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# IMD BASED NONLINEAR GALERKIN METHOD *

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**Abstract:** By taking example of the 2D Navier-Stokes equations, a kind of improved version of the nonlinear Galerkin method of Marion-Temam type based on the new concept of the inertial manifold with delay (IMD) is presented, which is focused on overcoming the defect that the feasibility of the M-T type nonlinear Galerkin method heavily depended on the least solving scale. It is shown that the improved version can greatly reduce the feasible conditions as well as preserve the superiority of the former version. Therefore, the version obtained here is an applicable, high performance and stable algorithm.

**Key words:** nonlinear Galerkin method; inertial manifold with delay; Navier-Stokes equation

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## Introduction

To make the long term simulation of the Navier-Stokes equations possible during the last decade, the so-called nonlinear Galerkin method (NGM), which is based upon the concepts of the Inertial Manifold (IM)[1] and the Approximate Inertial Manifold (AIM)[2], was constructed (see Refs. [3], [4]) and extensively studied (see Refs. [5], [6], [7], etc.). Among all sorts of NGMs, the M-T (Marion-Temam) type NGM is the most frequently discussed. Let us take the example of the dimensionless abstract Navier-Stokes equation confined in $\Omega = (-\pi, \pi)^2$ with periodic boundary condition

$$\begin{cases} \forall f \in H, \text{ find } u \in C(\mathbf{R}^+, V) \text{ such that} \\ \dfrac{du}{dt} + \lambda Au + B(u, u) = f, \\ u(0) = a. \end{cases} \tag{1}$$

Here $\lambda > 0$ is the kinetic viscosity, $A$ the Stokes operator on $H = \left\{ \phi \in (L^2_{\text{per}}(\Omega))^2, \int_\Omega \phi \, dx = 0, \operatorname{div} \phi = 0 \right\}$. For any $\alpha \in \mathbf{R}$, $A^\alpha$ denotes its power operator with domain $D(A^\alpha)$.

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And we denote $V = D(A^{1/2})$. If we denote by $(\cdot, \cdot)$ and $|\cdot|$ the $L^2$-inner product and norm, $|A^{\alpha}\cdot| = (A^{\alpha}\cdot, A^{\alpha}\cdot)^{1/2}$ is the $H^{2\alpha}$ equivalent norm of $D(A^{\alpha})$. $B(u,u) = P[(u\cdot\nabla)u]$, $P$ is the Leray projector from $(L^2_{\mathrm{per}}(\Omega))^2$ onto $H$. For given $n \in \mathbf{N}$, $V_n \subset V$ is an $n$ dimensional subspace, $P_n$ is a $L^2$-orthogonal projector from $V$ onto $V_n$ and we denote $Q_n = I - P_n$. By denoting $p = P_n u$ and $q = Q_n u$, IM believes that there must be some interactive relation between the large eddy components $p$ and the small eddy components $q$ of $u$ and AIM is some kind of approximation of this relation. From the point view of the M-T type AIM, the large and small eddy components satisfies the following approximate relation:

$$q \approx \Phi(p) = \lambda^{-1}A^{-1}Q_n[f - B(p,p)]. \qquad (2)$$

Then the Navier-Stokes equations can be approximated by the following finite dimensional system to some extend

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} + \lambda Av + P_nB(v+w, v+w) = P_nf, \\ \lambda Aw + Q_nB(v,v) = Q_nf, \\ v(0) = P_na. \end{cases} \qquad (3)$$

For $N \in \mathbf{N}$ and $N > n$, restricting $w$ of (3) in $(P_N - P_n)V$, we can get M-T type NGM.

In Ref. [7], the authors did some detailed analysis of Fourier NGM of M-T type and the results showed that the NGM possesses better numerical stability and computational efficiency than that of standard Galerkin method (SGM). Therefore, it is more suitable for long term and large scale simulation of the Navier-Stokes. But it is worth noticing that the M-T type NGM is based upon such kind of intuitive assumption, that is the small eddy component $q(t)$ together with its time derivative $\dot{q}(t)$ of $u$ are "small" quantities. Only when the ratios of $|q(t)|/|p(t)|$ and $|\dot{q}(t)|/|\dot{p}(t)|$ are quite small, the approximation of (1) by (3) is feasible. Or, it will cause massive loss of accuracy to keep some quantities while omitting some others of the same order. Now the question is whether these ratios will be very smalle or not. The answer is positive as long as we choose a sufficiently large n. We borrow several numerical results in Ref. [8] to illustrate this point.

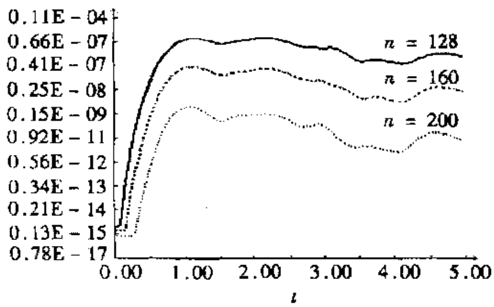


Fig. 1 Ratio of $|q(t)|/|p(t)|$

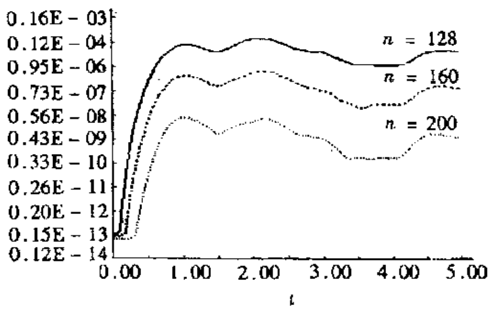


Fig. 2 Ratio of $|\dot{q}(t)|/|\dot{p}(t)|$

In the above examples, $N = 256$. For different $n$ (128, 160 and 200), they gave $|q(t)|/|p(t)|$ and $|\dot{q}(t)|/|\dot{p}(t)|$. Obviously when $n = N/2 = 128$, we can say that $|q(t)|/|p(t)| = o(\lambda^{-1}N^{-2})$. And at least we can regard $|\dot{q}(t)|/|\dot{p}(t)| = O(\lambda^{-1}N^{-2})$. And for $n = 160$ and $n = 200$, $|q(t)|/p(t)|$, $|\dot{q}(t)|/|\dot{p}(t)| =$

$o(\lambda^{-1}N^{-2})$, that is they are the higher or the same order small quantities compared with the space discretization error. Thus for such $N$ and $n$, M-T type NGM is feasible.

