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The uniform attractor for the 2D non-autonomous Navier–Stokes flow in some unbounded domain☆

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Abstract

By using the energy equation method, the existence of the uniform attractor for the 2D non-autonomous Navier–Stokes equations in some unbounded domains with usual settings is obtained without the restriction of the forcing term belonging to some weighted Sobolev space. And for the quasi-periodic forcing term, the estimation of the Hausdorff dimension of such attractors is also obtained.

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1. Introduction

It is well known that the long time behavior of dynamical systems generated by evolutionary partial differential equations of mathematical physics can be described in terms of attractors of the corresponding semigroups in many cases (see [3,13,20] and the references therein). The global attractors for the 2D autonomous Navier–Stokes equations were first obtained for bounded domains in the works of Ladyzhenskaya [14] and Foias

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and Temam [9]. The latter works show the finite dimensionality of such attractors in the sense of Hausdorff dimension (see [5,6,20]). Later on, the unbounded domain case was studied by Abergel [1] and Babin [2]. Yet because of the lack of compactness of the corresponding semigroups, one often makes the assumption that the forcing term and initial value belong to some weighted Sobolev space. But for certain unbounded domain on which the Poincaré inequality holds, Rosa [17] derived the existence and the dimension estimate of the global attractor without the forcing term restriction by using the energy equation method suggested by Ball [4]. This technique of energy equation method was applied to a weakly damped, driven KdV equation by Ghidaglia [10,11], and, subsequently, by several other authors in different contexts [22,21,15], especially for the so-called noncompact autonomous dynamical systems. Compared with the autonomous dynamical systems, the non-autonomous dynamical systems are less well understood. To our knowledge, the systematic study of the so-called uniform attractors of such kind of partial differential systems were discussed by Chepyzhov and Vishik [8]. And for quasi-periodic forcing terms, the finite dimensionality of the uniform attractors were obtained. Then Chepyzhov and Efendiev [7] studied the finite dimensionality of the uniform attractors of 2D non-autonomous Navier–Stokes equations in unbounded strip. The finite dimensionality is also based on the same forcing term restriction. Like what has been observed by Rosa [17] for the dimension estimate of the global attractor of the autonomous Navier–Stokes equations in unbounded domain, the dimension estimate of the uniform attractor of the non-autonomous Navier–Stokes equations in unbounded strip is also independent of the weighted norm of the forcing term. Therefore it is very natural to expect the existence of the uniform attractors of such systems for more general forces.

After we finished this paper, we learned that Moise et al. [16] derived the existence of the uniform attractor for an abstract noncompact non-autonomous system by the energy equation method and applied their results to flows past an obstacle in an infinite strip without the forcing term restriction. Comparing our result with the results in [16], the main difference is that the condition for the existence of the uniform attractor in this paper is weaker. In [16], the symbol space $\mathscr{F}$ is required to be precompact or the semigroup $T(h)$ on $\mathscr{F}$ is required to be asymptotically compact or the semiprocess $U_f(t,\tau)$ is required to be globally uniformly asymptotically smooth (uniformly asymptotically compact in [8]). All these requirements are either difficult to be satisfied or difficult to be verified (since it is very difficult to show the existence of a compact uniformly attracting set for the semiprocess). In this paper, we obtained the existence of the uniform attractors for the 2D non-autonomous Navier–Stokes equations with usual settings in the domains (bounded or unbounded) on which the Poincaré inequality holds by the energy equation method.

2. Preliminaries

We consider some incompressible viscous flow in a region $\Omega \subset \mathbf{R}^2$ with rigid boundary $\partial\Omega$ and governed by the following initial-boundary value problem of the non-autonomous Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t) \quad \text{in } \Omega\times\mathbf{R}_\tau^+, \\ \nabla\cdot u = 0 \quad \text{in } \Omega\times\mathbf{R}_\tau^+, \\ u(x,t) = 0 \quad \text{on } \partial\Omega\times\mathbf{R}_\tau^+, \\ u(x,\tau) = u_\tau(x) \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\mathbf{R}_\tau^+ = [\tau, +\infty)$, $u(x,t)\in\mathbf{R}^2$ and $p(x,t)\in\mathbf{R}$ denote the velocity and the pressure, respectively, at the point $x\in\Omega$ and at time $t\geqslant\tau$, $u_\tau(x)(\nabla\cdot u_\tau = 0)$ stands for the velocity field at the initial time $\tau\geqslant 0$, $\nu>0$ is the kinematic viscosity of the fluid and $f(x,t)\in\mathbf{R}^2$ is the time-dependent external force which drives the flow. The domain Ω can be an arbitrary bounded or unbounded open set in $\mathbf{R}^2$ without any regularity assumption on its boundary $\partial\Omega$. The only assumption on Ω is that the following Poincaré inequality holds on it:

$$\begin{cases} \text{There exists } \lambda_1 > 0 \text{ such that} \\ \displaystyle\int_\Omega \phi^2\,\mathrm{d}x \leqslant \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2\,\mathrm{d}x, \quad \forall\phi\in H_0^1(\Omega). \end{cases} \tag{2.2}$$

The mathematical settings of problem (2.1) is classical (see [19] for instance). Let

$$\mathbf{L}^2(\Omega) = (L^2(\Omega))^2, \quad \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^2,$$

equipped, respectively, with the inner products

$$(u,v) = \int_\Omega u\cdot v\,\mathrm{d}x, \quad \forall u,v\in\mathbf{L}^2(\Omega)$$

and

$$((u,v)) = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x, \quad \forall u,v\in\mathbf{H}_0^1(\Omega).$$

And the induced norms are

$$|\cdot| = (\cdot,\cdot)^{1/2}, \quad \|\cdot\| = ((\cdot,\cdot))^{1/2}.$$

Thanks to (2.2), the norm $\|\cdot\|$ is equivalent to the usual Sobolev norm in $\mathbf{H}_0^1(\Omega)$. Now we set

$$\begin{cases} \mathscr{V} = \{v\in(\mathscr{D}(\Omega))^2 : \nabla\cdot v = 0 \text{ in } \Omega\}, \\ H = \text{closure of } \mathscr{V} \text{ in } \mathbf{L}^2(\Omega), \\ V = \text{closure of } \mathscr{V} \text{ in } \mathbf{H}_0^1(\Omega), \end{cases}$$

with H and V endowed with the inner product and norm of, respectively, $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$. It follows from (2.2) that

$$|u|^2 \leqslant \frac{1}{\lambda_1}\|u\|^2, \quad \forall u\in V. \tag{2.3}$$

If we identify H with its dual H', we arrive at the inclusions

$$V\subset H\equiv H'\subset V',$$

where each space is dense in the following one and the injections are continuous. Different from the case of bounded domain, the injections above are no longer compact in the case of an unbounded domain. Now we assume that

$$u_\tau \in H, \quad f \in \mathbf{L}^\infty(\mathbf{R}_\tau^+; V'), \tag{2.4}$$

we then consider the following weak formulation of (2.1): find

$$u \in \mathbf{L}^\infty(\mathbf{R}_\tau^+; H) \cap \mathbf{L}^2(\tau, T; V), \quad \forall T > \tau,$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v) + \nu((u, v)) + b(u, u, v) = \langle f, v\rangle, \quad \forall v \in V,\ t > \tau \tag{2.5}$$

and

$$u(\tau) = u_\tau \text{ in } H. \tag{2.6}$$

Here $\langle\cdot,\cdot\rangle$ is the duality product between V' and V, $b: V \times V \times V \to \mathbf{R}$ is given by

$$b(u, v, w) = ((u \cdot \nabla)v, w).$$

Thanks to [19], (2.5) is equivalent to the functional equation

$$u' + \nu Au + B(u, u) = f \text{ in } V', \quad \forall t > \tau, \tag{2.7}$$

where $A: V \to V'$ is the Stokes operator defined by

$$\langle Au, v\rangle = ((u, v)), \quad \forall u, v \in V,$$

which is an isomorphism from V into V', and $B: V \times V \to V'$ defined by

$$\langle B(u, v), w\rangle = b(u, v, w), \quad \forall u, v, w \in V$$

and

$$\|B(u, u)\|_{V'} \leqslant 2^{\frac{1}{2}} |u| \|u\|, \quad \forall u \in V. \tag{2.8}$$

Now we state a classical result (see [19]) in the following

Theorem 2.1. *Given u_τ and f satisfying (2.4), there exists a unique solution*

$$u \in \mathbf{L}^\infty(\mathbf{R}_\tau^+; H) \cap \mathbf{L}^2(\tau, T; V), \quad \forall T > \tau,$$

such that (2.5) (hence (2.7)) and (2.6) hold. Moreover, $u' \in \mathbf{L}^2(\tau, T; V')$ for all $T > \tau$ and $u \in C(\mathbf{R}_\tau^+; H)$.

