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The global attractor of g-Navier–Stokes equations with linear dampness on R^2 [☆]

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ABSTRACT

In this paper, the long time behaviors of g-Navier–Stokes equations with linear dampness on R^2 were investigated. By using the energy equation method, the existence of the global attractor for the equations was proved without the restriction of the forcing term belonging to some weighted Sobolev space. Moreover, the estimation of the Hausdorff and Fractal dimensions of such attractors were also obtained.

1. Introduction

Autonomous and nonautonomous dynamical systems have been studied by many researchers (see e.g. [1–3,6,9,11,13–16,18,19]), one of the most important problem in infinite dimensional is to prove the existence of attractor and study the structure of attractors. Many authors have paid much attention to this problem for a quite long time and have made a lot of successfully progress [5–10,20–27].

Recently, the g-Navier–Stokes equations in spatial dimension 2 were introduced by Roh (see [12,26]) as

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x) \quad \text{in } \Omega\times(0,\infty),\\ &\nabla\cdot(gu) = 0 \quad \text{in } \Omega\times(0,\infty), \end{aligned} \tag{1.1}$$

where $0 < m_0 \leqslant g = g(x_1,x_2) \leqslant M_0$ and $g = g(x_1,x_2)$ is a suitable real-valued smooth function on the domain $\Omega \subset R^2$. here, ν and f are given and the velocity u and the pressure p are the unknowns. When $g = 1$, the Eq. (1.1) become the usual two-dimensional Navier–Stokes equations. Roh proved the existence of global solution and the global attractor on unbounded domain. In [26], Minkye Kwak estimates the dimension of the global attractor for the spatial Periodic and Dirichlet boundary conditions.

In this paper, by using the energy equation method, we prove the existence of a global attractor of 2D g-Navier–Stokes equation with linear dampness on R^2 and we prove the finite dimension of the global attractor.

This paper is organized as follows. In Section 2, we first introduce some notations and preliminary results for the g-Navier–Stokes equations. In Section 3, we prove the existence of the global attractors of 2D-g-Navier–Stokes equations with linear dampness on the whole R^2. In Section 4, we estimate the dimension of the global attractors.

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2. Preliminaries

We consider the flow of fluid enclosed in R^2 and governed by the g-Navier–Stokes equations. We denote by $u(x,t) \in R^2$ and $p(x,t) \in R$, respectively, the velocity and the pressure of initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \nu\Delta u + \alpha u + (u\cdot\nabla)u + \nabla p = f(x)x \in R^2, \quad t > 0,\\
&\nabla\cdot(gu) = 0 \quad \text{in } \Omega,\\
&\lim u(x,t) = 0 \quad \text{on } \partial\Omega,\\
&u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, $f = f(x) \in R^2$ is the external body force (assumed to be time independent), and the function $g = g(x_1,x_2)$ is positive real-valued smooth function. We assume that the function g satisfies

$$0 < m_0 \leqslant g(x_1,x_2) \leqslant M_0 \quad \text{for all } (x_1,x_2) \in R^2.$$

The mathematical frameworks of (2.1) is following:
let

$$\begin{aligned}
&L^2(g) = (L^2(R^2))^2 \text{ with the inner products,}\\
&(u,v) = \int_{R^2} u\cdot v g\,dx, \quad u,v \in L^2(g) \text{ and norms } |\cdot| = (\cdot,\cdot)^{1/2},\\
&H_0^1(g) = \left(H_0^1(R^2)\right)^2, \quad u = (u_1,u_2), \quad v = (v_1,v_2) \in H_0^1(g) \text{ with the inner products,}\\
&((u,v)) = \int_{R^2}\sum_{j=1}^{2}\nabla u_j\cdot\nabla v_j g\,dx \text{ and norms } \|\cdot\| = ((\cdot,\cdot))^{1/2}.
\end{aligned}$$

Let $D(R^2)$ be the space of $\mathscr{C}^\infty$ functions with compact support contained in R^2 and let

$$\begin{aligned}
&\mathscr{D} = \{v \in (D(R^2))^2 : \nabla\cdot gv = 0 \text{ in } R^2\},\\
&H = \text{closure of } \mathscr{D} \text{ in } L^2(g),\\
&V = \text{closure of } \mathscr{D} \text{ in } H_0^1(g).
\end{aligned}$$