Now we still have a problem of computing resources. The above numerical results of Ref. [8] were done on CRAY2. And the numerical simulation of similar scale under general computing environment is almost impossible because of its large computing scale and the large amount of memory it needs. Then we must reduce the solving scale to fit the restriction of the computing resources, that is to minish $N$ and $n$. What will the numerical results be if they are relatively small? We will show this by a numerical example, in which we take $N = 48$, $n = 24$, $\lambda = 0.005$, and use Euler backward difference quotient to approximate the time derivative with time step length $k = 0.0001$.

Figure 3 shows that $|q(t)|/|p(t)| = O(10^{-2}) = o(\lambda^{-1}N^{-2})$, that is $|q(t)|$ is a higher order small quantity compared with the error of space discretization. But Fig.4 indicates taht $\dot{q}(t)$ can not be regarded as "small" quantity. Intuitively, omitting $\dot{q}(t)$ under such solving scale can not guarantee the accuracy of M-T type NGM. In fact, the numerical results of M-T type NGM for such $N$ and $n$ are consistent with the above analysis.

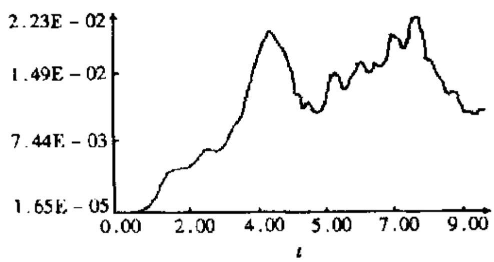


Fig. 3 Ratio of $|q(t)|/|p(t)|$

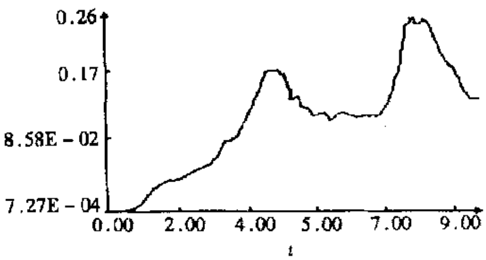


Fig. 4 Ratio of $|\dot{q}(t)|/|\dot{p}(t)|$

By the above numerical tests we can conclude that the M-T type NGM is feasible only when the solving scale (that is $n$) is large enough. That is the feasibility of M-T type NGM is heavily dependent on the dimension of the large eddy equation (which is called the least solving scale). It should satisfy

$$n \geq c\max\left\{\frac{|A^{1/2}u|^2 n^{1/4}}{\lambda^2}, \frac{|f|}{\lambda^2}\right\} \quad \text{in the case of periodic boundary condition.} \quad (4)$$

And for non-slip boundary condition, the requirement of $n$ is much more rigorous. This kind of rigorous requirement roots from the implicit base of IM and AIM, which regards that the interaction between the large and small eddy components is instantaneous. Recently, a more suitable concept of the inertial manifold with delay (IMD) (see Refs. [9] and [10] for detail) believe that this kind of interaction is not merely a simple instantaneous behavior but is in connection with the history of the evolution of the eddies. That is

$$q(t) = \Phi(p(t), q(t - T)) \qquad (T \text{ is certain time delay}). \quad (5)$$

It is shown that this kind of IMD broadly exists for dissipative systems and its existence almost has no restriction of the least dimension of the large eddy equation. We intend to improve the M-T type NGM based on the IMD like (5) in this paper. Our goal is to weaken the feasibility condition (4) as well as to keep the well stability and convergence rate.

## 1 Numerical Scheme and Its Numerical Stability

We can get the following approximation of (1) based upon the idea of IMD:

$$\begin{cases} \frac{dv}{dt} + \lambda Av + P_n B(v + w, v + w) = P_n f, \\ \frac{dw}{dt} + \lambda Aw + Q_n B(v, v) = Q_n f, \\ v(0) = P_n a, \ w(0) = Q_n a. \end{cases} \tag{6}$$

Actually, we can get an approximate IMD of type (5) from (6) according to the method in Ref.[10]. For given $n$, $N \in \mathbf{N}$ and $k > 0$, we fully discrete (6) by using a simple backward Euler difference quotient: denote $Q_{Nn} = P_N - P_n$, $V_N^n = Q_{Nn} V$

$$\begin{cases} \forall m \geq 0, \text{if } (v_m, w_m) \in V_n \times V_N^n \text{ is known, find } (v_{m+1}, w_{m+1}) \text{ such that} \\ v_{m+1} - v_m + \lambda k A v_{m+1} + k P_n B(v_m + w_m, v_m + w_m) = k P_n f, \\ w_{m+1} - w_m + \lambda k A w_{m+1} + k Q_{Nn} B(v_{m+1}, v_{m+1}) = k Q_{Nn} f, \\ v_0 = P_n a, \ w_0 = Q_{Nn} a. \end{cases} \tag{7}$$

Here $k > 0$ is the time step length. For simplicity, sometimes we denote $u_m = v_m + w_m$. (7) is the fully discrete form of (6) based on IMD, which is to be discussed in this section.