Following the terminology of [8], the system (2.5) (hence (2.7)) and (2.6) can be described by a family of two-parametric maps from H to H:

$$u(t) = U_f(t, \tau)u_\tau, \quad \forall t \geqslant \tau \geqslant 0,\ u(\tau) = u_\tau \in H.$$

Here $f \in \mathbf{L}^\infty(\mathbf{R}^+; V')$ is called the time symbol of the system. Thanks to Theorem 2.1, we can easily see that in H

$$\begin{cases} U_f(\tau, \tau) = Id(\text{identity}) \text{ and} \\ U_f(t, s) \circ U_f(s, \tau) = U_f(t, \tau) \text{ for all } t \geqslant s \geqslant \tau \geqslant 0. \end{cases} \tag{2.9}$$

Now we give some definitions (see [8] for instance).

Definition 2.1. For given time symbol $f \in \mathbf{L}^\infty(\mathbf{R}^+; V')$, a family of two-parametric maps $\{U_f(t,\tau)\}$ with $t \geqslant \tau \geqslant 0$ is called a semiprocess in H if (2.9) is satisfied.

For a given bounded close set $\mathscr{F} \subset \mathbf{L}^\infty(\mathbf{R}^+; V')$, we get a family of semiprocesses

$$U_f(t,\tau): H \to H, \quad \forall f \in \mathscr{F}, \quad t \geqslant \tau \geqslant 0,$$

where $\mathscr{F}$ is called the symbol space.

Definition 2.2. A family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, is said to be uniformly (with respect to (w.r.t.) $f \in \mathscr{F}$) bounded if for any bounded set B of H, the set $\bigcup_{f\in\mathscr{F}} \bigcup_{\tau\geqslant 0} \bigcup_{t\geqslant\tau} U_f(t,\tau)B$ is bounded.

Definition 2.3. A set $\mathscr{B}_0 \subset H$ is said to be uniformly (w.r.t. $f \in \mathscr{F}$) absorbing for the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, if for any bounded set B of H there exists $M = M(\tau, B) \geqslant \tau$ such that $\bigcup_{f\in\mathscr{F}} U_f(t,\tau)B \subset \mathscr{B}_0$ for all $t \geqslant M$.

Now let us define the uniform attractor of a family of semiprocesses.

Definition 2.4. A closed set $\mathscr{A}_{\mathscr{F}} \subset H$ is said to be the uniform (w.r.t. $f \in \mathscr{F}$) attractor of the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, if

[*Attracting property*]: for any bounded set B of H, $\mathscr{A}_{\mathscr{F}}$ satisfies;

$$\lim_{t\to+\infty} \sup_{f\in\mathscr{F}} \mathrm{dist}_H(U_f(t,\tau)B, \mathscr{A}_{\mathscr{F}}) = 0, \quad \forall \tau \geqslant 0;$$

[*Property of minimality*]: $\mathscr{A}_{\mathscr{F}} \subset P$ as long as the attracting property is valid for $P \subset H$.

Here we recall that for $X, Y \subset H$

$$\mathrm{dist}_H(X,Y) = \sup_{x\in X} \mathrm{dist}_H(x,Y) = \sup_{x\in X} \inf_{y\in Y} \|x-y\|_H.$$

Similar to the autonomous case (see [17]), we will give some properties of the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, corresponding to system (2.5) (hence (2.7)) and (2.6) in the next section.

3. Some properties

First of all, for a given bounded symbol space

$$\mathscr{F} \subset \{f \in \mathbf{L}^\infty(\mathbf{R}^+; V') : \|f\|_{\mathbf{L}^\infty(\mathbf{R}^+;V')} \leqslant \rho_{\mathscr{F}}\}$$

$$(\rho_{\mathscr{F}} > 0 \text{ is a certain constant}), \tag{3.1}$$

we will show that the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, is uniformly (w.r.t. $f \in \mathscr{F}$) bounded and possesses a uniformly (w.r.t. $f \in \mathscr{F}$) absorbing set.

Indeed, by using the well-known orthogonality property

$$b(u,v,v) = 0, \quad \forall u, v \in V,$$

we obtain the following energy equation:

$$\frac{\mathrm{d}|u|^2}{\mathrm{d}t} + 2\nu\|u\|^2 = 2\langle f, u\rangle, \quad \forall t > \tau. \tag{3.2}$$

Then one can easily deduce the classical estimate by using Poincaré inequality (2.2) that

$$\begin{cases} |u(t)|^2 \leqslant |u_\tau|^2 \mathrm{e}^{-\nu\lambda_1(t-\tau)} + \dfrac{1}{\nu^2\lambda_1}\|f\|^2_{\mathbf{L}^\infty(\mathbf{R}^+;V')}, & \forall t \geqslant \tau, \\ \dfrac{1}{t-\tau}\displaystyle\int_\tau^t \|u\|^2\,\mathrm{d}s \leqslant \dfrac{1}{\nu}\left(\dfrac{1}{t-\tau}|u_\tau|^2 + \dfrac{1}{\nu}\|f\|^2_{\mathbf{L}^\infty(\mathbf{R}^+;V')}\right), & \forall t > \tau. \end{cases} \tag{3.3}$$

This is sufficient to indicate that the corresponding family of semiprocesses is uniformly (w.r.t. $f \in \mathscr{F}$) bounded. Furthermore, the following set:

$$\mathscr{B}_0 = \left\{ v \in H : |v| \leqslant \rho_0 = \frac{1}{\nu}\sqrt{\frac{2}{\lambda_1}}\rho_{\mathscr{F}} \right\}, \tag{3.4}$$

is a uniformly (w.r.t. $f \in \mathscr{F}$) absorbing set in H. Actually for a given bounded set

$$B \subset B_0(r) = \{v \in H : |v| \leqslant r\} \ (r > 0 \text{ is a certain constant}),$$

if we take

$$M(\tau, B) = \frac{2}{\nu\lambda_1}\ln\frac{\nu r\sqrt{\lambda_1}}{\rho_{\mathscr{F}}} + \tau,$$

we can get

$$\bigcup_{f\in\mathscr{F}} U_f(t,\tau)B \subset \mathscr{B}_0, \quad \forall t \geqslant M(\tau, B).$$

Now we summarize the above properties in

Lemma 3.1. *For any bounded symbol space $\mathscr{F}$ given by (3.1), the family of semiprocesses associated with system (2.5)(hence (2.7)) and (2.6) is uniformly (w.r.t. $f \in \mathscr{F}$) bounded in $\mathbf{L}^\infty(\mathbf{R}^+_\tau; H)\cap\mathbf{L}^2(\tau, T; V)$ and possesses uniformly (w.r.t. $f \in \mathscr{F}$) absorbing sets in H.*

For further analysis, we will need the following weak continuity of this family of semiprocesses.