With H and V endowed with the inner product and norm of $L^2(g)$ and $H_0^1(g)$ respectively. where $H_0^1(R^2) = W^{1,2}(R^2)$ is usual soblev spaces with norm of $\|\cdot\|_1$, then we have

$$\|u\|_1^2 = |u|^2 + \|u\|^2, \quad \forall u \in H_0^1(g).$$

Now, we define a g-Laplacian operator as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla\cdot g\nabla)u = -\Delta u - \frac{1}{g}\nabla g\cdot\nabla u.$$

Using the g-Laplacian operator, we rewrite the first equation of (2.1) as follows:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{\nabla g}{g}\cdot\nabla u + \alpha u + (u,\nabla)u + \nabla p = f. \tag{2.2}$$

We define a g-orthogonal projection

$$P : L^2(g) \to H$$

and g-Stokes operator

$$Au = -P\left(\frac{1}{g}(\nabla\cdot(g\nabla u))\right).$$

When we apply the projection P into Eq. (2.2), we can obtain the following weak formulation of (2.1): let $f \in V$ and $u_0 \in H$, we find

$$u \in L^\infty(0,T;H)\cap L^2(0,T;V), \quad T > 0 \tag{2.3}$$

such that

$$\frac{d}{dt}(u,v) + \nu((u,v)) + \alpha(u,v) + b_g(u,u,v) + \nu(Ru,v) = \langle f,v\rangle \quad \forall v \in V, \quad \forall t > 0, \tag{2.4}$$

$$u(0) = u_0, \tag{2.5}$$

where $b_g : V \times V \times V \to R$ is given by

$$b_g(u, v, w) = \sum_{i,j=1}^{2} \int u_i \frac{\partial v_j}{\partial x} w_j g \, dx \tag{2.6}$$

and

$$Ru = P\left[\frac{1}{g}(\nabla g \cdot \nabla)u\right], \quad \forall u \in V.$$

Then, the weak formulation (2.4) is equivalent to the functional equation

$$\frac{du}{dt} + \nu Au + \alpha u + Bu + \nu Ru = f, \tag{2.7}$$

$$u(0) = u_0, \tag{2.8}$$

where $A : V \to V'$ is the g-Stokes operator defined by

$$\langle Au, v\rangle = ((u, v)), \quad \forall u, v \in V \tag{2.9}$$

and $B(u) = B(u, u) = P(u \cdot \nabla)u$ is a bilinear operator $B : V \times V \to V'$ defined by

$$\langle B(u, v), w\rangle = b_g(u, v, w), \quad \forall u, v, w \in V$$

The g-Stokes operator A is an isomorphism from V into V', while B and R satisfy the following inequalities (see Sell and You [14] and Roh [12]):

$$\|B(u)\|_{V'} \leqslant c|u|\|u\|, \quad \|Ru\|_{V'} \leqslant \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \|u\|, \quad \forall u \in V \tag{2.10}$$

We have the following result (see[4]).

Proposition 2.1. *Given $f \in L^2(g), u_0(x) \in H$, there exists a unique*

$$u(x, t) \in L^\infty(R^+; H) \cap L^2(0, T; V) \cap c(R^+; H) \quad (\forall T > 0)$$

such that (2.4), (2.5) hold.

Now let $u = u(t), t > 0$ be a solution given by Proposition 2.1. since $u \in L^2(0, T; V)$ and $u' \in L^2(0, T; V')$, we have

$$\frac{1}{2}\frac{d}{dt}|u|^2 = \langle u', u\rangle$$

so we have from (2.7) that

$$\frac{1}{2}\frac{d}{dt}|u|^2 = \langle f - \nu Au - \alpha u - Bu - \nu Ru, u\rangle = \langle f, u\rangle - \nu\|u\|^2 - \alpha|u|^2 - b_g(u, u, u) - \nu\left(\left(\frac{1}{g}\nabla g \cdot \nabla u\right), u\right).$$

Since $b_g(u, u, u) = 0, \ \forall u, v \in V$, we have

$$\frac{d}{dt}|u|^2 + 2\nu\|u\|^2 = 2\langle f, u\rangle - 2\alpha|u|^2 - 2\nu\left(\left(\frac{1}{g}\nabla g \cdot \nabla u\right), u\right). \tag{2.11}$$

Moreover, we can deduce

$$\frac{d}{dt}|u|^2 + 2\alpha|u|^2 + 2\nu\|u\|^2 = 2(f, u) - 2\nu\left(\left(\frac{1}{g}\nabla g \cdot \nabla u\right), u\right) \leqslant \frac{|f|^2}{\alpha} + \alpha|u|^2 + 2\nu\frac{|\nabla g|_\infty}{m_0 \lambda_0^{1/2}}\|u\|^2.$$