The frequently used trilinear form $b(u; v, w) = (B(u, v), w)$ in the rest has the following properties (see Ref.[11]):

$$\begin{cases} b(u; v, w) = - b(u; w, v) \quad (\forall u, v, w \in V), \\ b(u; u, Au) = 0 \quad (\forall u \in D(A)), \\ |b(u; v, w)| \leq c_1 |A^{s_1/2} u| |A^{(s_2+1)/2} v| |A^{s_3/2} w| \\ \qquad (s_i \geq 0, \sum_{i=1}^{3} s_i > 1 \text{ or } \sum_{i=1}^{3} s_i = 1 \text{ but } s_i \neq 1), \\ |A^{\alpha} B(u, v)| \leq \begin{cases} c_1 |A^{1/2} u| |Av| & (\forall \alpha < 1/2), \\ c_1 |A^{3/4} u| |Av| & (\forall \alpha = 1/2), \\ c_1 \left( |u|_{L^\infty} |A^{\alpha + 1/2} v| + |A^{\alpha} u| |A^{1/2} v|_{L^\infty} \right) / 2 \\ \qquad (\forall \alpha > 1/2). \end{cases} \end{cases} \tag{8}$$

In addition, we give some properties of the projectors $P_n$ and $Q_n$:

$$|P_n A^{\alpha} \phi| \leq n^{2(\alpha - \mu)} |A^{\mu} \phi|, \quad |Q_n A^{\mu} \phi| \leq n^{2(\mu - \alpha)} |A^{\alpha} \phi|$$
$$(\forall 0 \leq \mu \leq \alpha, \phi \in D(A^{\alpha})).$$

The following Gronwall lemma will also be used frequently.

Discrete Gronwall Lemma  If $\alpha$, $\beta$ and $h$ are positive constants, $\alpha \neq \beta$ and the positive sequences $\{d_l\}_{l=0}^{\infty}$ satisfy $\alpha d_{l+1} - \beta d_l \leq h$ for any $l \in \mathbf{N}$, we have

$$\forall l \in N, \quad d_l \leq \left( \frac{\beta}{\alpha} \right)^l \left( d_0 - \frac{h}{\alpha - \beta} \right) + \left( \frac{h}{\alpha - \beta} \right).$$

Theorem 1.1  Assume $a \in V$, $f \in H$ and set $R_1^2 = |A^{1/2} a|^2 + \frac{10 |f|^2}{\lambda^2}$. If $k$ and $n$ satisfy the following stability conditions

$$\begin{cases} \dfrac{50c_1^2 knR_1^2}{\lambda}, \quad 2c_1 kL_n nR_1 \le \dfrac{1}{3}, \quad \dfrac{n}{L_n} \ge \dfrac{80c_1 R_1}{3\lambda}, \\ k \le \dfrac{\lambda}{6c_1 L_n(|f|^2 + c_1(1+L_n)^2 R_1^2)^{1/2}}, \end{cases} \tag{9}$$

then the solution of (7) is uniformly stable in $V$ and

$$\forall m \ge 0, \quad |A^{1/2}(v_m + w_m)| \le R_1. \tag{10}$$

The $n$ dependent constant $L_n$ appeared in the theorem is the coefficient of the following Brezis-Gallouet inequality: $\forall \phi \in P_n V$

$$|\phi|_{L^\infty} \le L_n |A^{1/2}\phi|, \quad L_n \sim (1 + \ln n)^{1/2}.$$

Proof  We will prove this theorem by mathematical induction.

First of all, when $l = m$ we assume that

$$|A^{1/2}(v_m + w_m)| \le R_1.$$

Multiplying the first equation of (7) with $Av_{m+1}$ and integrating it on $\Omega$, we have

$$\begin{aligned} &|A^{1/2}v_{m+1}|^2 - |A^{1/2}v_m|^2 + |A^{1/2}(v_{m+1} - v_m)|^2 + 2k\lambda|Av_{m+1}|^2 = \\ &\quad 2k(f, Av_{m+1}) - 2b(u_m, u_m, Av_{m+1}) \le \\ &\quad 2k|f||Av_{m+1}| + 2k|b(v_m, u_m, Av_{m+1})| + 2k|b(w_m, v_m, Av_{m+1})| \le \\ &\quad k\lambda|Av_{m+1}|^2 + \frac{k|f|^2}{\lambda} + 2c_1 kR_1^2(L_n + 1)|Av_{m+1}| \le \\ &\quad 2k\lambda|Av_{m+1}|^2 + \frac{k(|f|^2 + c_1^2(1+L_n)^2 R_1^2)}{\lambda}. \end{aligned} \tag{11}$$

Firstly, we consider the case

$$|A^{1/2}v_{m+1}|^2 \ge |A^{1/2}v_m|^2,$$

and by using (11) we can get

$$|A^{1/2}(v_{m+1} - v_m)|^2 \le \frac{k(|f|^2 + c_1^2(1+L_n)^2 R_1^2)}{\lambda}. \tag{12}$$

Multiplying the first and second equation of (7) by $Av_{m+1}$ and $Aw_{m+1}$, respectively, and their summation leads to

$$\begin{aligned} &|A^{1/2}u_{m+1}|^2 - |A^{1/2}u_m|^2 + |A^{1/2}(u_{m+1} - u_m)|^2 + 2k\lambda|Au_{m+1}|^2 = \\ &\quad 2k(f, Au_{m+1}) - 2kb(v_m + w_m, v_m + w_m, Av_{m+1}) - 2kb(v_{m+1}, v_{m+1}, Aw_{m+1}). \end{aligned}$$

