodic case, we know that $|A^\alpha \cdot|$ and $|\cdot|_{2\alpha}$ are equivalent norms for any $\alpha \in \mathcal{R}$, and this equivalence property holds for the nonslip case at least for $\alpha \leq 1$.

To investigate the interaction between the large and small eddies, we define the finite-dimensional subspace H_m as in the Introduction for given $m \in \mathcal{N}$ as

$$H_m = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\},$$

and define the orthogonal projection P_m from H onto H_m as

$$u = \sum_{i=1}^\infty u_i \phi_i \in H, \quad P_m u = \sum_{i=1}^m u_i \phi_i, \quad u_i \in \mathcal{R}.$$

Also, we define $Q_m = I - P_m$. Projecting (1.1) by P_m and Q_m, we can get the coupled system (1.5)–(1.6).

For the later analysis, we recall the Agmon inequality, the Brézis–Gallouet inequality [2], and the Sobolev interpolation inequality in two dimensions: there exists a constant $c_0 > 0$ such that

$$(2.2)\quad \begin{cases} |u|_\infty \le c_0|u|^{\frac{1}{2}}\,|Au|^{\frac{1}{2}} \;\text{ or }\; c_0|A^{\frac{1}{4}}u|^{\frac{2}{3}}|Au|^{\frac{1}{3}} & \forall u \in D(A), \\ |u|_\infty \le c_0|A^{\frac{1}{2}}u|\left(1+\log\frac{|Au|}{\lambda_1^{\frac{1}{2}}|A^{\frac{1}{2}}u|}\right)^{\frac{1}{2}} & \forall u \in D(A), \\ |A^s u| \le c_0|u|^{(m-s)/m}|A^m u|^{s/m} & \forall u \in D(A^m),\ 0 < s < m \le 1. \end{cases}$$

As a result of the Brézis–Gallouet inequality, we have

$$(2.3)\qquad |u|_\infty \le c_0 L_m |A^{\frac{1}{2}}u| \quad \forall u \in H_m,$$

where L_m is defined in (1.4). To avoid having too many symbols, we always regard c_0 as 1 in the rest of this paper, and this will not cause any significant difference.

For the projections P_m and Q_m, the following properties hold (see [4]):

$$(2.4)\quad |A^\beta P_m u| \le \lambda_m^{\beta-\mu}|A^\mu u|, \quad |A^\mu Q_m u| \le \lambda_{m+1}^{\mu-\beta}|A^\beta u| \quad \forall \mu < \beta,\ u \in D(A^\beta).$$

In what follows, we often use the following orthogonal property:

$$(2.5)\qquad |A^s(P_m u + Q_m u)|^2 = |A^s P_m u|^2 + |A^s Q_m u|^2 \quad \forall u \in D(A^s), s \in \mathcal{R}.$$

3. Small eddy correction method and its boundedness. As shown in (1.7), the large eddy error or the total error of the SGM approximation can be controlled by the small eddy error, the truncation error of u_m. This suggests that we get a more accurate approximation by paying more attention to correcting the small eddy approximation successively. In this section, we will construct a small eddy correction method to NSE (1.1) by formally applying the Newton iteration to its small eddy equation (1.6).

First of all, we define

$$F(p,q) = \frac{dq}{dt} + \nu Aq + Q_m B(q,q) + Q_m B(q,p) + Q_m B(p,q) + Q_m B(p,p) - Q_m f.$$

Then (1.6) is equivalent to

$$F(p,q) = 0.$$

Suppose that the large eddy component p in the above abstract equation is known. Formally applying the Newton iteration to it, we can get the following iterative procedure: supposing the initial guess of the small eddy component $q^0 = 0$ and the kth approximation $q^k(t) \in Q_m H$ is known for some $k \in \mathcal{N}$, find the $(k+1)$th approximation $q^{k+1}(t) \in Q_m H$, which should be a more accurate approximation of $q(t)$, such that

$$D_q F(p,q^k)(q^{k+1} - q^k) = -F(p,q^k).$$

Simple calculation shows that it is the following small eddy correction procedure:

$$\begin{aligned} \frac{dq^{k+1}}{dt} &+ \nu A q^{k+1} + Q_m B(p,p) + Q_m B(p,q^{k+1}) + Q_m B(q^{k+1},p) \\ &+ Q_m B(q^k,q^{k+1}) + Q_m B(q^{k+1},q^k) = Q_m[f + B(q^k,q^k)]. \end{aligned}$$

Combining the above small eddy correction with the large eddy equation (1.5), we can obtain the desired small eddy correction method: with $w^0 = 0$,

$$\frac{dv}{dt} + \nu Av + P_m B(v + w^l, v + w^l) = P_m f \quad \text{for any fixed integer } l \geq 0, \tag{3.1}$$

$$\frac{dw^k}{dt} + \nu Aw^k + Q_m B(v, v) + Q_m B(v, w^k) + Q_m B(w^k, v) + Q_m B(w^{k-1}, w^k) + Q_m B(w^k, w^{k-1}) = Q_m[f + B(w^{k-1}, w^{k-1})] \quad \forall 1 \leq k \leq l, \tag{3.2}$$

$$v(0) = P_m a, \quad w^i(0) = Q_m a, \quad i = 1, 2, \ldots, l, \quad l \geq 1. \tag{3.3}$$

Remark 1. The small eddy correction scheme (3.1)–(3.3) can be regarded as a certain generalization of the SGM and ONG. In fact, if we take $l = 0$ in the above approximate procedure, (3.1) admits the SGM approximate equation (1.2). And if we take $l = 1$, (3.1)–(3.3) are the ONG approximate equations (1.11)–(1.12). For the sake of simplicity of expression, we give a symbol $u^l = v + w^l$ and call it the lth approximate solution to the NSE (1.1). Thus the SGM approximate solution is the 0th approximate solution, and the ONG approximate solution is the 1st approximate solution. Observing the above small eddy correction method, it is analogous to the classical Newton iteration for elliptic problems. We expect that it can have the second order convergence rate just as the usual Newton method does. Fortunately, our later analysis gives us a positive answer. It is worth mentioning that directly applying the Newton method to NSE (1.1) cannot reach such a second order convergence rate. The reason is that it does not take the advantage of the fast decay property of small eddies. For example, we refer readers to our estimates (4.11), (4.15), etc. Here (4.11) and (4.15) show that the accuracy of the small eddy approximation is always a half order higher ($\lambda_{m+1}^{-\frac{1}{2}}$) than that of the large eddy approximation, and the small eddy approximation converges very quickly, which is very important for us to derive the high order convergence rate, while the direct application of Newton iteration to NSE does not distinguish the small and large eddy components and cannot take this advantage.

In the rest of this section, we will show that the small eddy correction method (3.1)–(3.3) is a bounded procedure and can generate a bounded approximate solution. Before that, we define a trilinear form

$$b(u, v, w) = ((u \cdot \nabla)v, w) \quad \forall u, v, w \in (H^1(\Omega))^2$$

and state some properties (e.g., see [21]):

$$b(u, v, w) = -b(u, w, v) \quad \forall v, w \in (H^1(\Omega))^2,\ u \in V, \tag{3.4}$$

$$2|b(u, v, w)| \leq c_1 |u|_{s_1} |A^{\frac{1}{2}} v|_{s_2} |w|_{s_3}, \tag{3.5}$$

where $u \in (H^{s_1}(\Omega))^2$, $v \in (H^{s_2+1}(\Omega))^2$, $w \in (H^{s_3}(\Omega))^2$, and $s_1, s_2, s_3 \geq 0$, $s_1 + s_2 + s_3 \geq 1$, $(s_1, s_2, s_3) \neq (1, 0, 0)$,$(0, 1, 0)$,$(0, 0, 1)$, and $c_1 > 0$ is a constant independent of u, v, w. Especially, (3.5) is valid if we substitute the L^2-norm for any two of the three norms on the right-hand side and replace the rest of the norm by L^∞-norms.

Hereafter, we use $|f|$ to denote $|f|_{\mathcal{L}(\mathcal{R}^+, H)}$ and set some dimensionless constants throughout this section:

Reynold's number $\quad Re = \nu^{-1}|a| \quad$ if $\ |a| \neq 0$,
Grashof's number $\quad Gr = \lambda_1 \nu^{-2}|f|$.

THEOREM 3.1. *For any given nonnegative integer l, there exist constants M_0, M_1 independent of m, l, t and a constant $M_2(T)$ (for any given $T > 0$) independent of m and l such that, if m is so large that*

$$\lambda_{m+1} \geq 128 c_1^2 \lambda_1^{-1} \nu^{-2} L_m^2 M_1^2, \tag{3.6}$$

then (3.1)–(3.3) *has a bounded solution on* $[0, +\infty)$

$$U_m(t) = (v(t), w^1(t), \dots, w^l(t))$$

and, for any $0 \leq k \leq l$ *and* $T > 0$,

$$|v(t) + w^k(t)| \leq M_0, \quad |A^{\frac{1}{2}}(v(t) + w^k(t))| \leq M_1 \quad \forall t \geq 0,$$

$$\int_0^T |A(v(s) + w^k(s))|^2 ds \leq M_2(T),$$

where

$$M_0 = \sqrt{2\nu^2 Re^2 + 4\nu^2 Gr^2}, \qquad M_1 = 4\lambda_1^{\frac{1}{2}} \nu (Re + 2Gr) \exp(c_1^4 (2Re^2 + 5Gr^2)^2),$$

and

$$M_2(T) = 2\nu^{-1} M_1^2 (6 + c_1^4 \nu^{-3} M_0^2 M_1^2 T) + 32 \lambda_1^2 \nu^2 T Gr^2.$$

The complex form of the constant M_1 comes from the fact that we do not distinguish the types of boundary conditions. Actually, for the spatial periodic case, the corresponding M_1 will have a much simpler appearance (without the exponential factor).

Compared with NGM, PPGM, and ONG, in which the boundedness of the approximate solution is quite easy to obtain and the existence of the approximate solution is a direct result, the boundedness and the existence of the approximate solution is no longer obvious in our case, especially for $l \geq 2$.

To prove this theorem, we first consider the following Galerkin approximation of (3.1)–(3.3): for any given positive integer $M > m$, constant $T > 0$, and $k = 1, 2, \dots, l$ find

$$v_m = \sum_{i=1}^m g_{im}(t)\phi_i \in H_m, \qquad w_M^k = \sum_{i=m+1}^M g_{im}^k(t)\phi_i \in (P_M - P_m)H \stackrel{\triangle}{=} P_{mM}H$$

on time interval $[0, T]$ such that for any fixed integer $l \geq 0$ and $w_M^0 = 0$

$$\frac{dv_m}{dt} + \nu A v_m + P_m B(v_m + w_M^l, v_m + w_M^l) = P_m f, \tag{3.7}$$

$$\begin{aligned} \frac{dw_M^k}{dt} &+ \nu A w_M^k + P_{mM} B(v_m, v_m) + P_{mM} B(v_m, w_M^k) \\ &+ P_{mM} B(w_M^k, v_m) + P_{mM} B(w_M^{k-1}, w_M^k) + P_{mM} B(w_M^k, w_M^{k-1}) \\ &= P_{mM}[f + B(w_M^{k-1}, w_M^{k-1})] \quad \forall 1 \leq k \leq l, \end{aligned} \tag{3.8}$$

$$v_m(0) = P_m a, \quad w_M^i(0) = P_{mM} a, \quad i = 1, 2, \dots, l, \quad l \geq 1. \tag{3.9}$$

For simplicity of expression, we introduce the following notation for the rest of this section:

$$u_{mM}^k = v_m + w_M^k, \quad k = 0, \dots, l.$$

Noticing the orthogonal properties of $\{\phi_1, \dots, \phi_m, \dots\}$ in H and V, that is,

$$(\phi_i, \phi_j) = \delta_{ij}, \quad (A^{\frac{1}{2}}\phi_i, A^{\frac{1}{2}}\phi_j) = (A\phi_i, \phi_j) = \lambda_i \delta_{ij} \quad \forall i, j = 1, \dots, m, \dots,$$

we find that the system (3.7)–(3.9) is actually a first order ODE of $\{g_{im}(t)\}_{i=1}^m$ and $\{g_{im}^k(t)\}_{i=m+1}^M$, $k = 1, \dots, l$:

$$\begin{aligned}
&\dot{g}_{im} + \nu\lambda_i g_{im} + \sum_{j,n=1}^m \alpha_{nji} g_{jm} g_{nm} + \sum_{j=1}^m \sum_{n=m+1}^M \beta_{nji} g_{jm} g_{nm}^l \\
&\qquad + \sum_{j,n=m+1}^M \alpha_{nji} g_{jm}^l g_{nm}^l = f_i, \quad i = 1, \dots, m, \\
&\dot{g}_{im}^k + \nu\lambda_i g_{im}^k + \sum_{j,n=1}^m \alpha_{nji} g_{jm} g_{nm} + \sum_{j=1}^m \sum_{n=m+1}^M \beta_{nji} g_{jm} g_{nm}^k + \sum_{j,n=m+1}^M \beta_{nji} g_{jm}^k g_{nm}^{k-1} \\
&\qquad - \sum_{j,n=m+1}^M \alpha_{nji} g_{jm}^{k-1} g_{nm}^{k-1} = f_i \quad \forall i = m+1, \dots, M, \\
&g_{im}(0) = (a, \phi_i), \quad g_{jm}^k(0) = (a, \phi_j), \quad i = 1, \dots, m, \ j = m+1, \dots, M.
\end{aligned}$$

Here

$$\begin{cases} \alpha_{nji} = b(\phi_n, \phi_j, \phi_i), \ \beta_{nji} = b(\phi_n, \phi_j, \phi_i) + b(\phi_j, \phi_n, \phi_i), \\ f_i = (f, \phi_i), \ g_{km}^0 = 0 \ \text{ for } k = m+1, \dots, M. \end{cases}$$

Thanks to the theory of ODEs, the above nonlinear ODE has a maximal continuous solution defined on some interval $[0, T_M)$. If $T_M < T$, then $|A^{\frac{1}{2}} u_{mM}^k(t)|$, for some $0 \le k \le l$, must tend to $+\infty$ as $t \to T_M$. But we will show that this does not happen. Therefore $T_M \ge T$. Furthermore, both $|u_{mM}^k(t)|$ and $|A^{\frac{1}{2}} u_{mM}^k|$ are bounded by some constants independent of m, M, T, and l. To prove this, we need the following result.