Hence

$$\frac{d}{dt}|u|^2 + \alpha|u|^2 + 2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_0^{1/2}}\right)\|u\|^2 \leqslant \frac{|f|^2}{\alpha}. \tag{2.12}$$

With Grownwall inequality, we have

$$|u(t)|^2 \leqslant |u_0|^2 e^{-\alpha t} + \frac{1}{\alpha^2}(1 - e^{-\alpha t})|f|^2 \tag{2.13}$$

and

$$\frac{1}{t}\int_0^t \|u(s)\|_1^2 ds \leqslant \frac{1}{\alpha\beta}|f|^2 + \frac{1}{t\beta}|u_0|^2, \quad \forall t > 0 \quad \text{where } \beta = \min\left(\alpha, 2\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_0^{1/2}}\right)\right). \tag{2.14}$$

From Proposition 2.1, we can define a continuous semigroup $\{S(t)\}$ in H by

$$S(t)u_0 = u(t), \quad t > 0,$$

where $u(t)$ is a solution of (2.4) with $u(0) = u_0 \in H$, Moreover, it follows from (2.13) that the set

$$B = \left\{ u \in H : |u| \leqslant \rho_0 \equiv \frac{\sqrt{2}}{\alpha}|f| \right\} \tag{2.15}$$

is absorbing in H for the semigroup.

Then, we can prove the following weak continuity of the semigroup $\{S(t)\}_{t\geqslant 0}$.

Proposition 2.2. *Let $\{u_{0n}\}_n$ be a sequence in H converging weakly in H to an element $u_0 \in H$, Then*

$$(1) \quad S(t)u_{0n} \to S(t)u_0 \text{ weakly in } H, \quad \forall t \geqslant 0 \tag{2.16}$$

$$(2) \quad S(t)u_{0n} \to S(t)u_0 \text{ weakly in } L^2(0,T;V), \quad \forall T \geqslant 0 \tag{2.17}$$

Proof. Let $u_n(t) = S(t)u_{0n}$, and $u(t) = S(t)u_0$, for $\forall t \geqslant 0$. From (2.13), (2.14) we find that

$$u_n(t) = S(t)u_{0n} \text{ is bounded in } L^\infty(R^+;H) \cap L^2(0,T;V), \quad \forall T \geqslant 0 \tag{2.18}$$

$$\text{Since } u_n' = f - \nu A u_n - \alpha u_n - B(u_n) - \nu R u_n \tag{2.19}$$

and $A : V \to V'$ is bounded linear operator, and

$$\|B(u)\|_{V'} \leqslant c|u|\|u\|, \quad \|Ru\|_{V'} \leqslant \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\|u\|$$

it follows that $\{u_n'\}$ is bounded in $L^2(0,T;V')$, $\forall t \geqslant 0$. Then, for

$$\forall v \in V \quad \text{and} \quad 0 < t \leqslant t + a \leqslant T \quad \text{with} \quad T > 0,$$

$$(u_n(t+a) - u_n(t), v) = \int_t^{t+a} \langle u_n'(s), v\rangle ds \leqslant \|v\|a^{1/2}\|u_n'\|_{L^2(0,T;V')} \leqslant c_T\|v\|a^{1/2}, \tag{2.20}$$

where c_T is positive, independent of n. Then, for $v = u_n(t+a) - u_n(t)$, we find that

$$|u_n(t+a) - u_n(t)|^2 \leqslant c_T a^{1/2}\|u_n(t+a) - u_n(t)\|.$$

Hence

$$\int_0^{T-a} |u_n(t+a) - u_n(t)|^2 dt \leqslant c_T a^{1/2} \int_0^{T-a} \|u_n(t+a) - u_n(t)\| dt \leqslant 2c_T T^{1/2} a^{1/2} \left(\int_0^T \|u_n(t)\|^2 dt \right)^{1/2} \leqslant c_{\tilde{T}} a^{1/2},$$

where $\left(\int_0^T \|u_n(t)\|^2 dt\right)^{1/2}$ is bounded and positive constant $c_{\tilde{T}}$ is independent of n. so

$$\lim_{a\to 0} sup_n \int_0^{T-a} |u_n(t+a) - u_n(t)|^2_{L^2(B_r)} dt = 0, \tag{2.21}$$

where $B_r = \{x \in R^2, |x| \leqslant r\}$. Moreover, from (2.18),

$$\{u_n|_{B_r}\}_n \text{ is bounded in } L^\infty(R^+;L^2(B_r)) \cap L^2(0,T;H^1(B_r)).$$