Lemma 3.2. *For $\tau \geqslant 0$ fixed, let $\{u_{\tau_n}\}_n$ be a sequence in H converging weakly in H to an element $u_\tau \in H$, $\{f_n\}_n$ be a sequence in $\mathscr{F}$ converging weakly* in $\mathbf{L}^\infty(\mathbf{R}^+; V')$ to an element $f \in \mathscr{F}$. Then*

$$U_{f_n}(t,\tau)u_{\tau_n} \to U_f(t,\tau)u_\tau \ \textit{weakly in } H, \quad \forall t \geqslant \tau \tag{3.5}$$

and

$$U_{f_n}(\cdot,\tau)u_{\tau_n} \to U_f(\cdot,\tau)u_\tau \ \textit{weakly in } \mathbf{L}^2(\tau, T; V), \quad \forall T > \tau. \tag{3.6}$$

Proof. Let $u_n(t) = U_{f_n}(t,\tau)u_{\tau_n}$ and $u(t) = U_f(t,\tau)u_\tau$ for $t \geqslant \tau$. Then by Lemma 3.1 we know that

$$\{u_n\}_n \text{ is bounded in } \mathbf{L}^\infty(\mathbf{R}_\tau^+;H) \cap \mathbf{L}^2(\tau,T;V), \quad \forall T > \tau. \tag{3.7}$$

Therefore we can extract a subsequence $\{u_{n'}\}_{n'}$ such that

$$\begin{cases} u_{n'} \to \tilde{u} \text{ weak star in } \mathbf{L}^\infty(\mathbf{R}_\tau^+;H), \\ u_{n'} \to \tilde{u} \text{ weakly in } \mathbf{L}^2_{\mathrm{loc}}(\mathbf{R}_\tau^+;V), \end{cases} \tag{3.8}$$

for some

$$\tilde{u} \in \mathbf{L}^\infty(\mathbf{R}_\tau^+;H) \cap \mathbf{L}^2_{\mathrm{loc}}(\mathbf{R}_\tau^+;V).$$

To pass to the limit in the equation for $u_{n'}$ to show that $\tilde{u}$ is a solution of (2.5) (hence (2.7)) and (2.6), we have to establish the following strong convergence result

$$u_{n'} \to \tilde{u} \quad \text{strongly in } \mathbf{L}^2_{\mathrm{loc}}(\mathbf{R}_\tau^+;\mathbf{L}^2(\Omega_r)), \tag{3.9}$$

for any $r > 0$, where $\Omega_r = \Omega \cap \{x \in \mathbf{R}^2 : |x| < r\}$.

It follows from (2.7), (2.8), $f_n \in \mathbf{L}^\infty(\mathbf{R}_\tau^+;V') \subset \mathbf{L}^2_{\mathrm{loc}}(\mathbf{R}_\tau^+;V')$ and the boundedness of the Stokes operator A from V into V' that

$$\{u_n'\}_n \text{ is bounded in } \mathbf{L}^2(\tau,T;V'), \quad \forall T > \tau.$$

Thus for all $v \in V$ and $\tau \leqslant t \leqslant t+a \leqslant T$,

$$(u_n(t+a) - u_n(t), v) = \int_t^{t+a} \langle u_n'(s), v\rangle \,\mathrm{d}s \leqslant a^{\frac{1}{2}} \|v\| \, \|u_n'\|_{\mathbf{L}^2(0,T;V')} \leqslant c_T a^{\frac{1}{2}} \|v\|,$$

where $c_T > 0$ is a constant independent of n. Taking $v = u_n(t+a) - u_n(t)$, which belongs to V for almost every t, we find that

$$|u_n(t+a) - u_n(t)|^2 \leqslant c_T a^{\frac{1}{2}} \|u_n(t+a) - u_n(t)\|.$$

Thanks to (3.7), we can get from the above inequality that

$$\int_\tau^{T-a} |u_n(t+a) - u_n(t)|^2 \,\mathrm{d}t \leqslant \tilde{c}_T a^{\frac{1}{2}},$$

for another constant $\tilde{c}_T > 0$ independent of n. Particularly, we have

$$\lim_{a\to 0} \sup_n \int_\tau^{T-a} \|u_n(t+a) - u_n(t)\|^2_{\mathbf{L}^2(\Omega_r)} \,\mathrm{d}t = 0. \tag{3.10}$$

Consider a truncation function $\theta \in C^1(\mathbf{R}^+)$ satisfying $\theta(s) = 1$ for $s \in [0,1]$, $\theta(s) = 0$ for $s \in [2,+\infty)$ and $|\theta(s)| \leqslant 1$ for $s \in \mathbf{R}^+$. Let us set

$$v_{n,r}(x,t) = \theta(|x|^2/r^2) u_n(x,t), \quad \forall x \in \Omega_{2r}.$$

Then it follows from (3.10) that

$$\lim_{a\to 0} \sup_n \int_\tau^{T-a} \|v_{n,r}(t+a) - v_{n,r}(t)\|^2_{\mathbf{L}^2(\Omega_{2r})} \,\mathrm{d}t = 0. \tag{3.11}$$

And it is obvious that

$$\{v_{n,r}\}_n \text{ is bounded in } \mathbf{L}^2(\tau,T;\mathbf{H}^1_0(\Omega_{2r})) \cap \mathbf{L}^\infty(\tau,T;\mathbf{L}^2(\Omega_{2r})). \tag{3.12}$$

Combining (3.11) and (3.12), we can deduce that

$$\lim_{a\to 0}\sup_{n}\left\{\int_{\tau}^{T-a}\|v_{n,r}(t+a)-v_{n,r}(t)\|^2_{\mathbf{L}^2(\Omega_{2r})}\,\mathrm{d}t+\int_{T-a}^{T}\|v_{n,r}(t)\|^2_{\mathbf{L}^2(\Omega_{2r})}\,\mathrm{d}t\right.$$
$$\left.+\int_{\tau}^{\tau+a}\|v_{n,r}(t)\|^2_{\mathbf{L}^2(\Omega_{2r})}\,\mathrm{d}t\right\}=0,\quad \forall r>0.$$

By the compactness theorem in [18] (see Theorem 13.3) with $X=\mathbf{L}^2(\Omega_{2r})$, $Y=\mathbf{H}_0^1(\Omega_{2r})$ and $p=2$, we obtain

$$\{v_{n,r}\}_n \text{ is relatively compact in } \mathbf{L}^2(\tau,T;\mathbf{L}^2(\Omega_{2r})),\quad \forall T>\tau.$$

Therefore

$$\{u_n|_{\Omega_r}\}_n \text{ is relatively compact in } \mathbf{L}^2(\tau,T;\mathbf{L}^2(\Omega_r)),\quad \forall T>\tau. \tag{3.13}$$

Thanks to (3.8) and (3.13), we can extract a subsequence from $\{u_{n'}\}_{n'}$, which is still denoted by n', such that (3.9) is valid. Noticing that

$$f_n\to f \text{ weak star in } \mathbf{L}^\infty(\mathbf{R}_\tau^+;V'),$$

the passage to the limit in the equation for $u_{n'}$ shows that $\tilde{u}$ is a solution of (2.5) (hence (2.7)) and (2.6). By Theorem 2.1, we have $\tilde{u}=u$. If there was another subsequence $\{u_{n''}\}_{n''}$ converging in the sense of (3.8) and (3.9) to another element $\bar{u}$, it would lead to a contradiction to the uniqueness of u. This proves (3.6).

Because of (3.9), $\forall v\in\mathscr{V}$

$$(u_n(t),v)\to(u(t),v) \text{ for a.e. } t\in\mathbf{R}^+.$$

And it follows easily from (3.7) and (3.10) that $\{(u_n(t),v)\}_n$ is equibounded and equicontinuous on $[0,T]$ for all $T>0$. Then $\{(u_n(t),v)\}_n$ is precompact by Ascoli–Arzela theorem. Finally, we can obtain (3.5) by noticing that $\mathscr{V}$ is dense in H and the uniqueness of u. □

The key issue for proving the existence of the uniform (w.r.t. $f\in\mathscr{F}$) attractor for the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, is to establish the asymptotic compactness property. Such a family of semiprocesses is said to be asymptotically compact in H if

$$\{U_{f_n}(t_n,\tau)u_{\tau_n}\} \text{ is precompact in } H,$$

whenever

$$\{u_{\tau_n}\}_n \text{ is bounded in } H,\ \{f_n\}_n\subset\mathscr{F} \text{ and } t_n\to+\infty.$$

To show this we introduce a translation operator $T(h)$ on $\mathbf{L}^\infty(\mathbf{R}^+;V'):\forall f\in \mathbf{L}^\infty(\mathbf{R}^+;V')$

$$T(h)f(s)=f(s+h),\quad \forall h\geqslant 0,\ s\in\mathbf{R}^+. \tag{3.14}$$

Obviously

$$\|T(h)f\|_{\mathbf{L}^\infty(\mathbf{R}^+;V')}\leqslant\|f\|_{\mathbf{L}^\infty(\mathbf{R}^+;V')},\quad \forall h\geqslant 0,\ f\in\mathbf{L}^\infty(\mathbf{R}^+;V').$$

And $T(\cdot)$ is a positive invariant semigroup on $\mathscr{F}$, that is

$$T(h)\mathscr{F} \subset \mathscr{F}, \quad \forall h \geqslant 0. \tag{3.15}$$

It is shown in [8] that

$$U_{T(h)f}(t,\tau) = U_f(t+h,\tau+h), \quad \forall h \geqslant 0,\ t \geqslant \tau \geqslant 0. \tag{3.16}$$

Before we prove the asymptotic compactness of the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, in H, we need another form of the energy equation of (2.5).