LEMMA 3.1. *For any given $t_M > 0$, suppose*

$$|A^{\frac{1}{2}} u_{mM}^k(t)| \le M_1 \quad \forall 1 \le k \le l, \, t \in [0, t_M].$$

If the condition (3.6) *is valid, we have, for $k = 1, \dots, l$,*

$$|u_{mM}^k(t)| \le M_0 \quad \forall t \in [0, t_M],$$

$$\int_t^{t+r} |A^{\frac{1}{2}} u_{mM}^k(s)|^2 ds \le \frac{4M_0^2}{\nu} + \frac{2r|f|^2}{\lambda_1 \nu^2} \quad \textit{for any } t \ge 0, r > 0 \textit{ with } t + r \le t_M,$$

where $M_1, M_0 > 0$ are defined in Theorem 3.1.

Proof. The summation of (3.7) and (3.8) implies

$$\begin{aligned}
&\frac{du_{mM}^k}{dt} + \nu A u_{mM}^k + P_M B(v_m, v_m) + P_m[B(v_m, w_M^l) + B(w_M^l, v_m) + B(w_M^l, w_M^l)] \\
&\qquad + P_{mM}[B(v_m, w_M^k) + B(w_M^k, v_m) + B(w_M^{k-1}, w_M^k) \\
&\qquad\qquad + B(w_M^k, w_M^{k-1}) - B(w_M^{k-1}, w_M^{k-1})] = P_M f.
\end{aligned} \tag{3.10}$$

Multiplying (3.10) by $2u_{mM}^k$, integrating it on Ω, and using the property (3.4) of the trilinear form, we get

$$\frac{d|u_{mM}^k|^2}{dt} + 2\nu|A^{\frac{1}{2}}u_{mM}^k|^2 \le 2|b(v_m, v_m, w_M^k)| + 2|b(v_m, w_M^l, v_m)| + 2|b(w_M^l, w_M^l, v_m)|$$
$$+ 2|b(w_M^k, v_m, w_M^k)| + 2|b(w_M^k, w_M^{k-1}, w_M^k)| + 2|b(w_M^{k-1}, w_M^{k-1}, w_M^k)| + 2|(f, u_{mM}^k)|.$$

Due to (2.5) and (3.6), we know $|A^{\frac{1}{2}}v_m|, |A^{\frac{1}{2}}w_M^k| \le M_1$, and $c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \le \frac{\nu}{10}$. By using (2.2)–(2.4) and (3.4)–(3.5), we summarize the estimates of the seven terms on the right-hand side of the above inequality as follows:

$$\begin{aligned}
2|b(v_m, v_m, w_M^k)| &\le c_1|v_m|_\infty|A^{\frac{1}{2}}v_m|\,|w_M^k| \le \frac{c_1 M_1 L_m}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}v_m|^2 \le \frac{\nu}{10}|A^{\frac{1}{2}}v_m|^2,\\
2|b(v_m, w_M^l, v_m)| &= 2|b(v_m, v_m, w_M^l)| \le c_1|v_m|_\infty|A^{\frac{1}{2}}v_m|\,|w_M^l|\\
&\le \frac{c_1 M_1 L_m}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}v_m|^2 \le \frac{\nu}{10}|A^{\frac{1}{2}}v_m|^2,\\
2|b(w_M^l, w_M^l, v_m)| &= 2|b(w_M^l, v_m, w_M^l)| \le c_1|A^{\frac{1}{4}}w_M^l|^2|A^{\frac{1}{2}}v_m|\\
&\le \frac{c_1}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}v_m|\,|A^{\frac{1}{2}}w_M^l|^2,\\
2|b(w_M^k, v_m, w_M^k)| &\le c_1|A^{\frac{1}{4}}w_M^k|^2|A^{\frac{1}{2}}v_m| \le \frac{c_1 M_1}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}w_M^k|^2 \le \frac{\nu}{10}|A^{\frac{1}{2}}w_M^k|^2,\\
2|b(w_M^k, w_M^{k-1}, w_M^k)| &\le c_1|A^{\frac{1}{4}}w_M^k|^2|A^{\frac{1}{2}}w_M^{k-1}| \le \frac{c_1 M_1}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}w_M^k|^2 \le \frac{\nu}{10}|A^{\frac{1}{2}}w_M^k|^2,\\
2|b(w_M^{k-1}, w_M^{k-1}, w_M^k)| &= 2|b(w_M^{k-1}, w_M^k, w_M^{k-1})| \le \frac{c_1}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}w_M^k|\,|A^{\frac{1}{2}}w_M^{k-1}|^2,\\
2|(f, u_{mM}^k)| &\le 2|A^{-\frac{1}{2}}P_M f|\,|A^{\frac{1}{2}}u_{mM}^k| \le \nu|A^{\frac{1}{2}}u_{mM}^k|^2 + \frac{|f|^2}{\lambda_1\nu}.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
\frac{d|u_{mM}^k|^2}{dt} &+ \frac{4\nu}{5}|A^{\frac{1}{2}}u_{mM}^k|^2\\
&\le \frac{|f|^2}{\lambda_1\nu} + c_1\lambda_{m+1}^{-\frac{1}{2}}(|A^{\frac{1}{2}}w_m^k|\,|A^{\frac{1}{2}}w_M^{k-1}|^2 + |A^{\frac{1}{2}}v_m|\,|A^{\frac{1}{2}}w_M^l|^2).
\end{aligned} \tag{3.11}$$

Because of $|A^{\frac{1}{2}}u_{mM}^k|^2 \ge \lambda_1|u_{mM}^k|^2$, we have

$$\begin{aligned}
\frac{d|u_{mM}^k|^2}{dt} &+ \frac{11\lambda_1\nu}{20}|u_{mM}^k|^2 + \frac{\nu}{4}|A^{\frac{1}{2}}w_M^k|^2\\
&\le \frac{|f|^2}{\lambda_1\nu} + c_1\lambda_{m+1}^{-\frac{1}{2}}(|A^{\frac{1}{2}}w_m^k|\,|A^{\frac{1}{2}}w_M^{k-1}|^2 + |A^{\frac{1}{2}}v_m|\,|A^{\frac{1}{2}}w_M^l|^2).
\end{aligned}$$

Thanks to (2.5), $\max_{0\le t\le t_M}|A^{\frac{1}{2}}u_{mM}^k(t)| \le M_1$ implies

$$\max_{0\le t\le t_M}\left(\max_{1\le k\le l}|A^{\frac{1}{2}}w_M^k(t)|^2 + |A^{\frac{1}{2}}v_m(t)|^2\right) \le M_1^2.$$

Therefore

$$\max_{0\le t\le t_M}\left(\max_{1\le k\le l}|A^{\frac12}w_M^k(t)|+|A^{\frac12}v_m(t)|\right)\le\sqrt{2}M_1. \tag{3.12}$$

Integrating the above inequality on time interval $[0,t]$ and noticing (3.12), we obtain

$$\begin{aligned}|u_{mM}^k(t)|^2&+\frac{\nu}{4}\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^k(s)|^2ds\le|a|^2+\frac{20}{11\lambda_1^2\nu^2}|f|^2\\&+\frac{\sqrt{2}c_1M_1}{\lambda_{m+1}^{\frac12}}\max\left\{\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{k-1}(s)|^2ds,\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{l}(s)|^2ds\right\}.\end{aligned}$$

Because both terms on the left-hand side of the above inequality are positive, we have

$$\begin{aligned}|u_{mM}^k(t)|^2&\le|a|^2+\frac{20}{11\lambda_1^2\nu^2}|f|^2\\&+\frac{\sqrt{2}c_1M_1}{\lambda_{m+1}^{\frac12}}\max\left\{\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{k-1}(s)|^2ds,\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{l}(s)|^2ds\right\},\end{aligned}$$

$$\begin{aligned}\frac{\nu}{4}\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}&|A^{\frac12}w_M^k(s)|^2ds\le|a|^2+\frac{20}{11\lambda_1^2\nu^2}|f|^2\\&+\frac{\sqrt{2}c_1M_1}{\lambda_{m+1}^{\frac12}}\max\left\{\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{k-1}(s)|^2ds,\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^{l}(s)|^2ds\right\}.\end{aligned}$$

Taking the maximum value with respect to $1\le k\le l$ on both sides of the above two inequalities and noticing $w_M^0=0$, the summation of the results admits

$$\begin{aligned}\max_{1\le k\le l}|u_{mM}^k(t)|^2&+\frac{\nu}{4}\max_{1\le k\le l}\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^k(s)|^2ds\\&\le 2|a|^2+\frac{4}{\lambda_1^2\nu^2}|f|^2+2\sqrt{2}c_1M_1\lambda_{m+1}^{-\frac12}\max_{1\le k\le l}\int_0^t e^{-\frac{11\lambda_1\nu}{20}(t-s)}|A^{\frac12}w_M^k(s)|^2ds.\end{aligned} \tag{3.13}$$

Since we know $2\sqrt{2}c_1M_1\lambda_{m+1}^{-\frac12}\le\frac{\nu}{4}$ from the condition (3.6), then (3.13) implies

$$|u_{mM}^k(t)|\le\sqrt{2|a|^2+\frac{4|f|^2}{\lambda_1^2\nu^2}}=\sqrt{2\nu^2Re^2+4\nu^2Gr^2}=M_0\quad\forall t\in[0,t_M],\ 1\le k\le l.$$

From (3.11) and the result we just obtained, we have for any $t\ge0, r>0$, and $t+r\le t_M$

$$\frac{4\nu}{5}\int_t^{t+r}|A^{\frac12}u_{mM}^k|^2ds\le 2M_0^2+\frac{|f|^2r}{\lambda_1\nu}+2\sqrt{2}c_1M_1\lambda_{m+1}^{-\frac12}\max_{1\le i\le l}\int_t^{t+r}|A^{\frac12}w_M^i|^2ds.$$

Noticing that $|A^{\frac12}w_M^k|^2\le|A^{\frac12}u_{mM}^k|^2$ and that (3.6) guarantees $2\sqrt{2}c_1M_1\lambda_{m+1}^{-\frac12}\le\frac{\nu}{4}$, we can easily get from the above inequality that

$$\max_{1\le k\le l}\int_t^{t+r}|A^{\frac12}u_{mM}^k(s)|^2ds\le\frac{4M_0^2}{\nu}+\frac{2r|f|^2}{\lambda_1\nu^2}\quad\forall t\ge0,\ r>0,\ t+r\le t_M. \qquad\square$$

Now we are ready to prove that for the ODE system (3.7)–(3.9) there exists a global uniformly bounded solution.

LEMMA 3.2. *For any given $0 < T < \infty$, if m is so large that the condition* (3.6) *holds, we have*

$$|A^{\frac{1}{2}} u^k_{mM}(t)| \le M_1 \quad \forall t \in [0,T],\ 0 \le k \le l,$$

and

$$\int_0^T |Au^k_{mM}(s)|^2 ds \le M_2(T), \quad k = 0, 1, \dots, l,$$

where M_1 and $M_2(T)$ are given in Theorem 3.1.

Proof. Noticing what was mentioned before, that the ODE system (3.7)–(3.9) has a maximal continuous solution defined on some interval $[0, T_M)$, we will prove this lemma by contradiction.