Consider a truncation function $\tau \in C^1(R^+)$ with $\tau(s) = 1$ for $s \in [0,1]$ and $\tau(s) = 0$ for $s \in [2,+\infty)$. For $\forall r > 0$ and $x \in B_{2r}$, from (2.21), we find that

$$\lim_{a\to 0} sup_n \int_0^{T-a} |v_{n,r}(t+a) - v_{n,r}(t)|^2_{L^2(B_{2r})} dt = 0 \quad \forall T > 0,\ r > 0$$

and $\{v_{n,r}\}$ is bounded in $L^2(0,T;H_0^1(B_{2r})) \cap L^\infty(0,T;L^2(B_{2r}))$, $\forall T > 0, r > 0$ by a compactness theorem,

$$\{v_{n,r}\} \text{ is relatively compact in } L^2(0,T;L^2(B_{2r})), \quad \forall T > 0,\ r > 0 \tag{2.22}$$

It follows from (2.22) that

$$\{u_n|_{B_r}\} \text{ is relatively compact in } L^2(0,T;L^2(B_r)), \quad \forall T > 0,\ r > 0$$

so there exists a subsequence $\{u_{n_1}\}$ such that

$$\begin{aligned}&\{u_{n_1}\} \to \tilde{u} \text{ weak-star in } L^\infty(R^+, H)\\&\text{weakly in } L^2(0,T;V),\\&\text{strongly in } L^2(0,T;L^2(B_r))\end{aligned} \tag{2.23}$$

for some $\tilde{u} \in L^\infty(R^+;H) \cap L^2(0,T;V)$.

Then from (2.19) we can get

$$\begin{aligned}(u'_{n_1}, v) &= (f,v) - \nu((u_{n_1}, v)) - \alpha(u_{n_1}, v) - b_g(u_{n_1}, u_{n_1}, v) - \nu R(u_{n_1}, v)\\&= (f,v) - \nu((u_{n_1}, v)) - \alpha(u_{n_1}, v) - b_g(u_{n_1}, u_{n_1}, v) - \nu\left(\left(\frac{\nabla g}{g}\cdot\nabla u_{n_1}\right), v\right)\end{aligned}$$

We pass to the limit in the equation for u_{n_1}, then

$$(\tilde{u}, v) = (f,v) - \nu((\tilde{u}, v)) - \alpha(\tilde{u}, v) - b_g(\tilde{u}, \tilde{u}, v) - \nu\left(\left(\frac{\nabla g}{g}\cdot\nabla\tilde{u}\right), v\right), \quad v \in \mathscr{D}$$

since $\mathscr{D}$ is dense in H, so

$$(\tilde{u}, v) = (f,v) - \nu((\tilde{u}, v)) - \alpha(\tilde{u}, v) - b_g(\tilde{u}, \tilde{u}, v) - \nu\left(\left(\frac{\nabla g}{g}\cdot\nabla\tilde{u}\right), v\right), \quad \forall v \in \mathscr{D}$$

then $\tilde{u}$ is a solution of (2.4) with $\tilde{u}(0) = u_0$.

By the uniqueness of the solutions we must have $\tilde{u}(t) = u(t) = S(t)u_0$. then by a contradiction argument we deduce that the whole sequence $\{u_n\}$ converges to u in the sence of (2.23). This prove (2.17).

Since $u_n(t) = S(t)u_{0n}$ is bounded in $L^\infty(R^+;H) \cap L^2(0,t;V)$, $\forall T > 0$ and (2.20) we have

$$(u_n(t), v) \to (u(t), v), \quad \forall t \in R^+, \ v \in \mathscr{D}$$

for $\mathscr{D}$ is dense in H. we have

$$(u_n(t), v) \to (u(t), v), \quad \forall t \in R^+, \ v \in H$$

so we also have that $u_n(t) = S(t)u_{0n}$ converges weakly to $u(t) = S(t)u_0$ in H. □

3. Existence of the global attractor

For the existence of the global attractor, we will prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geqslant 0}$. A semigroup is said to be asymptotically compact in a given metric space if $\{S(t_n)u_n\}$ is precompact whenever $\{u_n\}$ is bounded and $t_n \to \infty$.