Similar to [17], we define $[\cdot,\cdot] : V \times V \to \mathbf{R}$ by

$$[u,v] = \nu((u,v)) - \frac{\nu\lambda_1}{2}(u,v), \quad \forall u,v \in V.$$

$[\cdot,\cdot]$ is bilinear and symmetric. Moreover, from (2.3),

$$[u]^2 = [u,u] = \nu\|u\|^2 - \frac{\nu\lambda_1}{2}|u|^2 \geqslant \nu\|u\|^2 - \frac{\nu}{2}\|u\|^2 = \frac{\nu}{2}\|u\|^2.$$

Thus $[\cdot,\cdot]$ defines an inner product in V with its associated norm $[\cdot] = [\cdot,\cdot]^{\frac{1}{2}}$ equivalent to $\|\cdot\|$. Now we rewrite (3.2) as

$$\frac{\mathrm{d}|u|^2}{\mathrm{d}t} + \nu\lambda_1|u|^2 + 2[u]^2 = 2\langle f,u\rangle,$$

for any solution $u = u(t) = U_f(t,\tau)u_\tau$, $u_\tau \in H$. Then by the variation of the constant formula, we have

$$|U_f(t,\tau)u_\tau|^2 = |u_\tau|^2 \mathrm{e}^{-\nu\lambda_1(t-\tau)} + 2\int_\tau^t \mathrm{e}^{-\nu\lambda_1(t-s)}(\langle f(s), U_f(s,\tau)u_\tau\rangle - [U_f(s,\tau)u_\tau]^2)\,\mathrm{d}s,$$

for all $u_\tau \in H$, $t \geqslant \tau \geqslant 0$. Alternatively substituting the integral variable s in the integral on the right-hand side with $s+\tau$ and noticing (3.14) and (3.16), we have

$$|U_f(t,\tau)u_\tau|^2 = |u_\tau|^2 \mathrm{e}^{-\nu\lambda_1(t-\tau)} + 2\int_0^{t-\tau} \mathrm{e}^{-\nu\lambda_1(t-\tau-s)}(\langle T(\tau)f(s), U_{T(\tau)f}(s,0)u_\tau\rangle - [U_{T(\tau)f}(s,0)u_\tau]^2)\,\mathrm{d}s, \tag{3.17}$$

for all $u_\tau \in H$ and $t \geqslant \tau \geqslant 0$.

Lemma 3.3. *Suppose that $\mathscr{F} \subset \mathrm{L}^\infty(\mathbf{R}^+;V')$ is a bounded closed symbol space satisfying* (3.15), *$\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, is the family of semiprocesses corresponding to system* (2.5) (hence (2.7)) and (2.6) *in H. Then it is asymptotically compact.*

Proof. Suppose $B \subset H$ is bounded, $\{u_{\tau_n}\}_n \subset B$, $\{f_n\}_n \subset \mathscr{F}$ and $\{t_n\}_n \subset \mathbf{R}^+$ satisfying $t_n \to +\infty$ as $n \to +\infty$. Thanks to Lemma 3.1, $\mathscr{B}_0$ defined in (3.4) is uniformly (w.r.t. $f \in \mathscr{F}$) absorbing, thus there exists a constant $M(B,\tau) > \tau$ such that

$$U_f(t,\tau)B \subset \mathscr{B}_0, \quad \forall t \geqslant M(B,\tau),\ f \in \mathscr{F}.$$

Then for t_n sufficiently large ($t_n \geqslant M(B,\tau)$),

$$U_{f_n}(t_n,\tau)B \subset \mathscr{B}_0.$$

This ensures that $\{U_{f_n}(t_n,\tau)u_{\tau_n}\}$ is weakly precompact in H and we can extract a subsequence n' such that

$$U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}} \rightarrow w \text{ weakly in } H, \tag{3.18}$$

for some $w \in \mathscr{B}_0 \subset H$.

Similarly for each $T > 0$ and $t_{n'} \geqslant T + M(B,\tau)$, it yields

$$U_{f_{n'}}(t_{n'} - T,\tau)u_{\tau_{n'}} \in \mathscr{B}_0. \tag{3.19}$$

Again, we can extract another subsequence from n', which is still denoted by n', such that

$$u_{t_{n'}} := U_{f_{n'}}(t_{n'} - T,\tau)u_{\tau_{n'}} \rightarrow w_T \text{ weakly in } H \text{ for every } T > 0, \tag{3.20}$$

for some $w_T \in \mathscr{B}_0$.

Noticing (2.9) and (3.16), we have

$$\begin{aligned} U_{f_{n'}}(t_{n'},\tau) &= U_{f_{n'}}(t_{n'},t_{n'} - T) \circ U_{f_{n'}}(t_{n'} - T,\tau) \\ &= U_{T(t_{n'}-T)f_{n'}}(T,0) \circ U_{f_{n'}}(t_{n'} - T,\tau). \end{aligned}$$

If we denote $g_{T,n'} = T(t_{n'} - T)f_{n'} \in \mathscr{F}$, we can derive

$$\begin{aligned} U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}} &= U_{g_{T,n'}}(T,0) \circ U_{f_{n'}}(t_{n'} - T,\tau)u_{\tau_{n'}} \\ &= U_{g_{T,n'}}(T,0)u_{t_{n'}}, \quad \forall T > 0. \end{aligned} \tag{3.21}$$

Since $\{g_{T,n'}\}_{n'} \subset \mathscr{F}$, we can extract a subsequence from n', which is denoted by n' again, such that

$$g_{T,n'} \rightarrow g_T \in \mathscr{F} \text{ weak star in } \mathbf{L}^\infty(\mathbf{R}^+;V') \text{ for every } T > 0. \tag{3.22}$$

Then by using (3.18), (3.20) and Lemma 3.2, we have

$$w = \lim_{n'\ H_w} U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}} = \lim_{n'\ H_w} U_{g_{T,n'}}(T,0)u_{t_{n'}} = U_{g_T}(T,0)w_T,$$

where $\lim_{n'\ H_w}$ denotes the weak limit in H. Therefore,

$$w = U_{g_T}(T,0)w_T \text{ for every } T > 0.$$

From (3.18) and (3.21), we find

$$|w| \leqslant \liminf_{n'} |U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}}| = \liminf_{n'} |U_{g_{T,n'}}(T,0)u_{t_{n'}}|. \tag{3.23}$$

If we can show

$$\limsup_{n'} |U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}}| \leqslant |w|,$$

we will get the result. To do this, we consider the following sequence:

$$f_n^k = \chi^k f_n \in \mathscr{F},$$

where $\chi^k(x) = \theta(|x|^2/k^2)$, $k \in \mathbf{N}$ and θ is the function defined in the proof of Lemma 3.2.

In the following, all subsequences are extracted from the previous subsequence n' and will be still denoted by n'. Similar to (3.18) and (3.22), we can get

$$U_{g^k_{T,n'}}(T,0)u_{t_{n'}} \to w^k \quad \text{weakly in } H, \tag{3.24}$$

for some $w^k \in H$, where

$$g^k_{T,n'} = \chi^k g_{T,n'} \to g^k_T = \chi^k g_T \in \mathscr{F} \text{ weak star in } \mathbf{L}^\infty(\mathbf{R}^+; V') \text{ for every } T > 0.$$

Finally, we have

$$w^k = U_{g^k_T}(T,0)w_T \text{ for every } T > 0.$$

For every $T > 0$ and $t_{n'} \geqslant T + M(B,\tau)$, by the same argument for deriving (3.17) we have

$$\begin{aligned}|U_{g^k_{T,n'}}(T,0)u_{t_{n'}}|^2 = {}& |u_{t_{n'}}|^2 \mathrm{e}^{-\nu\lambda_1 T}\\ &+ 2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}(\langle g^k_{T,n'}(s), U_{g^k_{T,n'}}(s,0)u_{t_{n'}}\rangle\\ &- [U_{g^k_{T,n'}}(s,0)u_{t_{n'}}]^2)\,\mathrm{d}s.\end{aligned} \tag{3.25}$$

Thanks to (3.19) and (3.4),

$$\limsup_{n'} (\mathrm{e}^{-\nu\lambda_1 T}|u_{t_{n'}}|^2) \leqslant \rho_0^2 \mathrm{e}^{-\nu\lambda_1 T}. \tag{3.26}$$