Now we summarize the estimates of each term of the above expression as follows

$$2k(f, Av_{m+1}) \le 2k|f||Av_{m+1}| \le \frac{k\lambda}{10}|Av_{m+1}|^2 + \frac{10k|f|^2}{\lambda},$$

$$\begin{aligned} &2kb(v_m, v_m, Av_{m+1}) = 2kb(v_m, v_m, A(v_{m+1} - v_m)) \le \\ &\qquad 2k|(A^{1/2}B(v_m, v_{m+1}), A^{1/2}(v_{m+1} - v_m))| + \\ &\qquad 2k|b(v_m, v_{m+1} - v_m, A(v_{m+1} - v_m))| \le \\ &\qquad 2c_1 kn^{1/2}B|A^{1/2}v_m||Av_{m+1}||A^{1/2}(v_{m+1} - v_m)| + \\ &\qquad 2c_1 kL_n n|A^{1/2}v_m||A^{1/2}(v_{m+1} - v_m)|^2 \le \\ &\qquad \frac{k\lambda}{10}|Av_{m+1}|^2 + \left(\frac{10c_1^2 knR_1^2}{\lambda} + 2c_1 kL_n nR_1\right)|A^{1/2}(v_{m+1} - v_m)|^2, \end{aligned}$$

$$2kb(v_m, w_m, Av_{m+1}) \le 2k|b(v_m, w_{m+1} - w_m, Av_{m+1})| + 2k|b(v_m, w_{m+1}, Av_{m+1})| \le$$

$$2c_1 kL_n R_1 \left| A^{1/2}(w_{m+1} - w_m) \right| \left| Av_{m+1} \right| + 2c_1 kL_n n^{-1} R_1 \left| Au_{m+1} \right|^2 \le$$

$$\left( \frac{k\lambda}{10} + 2c_1 kL_n n^{-1} R_1 \right) \left| Au_{m+1} \right|^2 + \frac{10 c_1^2 kL_n^2 R_1^2}{\lambda} \left| A^{1/2}(w_{m+1} - w_m) \right|^2,$$

$$2kb(w_m, v_m, Av_{m+1}) \le 2k \left| b(w_{m+1} - w_m, v_m, Av_{m+1}) \right| + 2k \left| b(w_{m+1}, v_m, Av_{m+1}) \right| \le$$

$$2c_1 kR_1 \left| A^{1/2}(w_{m+1} - w_m) \right| \left| Av_{m+1} \right| + 2c_1 kn^{-1} R_1 \left| Au_{m+1} \right|^2 \le$$

$$\left( \frac{k\lambda}{10} + 2c_1 kn^{-1} R_1 \right) \left| Au_{m+1} \right|^2 + \frac{10 c_1^2 kR_1^2}{\lambda} \left| A^{1/2}(w_{m+1} - w_m) \right|^2,$$

$$2kb(w_m, w_m, Av_{m+1}) \le 2k \left| b(w_{m+1} - w_m, w_m, Av_{m+1}) \right| + 2k \left| b(w_{m+1}, w_m, Av_{m+1}) \right| \le$$

$$2c_1 kR_1 \left| A^{1/2}(w_{m+1} - w_m) \right| \left| Av_{m+1} \right| + 2c_1 kn^{-1} R_1 \left| Au_{m+1} \right|^2 \le$$

$$\left( \frac{k\lambda}{10} + 2c_1 kn^{-1} R_1 \right) \left| Au_{m+1} \right|^2 + \frac{10 c_1^2 kR_1^2}{\lambda} \left| A^{1/2}(w_{m+1} - w_m) \right|^2,$$

$$2kb(v_{m+1}, v_{m+1}, Aw_{m+1}) \le 2k \left| b(v_{m+1} - v_m, v_{m+1}, \right| + 2k \left| b(v_m, v_{m+1}, Aw_{m+1}) \right| \le$$

$$2k \left| b(v_{m+1} - v_m, v_{m+1} - v_m, Aw_{m+1}) \right| +$$

$$2k \left| b(v_{m+1} - v_m, v_m, Aw_{m+1}) \right| + 2k \left| AB(v_m, v_{m+1}), w_{m+1}) \right| \le$$

$$2c_1 kL_n \left| A^{1/2}(v_{m+1} - v_m) \right|^2 \left| Aw_{m+1} \right| +$$

$$2c_1 kL_n R_1 \left| A^{1/2}(v_{m+1} - v_m) \right| \left| Aw_{m+1} \right| + 2c_1 kL_n n^{-1} R_1 \left| Aw_{m+1} \right|^2 \le$$

$$\frac{2k\lambda}{10} \left| Au_{m+1} \right|^2 + \frac{10 c_1^2 kL_n^2 \left| A^{1/2}(v_{m+1} - v_m) \right|^2}{\lambda} \left| A^{1/2}(v_{m+1} - v_m) \right|^2 +$$

$$\frac{10 c_1^2 kL_n^2 R_1^2}{\lambda} \left| A^{1/2}(v_{m+1} - v_m) \right|^2 + 2c_1 kL_n n^{-1} R_1 \left| Au_{m+1} \right|^2.$$