To prove that $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in H, we use the energy equation.

First, we define a function $[\cdot,\cdot] : V \times V \to R$ by

$$[u,v] = \alpha(u,v) + 2\nu((u,v)) \tag{3.1}$$

for all $u, v \in V$. Then $[\cdot,\cdot]$ is bilinear and symmetric.

$$[u]^2 = [u,u] = \alpha|u|^2 + 2\nu\|u\|^2.$$

Then we have

$$\beta\|u\|_1^2 \leqslant [u]^2 \leqslant \gamma\|u\|_1^2 \quad \text{where } \gamma = \max(\alpha, 2\nu).$$

Thus $[\cdot,\cdot]$ is an inner product on V with the norm $[\cdot] = [\cdot,\cdot]^{1/2}$ equivalent to $\|\cdot\|_1$. we obtain

$$\frac{d|u|^2}{dt} + [u]^2 + \alpha|u|^2 = 2\langle f, u\rangle - 2\nu\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right), u\right) \tag{3.2}$$

for any solution $u = u(t) = S(t)u_0, u_0 \in H$.

Then

$$|u(t)|^2 = |u_0|^2 e^{-\alpha t} + 2\int_0^t e^{-\alpha(t-s)}(\langle f, u(s)\rangle - [u(s)]^2 - 2\nu\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right), u\right))ds,$$

which can be written

$$|S(t)u_0|^2 = |u_0|^2 e^{-\alpha t} + 2\int_0^t e^{-\alpha(t-s)}\left(\langle f, S(s)u_0\rangle - [S(s)u_0]^2 - 2\nu\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right), S(s)u_0\right)\right)ds. \tag{3.3}$$

For all $u_0 \in H, t \geqslant 0$.

Lemma 3.1 (see [27]). *Let M be the uniformly convex Banach space, $\{u_n\} \subset M$ and $u_n \to u$ weakly in M, where*

$$\lim_{n\to\infty} |u_n|_M = |u|_M$$

$|\cdot|_M$ *is norm in M, then $u_n \to u$ strongly in M.*

Theorem 3.1. *Assume $\nu > 0, f \in V$, Then the semigroup $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact in H.*

Proof. Let $B \subset H$ be a bounded set and consider the sequence $\{u_n\} \subset B$ and $\{t_n\}$ such that $t_n \geqslant 0, t_n \to \infty$, Then the set $\{S(t_n)u_n\}$ is precompact in H. Since the set B defined in (2.15) is absorbing, there exists a time $T(B) > 0$ such that $S(t)B \subset B, \forall t \geqslant T(B)$, so that for t_n large enough $(t_n \geqslant T(B))$,

$$S(t_n)u_n \in B. \tag{3.4}$$

So $\{S(t_n)u_n\}$ is weakly compact in H and hence there are subsequences which relabeled as $u_{n'}$ and $t_{n'}$, and element $w \in B$ such that

$$S(t_{n'})u_{n'} \to \ w \text{ weakly in } H. \tag{3.5}$$

Similarly for each $T > 0$, we also have

$$S(t_{n'} - T)u_{n'} \in B \quad \text{for } t_{n'} \geqslant T + T(B). \tag{3.6}$$

Therefore, $S(t_{n\prime} - T)u_{n\prime}$ is also weakly precompact in H and hence there are further subsequences, which also relabeled as $\{S(t_{n'} - T)u_{n'}\}$ and an element $w_T \in B$ such that

$$S(t_{n'})u_{n'} \to w_T \text{ weakly in } H, \quad \forall T \in N. \tag{3.7}$$

From Proposition 2.2(1), we have

$$w = \lim_{n'\to\infty} H_w S(t_{n'})u_{n'} = \lim_{n'\to\infty} H_w S(T)S(t_{n'} - T)u_{n'} = S(T) \lim_{n'\to\infty} H_w S(t_{n'} - T)u_{n'} = S(T)w_T,$$

where $\lim H_w$ denotes the limit taken in the weak topology of H. Thus $w = S(T)w_T, \forall T \in N$. From (3.5), we have

$$|w| \leqslant \lim\inf_{n'\to\infty} |S(t_{n'})u_{n'}|$$

and we shall now prove that

$$\lim\sup_{n'\to\infty} |S(t_{n'})u_{n'}| \leqslant |w|.$$

For $T > 0$, and $t_n > T$ we have by (3.3)