Assuming that

$$T_M < T, \tag{3.14}$$

we then assert that $|A^{\frac{1}{2}} u^k_{mM}(t)|$ goes to ∞ as t tends to T_M for certain $0 \le k \le l$. Since $|A^{\frac{1}{2}} u^k_{mM}(t)|$ is a continuous function of t and $M_1 > |A^{\frac{1}{2}} a|$, there exists a constant $t_M < T_M$ such that

$$\max_{0\le k\le l} |A^{\frac{1}{2}} u^k_{mM}(t)| < M_1 \ \text{ on } [0, t_M) \quad \text{and} \quad \max_{0\le k\le l} |A^{\frac{1}{2}} u^k_{mM}(t_M)| = M_1. \tag{3.15}$$

Multiplying (3.10) by $2Au^k_{mM}$ and integrating it on Ω, we have

$$\begin{aligned}\frac{d|A^{\frac{1}{2}} u^k_{mM}|^2}{dt} + 2\nu |Au^k_{mM}|^2 &\le 2|b(v_m, v_m, Au^k_{mM})| + 2|b(v_m, w^l_M, Av_m)| \\ &+ 2|b(w^l_M, v_m, Av_m)| + 2|b(w^l_M, w^l_M, Av_m)| + 2|b(v_m, w^k_M, Aw^k_M)| \\ &+ 2|b(w^k_M, v_m, Aw^k_M)| + 2|b(w^{k-1}_M, w^k_M, Aw^k_M)| + 2|b(w^k_M, w^{k-1}_M, Aw^k_M)| \\ &+ 2|b(w^{k-1}_M, w^{k-1}_M, Aw^k_M)| + 2|(f, Au^k_{mM})|.\end{aligned}$$

Under the conditions of (3.6) and (3.15), Lemma 3.1 asserts that $|u^k_{mM}| \le M_0$ on $[0, t_M]$ for any $0 \le k \le l$. Of course, we know $|v_m|, |w^k_{mM}| \le M_0$ from (2.5). By using (2.2)–(2.4), (3.4)–(3.5), and the result of Lemma 3.1, we have the following estimates of the ten right-hand-side terms in the above inequality on $[0, t_M]$:

$$\begin{aligned}2|b(v_m, v_m, Au^k_{mM})| &\le c_1 |v_m|_\infty |A^{\frac{1}{2}} v_m|\,|Au^k_{mM}| \le c_1 |v_m|^{\frac{1}{2}} |A^{\frac{1}{2}} v_m|\,|Au^k_{mM}|^{\frac{3}{2}} \\ &\le \frac{3\nu}{4} |Au^k_{mM}|^2 + \frac{c_1^4 M_0^2}{4\nu^3} |A^{\frac{1}{2}} v_m|^4, \\ 2|b(v_m, w^l_M, Av_m)| &\le c_1 |v_m|_\infty |A^{\frac{1}{2}} w^l_M|\,|Av_m| \le \frac{c_1 L_m}{\lambda^{\frac{1}{2}}_{m+1}} |A^{\frac{1}{2}} v_m|\,|Aw^l_M|\,|Av_m| \\ &\le \frac{c_1 L_m}{4\lambda^{\frac{1}{2}}_{m+1}} |A^{\frac{1}{2}} v_m|\,|Aw^l_M|^2 + \frac{c_1 L_m}{\lambda^{\frac{1}{2}}_{m+1}} |A^{\frac{1}{2}} v_m|\,|Av_m|^2, \\ 2|b(w^l_M, v_m, Av_m)| &\le c_1 |w^l_M|\,|A^{\frac{1}{2}} v_m|_\infty |Av_m| \le \frac{c_1 L_m}{\lambda^{\frac{1}{2}}_{m+1}} |A^{\frac{1}{2}} w^l_M|\,|Av_m|^2,\end{aligned}$$

$$\begin{aligned}
2|b(w_M^l, w_M^l, Av_m)| &\le c_1|w_M^l|_\infty |A^{\frac12} w_M^l|\,|Av_m| \\
&\le c_1 |w_M^l|^{\frac12} |Aw_M^l|^{\frac12} |w_M^l|^{\frac12} |Aw_M^l|^{\frac12} |Av_m| \\
&= c_1 |w_M^l|\,|Aw_M^l|\,|Av_m| \le \frac{c_1}{\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^l|\,|Aw_M^l|\,|Av_m| \\
&\le \frac{c_1 L_m}{4\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^l|\,|Aw_M^l|^2 + \frac{c_1 L_m}{\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^l|\,|Av_m|^2, \\
2|b(v_m, w_M^k, Aw_M^k)| &\le c_1 |v_m|_\infty |A^{\frac12} w_M^k|\,|Aw_M^k| \le \frac{c_1 L_m}{\lambda_{m+1}^{\frac12}} |A^{\frac12} v_m|\,|Aw_M^k|^2, \\
2|b(w_M^k, v_m, Aw_M^k)| &\le c_1 |w_M^k|_\infty |A^{\frac12} v_m|\,|Aw_M^k| \le c_1 |w_M^k|^{\frac12} |Aw_M^k|^{\frac12} |A^{\frac12} v_m|\,|Aw_M^k| \\
&\le \frac{c_1}{\lambda_{m+1}^{\frac12}} |A^{\frac12} v_m|\,|Aw_M^k|^2, \\
2|b(w_M^{k-1}, w_M^k, Aw_M^k)| &\le c_1 |A^{\frac14} w_M^{k-1}|\,|A^{\frac34} w_M^k|\,|Aw_M^k| \le \frac{c_1}{\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2, \\
2|b(w_M^k, w_M^{k-1}, Aw_M^k)| &\le c_1 |w_M^k|_\infty |A^{\frac12} w_M^{k-1}|\,|Aw_M^k| \le c_1 |w_M^k|^{\frac12} |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^{\frac32} \\
&\le \frac{c_1}{\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2, \\
2|b(w_M^{k-1}, w_M^{k-1}, Aw_M^k)| &\le c_1 |w_M^{k-1}|_\infty |A^{\frac12} w_M^{k-1}|\,|Aw_M^k| \\
&\le c_1 |w_M^{k-1}|^{\frac12} |Aw_M^{k-1}|^{\frac12} |w_M^{k-1}|^{\frac12} |Aw_M^{k-1}|^{\frac12} |Aw_M^k| \\
&= c_1 |w_M^{k-1}|\,|Aw_M^{k-1}|\,|Aw_M^k| \le \frac{c_1}{\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^{k-1}|\,|Aw_M^{k-1}|\,|Aw_M^k| \\
&\le \frac{c_1}{2\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2 + \frac{c_1}{2\lambda_{m+1}^{\frac12}} |A^{\frac12} w_M^{k-1}|\,|Aw_M^{k-1}|^2, \\
2|(f, Au_{mM}^k)| &\le \frac{4}{\nu}|f|^2 + \frac{\nu}{4}|Au_{mM}^k|^2.
\end{aligned}$$

Combining the above estimates yields

$$\begin{aligned}
\frac{d|A^{\frac12} u_{mM}^k|^2}{dt} &+ \nu |Au_{mM}^k|^2 \le \frac{4}{\nu}|f|^2 + \frac{c_1^4 M_0^2}{4\nu^3} |A^{\frac12} v_m|^4 \\
&+ \frac14 c_1 L_m \lambda_{m+1}^{-\frac12} (|A^{\frac12} v_m| + |A^{\frac12} w_M^l|) |Aw_M^l|^2 \\
&+ c_1 L_m \lambda_{m+1}^{-\frac12} (|A^{\frac12} v_m|\,|Av_m|^2 + 2|A^{\frac12} w_M^l|\,|Av_m|^2 \\
&\qquad + 2|A^{\frac12} v_m|\,|Aw_M^k|^2 + 2|A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2) \\
&+ \frac{c_1}{2} \lambda_{m+1}^{-\frac12} |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2 + \frac{c_1}{2} \lambda_{m+1}^{-\frac12} |A^{\frac12} w_M^{k-1}|\,|Aw_M^{k-1}|^2.
\end{aligned}$$

Noticing (3.15) and (3.12), we have for $t \in [0, t_M]$

$$\begin{aligned}
|A^{\frac12} v_m|\,|Av_m|^2 &+ |A^{\frac12} w_M^l|\,|Av_m|^2 + |A^{\frac12} v_m|\,|Aw_M^k|^2 + |A^{\frac12} w_M^{k-1}|\,|Aw_M^k|^2 \\
&\le \sqrt{2} M_1 |Au_{mM}^k|^2.
\end{aligned}$$

The combination of the above estimations gives

$$\frac{d|A^{\frac{1}{2}}u^k_{mM}|^2}{dt} + \nu|Au^k_{mM}|^2 \le \frac{4}{\nu}|f|^2 + \frac{c_1^4 M_0^2}{4\nu^3}|A^{\frac{1}{2}}v_m|^4 + \frac{\sqrt{2}}{4}c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}}|Aw^l_M|^2 + \frac{2\sqrt{2}c_1 M_1 L_m}{\lambda_{m+1}^{\frac{1}{2}}}|Au^k_{mM}|^2 + \frac{c_1 M_1}{2\lambda_{m+1}^{\frac{1}{2}}}|Aw^k_M|^2 + \frac{c_1 M_1}{2\lambda_{m+1}^{\frac{1}{2}}}|Aw^{k-1}_M|^2.$$

Thanks to (3.6), we have $2\sqrt{2}c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \le \frac{\nu}{4}$. Then

$$\frac{d|A^{\frac{1}{2}}u^k_{mM}|^2}{dt} + \frac{3\nu}{4}|Au^k_{mM}|^2 \le \frac{4}{\nu}|f|^2 + \frac{c_1^4 M_0^2}{4\nu^3}|A^{\frac{1}{2}}v_m|^4 + \frac{c_1}{2}M_1 L_m \lambda_{m+1}^{-\frac{1}{2}}(|Aw^l_M|^2 + |Aw^k_M|^2 + |Aw^{k-1}_M|^2).$$

For any $t \ge 0$, $r > 0$ satisfying $t + r \le t_M$ and $\tau \in [t, t+r]$, integrating the above inequality on $[\tau, t+r]$ shows that for all $1 \le k \le l$

$$|A^{\frac{1}{2}}u^k_{mM}(t+r)|^2 + \frac{3\nu}{4}\int_\tau^{t+r}|Au^k_{mM}(s)|^2 ds \le \frac{4r}{\nu}|f|^2 + \int_\tau^{t+r}\frac{c_1^4 M_0^2}{4\nu^3}|A^{\frac{1}{2}}v_m(s)|^4 ds + \frac{c_1}{2}M_1 L_m \lambda_{m+1}^{-\frac{1}{2}}\int_\tau^{t+r}[|Aw^l_M(s)|^2 + |Aw^k_M(s)|^2 + |Aw^{k-1}_M(s)|^2]ds + |A^{\frac{1}{2}}u^k_{mM}(\tau)|^2.$$

Now, we define

$$y(s) = \max_{1\le i\le l}|A^{\frac{1}{2}}u^i_{mM}(s)|^2.$$

By noticing that (3.6) implies $3c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \le \frac{\nu}{2}$ and using a similar method for deriving (3.13),

$$y(t+r) + \frac{\nu}{4}\max_{1\le i\le l}\int_\tau^{t+r}|Au^i_{mM}(s)|^2 ds \le 2y(\tau) + \frac{8r}{\nu}|f|^2 + \int_\tau^{t+r}\frac{c_1^4 M_0^2}{2\nu^3}y^2(s)ds. \tag{3.16}$$

Thanks to Lemma 3.1, we know

$$\int_t^{t+r} y(s)ds \le \frac{4M_0^2}{\nu} + \frac{2r|f|^2}{\lambda_1\nu^2} \triangleq a_3(r) \quad \forall t \ge 0,\ r > 0,\ t + r \le t_M.$$

Also, we define

$$a_2(r) = \frac{8r}{\nu}|f|^2, \qquad a_1(r) = \frac{c_1^4 M_0^2}{2\nu^3}a_3(r).$$

It is obvious that $a_1(r)$, $a_2(r)$, and $a_3(r)$ are functions of r and independent of l and M_1. Omitting the second term on the left-hand side of (3.16) and combining it with the above three inequalities, we can use the idea of the proof of the uniform Gronwall inequality (see [22]) to get

$$y(t) \le \left(\frac{2a_3(r)}{r} + a_2(r)\right)\exp(a_1(r)) \triangleq B_{11}(r) \quad \forall r \le t \le t_M. \tag{3.17}$$

For $t \in [0, r]$, by using the ordinary Gronwall inequality, (3.16) implies

$$y(t) \le (2|A^{\frac{1}{2}}a|^2 + a_2(r))\exp(a_1(r)) \overset{\triangle}{=} B_{12}(r). \tag{3.18}$$

Now let us complete the proof in two cases.

Case 1. If $t_M > \frac{1}{\lambda_1 \nu}$, we take $r = \frac{1}{\lambda_1 \nu}$ in (3.17)–(3.18). This admits

$$\max_{1\le i\le l} |A^{\frac{1}{2}} u^i_{mM}(t)| \le \sqrt{\max\left\{B_{11}\left(\frac{1}{\lambda_1\nu}\right), B_{12}\left(\frac{1}{\lambda_1\nu}\right)\right\}} \quad \forall t \in [0, t_M].$$

It is easy to verify that

$$\sqrt{\max\left\{B_{11}\left(\frac{1}{\lambda_1\nu}\right), B_{12}\left(\frac{1}{\lambda_1\nu}\right)\right\}} < 4\lambda_1^{\frac{1}{2}}\nu(Re + 2Gr)\exp(c_1^4(2Re^2 + 5Gr^2)^2) = M_1,$$

and this is a contradiction of (3.15).