Without loss of generality, we always assume that $L_n^2 \le n$. Then the combination of the above six estimates and (12) leads to

$$\left| A^{1/2} u_{m+1} \right|^2 - \left| A^{1/2} u_m \right|^2 + k\lambda \left| Au_{m+1} \right|^2 \le \left( 1 - \frac{50 c_1^2 knR_1^2}{\lambda} - 2c_1 kL_n nR_1 - \right.$$

$$\left. \frac{10 c_1^2 k^2 L_n^2 \left( |f|^2 + c_1^2 (1 + L_n)^2 R_1^2 \right)}{\lambda^2} \right) \left| A^{1/2}(u_{m+1} - u_m) \right|^2 +$$

$$k\lambda \left( 1 - \frac{7}{10} - \frac{5 c_1 L_n n^{-1} R_1}{\lambda} \right) \left| Au_{m+1} \right|^2 \le \frac{10k|f|^2}{\lambda}.$$

As long as the condition (9) in the theorem is satisfied, it holds

$$(1 + k\lambda) \left| A^{1/2} u_{m+1} \right|^2 - \left| A^{1/2} u_m \right|^2 \le \frac{10k|f|^2}{\lambda}. \tag{13}$$

Otherwise, if

$$\left| A^{1/2} v_{m+1} \right| \le \left| A^{1/2} v_m \right|,$$

we can easily get (13).

It follows immediately from the discrete Gronwall inequality that

$$\left| A^{1/2} u_{m+1} \right|^2 \le (1 + k\lambda)^{-(m+1)} \left( |A^{1/2} a|^2 - \frac{10|f|^2}{\lambda^2} \right) + \frac{10|f|^2}{\lambda^2} \le$$

$$|A^{1/2} a|^2 + \frac{10|f|^2}{\lambda^2} \le R_1^2.$$

Then we can conclude the theorem by mathematical induction. □

From the stability conditions of the fully discrete NGM (7), we find that the stability only depends on $nL_n$, that is we demand $knL_n$ small enough, but the stability of SGM needs $kN^2$ small enough. Obviously, the stability of (7) is much better than that of SGM (especially when $N/n$ is more and more larger). In addition, the comparison of the condition of the least solving scale in theorem condition (9) and the restriction (4) of M-T type NGM shows that the restriction of the least solving scale of scheme (7) is much weaker than that of M-T type NGM.

## 2 Error Estimate

For the sake of simplicity, we assume $2n \leq N$. And it is easy to know that $Q_{Nn}B(v_m, v_m) = Q_nB(v_m, v_m)$. In the case of no confusion, we denote $|u_m| = \max_{0\leq i\leq m} |u_m|$ in the following discussion. And we use the same constant $R_1$ to denote the upper bound of the true solution $u(t)$ on $\mathbf{R}^+$, that is $|A^{1/2}u| \leq R_1$.

Before we analyse the error of scheme (7), we give some properties of the following classical Galerkin approximation of (1):

$$\begin{cases} u^{m+1} - u^m + k\lambda Au^{m+1} + kP_NB(u^{m+1}, u^{m+1}) = kP_Nf, \\ u^0 = P_Na. \end{cases} \tag{14}$$

**Lemma 2.1** If $a \in D(A)$, $f \in D(A^{1/2})$, the solution to (1) satisfies

$$u \in C(\mathbf{R}^+, D(A)), \quad \dot{u} \in C(\mathbf{R}^+, H),$$

and there exist constants $R_2 = R_2(\lambda, |f|, |Aa|) > 0$ and $M_0(t) > 0$ such that

$$|Au^m|, |Au(t)| \leq R_2, \quad |u(mk) - u^m|^2 \leq M_0(t)(k^2 + N^{-4})$$
$$(\forall m \geq 0, t \in \mathbf{R}^+).$$

**Theorem 2.1** Assume the condition in Theorem 1.1 and Lemma 2.1 are valid. Then $\forall m \in \mathbf{N}$,

$$|u(mk) - u_m|^2 \leq M_1(t)k^2 + M_2(t)\lambda^{-2}n^{-6} + M_0(t)N^{-4}, \tag{15}$$

where

$$M_1(t) = 4C_1^2E^m + M_0(t), \quad M_2(t) = (22E^m + 8)c_1^2R_1^2R_2^2,$$
$$E = 1 + 7c_1^2R_1R_2k\lambda^{-1},$$

and $C_1$ is a positive constant which will be given in the proof of the theorem.

Proof Since

$$|u(mk) - u_m| \leq |u(mk) - u_m| + |u^m - u_m|,$$

and Lemma 2.1, to obtain the result of the theorem we only have to estimate the second term in the above inequality.

In the proof, we will estimate the errors of the small eddy and large eddy components respectively. This is the key point of the proof because we can adequately use the properties of the small eddy equation.

First of all, we denote

$$v^m = P_nu^m, \quad w^m = Q_nu^m,$$

then

$$\begin{cases} v^{m+1} - v^m + k\lambda Av^m + kP_nB(v^{m+1} + w^{m+1}, v^{m+1} + w^{m+1}) = kP_nf, \\ w^{m+1} - w^m + k\lambda Aw^m + kQ_{N_n}B(v^{m+1} + w^{m+1}, v^{m+1} + w^{m+1}) = kQ_{N_n}f. \end{cases} \tag{16}$$

Denote

$$V_m = v^m - v_m, \quad W_m = w^m - w_m.$$

Subtracting the second expression of (7) from the second expression of (16), multiplying the difference with $W_{m+1}$ and then integrating it on $\Omega$, we have

$$|W_{m+1}|^2 - |W_m|^2 + |W_{m+1} - W_m|^2 + 2k\lambda |A^{1/2} W_{m+1}|^2 \leq$$
$$2k|b(w^{m+1}, u^{m+1}, W_{m+1})| + 2k|b(v^{m+1}, w^{m+1}, W_{m+1})| +$$
$$2k|b(V_{m+1}, v^{m+1}, W_{m+1})| + 2k|b(v_{m+1}, V_{m+1}, W_{m+1})|.$$