Besides, it is easy to show that

$$\left(\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[\,\cdot\,]^2\,\mathrm{d}s\right)^{\frac{1}{2}},$$

is equivalent to the usual norm in $\mathbf{L}^2(0,T;V)$. Then from Lemma 3.2 and (3.24) we deduce that

$$\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[U_{g^k_T}(s,0)w_T]^2\,\mathrm{d}s \leqslant \liminf_{n'} \int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[U_{g^k_{T,n'}}(s,0)u_{t_{n'}}]^2\,\mathrm{d}s.$$

Hence

$$\begin{aligned}&\limsup_{n'} -2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[U_{g^k_{T,n'}}(s,0)u_{t_{n'}}]^2\,\mathrm{d}s\\ &\quad = -2\liminf_{n'} \int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[U_{g^k_{T,n'}}(s,0)u_{t_{n'}}]^2\,\mathrm{d}s\\ &\quad \leqslant -2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}[U_{g^k_T}(s,0)w_T]^2\,\mathrm{d}s.\end{aligned} \tag{3.27}$$

Thanks to Lemma 3.2 and its proof, in particular (3.13), we realize that for every $0 < T < \infty$

$$\chi^k U_{g^k_{T,n'}}(s,0)u_{t_{n'}} \to \chi^k U_{g^k_T}(s,0)w_T \quad \text{in } \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)).$$

Noticing that

$$\langle g^k_{T,n'}(s), U_{g^k_{T,n'}}(s,0)u_{t_{n'}}\rangle = \langle g_{T,n'}(s), \chi^k U_{g^k_{T,n'}}(s,0)u_{t_{n'}}\rangle,$$

we can easily find

$$\lim_{n'\to\infty} \int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}\langle g^k_{T,n'}(s), U_{g^k_{T,n'}}(s,0)u_{t_{n'}}\rangle\,\mathrm{d}s$$

$$= \int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}\langle g^k_T(s), U_{g^k_T}(s,0)w_T\rangle\,\mathrm{d}s \tag{3.28}$$

Combination of (3.25)–(3.28) leads to

$$\limsup_{n'} |U_{g^k_{T,n'}}(T,0)u_{t_{n'}}|^2 \leqslant \rho_0^2\mathrm{e}^{-\nu\lambda_1 T}$$

$$+2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}(\langle g^k_T(s), U_{g^k_T}(s,0)w_T\rangle - [U_{g^k_T}(s,0)w_T]^2)\,\mathrm{d}s.$$

On the other hand, it follows from $w^k = U_{g^k_T}(T,0)w_T$ that

$$|w^k|^2 = |U_{g^k_T}(T,0)w_T|^2$$

$$= \mathrm{e}^{-\nu\lambda_1 T}|w_T|^2 + 2\int_0^T \mathrm{e}^{-\nu\lambda_1(T-s)}(\langle g^k_T(s), U_{g^k_T}(s,0)w_T\rangle - [U_{g^k_T}(s,0)w_T]^2)\,\mathrm{d}s.$$

Combination of the two above inequalities admits

$$\limsup_{n'} |U_{g^k_{T,n'}}(T,0)u_{t_{n'}}|^2 \leqslant |w^k|^2 + (\rho_0^2 - |w_T|^2)\mathrm{e}^{-\nu\lambda_1 T} \leqslant |w^k|^2 + \rho_0^2\mathrm{e}^{-\nu\lambda_1 T},$$

for every $T > 0$. It is easy to verify that for any $f \in \mathscr{F}$ and given $0 < T < \infty$

$$\chi^k f \to f \text{ in } \mathbf{L}^\infty(0,T;V').$$

Therefore

$$w^k \to w \text{ in } H. \tag{3.29}$$

Thus there exists a constant $\varepsilon(k)$, where $\varepsilon(k) \to 0$ as $k \to \infty$, such that

$$|w^k|^2 \leqslant |w|^2 + \varepsilon(k).$$

Then we have

$$\limsup_{n'} |U_{g^k_{T,n'}}(T,0)u_{t_{n'}}| \leqslant |w|^2 + \varepsilon(k) + \rho_0^2\mathrm{e}^{-\nu\lambda_1 T}.$$

And for any $m \in \mathbf{N}$, there exists a constant $\delta(m)$, $\delta(m) \to 0$ as $m \to \infty$, such that

$$\sup_{n'\geqslant m} |U_{g^k_{T,n'}}(T,0)u_{t_{n'}}|^2 \leqslant |w|^2 + \varepsilon(k) + \delta(m) + \rho_0^2\mathrm{e}^{-\nu\lambda_1 T}. \tag{3.30}$$

As in (3.29),

$$\lim_{k\to\infty} |U_{g^k_{T,n'}}(T,0)u_{t_{n'}}|^2 = |U_{g_{T,n'}}(T,0)u_{t_{n'}}|^2.$$

Then from (3.30) we have

$$\sup_{n'\geqslant m} |U_{g_{T,n'}}(T,0)u_{t_{n'}}|^2 \leqslant |w|^2 + \delta(m) + \rho_0^2 e^{-\nu\lambda_1 T}.$$

Alternatively, thanks to (3.21)

$$\sup_{n'\geqslant m} |U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}}|^2 = \sup_{n'\geqslant m} |U_{g_{T,n'}}(T,0)u_{t_{n'}}|^2 \leqslant |w|^2 + \delta(m) + \rho_0^2 e^{-\nu\lambda_1 T}.$$

Let m and T tend to infinity, we can eventually get

$$\limsup_{n'} |U_{f_{n'}}(t_{n'},\tau)u_{\tau_{n'}}|^2 \leqslant |w|^2. \tag{3.31}$$

Combination of (3.23) and (3.31) leads to the result of this lemma. □

If $T(h)$ is an asymptotically compact semigroup on $\mathscr{F}$ as required in [16], the proof of (3.31) will be much easier and it is not necessary to introduce the auxiliary function χ^k.

4. The uniform attractor

Consider the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, in a Hilbert space H generated by the Navier–Stokes equation (2.5) (hence (2.7)) and (2.6), where $\mathscr{F}$ is the bounded symbol space defined by (3.1) which satisfies (3.15). By analogy with the definition of the ω-lim set of semigroup, we introduce the corresponding ω-lim set of such a family of semiprocesses. For an arbitrary bounded set $B\subset H$ we define the uniform (w.r.t. $f\in\mathscr{F}$) ω-lim set $\omega_{\tau,\mathscr{F}}(B)$ with an origin at τ for the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$ [8]:

$$\omega_{\tau,\mathscr{F}}(B) = \bigcap_{t\geqslant\tau} \overline{\bigcup_{f\in\mathscr{F}} \bigcup_{s\geqslant t} U_f(s,\tau)B}. \tag{4.1}$$

It follows from (4.1) that

$$\begin{cases} v\in\omega_{\tau,\mathscr{F}}(B) \Leftrightarrow \exists \text{ sequences } \{v_n\}_n\subset B,\ \{f_n\}_n\subset\mathscr{F} \text{ and } \{t_n\}_n\subset[\tau,+\infty) \\ \qquad\qquad\qquad \text{satisfying } t_n\to+\infty \text{ as } n\to+\infty, \\ \qquad\qquad\qquad \text{such that } U_{f_n}(t_n,\tau)v_n\to v \text{ in } H \text{ (as } n\to\infty). \end{cases} \tag{4.2}$$

Proposition 4.1. *If the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, is uniformly (w.r.t. $f\in\mathscr{F}$) bounded and asymptotically compact in H, its ω-*lim *set $\omega_{\tau,\mathscr{F}}(B)$ is a nonempty compact set in H.*

Proof. The result that $\omega_{\tau,\mathscr{F}}(B)\neq\phi$ is a direct result of the asymptotic compactness of the family of the semiprocesses and (4.2). Now we give the proof of the compactness of $\omega_{\tau,\mathscr{F}}(B)$.