Case 2. If $t_M \le \frac{1}{\lambda_1\nu}$, it is quite easy to verify that (3.16) is still valid with $t = \tau = 0$ and any $r \le t_M$. Then we can use the ordinary Gronwall inequality on (3.16) to get

$$\max_{1\le i\le l} |A^{\frac{1}{2}} u^i_{mM}(r)| \le \sqrt{B_{12}\left(\frac{1}{\lambda_1\nu}\right)} < M_1 \quad \forall r \in [0, t_M].$$

In particular, $\max_{1\le i\le l} |A^{\frac{1}{2}} u^i_{mM}(t_M)| < M_1$. This also leads to a contradiction with (3.15).

Therefore the assumption (3.14) is invalid. We can thus get $T_M \ge T$; that is, the solution of (3.7)–(3.9) will not blow up in any bounded time interval. And the proof shows that its solution is uniformly (with respect to m, M, l, and T) bounded with bound M_1.

Furthermore, from (3.16) we can get, for $k = 1, \dots, l$,

$$\begin{aligned}\int_0^T |Au^k_{mM}(s)|^2 ds &\le 12\nu^{-1}M_1^2 + 32T\nu^{-2}|f|^2 + \frac{2c_1^4 M_0^2 M_1^4 T}{\nu^4} \\ &\le 2\nu^{-1}M_1^2(6 + c_1^4\nu^{-3}M_0^2M_1^2T) + 32\lambda_1^2\nu^2 T Gr^2 = M_2^2(T). \qquad \square\end{aligned}$$

Proof of Theorem 3.1. By the results of Lemma 3.2, we claim that for any fixed m the sequence

$$U_{mM} \overset{\triangle}{=} \{v_m, w^1_M, \dots, w^l_M\}, \quad M > m, \tag{3.19}$$

remains in a bounded set of $\mathcal{L}^2(0, T; \mathbf{D(A)}) \cap \mathcal{L}^\infty(0, T; \mathbf{V})$,

where $\mathbf{D(A)} = P_m D(A) \times (Q_m D(A))^l$, $\mathbf{V} = P_m V \times (Q_m V)^l$. And it is easy for us to verify that

$$U'_{mM} = \frac{dU_{mM}}{dt} \quad \text{remains in a bounded set of } \mathcal{L}^2(0, T; \mathbf{H}). \tag{3.20}$$

Here $\mathbf{H} = H_m \times (Q_m H)^l$.

Now we define $X_0 = \mathbf{D(A)}$, $X = \mathbf{V}$, $X_1 = \mathbf{H}$, and

$$\mathcal{Y} = \{U \in \mathcal{L}^2(0, T; X_0),\ U' \in \mathcal{L}^2(0, T; X_1)\}.$$

From (3.20) and the second result of Lemma 3.2, we have that the sequence $\{U_{mM}\}_{M>m}$ remains in a bounded set of $\mathcal{Y}$. Thanks to (3.19) and the compactness theorem (see Theorem 2.1 in Chapter III of [21]), we can assert the existence of an element $U_m = \{\tilde{v}_m, \tilde{w}^1, \dots, \tilde{w}^l\} \in \mathcal{L}^\infty(0,T;\mathbf{V})$ and a subsequence $\{U_{mM'}\}_{M'>m}$ such that

$$\begin{cases} U_{mM'} \to U_m \text{ in } \mathcal{L}^2(0,T;\mathbf{D(A)}) \text{ weakly, in} \\ \mathcal{L}^\infty(0,T;\mathbf{V}) \text{ weak star, and in} \\ \mathcal{L}^2(0,T;\mathbf{V}) \text{ strongly, as } M' \to \infty. \end{cases}$$

We can certainly get $U_{mM'} \to U_m$ weakly in $\mathcal{L}^2(0,T;\mathbf{V})$, weak star in $\mathcal{L}^\infty(0,T;\mathbf{H})$, and strongly in $\mathcal{L}^2(0,T;\mathbf{H})$. Noticing the continuity property of the trilinear form $b(\cdot,\cdot,\cdot)$ (see Lemma 3.2 in Chapter III of [21]), the passage to the limit shows that U_m is the solution of system (3.1)–(3.3), which shares the same bound of U_{mM}. ◻

In the following theorem, we will show that the small eddy components, namely $|w_M^k|$ and $|A^{\frac{1}{2}}w_M^k|$, are bounded by some small quantities. This result is very important for the error estimates in the next section.

THEOREM 3.2. *If $|A^{\frac{1}{2}}u_{mM}^k(t)| \le M_1$ for any $t \ge 0$ and $1 \le k \le l$ and m is large enough such that* (3.6) *holds, then there exists a positive constant $T_0 = T_0(a,f,\nu)$ such that*

$$|w_M^k(t)| \le \frac{K_0 L_m}{\lambda_{m+1}}, \qquad |A^{\frac{1}{2}}w_M^k(t)| \le \frac{K_1 L_m}{\lambda_{m+1}^{\frac{1}{2}}} \quad \forall t \ge T_0, 1 \le k \le l,$$

where

$$K_0 = \frac{2(c_1M_1^2 + 2|P_{mM}f|L_m^{-1})}{\nu}, \qquad K_1 = \frac{5(c_1M_1^2 + |P_{mM}f|L_m^{-1})}{\nu}.$$

Proof. Multiplying (3.8) with $2w_M^k$, integrating it on Ω, and using (3.4), we get

$$\begin{aligned} &\frac{d|w_M^k|^2}{dt} + 2\nu|A^{\frac{1}{2}}w_M^k|^2 + 2b(v_m,v_m,w_M^k) + 2b(w_M^k, v_m + w_M^{k-1}, w_M^k) \\ &\quad = 2(f,w_M^k) + 2b(w_M^{k-1}, w_M^{k-1}, w_M^k). \end{aligned}$$

For the trilinear forms and the force term above, using (2.2)–(2.5), (3.4)–(3.5), and the assumption $|A^{\frac{1}{2}}(v_m + w_M^k)| \le M_1$, we have

$$\begin{aligned} 2b(v_m,v_m,w_M^k) &\le c_1|v_m|_\infty|A^{\frac{1}{2}}v_m|\,|w_M^k| \le c_1L_m\lambda_{m+1}^{-\frac{1}{2}}|A^{\frac{1}{2}}v_m|^2|A^{\frac{1}{2}}w_M^k| \\ &\le \frac{\nu}{4}|A^{\frac{1}{2}}w_M^k|^2 + \frac{c_1^2L_m^2|A^{\frac{1}{2}}v_m|^4}{\nu\lambda_{m+1}}, \\ 2b(w_M^k, v_m + w_M^{k-1}, w_M^k) &\le c_1|A^{\frac{1}{4}}w_M^k|\,|A^{\frac{1}{2}}(v_m + w_M^{k-1})|\,|A^{\frac{1}{4}}w_M^k| \le \frac{c_1M_1}{\lambda_{m+1}^{\frac{1}{2}}}|A^{\frac{1}{2}}w_M^k|^2, \\ 2b(w_M^{k-1}, w_M^{k-1}, w_M^k) &= -2b(w_M^{k-1}, w_M^k, w_M^{k-1}) \le c_1|A^{\frac{1}{4}}w_M^{k-1}|\,|A^{\frac{1}{2}}w_M^k|\,|A^{\frac{1}{4}}w_M^{k-1}| \\ &\le c_1\lambda_{m+1}^{-\frac{1}{2}}|A^{\frac{1}{2}}w_M^{k-1}|^2|A^{\frac{1}{2}}w_M^k| \le \frac{\nu}{4}|A^{\frac{1}{2}}w_M^k|^2 + \frac{c_1^2|A^{\frac{1}{2}}w_M^{k-1}|^4}{\nu\lambda_{m+1}}, \\ 2(f,w_M^k) &\le 2\lambda_{m+1}^{-\frac{1}{2}}|P_{mM}f|\,|A^{\frac{1}{2}}w_M^k| \le \frac{\nu}{4}|A^{\frac{1}{2}}w_M^k|^2 + \frac{4}{\nu\lambda_{m+1}}|P_{mM}f|^2. \end{aligned}$$

Thanks to (2.5), $|A^{\frac{1}{2}}v_m|^4 + |A^{\frac{1}{2}}w_M^{k-1}|^4 \le |A^{\frac{1}{2}}(v_m + w_M^{k-1})|^4 \le M_1^4$. By (3.6), we see that $c_1M_1\lambda_{m+1}^{-\frac{1}{2}} \le \frac{\nu}{4}$. Then

$$\frac{d|w_M^k|^2}{dt} + \nu\lambda_{m+1}|w_M^k|^2 \le \frac{c_1^2M_1^4L_m^2 + 4|P_{mM}f|^2}{\nu\lambda_{m+1}}.$$

Integrating the above inequality on $[0,t]$ yields

$$|w_M^k(t)| \le \frac{L_m}{\lambda_{m+1}} \cdot \frac{c_1 M_1^2 + 2|P_{mM} f| L_m^{-1}}{\nu} + e^{-\nu\lambda_{m+1} t/2} |P_{mM} a| \quad \forall t \ge 0. \tag{3.21}$$

Similarly, by using equation (3.8) again, we have

$$\begin{aligned} \frac{d|A^{\frac{1}{2}} w_M^k|^2}{dt} &+ 2\nu |A w_M^k|^2 + 2b(v_m, v_m, A w_M^k) + 2b(v_m + w_M^{k-1}, w_M^k, A w_M^k) \\ &+ 2b(w_M^k, v_m + w_M^{k-1}, A w_M^k) - 2b(w_M^{k-1}, w_M^{k-1}, A w_M^k) = 2(f, A w_M^k). \end{aligned}$$

Applying (2.2)–(2.5) and (3.5), the following estimates hold:

$$\begin{aligned} 2b(v_m, v_m, A w_M^k) &\le c_1 |v_m|_\infty |A^{\frac{1}{2}} v_m| \, |A w_M^k| \\ &\le c_1 L_m |A^{\frac{1}{2}} v_m|^2 |A w_M^k| \le \frac{\nu}{8} |A w_M^k|^2 + \frac{4 c_1^2 L_m^2 M_1^4}{\nu}, \\ 2b(v_m + w_M^{k-1}, w_M^k, A w_M^k) &\le c_1 |v_m|_\infty |A^{\frac{1}{2}} w_M^k| \, |A w_M^k| + c_1 |A^{\frac{1}{4}} w_M^{k-1}| \, |A^{\frac{3}{4}} w_M^k| \, |A w_M^k| \\ &\le \frac{c_1}{\lambda_{m+1}^{\frac{1}{2}}} (L_m |A^{\frac{1}{2}} v_m| + |A^{\frac{1}{2}} w_M^{k-1}|) |A w_M^k|^2 \\ &\le \frac{\sqrt{2} c_1 M_1 L_m}{\lambda_{m+1}^{\frac{1}{2}}} |A w_M^k|^2, \\ 2b(w_M^k, v_m + w_M^{k-1}, A w_M^k) &\le c_1 |w_M^k|_\infty |A^{\frac{1}{2}} (v_m + w_M^{k-1})| \, |A w_M^k| \\ &\le c_1 M_1 |w_M^k|^{\frac{1}{2}} |A w_M^k|^{\frac{3}{2}} \le \frac{c_1 M_1}{\lambda_{m+1}^{\frac{1}{2}}} |A w_M^k|^2, \\ 2b(w_M^{k-1}, w_M^{k-1}, A w_M^k) &\le c_1 |w_M^{k-1}|_\infty |A^{\frac{1}{2}} w_M^{k-1}| \, |A w_M^k| \\ &\le c_1 |w_M^{k-1}|^{\frac{1}{2}} |A w_M^{k-1}|^{\frac{1}{2}} |A^{\frac{1}{2}} w_M^{k-1}| \, |A w_M^k| \\ &\le \frac{c_1}{\lambda_{m+1}^{\frac{1}{2}}} |A^{\frac{1}{2}} w_M^{k-1}| \, |A w_M^{k-1}| \, |A w_M^k| \\ &\le \frac{\nu}{8} |A w_M^k|^2 + \frac{4 c_1^2 M_1^2}{\nu \lambda_{m+1}} |A w_M^{k-1}|^2, \\ 2(f, A w_M^k) &\le 2 |P_{mM} f| \, |A w_M^k| \le \frac{\nu}{4} |A w_M^k|^2 + \frac{4}{\nu} |P_{mM} f|^2. \end{aligned}$$

Thanks again to (3.6), $\sqrt{2} c_1 M_1 L_m \lambda_{m+1}^{-\frac{1}{2}} \le \frac{\nu}{4}$. Then we get

$$\frac{d|A^{\frac{1}{2}} w_M^k|^2}{dt} + \nu |A w_M^k|^2 \le \frac{4}{\nu} |P_{mM} f|^2 + \frac{4 c_1^2 M_1^2}{\nu \lambda_{m+1}} |A w_M^{k-1}|^2 + \frac{4 c_1^2 M_1^4 L_m^2}{\nu}. \tag{3.22}$$