We give the estimates of the above four terms on the right hand side in the following:

$$2k|b(w^{m+1}, u^{m+1}, W_{m+1})| \leq 2c_1 k |w^{m+1}|_{1/2} |A^{1/2} u^{m+1}| |W_{m+1}|_{1/2} \leq$$
$$2c_1 kn^{-2} R_1 R_2 |A^{1/2} W_{m+1}| \leq \frac{k\lambda}{4} |A^{1/2} W_{m+1}|^2 + \frac{4c_1^2 kR_1^2 R_2^2}{\lambda n^4},$$
$$2k|b(v^{m+1}, w^{m+1}, W_{m+1})| \leq 2c_1 k |v^{m+1}|_{L^\infty} |A^{1/2} w^{m+1}| |W_{m+1}| \leq$$
$$2c_1 kn^{-2} L_n R_1 R_2 |A^{1/2} W_{m+1}| \leq \frac{k\lambda}{4} |A^{1/2} W_{m+1}|^2 + \frac{4c_1^2 kL_n^2 R_1^2 R_2^2}{\lambda n^4},$$
$$2k|b(V_{m+1}, v^{m+1}, W_{m+1})| \leq 2c_1 k |V_{m+1}|_{L^\infty} |A^{1/2} v^{m+1}| |W_{m+1}| \leq$$
$$2c_1 kL_n R_1 |V_{m+1}| |A^{1/2} W_{m+1}| \leq \frac{k\lambda}{4} |A^{1/2} W_{m+1}|^2 + \frac{4c_1^2 kL_N^2 R_1^2}{\lambda} |V_{m+1}|^2,$$
$$2k|b(v_{m+1}, V_{m+1}, W_{m+1})| \leq 2c_1 k |v^{m+1}|_{L^\infty} |A^{1/2} V_{m+1}| |W_{m+1}| \leq$$
$$2c_1 kL_n R_1 |V_{m+1}| |A^{1/2} W_{m+1}| \leq \frac{k\lambda}{4} |A^{1/2} W_{m+1}|^2 + \frac{4c_1^2 kL_n^2 R_1^2}{\lambda} |V_{m+1}|^2.$$

The combination of the above estimates give

$$(1 + \lambda kn^2) |W_{m+1}|^2 - |W_m|^2 \leq \frac{8c_1^2 kR_1^2 R_2^2}{\lambda n^4} + \frac{8c_1^2 kL_n^2 R_1^2}{\lambda} |V_{m+1}|^2.$$

Applying the discrete Gronwall inequality admits

$$|W_{m+1}|^2 \leq \frac{8c_1^2 R_1^2 R_2^2}{\lambda^2 n^6} + \frac{8c_1^2 L_n^2 R_1^2}{\lambda^2 n^2} |\bar{V}_{m+1}|^2, \tag{17}$$

where $|\bar{V}_{m+1}| = \sup_{0 \leq l \leq m+1} |V_l|$.

Now let us begin to do the large eddy estimate. Denote

$$\Delta u^{m+1} = u^{m+1} - u^m,$$

and from (14) we know there must exist constant $C_1 = \lambda R_2 + cR_1^{3/2} R_2^{1/2} + |f|$ such that

$$|\Delta u^{m+1}| \leq C_1 k. \tag{18}$$

Subtracting the first expression of (7) from the first expression of (16) yields

$$V_{m+1} - V_m + k\lambda AV_{m+1} + kP_n B(\Delta u^{m+1}, u^{m+1}) + kP_n B(u^m, \Delta u^{m+1}) +$$
$$kP_n B(V_m + W_m, u^m) + kP_n B(u_m, V_m + W_m) = 0.$$

Multiplying the above equation with $V_{m+1}$ and integrating it on $\Omega$, we obtain

$$|V_{m+1}|^2 - |V_m|^2 + |V_{m+1} - V_m|^2 + 2k\lambda |A^{1/2} V_{m+1}|^2 \leq$$
$$2k|b(\Delta u^{m+1}, u^{m+1}, V_{m+1})| + 2k|b(u^m, \Delta u^{m+1}, V_{m+1})| +$$
$$2k|b(V_m, u^m, V_{m+1})| + 2k|b(u_m, V_m, V_{m+1})| +$$

$$2k|\ b(W_m, u^m, V_{m+1})\ | + 2k|\ b(u_m, W_m, V_{m+1})\ |.$$

For each term on the right hand side, we have the following estimates

$$2k|\ b(\Delta u^{m+1}, u^{m+1}, V_{m+1})\ | \le 2c_1 k|\ \Delta u^{m+1}\ ||\ u^{m+1}\ |_{3/2}|\ A^{1/2}V_{m+1}\ | \le$$

$$2c_1 C_1 k^2 R_1^{1/2} R_2^{1/2} |\ A^{1/2}V_{m+1}\ | \le \frac{k\lambda}{6}|\ A^{1/2}V_{m+1}\ |^2 + \frac{6c_1^2 C_1^2 R_1 R_2 k^3}{\lambda},$$

$$2k|\ b(u^m, \Delta u^{m+1}, V_{m+1})\ | \le 2c_1 k|\ u^m\ |_{3/2}|\ \Delta u^{m+1}\ ||\ A^{1/2}V_{m+1}\ | \le$$

$$2c_1 C_1 k^2 R_1^{1/2} R_2^{1/2} |\ A^{1/2}V_{m+1}\ | \le \frac{k\lambda}{6}|\ A^{1/2}V_{m+1}\ |^2 + \frac{6c_1^2 C_1^2 R_1 R_2 k^3}{\lambda},$$