$$\begin{aligned} |S(t_{n'})u_{n'}|^2 = |S(T)S(t_{n'} - T)u_{n'}|^2 = |S(t_{n'} - T)u_{n'}|^2 e^{-\alpha t} + 2\int_0^T e^{-\alpha(t-s)}\langle f, S(s)S(t_{n'} - T)u_{n'}\rangle ds \\ - 2\int_0^T e^{-\alpha(t-s)}[S(s)S(t_{n'} - T)u_{n'}]^2 ds - 2\int_0^T e^{-\alpha(t-s)}[\nu\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right), S(s)S(t_{n'} - T)u_{n'}\right) ds. \end{aligned} \tag{3.8}$$

From (3.6), we obtain

$$\lim\sup_{n'\to\infty} e^{-\alpha T}|S(t_{n'} - T)u_{n'}|^2 \leqslant \rho_0^2 e^{-\alpha T}. \tag{3.9}$$

By the weak continuity Proposition 2.2(2), we obtain

$$S(s)S(t_{n'} - T)u_{n'} \to S(s)w_T \text{ weakly in } L^2(0, T; V). \tag{3.10}$$

Then

$$\lim_{n'\to\infty} \int_0^T e^{-\alpha T}\langle f, S(s)S(t_{n'} - T)u_{n'}\rangle ds = \int_0^T e^{-\alpha T}\langle f, S(s)w_T\rangle ds. \tag{3.11}$$

Moreover, since $[\cdot]$ is a norm on V equivalent $\|\cdot\|_1, 0 < e^{-\alpha T} \leqslant e^{-\alpha(T-s)} \leqslant 1, s \in [0, T]$ so $\left(\int_0^T e^{-\alpha(T-s)}[\cdot]^2 ds\right)^{1/2}$ is equivalent to the usual norm in $L^2(0, T; V)$.

Therefore, we have

$$\int_0^T e^{-\alpha(T-s)}[S(s)w_T]^2 ds \leqslant \lim_{n'\to\infty}\inf \int_0^T e^{-\alpha(T-s)}[S(s)S(t_{n'} - T)u_{n'}]^2 ds. \tag{3.12}$$

Hence

$$\begin{aligned}
&\lim_{n'\to\infty}\sup\left(-\int_0^T e^{-\alpha(T-s)}[S(s)S(t_{n'}-T)u_{n'}]^2 ds\right)\\
&\quad=-\lim_{n'\to\infty}\inf\left(\int_0^T e^{-\alpha(T-s)}[S(s)S(t_{n'}-T)u_{n'}]^2 ds\right)\\
&\quad\leqslant-\int_0^T e^{-\alpha(T-s)}[S(s)w_T]^2 ds,\\
&\lim_{n'\to\infty}\sup\left(-\int_0^T e^{-\alpha(T-s)}[2v\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right),S(s)S(t_{n'}-T)u_{n'}\right)]ds\right)\\
&\quad=-\lim_{n'\to\infty}\inf\left(\int_0^T e^{-\alpha(T-s)}\left[2v\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right),S(s)S(t_{n'}-T)u_{n'}\right)\right]ds\right)\\
&\quad\leqslant-\int_0^T e^{-\alpha(T-s)}\left[2v\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right),S(s)w_T\right)\right]ds.
\end{aligned}\tag{3.13}$$

We obtain

$$\lim_{n'\to\infty}\sup|S(t_{n'})u_{n'}|^2\leqslant|\rho_0|^2e^{-\alpha T}+\int_0^T e^{-\alpha(T-s)}\left(2\langle f,S(s)w_T\rangle-[S(s)w_T]^2-\left[2v\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right),S(s)w_T\right)\right]\right)ds.\tag{3.14}$$

On the other hand, we obtain from (3.3)

$$|w|^2=|S(s)w_T|^2=|w_T|^2e^{-\alpha T}+2\int_0^T e^{-\alpha(T-s)}\left(\langle f,S(s)w_T\rangle-[S(s)w_T]^2-\left[2v\left(\left(\frac{1}{g}\nabla g\cdot\nabla u\right),S(s)w_T\right)\right]\right)ds.\tag{3.15}$$

Hence, we have

$$\lim_{n'\to\infty}\sup|S(t_{n'})u_{n'}|^2\leqslant|w|^2+\left(\rho_0^2-|w_T|^2\right)e^{-\alpha T}\leqslant|w|^2+\rho_0^2e^{-\left(\alpha+\frac{v\lambda_1}{2}\right)T},\quad\forall T>0.\tag{3.16}$$