By using (2.4), we have $|A w_M^k|^2 \ge \frac{1}{2} |A w_M^k|^2 + \frac{\lambda_{m+1}}{2} |A^{\frac{1}{2}} w_M^k|^2$. Then it follows that

$$\begin{aligned} \frac{d|A^{\frac{1}{2}} w_M^k|^2}{dt} &+ \frac{\nu \lambda_{m+1}}{2} |A^{\frac{1}{2}} w_M^k|^2 + \frac{\nu}{2} |A w_M^k|^2 \\ &\le \frac{4}{\nu} |P_{mM} f|^2 + \frac{4 c_1^2 M_1^2}{\nu \lambda_{m+1}} |A w_M^{k-1}|^2 + \frac{4 c_1^2 M_1^4 L_m^2}{\nu}. \end{aligned}$$

Integrating this inequality on $[0,t]$ admits for $1 \le k \le l$

$$|A^{\frac12} w_M^k(t)|^2 + \frac{\nu}{2}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw_M^k(s)|^2 ds \le \frac{8(|P_{mM}f|^2 + c_1^2 M_1^4 L_m^2)}{\nu^2\lambda_{m+1}} + e^{-\nu\lambda_{m+1}t/2}|A^{\frac12}P_{mM}a|^2 + \frac{4c_1^2M_1^2}{\nu\lambda_{m+1}}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw_M^{k-1}(s)|^2 ds.$$

By using the same method for deriving (3.15) again, we have

$$|A^{\frac12} w_M^k(t)|^2 + \frac{\nu}{2}\max_{1\le i\le l}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw_M^i(s)|^2 ds \le \frac{16(|P_{mM}f|^2 + c_1^2 M_1^4 L_m^2)}{\nu^2\lambda_{m+1}} + 2e^{-\nu\lambda_{m+1}t/2}|A^{\frac12}P_{mM}a|^2 + \frac{8c_1^2M_1^2}{\nu\lambda_{m+1}}\max_{1\le i\le l}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw_M^i(s)|^2 ds.$$

If m is so large that (3.6) is satisfied, we have $\frac{8c_1^2M_1^2}{\nu\lambda_{m+1}} \le \frac{\nu}{2}$. Therefore, we can obtain for $t \ge 0$

$$|A^{\frac12} w_M^k(t)| \le \frac{L_m}{\lambda_{m+1}^{\frac12}}\cdot\frac{4(|P_{mM}f|L_m^{-1} + c_1M_1^2)}{\nu} + \sqrt{2}e^{-\nu\lambda_{m+1}t/4}|A^{\frac12}P_{mM}a|. \tag{3.23}$$

Taking into account (3.21) and (3.23), there must exist a constant $T_0(a,f,\nu) > 0$ such that the results of the theorem are valid. □

Comparing this result with the small eddy estimates of the NSE given in [8], we find that the small eddy components obtained in the small eddy correction method share the same properties as those of the NSE, which will be listed in the next section.

4. Convergence analysis. First of all, we recall the following property of the NSE (see [8]). There exist constants, which will be also denoted by M_0, M_1, K_0, K_1, and $T_0 = T_0(a,f,\nu)$ appearing in Theorem 3.1 and 3.2, such that for any solution $u = p + q$ of (1.1) or (1.5)–(1.6),

$$|u(t)| \le M_0, \quad |A^{\frac12}u(t)| \le M_1 \quad \forall t \ge 0, \tag{4.1}$$

and

$$|q(t)| \le K_0 L_m \lambda_{m+1}^{-1}, \quad |A^{\frac12}q| \le K_1 L_m \lambda_{m+1}^{-\frac12} \quad \forall t \ge T_0. \tag{4.2}$$

LEMMA 4.1. *Under the conditions of Theorem* 3.1, *there exists a constant* $T_0' = T_0'(a,f,\nu) > 0$ *such that for any* $t \ge T_0'$

$$\nu\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw^1(s)|^2 ds \le \frac{K_1^2L_m^2}{\lambda_{m+1}},$$
$$\nu\int_0^t e^{-\nu(t-s)/2}|Aw^1(s)|^2 ds \le K_1^2L_m^2.$$

Proof. Consider $k = 1$ in (3.2). From (3.10), we can get

$$\frac{d|A^{\frac12}w^1|^2}{dt} + \nu|Aw^1|^2 \le \frac{3}{\nu}|Q_mf|^2 + \frac{3c_1^2L_m^2M_1^4}{\nu}.$$

Since $\nu|Aw^1|^2 \ge \frac{\nu\lambda_{m+1}}{2}|A^{\frac12}w^1|^2 + \frac{\nu}{2}|Aw^1|^2$, we have

$$\frac{d|A^{\frac12}w^1|^2}{dt} + \frac{\nu}{2}|Aw^1|^2 + \frac{\nu\lambda_{m+1}}{2}|A^{\frac12}w^1|^2 \le \frac{3}{\nu}|Q_mf|^2 + \frac{3c_1^2L_m^2M_1^4}{\nu}. \tag{4.3}$$

Integrating the above inequality leads to

$$\begin{aligned}
&\nu \int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|Aw^1(s)|^2 ds \\
&\le 2e^{-\nu\lambda_{m+1}t/2}|A^{\frac{1}{2}}Q_m a|^2 + \frac{6|Q_m f|^2}{\nu^2\lambda_{m+1}} + \frac{6c_1^2 L_m^2 M_1^4}{\nu^2\lambda_{m+1}} \\
&\le 2e^{-\nu\lambda_{m+1}t/2}|A^{\frac{1}{2}}Q_m a|^2 + \frac{L_m^2}{6\lambda_{m+1}} \times \frac{36(L_m^{-2}|Q_m f|^2 + c_1^2 M_1^4)}{\nu^2} \\
&\le 2e^{-\nu\lambda_{m+1}t/2}|A^{\frac{1}{2}}Q_m a|^2 + \frac{L_m^2}{6\lambda_{m+1}}\left(\frac{6(L_m^{-1}|Q_m f| + c_1 M_1^2)}{\nu}\right)^2 \\
&= 2e^{-\nu\lambda_{m+1}t/2}|A^{\frac{1}{2}}Q_m a|^2 + \frac{L_m^2}{6\lambda_{m+1}} K_1^2.
\end{aligned}$$

Not enlarging the third term on the left-hand side of (4.3) to $\frac{\nu\lambda_{m+1}}{2}|A^{\frac{1}{2}}w^1|^2$, we can use the same procedure to get

$$\nu \int_0^t e^{-\nu(t-s)/2}|Aw^1(s)|^2 ds \le 2e^{-\nu t/2}|A^{\frac{1}{2}}Q_m a|^2 + \frac{L_m^2}{6} K_1^2.$$

Certainly, there is a $T_0'(a,f,\nu) > 0$ such that the results are valid. ∎

Now, let us introduce some notation:

$$e(t) = u(t) - u^l(t), \quad \varepsilon^k(t) = w^k(t) - w^{k-1}(t), \quad ||| \cdot (t)||| = \sup_{0\le s\le t} |\cdot(s)|.$$

In particular, $\varepsilon^1(t) = w^1(t)$.

THEOREM 4.1. *Under the condition of Theorem* 3.1, *that is, with m large enough such that* (3.6) *is valid, we assume that the results of Theorem* 3.2, (4.1)–(4.2), *and Lemma* 4.1 *hold for* $T_0 = 0$ *and* $T_0' = 0$. *Then we have for* $l \ge 1$

$$|e(t)| \le \frac{\nu K_0 L_m^2}{2^{\frac{l-4}{2}} c_1 K_1}\left(\frac{2^{\frac{1}{4}} c_1 K_1 L_m}{\nu\lambda_{m+1}}\right)^{2^l} \exp(c_1^2 M_1^2 \nu^{-1} t/4) \quad \forall t \ge 0,$$

where the constants M_1, K_0, *and* K_1 *are defined in Theorems* 3.1 *and* 3.2.

Proof. Combine (3.1) and (3.2) to see that

$$\frac{du^l}{dt} + \nu A u^l + B(u^l, u^l) - Q_m B(\varepsilon^l, \varepsilon^l) = f. \tag{4.4}$$

Subtracting (4.4) from (1.1) yields

$$\frac{de}{dt} + \nu A e + B(e,u) + B(u^l, e) + Q_m B(\varepsilon^l, \varepsilon^l) = 0. \tag{4.5}$$

Next, we give a rough estimate of $|e|^2$ in terms of $|\varepsilon^l|^2$ and $|A^{\frac{1}{2}}\varepsilon^l|^2$. Multiplying (4.5) by $2e$ and integrating it on Ω, we have

$$\frac{d|e|^2}{dt} + 2\nu|A^{\frac{1}{2}}e|^2 \le 2|b(e,u,e)| + 2|b(\varepsilon^l, \varepsilon^l, Q_m e)|.$$

For the two terms on the right-hand side of the above inequality, we have

$$
\begin{aligned}
2|b(e,u,e)| &\le c_1|A^{\frac14}e|^2|A^{\frac12}u| \le c_1|e|\,|A^{\frac12}e|\,|A^{\frac12}u| \\
&\le c_1M_1|e|\,|A^{\frac12}e| \le \nu|A^{\frac12}e|^2 + \frac{c_1^2M_1^2}{4\nu}|e|^2,
\end{aligned}
$$

$$
2|b(\varepsilon^l,\varepsilon^l,Q_me)| = 2|b(\varepsilon^l,Q_me,\varepsilon^l)| \le c_1|\varepsilon^l|\,|A^{\frac12}\varepsilon^l|\,|A^{\frac12}e| \le \nu|A^{\frac12}e|^2 + \frac{c_1^2}{4\nu}|\varepsilon^l|^2|A^{\frac12}\varepsilon^l|^2.
$$

Here we applied the Sobolev interpolation inequality (2.2) (the third equation) to the second inequality of both of the above two expressions with $s=\frac14$, $m=\frac12$. Therefore, we get

$$
\frac{d|e|^2}{dt} \le \frac{c_1^2M_1^2}{4\nu}|e|^2 + \frac{c_1^2}{4\nu}|\varepsilon^l|^2|A^{\frac12}\varepsilon^l|^2.
$$

Integrate the above inequality on $[0,t]$ to obtain

$$
|e(t)|^2 \le M_1^{-2}e^{(c_1^2M_1^2\nu^{-1}t/4)}\,|||\varepsilon^l(t)|||^2|||A^{\frac12}\varepsilon^l(t)|||^2. \tag{4.6}
$$

To take advantage of the fast decay property of the small eddy components, we estimate the large and small eddy errors $|e_p|^2$ and $|e_q|^2$ in terms of $|\varepsilon^l|^2$ and $|A^{\frac12}\varepsilon^l|^2$ by using (4.6).

Projecting (4.5) onto H_m and Q_mH, respectively, we have

$$
\frac{de_p}{dt} + \nu Ae_p + P_mB(e_p+e_q,u) + P_mB(u^l,e_p+e_q) = 0, \tag{4.7}
$$

$$
\frac{de_q}{dt} + \nu Ae_q + Q_mB(e_p+e_q,u) + Q_mB(u^l,e_p+e_q) + Q_mB(\varepsilon^l,\varepsilon^l) = 0. \tag{4.8}
$$

Multiplying (4.7) by $2e_p$, integrating it on Ω, and noticing (3.4) lead to

$$
\frac{d|e_p|^2}{dt} + 2\nu|A^{\frac12}e_p|^2 \le 2|b(e_p,u,e_p)| + 2|b(e_q,u,e_p)| + 2|b(u^l,e_q,e_p)|.
$$

Thanks to (2.3)–(2.4), (3.5), and Lemma 1 in [8], we know that

$$
2b(u^l,e_p,e_q) = 2b(P_mu^l,e_p,e_q) + 2b(Q_mu^l,e_p,e_q) \le c_1M_1L_m|A^{\frac12}e_p|\,|e_q|. \tag{4.9}
$$

By using (2.2)–(2.4), (3.4)–(3.5), and (4.9), we have

$$
\begin{aligned}
&2b(e_p,u,e_p) \le c_1M_1|e_p|\,|A^{\frac12}e_p| \le \nu|A^{\frac12}e_p|^2 + \frac{c_1^2M_1^2}{4\nu}|e_p|^2,\\
&2b(e_q,u,e_p) \le c_1M_1|e_q|\,|e_p|_\infty \le c_1M_1L_m|e_q|\,|A^{\frac12}e_p| \le \frac{\nu}{2}|A^{\frac12}e_p|^2 + \frac{c_1^2M_1^2L_m^2}{2\nu}|e_q|^2,\\
&2b(u^l,e_q,e_p) \le c_1M_1L_m|A^{\frac12}e_p|\,|e_q| \le \frac{\nu}{2}|A^{\frac12}e_p|^2 + \frac{c_1^2M_1^2L_m^2}{2\nu}|e_q|^2.
\end{aligned}
$$

Finally, we obtain

$$
\frac{d|e_p|^2}{dt} \le \frac{c_1^2M_1^2}{4\nu}|e_p|^2 + \frac{c_1^2M_1^2L_m^2}{\nu}|e_q|^2.
$$

Integrating the above inequality on $[0, t]$ admits

$$|e_p(t)|^2 \leq 4L_m^2 \exp(c_1^2 M_1^2 \nu^{-1} t/4)|||e_q(t)|||^2. \tag{4.10}$$