$$2k|\ b(V_m, u^m, V_{m+1})\ | \le 2c_1 k|\ V_m\ ||\ A^{1/2}V_{m+1}\ ||\ u^m\ |_{3/2} \le$$

$$2c_1 k R_1^{1/2} R_2^{1/2} |\ V_m\ ||\ A^{1/2}V_{m+1}\ | \le \frac{k\lambda}{6}|\ A^{1/2}V_{m+1}\ |^2 + \frac{6c_1^2 R_1 R_2 k}{\lambda}|\ V_m\ |^2,$$

$$2k|\ b(u_m, V_m, V_{m+1})\ | = 2k|\ b(u_m, V_{m+1}, V_{m+1} - V_m)\ | \le$$

$$2c_1 k L_n R_1 |\ A^{1/2}V_{m+1}\ ||\ V_{m+1} - V_m\ | \le$$

$$\frac{k\lambda}{6}|\ A^{1/2}V_{m+1}\ |^2 + \frac{6c_1^2 R_1^2 k L_n^2}{\lambda}|\ V_{m+1} - V_m\ |^2,$$

$$2k|\ b(W_m, u^m, V_{m+1})\ | \le 2c_1 k|\ W_m\ ||\ A^{1/2}V_{m+1}\ ||\ u^m\ |_{3/2} \le$$

$$2c_1 k R_1^{1/2} R_2^{1/2} |\ W_m\ ||\ A^{1/2}V_{m+1}\ | \le \frac{k\lambda}{3}|\ A^{1/2}V_{m+1}\ |^2 + \frac{3c_1^2 R_1 R_2 k}{\lambda}|\ W_m\ |^2,$$

$$2k|\ b(u_m, W_m, V_{m+1})\ | = 2k|\ b(V_{m+1}, W_m, V_{m+1}) + b(W_m, W_m, V_{m+1}) -$$

$$b(V_{m+1} - V_m, W_m, V_{m+1}) - b(u^m, W_m, V_{m+1})\ | \le$$

$$2c_1 k n^{-1} L_n |\ A^{1/2}W_m\ ||\ A^{1/2}V_{m+1}\ |^2 + 2c_1 k|\ A^{1/2}W_m\ ||\ W_m\ ||\ A^{1/2}V_{m+1}\ | +$$

$$2c_1 k L_n |\ A^{1/2}W_m\ ||\ V_{m+1} - V_m\ ||\ A^{1/2}V_{m+1}\ | + 2c_1 k|\ u^m\ |_{3/2}|\ W_m\ ||\ A^{1/2}V_{m+1}\ | \le$$

$$2c_1 k n^{-1} L_n R_1 |\ A^{1/2}V_{m+1}\ |^2 + \frac{5k\lambda}{6}|\ A^{1/2}V_{m+1}\ |^2 + \frac{3c_1^2 R_1^2 k}{\lambda}|\ W_m\ |^2 +$$

$$\frac{6c_1^2 R_1^2 L_n^2 k}{\lambda}|\ V_{m+1} - V_m\ |^2 + \frac{3c_1^2 R_1 R_2 k}{\lambda}|\ W_m\ |^2.$$

We used the Sobolev interpolation inequality several times in the above estimates. To avoid introducing more symbols, we regard the coefficient in the inequality as 1 and this will not cause any significant differences.

Under the theorem condition we have $2c_1 k L_n n^{-1} R_1 \le 1/6$. Combining the above estimates with (17) yields

$$|\ V_{m+1}\ |^2 - |\ V_m\ |^2 \le \frac{12c_1^2 C_1^2 R_1 R_2 k^3}{\lambda} + \frac{6c_1^2 R_1 R_2 k}{\lambda}|\ V_m\ |^2 + \frac{9c_1^2 R_1 R_2 k}{\lambda}|\ W_m\ |^2 \le$$

$$\frac{12c_1^2 C_1^2 R_1 R_2 k^3}{\lambda} + \frac{6c_1^2 R_1 R_2 k}{\lambda}|\ V_m\ |^2 + \frac{72c_1^4 R_1^3 R_2^3 k}{\lambda^3 n^6} + \frac{72c_1^4 R_1^3 R_2 L_n^2 k}{\lambda^3 n^2}|\ \bar{V}_m\ |^2.$$

Thanks to the conditions of the theorem we have $\dfrac{72c_1^4 R_1^3 R_2 L_n^2 k}{\lambda^3 n^2} \le \dfrac{c_1^2 R_1 R_2 k}{\lambda}$. Therefore

$$|\ \bar{V}_{m+1}\ |^2 \le \frac{12c_1^2 C_1^2 R_1 R_2 k^3}{\lambda} + \frac{72c_1^4 R_1^3 R_2^3 k}{\lambda^3 n^6} + \left(1 + \frac{7c_1^2 R_1 R_2 k}{\lambda}\right)|\ \bar{V}_m\ |^2.$$

By using the discrete Gronwall inequality we can finally get

$$|\bar{V}_m|^2 \le \left(1 + \frac{7c_1^2 R_1 R_2 k}{\lambda}\right)^m \left(2C_1^2 k^2 + \frac{11 c_1^2 R_1^2 R_2^2}{\lambda^2 n^6}\right). \quad (19)$$

The combination of (17), (19) and Lemma 2.1 leads to (15). □

The results of the Theorems 1.1 and 2.1 show that the modified NGM scheme (7) keeps the stability and accuracy of M-T type NGM. At the same time, we find from the condition (9) of the Theorem 1.1 and the conditions of the Theorem 2.1 that its demand on $n$ is much weaker than the condition (4). That is to say that the modified scheme (7) can preserve the accuracy of the M-T type NGM and improve the stability of the scheme as well as loosen the restriction of the so-called least solving scale.