Since the family of semiprocesses under consideration is uniformly (w.r.t. $f \in \mathscr{F}$) bounded, we find that $\omega_{\tau,\mathscr{F}}(B)$ is a bounded set in H. Suppose there is a sequence $\{w_n\}_n \subset \omega_{\tau,\mathscr{F}}(B)$. Thanks to (4.2), for any $w_n \in \omega_{\tau,\mathscr{F}}(B)$ one can find sequences $\{w_m^n\}_m \subset B$, $\{f_m^n\}_m \subset \mathscr{F}$ and $\{t_m^n\}_m \subset [\tau, +\infty)$ satisfying $t_m^n \to +\infty$ such that $U_{f_m^n}(t_m^n, \tau)w_m^n \to w_n$ as $m \to +\infty$. For any fixed $n \in \mathbf{N}$ and $\varepsilon_n = 1/n$, there exists some $m_n \in \mathbf{N}$ such that

$$|U_{f_{m_n}^n}(t_{m_n}^n, \tau)w_{m_n}^n - w_n| < \varepsilon_n. \tag{4.3}$$

We set

$$f^n = f_{m_n}^n, \;\; t^n = t_{m_n}^n, \;\; w^n = w_{m_n}^n \text{ for every } n \in \mathbf{N}.$$

Now we can get sequences $\{w^n\}_n \subset B$, $\{f^n\}_n \subset \mathscr{F}$, $\{t^n\}_n \subset [\tau, +\infty)$. Without loss of generality, we assume $\{t^n\}_n$ is a increasing sequence converging to infinity.

Because $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, is asymptotically compact in H, we can extract a subsequence n' such that

$$U_{f^{n'}}(t^{n'}, \tau)w^{n'} \to w \text{ strongly in } H.$$

Thanks to (4.2), $w \in \omega_{\tau,\mathscr{F}}(B)$. From (4.3) we have

$$|U_{f^{n'}}(t^{n'}, \tau)w^{n'} - w_{n'}| \leqslant \varepsilon_{n'}.$$

Then we can get a subsequence $\{w_{n'}\}_{n'} \subset \{w_n\}_n$ such that

$$w_{n'} \to w \text{ stronly in } H.$$

This proves the proposition. □

Proposition 4.2. *Suppose the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, is asymptotically compact in H, then for any bounded set $B \subset H$*

$$\sup_{f \in \mathscr{F}} \operatorname{dist}_H(U_f(t,\tau)B, \omega_{\tau,\mathscr{F}}(B)) \to 0 \text{ as } t \to +\infty.$$

Proof. This proposition is quite easy to be proven by contradiction argument.

Suppose for certain bounded set $B \subset H$

$$\sup_{f \in \mathscr{F}} \operatorname{dist}_H(U_f(t,\tau)B, \omega_{\tau,\mathscr{F}}(B)) \text{ does not converge to 0 as } t \to \infty.$$

Then there exists a constant $\delta > 0$, sequences $\{v_n\}_n \subset B$, $\{f_n\}_n \subset \mathscr{F}$, $\{t_n\}_n \subset [\tau, +\infty)$ $(t_n \to +\infty)$ such that

$$\operatorname{dist}_H(U_{f_n}(t_n, \tau)v_n, \omega_{\tau,\mathscr{F}}(B)) \geqslant \delta, \quad \forall n \in \mathbf{N}. \tag{4.4}$$

On the other hand, we can extract a subsequence n' such that

$$U_{f_{n'}}(t_{n'}, \tau)v_{n'} \to v \text{ strongly in } H,$$

for certain $v \in H$ by the asymptotic compactness. By (4.2), $v \in \omega_{\tau,\mathscr{F}}(B)$. This is a contradiction to (4.4). □

Proposition 4.3. *Suppose* (3.15) *is satisfied by* $\mathscr{F}$ *and the family of semiprocesses* $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, *is uniformly* (*w.r.t.* $f\in\mathscr{F}$) *absorbing in* H. *Then*

(i) *for any* $\tau>0$ *and bounded set* B *in* H:

$$\omega_{\tau,\mathscr{F}}(B)\subset\omega_{0,\mathscr{F}}(B);$$

(ii) *for any uniform* (*w.r.t.* $f\in\mathscr{F}$) *absorbing sets* $B_1,B_2\subset H$:

$$\omega_{0,\mathscr{F}}(B_1)=\omega_{0,\mathscr{F}}(B_2);$$

(iii) *for any bounded set* $B\subset H$ *and any absorbing set* $B_1\subset H$:

$$\omega_{0,\mathscr{F}}(B)\subset\omega_{0,\mathscr{F}}(B_1).$$

Proof. (i) Having (3.16), we have, for any bounded set $B\subset H$ and $\tau\geqslant 0$

$$U_f(s,\tau)B=U_f(s-\tau+\tau,\tau-\tau+\tau)B=U_{T(\tau)f}(s-\tau,0)B.$$

Then by (3.15)

$$\begin{aligned}\omega_{\tau,\mathscr{F}}(B)&=\bigcap_{t\geqslant\tau}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t}U_f(s,\tau)B}=\bigcap_{t\geqslant\tau}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t}U_{T(\tau)f}(s-\tau,0)B}\\&=\bigcap_{t\geqslant\tau}\overline{\bigcup_{f\in T(\tau)\mathscr{F}}\bigcup_{s\geqslant t}U_f(s-\tau,0)B}=\bigcap_{t\geqslant\tau}\overline{\bigcup_{f\in T(\tau)\mathscr{F}}\bigcup_{l\geqslant t-\tau}U_f(l,0)B}\\&=\bigcap_{r\geqslant 0}\overline{\bigcup_{f\in T(\tau)\mathscr{F}}\bigcup_{l\geqslant r}U_f(l,0)B}=\bigcap_{t\geqslant 0}\overline{\bigcup_{f\in T(\tau)\mathscr{F}}\bigcup_{s\geqslant t}U_f(s,0)B}\\&\subset\bigcap_{t\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t}U_f(s,0)B}=\omega_{0,\mathscr{F}}(B).\end{aligned}$$

That is $\omega_{\tau,\mathscr{F}}(B)\subset\omega_{0,\mathscr{F}}(B)$.

(ii) We assume that $B_1,B_2\subset H$ are two uniformly (w.r.t. $f\in\mathscr{F}$) absorbing sets of the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$. Because of the uniform (w.r.t. $f\in\mathscr{F}$) absorbing property, we claim that there must be certain $t_2>0$ such that

$$B_2'=\overline{\bigcup_{f\in\mathscr{F}}U_f(t_2,0)B_2}\subset B_1.$$

Therefore

$$\omega_{0,\mathscr{F}}(B_2')\subset\omega_{0,\mathscr{F}}(B_1).\tag{4.5}$$

On the other hand,

$$\begin{aligned}\omega_{0,\mathscr{F}}(B_2)&=\bigcap_{t\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t}U_f(s,0)B_2}\subset\bigcap_{t\geqslant t_2}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t}U_f(s,0)B_2}\\&=\bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant l+t_2}U_f(s,0)B_2}=\bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l}U_f(r+t_2,0)B_2}\end{aligned}$$

$$
\begin{aligned}
&= \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_f(r+t_2,t_2)\circ U_f(t_2,0)B_2}\\
&\subset \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_f(r+t_2,t_2)B_2'} = \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_{T(t_2)f}(r,0)B_2'}\\
&= \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in T(t_2)\mathscr{F}}\bigcup_{r\geqslant l} U_f(r,0)B_2'} \subset \bigcap_{t\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t} U_f(s,0)B_2'}\\
&= \omega_{0,\mathscr{F}}(B_2').
\end{aligned}
$$

And noticing (4.5), we can get

$$\omega_{0,\mathscr{F}}(B_2)\subset\omega_{0,\mathscr{F}}(B_1).$$

By the similar procedure, we can also prove

$$\omega_{0,\mathscr{F}}(B_1)\subset\omega_{0,\mathscr{F}}(B_2),$$

that is $\omega_{0,\mathscr{F}}(B_1)=\omega_{0,\mathscr{F}}(B_2)$.