Multiplying (4.8) with $2e_q$ and integrating it on Ω, we get

$$\frac{d|e_q|^2}{dt} + 2\nu|A^{\frac{1}{2}}e_q|^2 \leq 2|b(e_p, u, e_q)| + 2|b(e_q, u, e_q)| + 2|b(u^l, e_p, e_q)| + 2|b(\varepsilon^l, \varepsilon^l, e_q)|.$$

For the right-hand side terms, we have

$$\begin{aligned}
2b(e_p, u, e_q) &\leq c_1 M_1 |A^{\frac{1}{4}}e_p|\,|A^{\frac{1}{4}}e_q| \leq c_1 M_1 |A^{\frac{1}{2}}e_q|\,|e_p| \leq \frac{\nu}{4}|A^{\frac{1}{2}}e_q|^2 + \frac{c_1^2 M_1^2}{\nu}|e_p|^2,\\
2b(e_q, u, e_q) &\leq c_1 M_1 |A^{\frac{1}{4}}e_q|^2 \leq c_1 M_1 |A^{\frac{1}{2}}e_q|\,|e_q| \leq \frac{\nu}{4}|A^{\frac{1}{2}}e_q|^2 + \frac{c_1^2 M_1^2}{\nu}|e_q|^2,\\
2b(u^l, e_p, e_q) &\leq c_1 M_1 L_m |A^{\frac{1}{2}}e_p|\,|e_q| \leq c_1 M_1 L_m |e_p|\,|A^{\frac{1}{2}}e_q|\\
&\leq \frac{\nu}{4}|A^{\frac{1}{2}}e_q|^2 + \frac{c_1^2 M_1^2 L_m^2}{\nu}|e_p|^2,\\
2b(\varepsilon^l, \varepsilon^l, e_q) &\leq c_1 |A^{\frac{1}{4}}\varepsilon^l|^2 |A^{\frac{1}{2}}e_q| \leq c_1 |\varepsilon^l|\,|A^{\frac{1}{2}}\varepsilon^l|\,|A^{\frac{1}{2}}e_q| \leq \frac{\nu}{4}|A^{\frac{1}{2}}e_q|^2 + \frac{c_1^2}{\nu}|\varepsilon^l|^2|A^{\frac{1}{2}}\varepsilon^l|^2.
\end{aligned}$$

Then we derive

$$\frac{d|e_q|^2}{dt} + \nu\lambda_{m+1}|e_q|^2 \leq \frac{3c_1^2 M_1^2 L_m^2}{\nu}|e_p|^2 + \frac{c_1^2}{\nu}|\varepsilon^l|^2|A^{\frac{1}{2}}\varepsilon^l|^2.$$

Integrating this inequality on $[0, t]$ yields

$$|e_q(t)|^2 \leq \frac{3c_1^2 M_1^2 L_m^2}{\nu^2 \lambda_{m+1}}|||e_p(t)|||^2 + \frac{c_1^2}{\nu^2 \lambda_{m+1}}|||\varepsilon^l(t)|||^2|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2. \tag{4.11}$$

Thanks to (4.6), we obtain

$$|e_q(t)|^2 \leq \frac{4c_1^2 L_m^2 e^{(c_1^2 M_1^2 \nu^{-1} t)/4}}{\nu^2 \lambda_{m+1}}|||\varepsilon^l(t)|||^2|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2. \tag{4.12}$$

Then the combination of (4.12) and (4.10) admits

$$|e(t)|^2 \leq \frac{20c_1^2 L_m^4 e^{(c_1^2 M_1^2 \nu^{-1} t)/2}}{\nu^2 \lambda_{m+1}}|||\varepsilon^l(t)|||^2|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2. \tag{4.13}$$

To complete the proof, we have to estimate the two factors on the right-hand side of (4.13) for $2 \leq k \leq l$. First of all, from (3.2) we find that $\varepsilon^k(t)$ satisfies

$$\begin{aligned}
&\frac{d\varepsilon^k}{dt} + \nu A\varepsilon^k + Q_m B(v, \varepsilon^k) + Q_m B(\varepsilon^k, v) + Q_m B(w^{k-1}, \varepsilon^k)\\
&\quad + Q_m B(\varepsilon^k, w^{k-1}) + Q_m B(\varepsilon^{k-1}, \varepsilon^{k-1}) = 0 \quad \forall 2 \leq k \leq l.
\end{aligned} \tag{4.14}$$

Multiplying (4.14) by $2\varepsilon^k$ and integrating it on Ω gives

$$\frac{d|\varepsilon^k|^2}{dt} + 2\nu|A^{\frac{1}{2}}\varepsilon^k|^2 \leq 2|b(\varepsilon^k, v, \varepsilon^k)| + 2|b(\varepsilon^k, w^{k-1}, \varepsilon^k)| + 2|b(\varepsilon^{k-1}, \varepsilon^{k-1}, \varepsilon^k)|.$$

We can summarize the estimates of the right-hand-side terms of this inequality as

$$
\begin{aligned}
2|b(\varepsilon^k, v, \varepsilon^k)| &\le c_1|A^{\frac14}\varepsilon^k|^2|A^{\frac12}v| \le c_1 M_1\lambda_{m+1}^{-\frac12}|A^{\frac12}\varepsilon^k|^2,\\
2|b(\varepsilon^k, w^{k-1}, \varepsilon^k)| &\le c_1|A^{\frac14}\varepsilon^k|^2|A^{\frac12}w^{k-1}| \le c_1 M_1\lambda_{m+1}^{-\frac12}|A^{\frac12}\varepsilon^k|^2,\\
2|b(\varepsilon^{k-1}, \varepsilon^{k-1}, \varepsilon^k)| &\le c_1|\varepsilon^{k-1}|\,|A^{\frac12}\varepsilon^{k-1}|\,|A^{\frac12}\varepsilon^k| \le \frac{\nu}{3}|A^{\frac12}\varepsilon^k|^2 + \frac{3c_1^2}{4\nu}|\varepsilon^{k-1}|^2|A^{\frac12}\varepsilon^{k-1}|^2,
\end{aligned}
$$

where, in the first inequality of the last expression above, we used the interpolation inequality in (2.2) with $s = \frac14$ and $m = \frac12$.

By choosing m large enough such that (3.6) is valid, we have

$$\frac{d|\varepsilon^k|^2}{dt} + \nu|A^{\frac12}\varepsilon^k|^2 \le \frac{3c_1^2}{4\nu}|\varepsilon^{k-1}|^2|A^{\frac12}\varepsilon^{k-1}|^2.$$

Finally, we get

$$|\varepsilon^k(t)|^2 \le \frac{3c_1^2}{4\nu^2\lambda_{m+1}}|||\varepsilon^{k-1}(t)|||^2|||A^{\frac12}\varepsilon^{k-1}(t)|||^2 \quad \forall t \ge 0,\ 2 \le k \le l. \tag{4.15}$$

Now, let us estimate $|A^{\frac12}\varepsilon^k(t)|^2$. Multiplying (4.14) by $2A\varepsilon^k$ and integrating it on Ω, we obtain

$$
\begin{aligned}
&\frac{d|A^{\frac12}\varepsilon^k|^2}{dt} + 2\nu|A\varepsilon^k|^2\\
&\qquad \le 2|b(v + w^{k-1}, \varepsilon^k, A\varepsilon^k)| + 2|b(\varepsilon^k, v + w^{k-1}, A\varepsilon^k)| + 2|b(\varepsilon^{k-1}, \varepsilon^{k-1}, A\varepsilon^k)|.
\end{aligned}
$$

For the three terms on the right-hand side, we have

$$
\begin{aligned}
2|b(v + w^{k-1}, \varepsilon^k, A\varepsilon^k)| &\le c_1(|v|_\infty|A^{\frac12}\varepsilon^k| + |A^{\frac14}w^{k-1}|\,|A^{\frac34}\varepsilon^k|)|A\varepsilon^k| \le \frac{c_1 M_1 L_m}{\lambda_{m+1}^{\frac12}}|A\varepsilon^k|^2,\\
2|b(\varepsilon^k, v + w^{k-1}, A\varepsilon^k)| &\le c_1|\varepsilon^k|_\infty|A^{\frac12}(v + w^{k-1})|\,|A\varepsilon^k| \le c_1 M_1\lambda_{m+1}^{-\frac12}|A\varepsilon^k|^2,\\
2|b(\varepsilon^{k-1}, \varepsilon^{k-1}, A\varepsilon^k)| &\le c_1|\varepsilon^{k-1}|_\infty|A^{\frac12}\varepsilon^{k-1}|\,|A\varepsilon^k| \le c_1\lambda_{m+1}^{-\frac12}|A^{\frac12}\varepsilon^{k-1}|\,|A\varepsilon^{k-1}|\,|A\varepsilon^k|\\
&\le \frac{\nu}{3}|A\varepsilon^k|^2 + \frac{3c_1^2}{4\nu\lambda_{m+1}}|A^{\frac12}\varepsilon^{k-1}|^2|A\varepsilon^{k-1}|^2.
\end{aligned}
$$

Thanks to (3.6) again, we derive

$$\frac{d|A^{\frac12}\varepsilon^k|^2}{dt} + \frac{\nu\lambda_{m+1}}{2}|A^{\frac12}\varepsilon^k|^2 + \frac{\nu}{2}|A\varepsilon^k|^2 \le \frac{3c_1^2}{4\nu\lambda_{m+1}}|A^{\frac12}\varepsilon^{k-1}|^2|A\varepsilon^{k-1}|^2. \tag{4.16}$$

Integrating the above inequality on $[0, t]$ yields

$$
\begin{aligned}
&|A^{\frac12}\varepsilon^k(t)|^2 + \frac{\nu}{2}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^k(s)|^2 ds\\
&\quad \le e^{-\nu\lambda_{m+1}t/2}|A^{\frac12}\varepsilon^k(0)|^2 + \frac{3c_1^2|||A^{\frac12}\varepsilon^{k-1}(t)|||^2}{4\nu\lambda_{m+1}}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^{k-1}(s)|^2 ds.
\end{aligned}
$$

Thanks to (3.3), we have for $2 \le k \le l$

$$\left\{\begin{array}{l} |A^{\frac{1}{2}}\varepsilon^k(t)|^2 \le \dfrac{c_1^2 |||A^{\frac{1}{2}}\varepsilon^{k-1}(t)|||^2}{\nu\lambda_{m+1}} \displaystyle\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^{k-1}(s)|^2 ds, \\ \dfrac{\nu}{2}\displaystyle\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^k(s)|^2 ds \\ \qquad \le \dfrac{c_1^2 |||A^{\frac{1}{2}}\varepsilon^{k-1}(t)|||^2}{\nu\lambda_{m+1}} \displaystyle\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^{k-1}(s)|^2 ds. \end{array}\right. \tag{4.17}$$

By (4.15) and (4.17), if we define

$$\begin{aligned} &a_k = |||\varepsilon^k|||^2, \quad b_k = |||A^{\frac{1}{2}}\varepsilon^k|||^2, \\ &c_k = \nu\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^k(s)|^2 ds, \quad \alpha = \frac{c_1^2}{\nu^2\lambda_{m+1}} < 1, \end{aligned}$$

we have for $2 \le k \le l$

$$a_k \le \alpha a_{k-1} b_{k-1}, \quad b_k \le \alpha b_{k-1} c_{k-1}, \quad c_k \le 2\alpha b_{k-1} c_{k-1}. \tag{4.18}$$

From the last two inequalities in (4.18), we see that

$$b_k c_k \le 2\alpha^2 (b_{k-1}c_{k-1})^2.$$

So we get for $2 \le k \le l$

$$b_k \le \left\{\begin{array}{ll} \alpha b_1 c_1 & (k=2) \\ \alpha \left[\displaystyle\prod_{i=0}^{k-3}(2\alpha^2)^{2^i}\right](b_1c_1)^{2^{k-2}} & (k \ge 3) \end{array}\right\} \le 2^{-\frac{1}{2}}(\sqrt{2}\alpha)^{2^{k-1}-1}(b_1c_1)^{2^{k-2}}. \tag{4.19}$$

The inequality for b_2 is obvious, and the one for b_k is obtained as follows: for $3 \le k \le l$

$$\begin{aligned} b_k &\le \alpha b_{k-1}c_{k-1} \le \alpha\cdot(2\alpha^2)^{2^0}(b_{k-2}c_{k-2})^{2^1} \le \alpha\cdot(2\alpha^2)^{2^0}\cdot(2\alpha^2)^{2^1}(b_{k-3}c_{k-3})^{2^2} \le \cdots \\ &\le \alpha\cdot(2\alpha^2)^{2^0+2^1+\cdots+2^{k-3}}(b_1c_1)^{2^{k-2}} = \alpha\left[\prod_{i=0}^{k-3}(2\alpha^2)^{2^i}\right](b_1c_1)^{2^{k-2}}. \end{aligned}$$

Thanks to inequality (4.18), we have for $2 \le k \le l$

$$a_k b_k \le \alpha a_{k-1}b_{k-1}b_k \le \alpha^2 a_{k-2}b_{k-2}b_{k-1}b_k \le \cdots \le \alpha^{k-1}\left[\prod_{i=2}^k b_i\right](a_1b_1).$$

Then by using (4.19), we can finally get

$$\begin{aligned} a_k b_k &\le \alpha^{k-1}\prod_{i=2}^k\left[2^{-\frac{1}{2}}(\sqrt{2}\alpha)^{2^{i-1}-1}(b_1c_1)^{2^{i-2}}\right](a_1b_1) \\ &= \alpha^{k-1}\cdot 2^{-\frac{k-1}{2}}\cdot(\sqrt{2}\alpha)^{\sum_{i=2}^k(2^{i-1}-1)} \\ &\quad \times(b_1c_1)^{\sum_{i=2}^k 2^{i-2}}\cdot(a_1b_1) \le 2^{1-k}(\sqrt{2}\alpha)^{2^k-2}a_1b_1(b_1c_1)^{2^{k-1}-1}. \end{aligned} \tag{4.20}$$

From the results of Theorem 3.2 and Lemma 4.1, we can finally get the result of the theorem from (4.13) and (4.18)–(4.20) by simple calculations. ∎

For H^1-error estimates, we have the following result.