Let T go to infinity in (3.16), we obtain

$$\lim_{n'\to\infty}\sup|S(t_{n'})u_{n'}|^2\leqslant|w|^2.\tag{3.17}$$

Since H is a Hilbert space, From Lemma 3.1, (3.17) together with (3.5), we can prove

$$S(t_{n'})u_{n'}\to w\ \text{strongly in}\ H.$$

This shows that $\{s(t_n)u_n\}$ is precompact in H, and hence that $\{S(t)\}$ is asymptotically compact in H. Since $\{S(t)\}$ has a bounded absorbing set B in H, so we prove the existence of the global attractor. □

Theorem 3.2. *Assume $v>0, f\in V$, Then the dynamical system $\{S(t)\}_{t\geqslant 0}$ associated to the evolution equation (2.4) possesses a global attractor in H, i.e., a compact invariant set $A\subset H$ which attracts all bounded sets in H.*

4. The dimension of the global attractor

In this section, we want to estimate the dimension of the global attractor A of 2D g-Navier–Stokes equations with linear dampness on the whole R^2. We use the general theory given by Temam[17].

Let $u_0\in A$ and set $u(t)=S(t)u_0$, for $t\geqslant 0$, From (2.9) we see that the linearized flow around u is given by the equation

$$\frac{\partial U}{\partial t}+vAU+\alpha U+B(u,U)+B(U,u)+vRU=0,\tag{4.1}$$

$$U(0)=\psi,\tag{4.2}$$

$\forall\psi\in H$, *there exists a unique* $U\in L^2(0,T;V)\cap C([0,T];H)$ *satisfying* (4.1) $\forall T>0$

We can define a linear map $L(t;u_0):H\to H$ by setting $L(t;u_0)\zeta=U(t)$. It can also be proven that $L(t;u_0)$ is bounded and that $\{S(t)\}_{t\geqslant o}$ is uniformly differentiable on A, i.e

$$\lim_{\varepsilon\to 0}\sup_{u_0,v_0\in A,0<|u_0-v_0|\leqslant\varepsilon}\frac{|S(t)v_0-S(t)u_0-L(t;u_0)\cdot(v_0-u_0)|}{|v_0-u_0|}=0.\tag{4.3}$$

Let

$$F'(u)U==-vAU-\alpha U-B(u,U)-B(U,u)-vRU$$

write (4.1) as

$$U' = F'(u)U = -\nu AU - \alpha U - B(u,U) - B(U,u) - \nu RU \tag{4.4}$$

and define numbers $q_m, m \in N$, by

$$q_m = \lim_{t\to\infty} \sup_{u_0 \in A} \sup_{\psi_i \in H, |\psi_i| \leqslant 1, i=1,\cdots,m} \frac{1}{t}\int_0^t Tr(F'(S(\tau)u_0 \cdot Q_m(\tau))d\tau, \tag{4.5}$$

where $Q_m(\tau) = Q_m(\tau; u_0, \psi_1, \cdots, \psi_m)$ is the orthogonal projector in H onto the space spanned by $L(t;u_0)\psi_1, \cdots, L(t;u_0)\psi_m$, $\forall \psi_1, \cdots, \psi_m$ is linearly independent in H.

Lemma 4.1. *Let A is the global attractor of (1.1), if $q_n < 0$, for some $n \in N$. then A has finite Hausdorff and fractal dimensions estimated respectively as*

$$dim_H(A) \leqslant n,$$
$$dim_F(A) \leqslant n\left(1 + \max_{1\leqslant j \leqslant n-1} \frac{(q_j)_+}{|q_n|}\right).$$

In order to estimate the numbers q_m, Let $u_0 \in A$ and set $u(t) = S(t)u_0$ and $U_j(t) = L(t;u_0)\psi_j$, $t \geqslant 0$. Let $\phi_i(t)(i = 1 \cdots, m)$ be an orthonormal basis in H.