(iii) For any bounded set $B\subset H$, absorbing set $B_1\subset H$ and $t_2>0$,

$$
\begin{aligned}
\omega_{0,\mathscr{F}}(B) &= \bigcap_{t\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t} U_f(s,0)B} \subset \bigcap_{t\geqslant t_2}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t} U_f(s,0)B}\\
&= \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant l+t_2} U_f(s,0)B} = \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_f(r+t_2,0)B}\\
&= \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_f(r+t_2,t_2)\circ U_f(t_2,0)B}\\
&\qquad\text{(for sufficiently large } t_2)\\
&\subset \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_f(r+t_2,t_2)B_1} = \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{r\geqslant l} U_{T(t_2)f}(r,0)B_1}\\
&= \bigcap_{l\geqslant 0}\overline{\bigcup_{f\in T(t_2)\mathscr{F}}\bigcup_{r\geqslant l} U_f(r,0)B_1} \subset \bigcap_{t\geqslant 0}\overline{\bigcup_{f\in\mathscr{F}}\bigcup_{s\geqslant t} U_f(s,0)B_1}\\
&= \omega_{0,\mathscr{F}}(B_1). \qquad\square
\end{aligned}
$$

Now we give the existence of the uniform attractor for the family of semiprocesses in

Theorem 4.1. *Suppose that $\mathscr{F}$ is a bounded symbol space defined by* (3.1), *which is positively invariant under translation, $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, is the family of semiprocesses in the Hilbert space H generated by the Navier–Stokes equation* (2.5) (*hence* (2.7)) *and* (2.6). *Then the family of semiprocesses $\{U_f(t,\tau)\}$, $f\in\mathscr{F}$, possesses* *a unique compact uniform* (*w.r.t.* $f \in \mathscr{F}$) *attractor* $\mathscr{A}_{\mathscr{F}} \subset H$ *given by*

$$\mathscr{A}_{\mathscr{F}} = \omega_{0,\mathscr{F}}(\mathscr{B}_0),$$

where $\mathscr{B}_0 \subset H$ *is an arbitrary uniformly* (*w.r.t.* $f \in \mathscr{F}$) *absorbing set of the family of semiprocesses under consideration.*

Proof. The result of Proposition 4.2 applies to the family of semiprocesses under consideration because the family of semiprocesses is asymptotically compact in H. That is the attracting property of $\mathscr{A}_{\mathscr{F}}$ is valid. To show that $\mathscr{A}_{\mathscr{F}}$ is the uniform (w.r.t. $f \in \mathscr{F}$) attractor, we have to prove that the property of minimality is also valid. Here we will use a contradiction argument.

Assume that there exists a closed bounded set $P \subset H$ satisfying

$$\lim_{t\to\infty} \sup_{f\in\mathscr{F}} \mathrm{dist}_H(U_f(t,\tau)B, P) = 0, \tag{4.6}$$

for any $\tau \geqslant 0$ and any bounded set $B \subset H$ such that $\mathscr{A}_{\mathscr{F}}$ is not included in P. This means

$$\text{there exists at least one point } v \in \mathscr{A}_{\mathscr{F}} \text{ such that } v \notin P. \tag{4.7}$$

Thanks to $v \in \mathscr{A}_{\mathscr{F}} = \omega_{0,\mathscr{F}}(\mathscr{B}_0)$, we deduce from (4.2) that there exist sequences $\{v_n\}_n \subset \mathscr{B}_0$, $\{f_n\}_n \subset \mathscr{F}$, $\{t_n\}_n \in [\tau, +\infty)$ with $t_n \to +\infty$ as n goes to infinity such that

$$U_{f_n}(t_n, 0)v_n \to v \text{ as } n \to +\infty.$$

If we set

$$\bar{v}_n = U_{f_n}(\tau, 0)v_n,$$

it follows that

$$U_{f_n}(t_n, \tau)\bar{v}_n \to v \text{ as } n \to +\infty. \tag{4.8}$$

Without loss of generality, we assume $\{\bar{v}_n\}_n \subset B$. Since $P \subset H$ is a closed set, we can deduce from (4.6) and (4.8) that $v \in P$, which is a contradiction to (4.7). So we have $\mathscr{A}_{\mathscr{F}} \subset P$. This proves the property of minimality of $\mathscr{A}_{\mathscr{F}}$. That is, $\mathscr{A}_{\mathscr{F}}$ is a uniformly (w.r.t. $f \in \mathscr{F}$) attractor for the family of the semiprocesses. The uniqueness is a direct consequence of the property of minimality. □

Theorem 4.2. *Under the condition of Theorem* 4.1, *the uniform attractor* $\mathscr{A}_{\mathscr{F}}$ *of the family of semiprocesses* $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, *corresponding to Eq.* (2.5) (*hence* (2.7)) *and* (2.6) *is negatively invariant. That is for any* $t \geqslant \tau \geqslant 0$,

$$\bigcup_{f\in\mathscr{F}} U_f(t,\tau)\mathscr{A}_{\mathscr{F}} \supset \mathscr{A}_{\mathscr{F}}.$$

Proof. To prove this theorem, we have and only have to show for any $v \in \mathscr{A}_{\mathscr{F}}$, there exist $f \in \mathscr{F}$ and $\tilde{v} \in \mathscr{A}_{\mathscr{F}}$ such that

$$v = U_f(t,\tau)\tilde{v}.$$

Notice the definition of $\mathscr{A}_{\mathscr{F}} = \omega_{0,\mathscr{F}}(\mathscr{B}_0)$ in Theorem 4.1 and (4.2),

$$\begin{cases} v \in \omega_{0,\mathscr{F}}(B) \Leftrightarrow \exists \text{ sequences } \{v_n\}_n \subset B,\ \{f_n\}_n \subset \mathscr{F} \text{ and } \{t_n\}_n \subset [0,+\infty) \\ \qquad\qquad\qquad \text{satisfying } t_n \to +\infty \text{ as } n \to +\infty, \\ \qquad\qquad\qquad \text{such that } U_{f_n}(t_n,0)v_n \to v \text{ in } H \ (n \to \infty). \end{cases} \tag{4.9}$$

Without loss of generality, we suppose that $t_n > t - \tau$. Then from (2.9) we have

$$U_{f_n}(t_n,0) = U_{f_n}(t_n, t_n - (t-\tau)) \circ U_{f_n}(t_n - (t-\tau), 0).$$

Thanks to Lemma 3.3, the family of semiprocesses $\{U_f(t,\tau)\}$, $f \in \mathscr{F}$, corresponding to the system (2.5) (hence (2.7)) and (2.6) is asymptotically compact in H. Therefore we can extract a subsequence n' from n such that

$$\tilde{v}_{n'} = U_{f_{n'}}(t_n - (t-\tau), 0)v_{n'} \to \tilde{v} \quad \text{in } H. \tag{4.10}$$

Thus $\tilde{v} \in \mathscr{A}_{\mathscr{F}}$.

On the other hand, from property (3.22) we have

$$U_{f_{n'}}(t_{n'}, t_{n'} - (t-\tau)) = U_{\tilde{f}_{n'}}(t,\tau), \tag{4.11}$$

where $\tilde{f}_{n'} = T(t_{n'} - t)f_{n'} \in \mathscr{F}$.

Combining (4.9)–(4.11), we obtain

$$U_{\tilde{f}_{n'}}(t,\tau)\tilde{v}_{n'} \to v \quad \text{in } H.$$

Now we extract again a subsequence n'' from n' such that

$$\tilde{f}_{n''} \to f \in \mathscr{F} \quad \text{weakly}^* \text{ in } H.$$

Thanks to Lemma 3.2 we know

$$U_{\tilde{f}_{n''}}(t,\tau)\tilde{v}_{n''} \to U_f(t,\tau)\tilde{v} \quad \text{weakly in } H.$$

Therefore from (4.9) we can get

$$U_f(t,\tau)\tilde{v} = v.$$

This ends the proof. □

5. Hausdorff dimension estimation for quasi-periodical external force case

Following the method in [8], the family of semiprocesses in the previous section is equivalent to the autonomous system

$$\begin{cases} u_t = -\nu A u - B(u,u) + f(x,t), \\ f(x,t) = T(t)f_0, \end{cases}$$

where $f_0 \in \mathscr{F}$. Thanks to [8], if $\mathscr{F}$ is compact, the above system possesses a global attractor $\mathscr{A}$, which is invariant, and $\Pi\mathscr{A} = \mathscr{A}_{\mathscr{F}}$, where Π is the projector from $H \times \mathbf{L}^\infty(\mathbf{R}^+; V')$ onto H.

For the non-autonomous Navier–Stokes equations with quasi-periodical external force

$$f(x,t) = f(x, \omega_1(t), \omega_2(t), \dots, \omega_k(t)),$$

the finite dimensionality of the uniform attractor were derived by Chepyzhov and Vishik [8] and Chepyzhov and Efendiev [7] for a bounded domain, an unbounded strip, respectively. Here

$$\omega_1(t+\alpha_1) = \omega_1(t),\ \ \omega_2(t+\alpha_2) = \omega_2(t), \dots, \omega_k(t+\alpha_k) = \omega_k(t)$$

and $\alpha_1, \alpha_2, \dots, \alpha_k$ are rational independent.