THEOREM 4.2. *Under the same conditions of Theorem* 4.1, *we have for* $l \geq 1$

$$|A^{\frac{1}{2}}e(t)| \leq 4M_1(c_1K_1L_m^5\nu^{-1})^{\frac{1}{2}}\left(\frac{2^{\frac{1}{4}}c_1K_1L_m}{\nu\lambda_{m+1}}\right)^{2^l-\frac{1}{2}} \exp(c_3t) \quad \forall t \geq 0,$$

where $c_3 = 2c_1^3\nu^{-2}M_1^3 + 2^{-1}\nu$ *and* M_1, K_1 *are constants defined in Theorems* 3.1 *and* 3.2.

Proof. Multiplying (4.5) by $2Ae$ and integrating it on Ω yields

$$\frac{d|A^{\frac{1}{2}}e|^2}{dt} + 2\nu|Ae|^2 \leq 2|b(e,u,Ae)| + 2|b(u^l,e,Ae)| + 2|b(\varepsilon^l,\varepsilon^l,Ae)|.$$

Thanks to (2.2), (2.4), and (3.5), we have

$$\begin{aligned}
2|b(e,u,Ae)| + 2|b(u^l,e,Ae)| &\leq c_1M_1|e|_\infty|Ae| + c_1M_1|A^{\frac{1}{2}+\frac{1}{3}}e|\,|Ae| \\
&\leq 2c_1M_1|A^{\frac{1}{2}}e|^{\frac{2}{3}}|Ae|^{\frac{4}{3}} \leq \nu|Ae|^2 + \frac{2c_1^3M_1^3}{\nu^2}|A^{\frac{1}{2}}e|^2, \\
2|b(\varepsilon^l,\varepsilon^l,Ae)| &\leq c_1|\varepsilon^l|_\infty|A^{\frac{1}{2}}\varepsilon^l|\,|Ae| \leq c_1|\varepsilon^l|^{\frac{1}{2}}|A\varepsilon^l|^{\frac{1}{2}}|A^{\frac{1}{2}}\varepsilon^l|\,|Ae| \\
&\leq c_1\lambda_{m+1}^{-\frac{1}{2}}|A^{\frac{1}{2}}\varepsilon^l|\,|A\varepsilon^l|\,|Ae| \leq \nu|Ae|^2 + \frac{c_1^2}{4\nu\lambda_{m+1}}|A^{\frac{1}{2}}\varepsilon^l|^2|A\varepsilon^l|^2.
\end{aligned}$$

Therefore

$$\frac{d|A^{\frac{1}{2}}e|^2}{dt} - \frac{2c_1^3M_1^3}{\nu^2}|A^{\frac{1}{2}}e|^2 \leq \frac{c_1^2}{4\nu\lambda_{m+1}}|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2|A\varepsilon^l(t)|^2.$$

Integrating the above inequality admits

$$|A^{\frac{1}{2}}e(t)|^2 \leq \frac{c_1^2|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2}{4\nu\lambda_{m+1}}\int_0^t e^{\frac{2c_1^3M_1^3(t-s)}{\nu^2}}|A\varepsilon^l(s)|^2ds.$$

Defining $c_3 = 2c_1^3\nu^{-2}M_1^3 + 2^{-1}\nu$, we can deduce from the above inequality that

$$|A^{\frac{1}{2}}e(t)|^2 \leq \frac{c_1^2e^{c_3t}|||A^{\frac{1}{2}}\varepsilon^l(t)|||^2}{4\nu\lambda_{m+1}}\int_0^t e^{-\frac{\nu(t-s)}{2}}|A\varepsilon^l(s)|^2ds. \tag{4.21}$$

Similarly to the proof of Theorem 4.1, we estimate $|A^{\frac{1}{2}}e_p|^2$ and $|A^{\frac{1}{2}}e_q|^2$, respectively. Noticing (2.4) and (4.10), we have

$$\begin{aligned}
|A^{\frac{1}{2}}e_p(t)|^2 \leq \lambda_{m+1}|e_p|^2 &\leq 4L_m^2\lambda_{m+1}e^{\frac{c_1^2M_1^2\nu^{-1}t}{4}}|||e_q(t)|||^2 \\
&\leq 4L_m^2e^{\frac{c_1^2M_1^2\nu^{-1}t}{4}}|||A^{\frac{1}{2}}e_q(t)|||^2.
\end{aligned} \tag{4.22}$$

Multiplying (4.8) with $2Ae_q$, integrating it on Ω, doing some estimates of the corresponding trilinear terms, and then integrating the final differential inequality of $|A^{\frac{1}{2}}e_q|^2$ on $[0,t]$, we can finally get

$$\begin{aligned}
|A^{\frac{1}{2}}e_q|^2 \leq{}& \frac{5c_1^2L_m^2M_1^2}{\nu^2\lambda_{m+1}}|||A^{\frac{1}{2}}e_p(t)|||^2 \\
&+ \frac{5c_1^2|||A^{\frac{1}{2}}\varepsilon(t)^l|||^2}{\nu\lambda_{m+1}}\int_0^t e^{-\nu\lambda_{m+1}(t-s)/2}|A\varepsilon^l(s)|^2ds.
\end{aligned} \tag{4.23}$$

Combination of (4.21)–(4.23) admits

$$|A^{\frac{1}{2}}e(t)|^2 \leq \frac{10c_1^4 M_1^2 L_m^4 e^{2c_3 t}\||A^{\frac{1}{2}}\varepsilon^l(t)|\|^2}{\nu^3\lambda_{m+1}^2} \times \left(\int_0^t e^{-\frac{\nu(t-s)}{2}}|A\varepsilon^l(s)|^2 ds + \lambda_{m+1}\int_0^t e^{-\frac{\nu\lambda_{m+1}(t-s)}{2}}|A\varepsilon^l(s)|^2 ds\right). \tag{4.24}$$

Let us define

$$b_k = \||A^{\frac{1}{2}}\varepsilon^l(t)|\|^2, \quad c_k = \nu\int_0^t e^{-\frac{\nu\lambda_{m+1}(t-s)}{2}}|A\varepsilon^k(s)|^2 ds, \quad \alpha = \frac{c_1^2}{\nu^2\lambda_{m+1}},$$

and

$$d_k = \nu\int_0^t e^{-\nu(t-s)/2}|A\varepsilon^k(s)|^2 ds.$$

We have from (4.17)–(4.18) and (4.24) that

$$|A^{\frac{1}{2}}e(t)|^2 \leq 10M_1^2 L_m^4 e^{2c_3 t}\alpha^2 b_l(d_l + \lambda_{m+1}c_l),$$

$$b_k \leq \alpha b_{k-1}c_{k-1}, \quad c_k \leq 2\alpha b_{k-1}c_{k-1}, \quad d_k \leq 2\alpha b_{k-1}d_{k-1}.$$

The last inequality is obvious if we substitute $\frac{\nu\lambda_{m+1}}{2}|A^{\frac{1}{2}}\varepsilon^k|^2$ with $\frac{\nu}{2}|A^{\frac{1}{2}}\varepsilon^k|^2$ in (4.16). From the last two inequalities we have for $l \geq 1$

$$\begin{aligned} d_l + \lambda_{m+1}c_l &\leq 2\alpha b_{l-1}d_{l-1} + 2\alpha\lambda_{m+1}b_{l-1}c_{l-1} = 2\alpha b_{l-1}(d_{l-1} + \lambda_{m+1}c_{l-1}) \\ &\leq (2\alpha)^2 b_{l-1}b_{l-2}(d_{l-2} + \lambda_{m+1}c_{l-2}) \leq \cdots \leq (2\alpha)^{l-1}\left[\prod_{i=1}^{l-1} b_i\right](d_1 + \lambda_{m+1}c_1). \end{aligned}$$

Now it is easy to get for $l \geq 1$

$$b_l(d_l + \lambda_{m+1}c_l) \leq (2\alpha)^{l-1}\left(\prod_{i=1}^{l} b_i\right)(d_1 + \lambda_{m+1}c_1). \tag{4.25}$$

Thanks to (4.19) and (4.25) we have for $l \geq 1$

$$\begin{aligned} b_l(d_l + \lambda_{m+1}c_l) &\leq (2\alpha)^{l-1}b_1\left[\prod_{i=2}^{l} 2^{-\frac{1}{2}}(\sqrt{2}\alpha)^{2^{i-1}-1}(b_1c_1)^{2^{i-2}}\right](d_1 + \lambda_{m+1}c_1) \\ &\leq \sqrt{2}(2\alpha)^{l-1}b_1(b_1c_1)^{-2^{-1}}\left[\prod_{i=1}^{l} 2^{-\frac{1}{2}}(\sqrt{2}\alpha)^{2^{i-1}-1}(b_1c_1)^{2^{i-2}}\right](d_1 + \lambda_{m+1}c_1) \\ &\leq 2^{-\frac{1}{2}}(\sqrt{2}\alpha)^{2^l-2}(b_1c_1)^{2^{l-1}-\frac{1}{2}}(d_1 + \lambda_{m+1}c_1). \end{aligned}$$

Combining the above inequality with the results of Theorem 3.2, Lemma 4.1, and (4.24) leads to the result of the theorem. ∎

Remark 2. Compared with NGM (1.8) and PPGM (1.10), the small eddy correction method (3.1)–(3.3) involves the self evolution of the small eddy components as well as the interaction between the large and small eddies. Therefore it is suitable for approximating the NSE whenever the small eddy components change slowly or rapidly (in this case, approximating the small eddy equation with the steady Stokes equation in both NGM (1.8) and PPGM (1.10) is not suitable) and can be expected to yield a more accurate approximation. Actually, the result of Theorem 4.1 shows that u^1 can improve the L^2-accuracy of both NGM and PPGM approximations for almost half order ($\lambda_{m+1}^{-\frac{1}{2}}$).

As is said in Remark 1, u^1 is the ONG-approximate solution given in [11], in which the authors imposed some rigorous conditions on the data of the NSE, for example, $a \in D(A)$ and $f \in L^\infty(\mathcal{R}^+, V)$, and for the periodic case proved for the semidiscrete formula (3.7)–(3.9) (see (101) and (102) in [11]) that

$$|u - u_{mM}^1| \le c(t)(\lambda_{m+1}^{-2} + \lambda_{M+1}^{-1}).$$

From the result of Theorem 4.1, it is obvious that our conditions on the data are much weaker. Indeed we demand only that $a \in V$ and $f \in L^\infty(\mathcal{R}^+, H)$. And we get almost the same estimate for both periodic and nonslip boundary conditions except for the logarithm term L_m^4, which increases very slowly as $m \to \infty$ and can be regarded as a constant in general circumstances. A more important thing is that we provide a successive correction procedure which can double the convergence rate of the previous approximate solution just as the Newton method does for elliptic problems. Of course, the larger l is, the more accurate the approximate solution is. On the other hand, Theorem 4.1 tells us that for any fixed $T > 0$ and $t \in [0, T]$

$$|u(t) - u^l(t)| \to 0 \qquad \text{as } l \to \infty.$$

That is, to ensure that $u^l(t)$ approximates $u(t)$ uniformly (with respect to certain prescribed error bound) in certain fixed time interval $[0, T]$, we have two choices: enlarge m or choose a large l.

Remark 3. In this paper, we consider only the continuous small eddy correction method. But some intermediate steps in the proofs of Theorems 4.1 and 4.2, i.e., (4.10)–(4.11) and (4.22)–(4.23), give us some suggestions for constructing its full discrete form. Actually, the four inequalities listed above show that the total error $|e|$ (or $|A^{\frac{1}{2}}e|$) of the scheme can be controlled by its small eddy error $|e_q|$ (or $|A^{\frac{1}{2}}e_q|$), and the accuracy of $|e_q|$ (or $|A^{\frac{1}{2}}e_q|$) is always a half-order higher than the large eddy error $|e_p|$ (or $|A^{\frac{1}{2}}e_p|$). To balance this kind of difference and make the full discrete algorithm more effective, one possible choice is to use different time step lengths for large eddy and small eddy equations.