Since

$$\begin{aligned} Tr(F'(u(\tau)) \cdot Q_m(\tau)) &= \sum_{i=1}^m \langle F'(u(\tau))\phi_i, \phi_i\rangle = \sum_{i=1}^m \langle -\nu A\phi_i - \alpha\phi_i - B(u,\phi_i) - B(\phi_i,u) - \nu R\phi_i, \phi_i\rangle \\ &= \sum_{i=1}^m \left(-\nu\|\phi_i\|^2 - \alpha|\phi_i| - b_g(\phi_i,u,\phi_i) - \frac{\nu}{g}(\nabla g \cdot \nabla\phi_i)\phi_i\right). \end{aligned}$$

Now

$$\begin{aligned} \left|\sum_{i=1}^m b_g(\phi_i,u,\phi_i)\right| &= \left|\int_\Omega \sum_{i=1}^m \sum_{j,k=1}^2 \phi_{ij}D_j u_k(x)\phi_{ik}(x)g(x)\,dx\right| \leqslant \int_\Omega |gradu(x)|\rho(x)\,dx \leqslant \text{ (with the Schwarz inequality)} \\ &\leqslant \|u\||\rho| \text{ where } \rho(x) = \sum_{i=1}^m |\sqrt{g}\phi_i(x)|^2. \end{aligned}$$

We have the Lieb–Thirring inequality

$$|\rho(\tau)|^2 = \int_\Omega \rho^2(x,\tau)g(x)\,dx \leqslant c\sum_{i=1}^m \|\phi_i\|^2,$$
$$\left|\sum_{i=1}^m \left(\frac{\nu}{g}(\nabla g \cdot \nabla\phi_i)\phi_i\right)\right| \leqslant \sum_{i=1}^m \frac{\nu|\nabla g|_\infty}{m_0}\|\phi_i\||\phi_i|.$$

Hence

$$\begin{aligned} Tr(F'(u(\tau)) \cdot Q_m(\tau)) &= -\alpha m - \sum_{i=1}^m (\nu\|\phi_i\|^2 + b_g(\phi_i,u,\phi_i) + \frac{\nu}{g}(\nabla g \cdot \nabla\phi_i)\phi_i) \\ &\leqslant -\nu\sum_{i=1}^m \|\phi_i\|^2 + \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\sum_{i=1}^m \|\phi_i\||\phi_i| + \|u\||\rho| - \alpha m \\ &\leqslant -\nu\sum_{i=1}^m \|\phi_i\|^2 + \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\sum_{i=1}^m \|\phi_i\||\phi_i| + \|u\|\left(c\sum_{i=1}^m \|\phi_i\|^2\right)^{1/2} - \alpha m \leqslant \text{(with the Schwarz inequality)} \\ &\leqslant -\nu\sum_{i=1}^m \|\phi_i\|^2 + \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\sum_{i=1}^m \|\phi_i\||\phi_i| + \frac{c}{2\nu}\|u\|^2 - \alpha m. \end{aligned}$$

From (2.12) we obtain $\frac{1}{t}\int_0^t \|u(s)\|^2 ds \leqslant \frac{1}{2\nu\alpha\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}|f|^2 + \frac{1}{t}|u_0|^2$. For sufficiently small $|\nabla g|_\infty$, we let $1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} = \tilde{m}$, then we have

$$q_m \leqslant -\alpha m + \frac{c}{2\nu^2\alpha\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)}|f|^2 = -\alpha m + \frac{c}{2\nu^2\alpha\tilde{m}}|f|^2,$$

Let $m_0 = \left[\frac{c}{2\nu^2\alpha^2\tilde{m}}|f|^2\right]_* + 1$, where $[\cdot]_*$ is rounding function, then $q_{m_0} \leqslant 0$, so we obtain

$$d_H(A) \leqslant \frac{c}{2\nu^2\alpha^2\tilde{m}}|f|^2 + 1.$$

If we let $m_1 = \left[\frac{c}{\nu^2\alpha^2\tilde{m}}|f|^2\right]_* + 1$, then we have $q_{m_1} \leqslant 0$ and

$$\max_{1\leqslant j\leqslant m_1-1} \frac{(q_j)_+}{|q_{m_1}|} \leqslant 1$$

therefore

$$d_F(A) \leqslant \frac{c}{\nu^2\alpha^2\tilde{m}}|f|^2 + 2.$$

Theorem 4.1. *we consider the two-dimensional g-Navier–Stokes equations with the linear dampness, when* $|\nabla g|$ *is sufficiently small, we define* $\tilde{m} = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}$, *then*

$$d_H(A) \leqslant \frac{c}{2\nu^2\alpha^2\tilde{m}}|f|^2 + 1,$$

$$d_F(A) \leqslant \frac{2c}{\nu^2\alpha^2\tilde{m}}|f|^2 + 2.$$

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