Following the method in [8] and [7], the family of semiprocesses in the previous section is equivalent to the autonomous system

$$\begin{cases} u_t = -\nu A u - B(u,u) + f(x,\omega), \\ \omega_t = \alpha, \end{cases} \tag{5.1}$$

where $\omega = (\omega_1(t), \dots, \omega_k(t))$, $\alpha = (\alpha_1, \dots, \alpha_k)$. In this case, the symbol space is $\mathbf{T}^k$, a k-dimensional torus and it is of course compact.

Suppose that $(u,\omega) \in H \times \mathbf{T}^k$ is the solution of (5.1). Then the linearized system around (u,ω) is

$$\begin{cases} v_t = -\nu A v - B(v,u) - B(u,v) + f_\omega(x,\omega)\cdot\eta, \\ \eta_t = 0. \end{cases}$$

Or in short form,

$$z_t = M(u,\omega) z, \quad z = (v,\eta) \in H \times \mathbf{T}^k.$$

For any $z = (v,\eta) \in H \times \mathbf{T}^k$, we have for any positive constants b and δ

$$\begin{aligned} (Mz,z) &= -\nu(Av,v) - b(v,u,v) + \int_\Omega f_\omega \cdot \eta v \,\mathrm{d}x \\ &\leqslant -\frac{\nu}{2}(Av,v) + \frac{|\eta|^2}{2b}\int_\Omega |f_\omega|^{1+\delta}\,\mathrm{d}x - \frac{\nu}{2}(Av,v) \\ &\quad + \int_\Omega |\nabla u||v|^2\,\mathrm{d}x + \frac{b}{2}\int_\Omega |f_\omega|^{1-\delta}|v|^2\,\mathrm{d}x \\ &:= (M_1 z, z) + (M_2 z, z), \end{aligned}$$

where

$$M_1 = \begin{Bmatrix} -\frac{\nu}{2}A & 0 \\ 0 & \frac{1}{2b}|f_\omega|^{1+\delta} \end{Bmatrix}, \quad M_2 = \begin{Bmatrix} -\frac{\nu}{2}A + (|\nabla u| + \frac{b}{2}|f_\omega|^{1-\delta})P & 0 \\ 0 & 0 \end{Bmatrix},$$

P is the Leray orthogonal projection from $\mathbf{L}^2(\Omega)$ onto H. For any given positive integer d, denote by Q_d the orthogonal projection from $H \times \mathbf{T}^k$ onto its d-dimensional subspace, we find

$$\sup_{Q_d} \operatorname{Tr}(M \circ Q_d) \leqslant \sup_{Q_d} \operatorname{Tr}(M_1 \circ Q_d) + \sup_{Q_d} \operatorname{Tr}(M_2 \circ Q_d).$$

Notice that M_1 and M_2 are self-adjoint operators on $H \times \mathbf{T}^k$ and (2.2), we have

$$\sup_{Q_d} \operatorname{Tr}(M_1 \circ Q_d) \leqslant -\frac{v\lambda_1}{2}(d-k) + \frac{G_1}{2b}k,$$

where $G_1 = \sup_{\omega\in\mathbf{T}^k} \int_\Omega |f_\omega|^{1+\delta}\,\mathrm{d}x$. If we denote by v_j, $j = 1,2,\ldots,d-k_1$ ($k_1 \leqslant k$), the first $d-k_1$ eigenvectors of operator $-\frac{v}{2}A + (|\nabla u| + \frac{b}{2}|f_\omega|^{1-\delta})P$, whose corresponding eigenvalues are greater than zero, and denote $\rho(x) = \sum_{j=1}^{d-k_1} v_j(x)$ and $G_2 = \sup_{\omega\in\mathbf{T}^k} \int_\Omega |f_\omega|^{2-2\delta}\,\mathrm{d}x$, by using the Lieb–Thirring inequality (see Ghidaglia et al. [12] and [17]) and Young's inequality we have

$$\begin{aligned}\sup_{Q_d} \operatorname{Tr}(M_2 \circ Q_d) &\leqslant -\frac{v}{2}\sum_{j=1}^{d-k_1} |A^{\frac{1}{2}} v_j|^2 + \int_\Omega |\nabla u|\rho(x)\,\mathrm{d}x + \frac{b}{2}\int_\Omega |f_\omega|^{1-\delta}\rho(x)\,\mathrm{d}x \\ &\leqslant -\frac{v}{2\kappa}\int_\Omega \rho(x)^2\,\mathrm{d}x + \int_\Omega |\nabla u|\rho(x)\,\mathrm{d}x \\ &\quad + \frac{b}{2}\left(\varepsilon\int_\Omega \rho(x)^2\,\mathrm{d}x + \frac{G_2}{4\varepsilon}\right) \\ &\leqslant \int_\Omega \left[|\nabla u|\rho(x) - \frac{v}{2\kappa}\rho^2(x) + \frac{b\varepsilon}{2}\rho^2(x)\right]\mathrm{d}x + \frac{G_2 b}{8\varepsilon}.\end{aligned}$$

Here $\kappa > 0$ is an absolute constant appearing in the Lieb–Thirring inequality. Choose ε small enough such that $b\varepsilon = v/2\kappa$, we get

$$|\nabla u|\rho(x) - \frac{v}{2\kappa}\rho^2(x) + \frac{b\varepsilon}{2}\rho^2(x) = |\nabla u(x)|\rho(x) - \frac{v}{4\kappa}\rho^2(x).$$

And the maximum value of the above quadratic form is $\frac{\kappa}{v}|\nabla u(x)|^2$. Therefore

$$\sup_{Q_d} \frac{1}{t}\int_0^t \operatorname{Tr}(M \circ Q_d)\,\mathrm{d}t \leqslant -\frac{v\lambda_1}{2}(d-k) + \frac{G_1}{2b}k + \frac{\kappa b^2}{4v}G_2 + \frac{\kappa}{v}\frac{1}{t}\int_0^t \|u\|^2\,\mathrm{d}t.$$

Thanks to (3.3), for sufficiently large $t > 0$ we have

$$\sup_{Q_d} \frac{1}{t}\int_0^t \operatorname{Tr}(M \circ Q_d)\,\mathrm{d}t \leqslant -\frac{v\lambda_1}{2}(d-k) + \frac{G_1}{2b}k + \frac{\kappa b^2}{4v}G_2 + \frac{2\kappa}{v^3}\|f\|^2_{\mathbf{L}^\infty(\mathbf{R}^+;V')}.$$

To ensure that the right-hand side of the above inequality is less than zero, we have to choose d large enough such that

$$d > k + \frac{2v^2\lambda_1}{G_2\kappa}b^2 + \frac{G_1 k}{v\lambda_1}\frac{1}{b} + \frac{4\kappa}{v^4\lambda_1}\|f\|^2_{\mathbf{L}^\infty(\mathbf{R}^+;V')}.$$

If we take $b = (vG_1G_2^{-1}\kappa^{-1}k)^{\frac{1}{3}}$, the right-hand side of the above inequality reaches its minimum value. Thus

$$d > \frac{4\kappa}{v^4\lambda_1}\|f\|^2_{\mathbf{L}^\infty(\mathbf{R}^+;V')} + \frac{3G_1^{2/3}G_2^{1/3}\kappa^{1/3}}{2v^{4/3}\lambda_1}k^{2/3} + k.$$

If we define the generalized Reynolds number as

$$Re = \frac{\|f\|^{1/2}_{\mathbf{L}^\infty(\mathbf{R}^+;V')}}{v\lambda_1^{1/4}},$$

we summarize the above result in

Theorem 5.1. *Suppose* $f \in \mathbf{L}^\infty(\mathbf{R}^+; V')$ *is a* k*-dimensional quasi-periodic external force and* $G = \sup_{\omega\in\mathbf{T}^k} \int_\Omega |f_\omega|^{4/3}\,\mathrm{d}x < \infty$. *Then we have*

$$\dim_H(\mathscr{A}_{\mathscr{F}}) \leqslant 4\kappa\, Re^4 + \frac{3\kappa^{1/3}G}{2\nu^{4/3}\lambda_1}k^{2/3} + k.$$

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