5. Numerical examples. In this section, we will present some numerical examples of the small eddy correction method for dissipative evolutionary PDEs.

The small eddy correction method (3.1)–(3.3), proposed in section 3, is a time continuous scheme and is defined in an infinite-dimensional Hilbert space. In practice, we have to construct its full discrete formulations, that is, to restrict it in a finite-dimensional subspace (for example, see (3.7)–(3.9)) and do time discretization by a finite difference scheme. Of course, we have to investigate its corresponding numerical stability, error analysis, and possible multilevel scheme in time discretization, as was pointed out in Remark 3. We will address these questions elsewhere.

As the first numerical example, we integrate the following one-dimensional Burger equation with the homogeneous Dirichlet boundary condition on $[0, \pi]$. Using notation similar to that of the NSE, we have

$$\frac{du}{dt} + \nu A u + B(u, u) = f, \qquad u(0) = u_0, \tag{5.1}$$

where, in this case, $A = -\frac{\partial^2}{\partial x^2}$ with domain $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$ and $B(u,v) = \frac{2}{3}uv_x + \frac{1}{3}u_x v$ for $u, v \in H_0^1(0,\pi)$. The eigenfunctions of A are $\phi_i = \sqrt{2/\pi}\sin(ix)$ with corresponding eigenvalues $\lambda_i = i^2$, $i = 1, 2, \dots$.

We choose an exact solution $u_e(x,t)$ and then compute the time dependent forcing term $f(x,t)$. This makes it possible to compare the numerical solutions with the exact solution without computing a large Galerkin approximation as an "exact" solution. We choose

$$u_e(x,t) = \sum_{j=1}^{\infty} \frac{a_j(t)}{j^3} \sin(jx), \quad a_j(t) = 1 + \gamma \sin(j^2 t).$$

To give a numerical implementation of the small eddy correction for (5.1), we use a spectral method for the space discretization, and the Euler backward scheme for the time discretization, with time step length $\tau > 0$. For any two positive integers $M \gg m$, we have the following two finite-dimensional subspaces:

$$H_m = P_m H = \{\phi_1, \phi_2, \dots, \phi_m\} \quad \text{and}$$
$$\hat{H}_{mM} = P_{mM} H = \{\phi_{m+1}, \phi_{m+2}, \dots, \phi_M\}.$$

Then the corresponding numerical scheme for (5.1) reads: find $v_{n+1} \in H_m$ and $w_{n+1}^k \in \hat{H}_{mM}$, for $k = 1, 2, \dots, l$, such that

$$v_{n+1} - v_n + \nu\tau A v_{n+1} + \tau P_m B(v_{n+1} + w_n^l, v_{n+1} + w_n^l) = \tau P_m f((n+1)\tau), \tag{5.2}$$
$$\begin{aligned} w_{n+1}^k - w_n^k + \nu\tau A w_{n+1}^k + \tau P_{mM}[&B(v_{n+1}, v_{n+1}) + B(v_{n+1}, w_{n+1}^k) \\ &+ B(w_{n+1}^k, v_{n+1}) + B(w_{n+1}^{k-1}, w_{n+1}^k) + B(w_{n+1}^k, w_{n+1}^{k-1})] \\ &= \tau P_{mM}[f((n+1)\tau) + B(w_{n+1}^{k-1}, w_{n+1}^{k-1})] \qquad \forall 1 \le k \le l, \end{aligned} \tag{5.3}$$
$$v_0 = P_m u_0,\ w_0^i = P_{mM} u_0,\ i = 1, 2, \dots, l,\ l \ge 1, \ \text{ and } \ w_n^0 = 0 \ \text{ for } \ n \ge 0. \tag{5.4}$$

Owing to our limited computing resources, we compute only the u^0 on H_m, u^1 and u^2 on $[H_m, H_M]$. For fixed $m = 2$, we choose a suitable $M > m$ according to the principle that the error of u^1 on $[H_m, H_M]$ will decrease no further when M increases. Here we choose $M = 254$. The time step length is $\tau = 0.001$, and it is determined in a similar manner. That is, for fixed m and M, decreasing k will not improve the accuracy of u^1 any further. Then we can regard the error as mainly determined by the space discretization. We computed the u^0, u^1, and u^2 in time interval $[0, 2]$ for $\nu = 1.0$. Furthermore, as a comparison, we also computed another u^0 on H_{m+M} in this interval. Following are the L^2-error comparison graph (Figure 5.1) and the CPU time table (Table 5.1), which indicates the CPU time used for deriving u^0, u^1, u^2 on H_m, $[H_m, H_M]$ and a large scale u^0 on H_{m+M}.

Note that the curves which represent the error of u^2 on $[H_m, H_M]$ and u^0 on H_{m+M} coincide.

As the second numerical example, we consider the following two-dimensional NSEs in the bounded domain $\Omega = [0,1] \times [0,1]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \text{div } u = 0, \quad u|_{t=0} = 0, \\ \text{periodic boundary condition,} \end{cases} \tag{5.5}$$

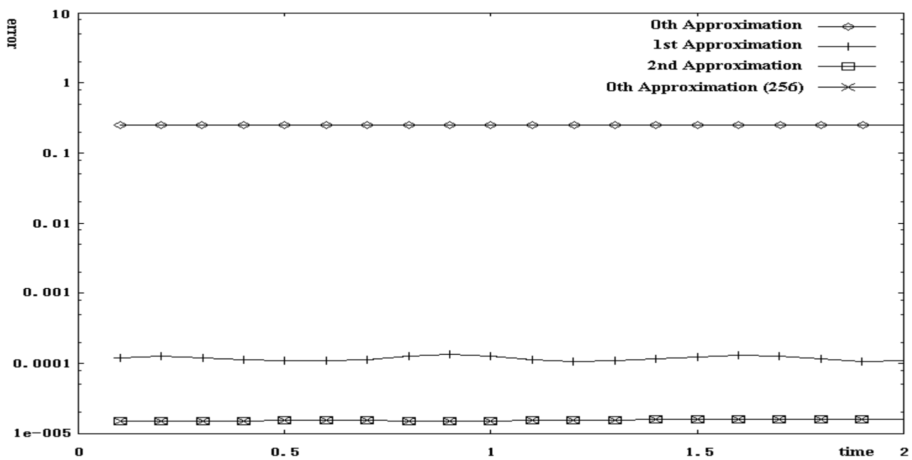

FIG. 5.1. *L^2-error comparison.*

TABLE 5.1
CPU time comparison.

u^0 on H_m	u^1 on $[H_m, H_M]$	u^2 on $[H_m, H_M]$	u^0 on H_{m+M}
0.44 sec	1328.83 sec	7469.48 sec	10054.39 sec

where $\nu = 0.01$ is the kinetic viscosity and $f(t, x, y) = f_1(x, y)(2 + \cos(t))/3$ is the density of the external forces, where $f_1(x, y) = f_1(r, \phi) = (0, f_\phi)^T$,

$$f_\phi(r,\phi) = \begin{cases} \dfrac{1}{8r_+}\displaystyle\int_0^{r_+} \rho(1+\cos(4\rho))^2 d\rho & \text{if } r_+ < \frac{1}{8}, \\ -\dfrac{1}{8r_-}\displaystyle\int_0^{r_-} \rho(1+\cos(4\rho))^2 d\rho & \text{if } r_- < \frac{1}{8}, \\ 0 & \text{otherwise,} \end{cases} \tag{5.6}$$

$r_\pm = |x + iy - (\frac{1}{2}(1+i) \pm \frac{1}{4}e^{i\theta})|$, and $\theta = 0.7$. This external force f represents stirring the fluid in opposite directions at the locations $\frac{1}{2}(1+i) \pm \frac{1}{4}e^{i\theta}$.

In this particular case, if we denote $k = (k_1, k_2)^T \in Z^2$,

$$L^2(\Omega)^2 = \left\{ \phi = \sum_{k\in Z^2, k\neq 0} c_k e^{2\pi i k\cdot x},\ c_k = \overline{c_{-k}},\ \sum_{k\in Z^2,k\neq 0} |c_k|^2 < \infty \right\},$$

$H = PL^2(\Omega)^2$, and $H_m = P_m H$, where, for any $\phi = \sum_{k\in Z^2,k\neq 0} c_k e^{2\pi i k\cdot x} \in L^2(\Omega)^2$,

$$P\phi = \sum_{k\in Z^2,k\neq 0} \left(I - \frac{k\cdot k^T}{|k|^2}\right) c_k e^{2\pi i k\cdot x} \quad \text{and} \quad P_m P\phi = \sum_{0<|k|^2\leq m} \left(I - \frac{k\cdot k^T}{|k|^2}\right) c_k e^{2\pi i k\cdot x}.$$

Projecting the above equations onto H, we can get its functional form (1.1), and its fully discrete form is completely the same as (5.2)–(5.4). Considering the computing scale, we take

$$m = 9^2, \quad M = 19^2, \ \text{ and } \ \tau = 0.005.$$

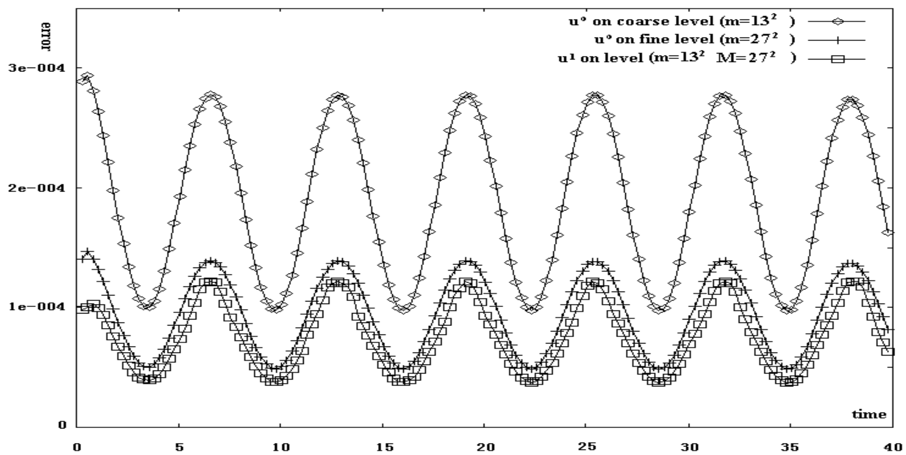

FIG. 5.2. *H^1-error comparison.*

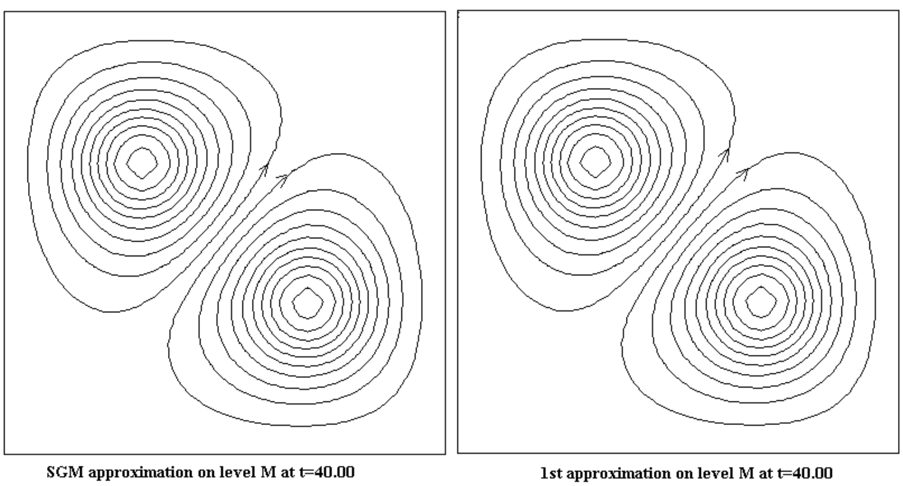

FIG. 5.3. *H^1-streamline graphs.*

We compute only u^0 (with $|k_1|, |k_2| \leq \sqrt{m}$) and u^1 (with $|k_1|, |k_2| \leq \sqrt{M}$). To get the error of these numerical results, we compute another u^0 (with $|k_1|, |k_2| \leq \sqrt{\tilde{M}}$) on a larger finite-dimensional subspace $H_{\tilde{M}}$ with $\tilde{M} = 39^2$ and regard it as the "exact" solution.

In Figures 5.2 and 5.3, we give the H^1-error curves of u^0 on H_m and H_M and u^1 on $[H_m, H_M]$, and the streamline graphs of u^0 on H_M and u^1 on $[H_m, H_M]$ at $t = 40$. Here the CPU time used by u^1 on $[H_m, H_M]$ is less than one-half of the CPU time used by u^0 on H_M.

From the error comparison of Figures 5.1 and 5.2, we can easily find that both u^1 and u^2 can greatly improve the accuracy of u^0, the SGM approximation, with less CPU time than the large scale SGM approximation. If the CPU time is what we care about, we would prefer the 1st approximation u^1. However, if we care more about the accuracy of the approximate solution, we prefer a higher order approximation like u^2.

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