## 3 Numerical Experiments

In the two numerical experiments in this section, we always take $N = 48$ and $n = 24$ in the scheme (7). That is, we have to solve 576 large eddy equations and 1728 small eddy equations. In addition, the physical quantities appears in this section (such as time $t$ and velocity $u$) are all dimensionless. The computational work is mainly concentrated on the computing of the bilinear operators $P_nB(v_m + w_m, v_m + w_m)$ and $(Q_{Nn}B(v_m, v_m)$ when we do the numerical simulation of (7). Here we will use the standard vector fast Fourier transformation (VFFT) to do these computations. On the other hand, to avoid the so-called Aliasing Error (see Ref. [12]), we will use the (3/2)-rule to treat the two terms and this will cause some non-necessary computations. Although it is only necessary for us to compute the large and small eddy components, respectively, the matrices involved in the computation of VFFT are all order $(3N/2) \times (3N/2)$ and, of course, the result matrices are the same order. We have to spend some extra CPU time on some quantities we do not care about and this is caused by the generality of the standard VFFT package. It is not difficult to find that the amount of work of the computation in single time step in (7) is as much as that of two time steps in SGM with the same solving scale (this is the case when using VFFT; in the case of direct computation, for example the finite element NGM case, the amount of the computation in each time step of (7) is much less than the classical method). Nevertheless, the numerical results show that NGM can still be a numerical scheme which can save a lot of CPU time and the reason will be talked after the numerical results.

First of all, we will solve the Navier-Stokes equation whose exact solution is known by using the NGM scheme (7) in this paper and the SGM scheme and this makes it possible to compare their error and the evolution of their energy.

The Navier-Stokes equation whose exact solution is known means we assign a known function

$$u(t) = \sum_{l \in Z^2, l \neq 0} a_l \cos(\omega_0 + |l_1| t + |l_2| t) e^{il \cdot x}, \quad (20)$$

as the solution of the systems (1) and then we can compute the external force $f$, initial value $a = u(x, 0)$. Then we numerically solve $u$ for such $a$ and $f$. In this example, we demand

$$|a_l| \sim |l|^{-5}. \quad (21)$$

For SGM scheme, we take the time step length $k_{SGM} = 0.0001$. And for NGM scheme (7), we take $k_{NGM} = 0.0005$. From Fig. 5, we find that the NGM scheme (7) have the same accuracy as SGM scheme in the beginning time interval but the long-term behaviors show that (7) has better accuracy. And the Fig. 6 shows that the classical method divergences very quickly when $k_{SGM} = k_{NGM}$. This is sufficient for us to conclude that the IMD based NGM (7) is much stable than SGM and therefore it is more suitable for long-term simulation.

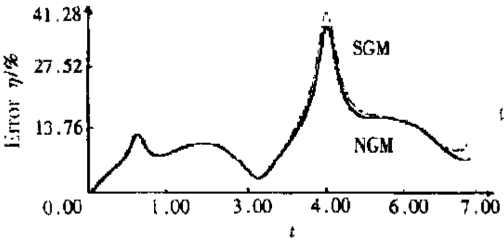


Fig. 5 Evolution of the relative error

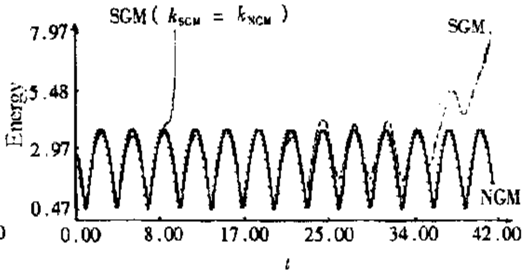


Fig. 6 Evolution of the norms

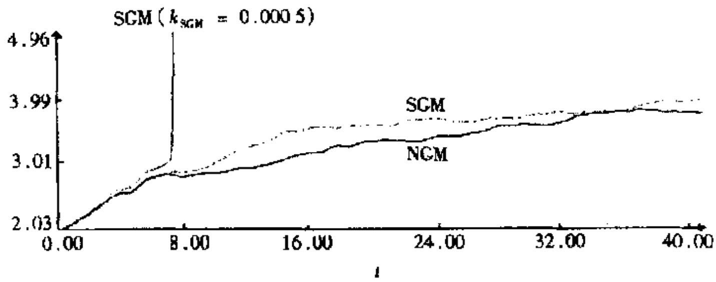


Fig. 7 Evolution of the norm

In the next example, we will numerically solve NGM scheme (7) and SGM scheme for certain given initial value $a$ and external force $f$ and give the evolution of the fluid energy ( $L^2$ norm of $(u(x,t))$. Here we have the same demand (21) for $a$. And we take $k_{NGM} = 0.0005$ in NGM (7) while $k_{SGM} = 0.0001$ and $k_{SGM} = k_{NGM} = 0.0005$ in SGM, respectively. In this example, we take $\lambda = 0.005$.

From the above two numerical examples, we can conclude that the stability and accuracy of the IMD based NGM (7) is superior to SGM. It is worth pointing out that this kind of better stability of (7) makes it possible to choose larger time step. For example, $k_{SGM} = 5k_{NGM}$ in our simulation. Taking into account of the additional computation by using VFFT in the numerical simulation of the Navier-Stokes equation by using NGM and SGM respectively, the CPU time used by NGM (7) should be no more than half of the CPU time used by SGM. The actual computation just verified this. Obviously, NGM scheme is more suitable for long-term simulation and its application will help people in studying the long-term behavior of the Navier-Stokes equation.

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