Interactions of incoherent localized beams
in a photorefractive medium

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We numerically investigate interactions between two bright or dark incoherent localized beams in a strontium barium niobate photorefractive crystal in one dimension, using the coherent density method. For the case of bright beams, if the interacting beams are in-phase, they attract each other during propagation and form bound breathers; if out-of-phase, the beams repel each other and fly away. The bright incoherent beams do not radiate much and form long-lived, well-defined breathers or quasi-stable solitons. If the phase difference is $\pi/2$, the interacting beams may both attract or repel each other, depending on the interval between the two beams, the beam widths, and the degree of coherence. For the case of dark incoherent beams, in addition to the above, the interactions also depend on the symmetry of the incident beams. As already known, an even-symmetric incident beam tends to split into a doublet, whereas an odd-symmetric incident beam tends to split into a triplet. When launched in pairs, the dark beams display dynamics consistent with such a picture and in general obey soliton-like conservation laws, so that the collisions are mostly elastic, leading to little energy and momentum exchange. But they also radiate and breathe while propagating. In all the cases, the smaller the interval between the two interacting beams, the stronger the mutual interaction. On the other hand, the larger the degree of incoherence, the weaker the interaction.   © 2014 Optical Society of America

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1. INTRODUCTION

Incoherent spatial solitons in photorefractive (PR) media have come into research focus and attracted much attention in the last two decades [1–24]. Lately, many types of incoherent solitons have been reported in both theory and experiment. This remarkable progress not only opened up new research fields in soliton science and nonlinear optics but also broadened the potential applications of optical solitons.

The progress was facilitated by the appearance of new methods for the treatment of incoherent localized beams in PR media: the coherent density method [4,5,9,14], the self-consistent multimode method [2,7,8,11,13,15,18,19], and the mutual coherence function method [6,12]. They were developed independently to describe exactly such sorts of solitary waves in theory; however, it was soon demonstrated that these three seemingly different theoretical methods are equivalent to each other in inertial nonlinear media [21]. Recently, the coherent density method seems to be employed more often than the other in papers on incoherent solitons [20,22–24]. Thus, incoherent bright solitons interacting with incoherent dark solitons or coherent dark solitons (which would enhance the spatial coherence) have been investigated with the coherent density method [20]. In addition, interactions of incoherent solitons were also considered in [22,25]. However, there are still topics related with the interactions of copropagating

incoherent solitons that are worth investigating. For example, interactions of dark incoherent beams have never, to our knowledge, been discussed before. This is done here.

Thus far, interactions of bright coherent solitons have been investigated thoroughly [26]. Fusion, fission, and repulsion of solitons have been reported. On the other hand, interactions of dark coherent solitons [27–29] have been less investigated yet are quite different from those of bright solitons. From this point of view, interactions of incoherent dark solitons may show interesting phenomena and deserve further attention.

In this paper, we first introduce the coupled nonlinear Schrödinger-like integrodifferential equations that provide a general description of the incoherent solitary waves in PR media. Then we numerically solve them by employing the coherent density method, combined with the beam propagation method. Following this procedure, we investigate the interactions between incoherent bright and dark solitary waves. The target medium used in our numerical simulations is the biased strontium barium niobate (SBN) PR crystal, which is often employed in soliton experiments [30–34]. We refrain from calling these localized beams solitons, because they sometimes display inelastic collisions, breathe, and radiate, but propagate quasi-stably over considerable lengths that outdistance typical crystal thicknesses.

The paper is organized as follows: in Section 2, we briefly introduce the theoretical model; in Section 3, we discuss the interactions of bright and dark incoherent beams in detail; in Section 4, we conclude the paper.

2. THEORETICAL MODEL

We assume that the beams propagate along the z axis and are allowed to diffract along the x axis. Under these conditions and according to the theory developed in [5,20], in a biased SBN PR crystal with the optical c axis oriented in the x direction, the two incoherent light beams evolve according to the equation

$$i\left(\frac{\partial f}{\partial z} + \theta\frac{\partial f}{\partial x}\right) + \frac{1}{2k}\frac{\partial^2 f}{\partial x^2} - \frac{k_0}{2}n_e^3 r_{33}E_0(1+\rho)\frac{f}{1+I_f(x,z)} = 0, \quad (1)$$

from an initial condition at $z = 0$,

$$f(z = 0, x, \theta) = \sqrt{r_f G_N(\theta)}\phi_0(x), \quad (2)$$

where

$$I_f(x,z) = \int_{-\infty}^{+\infty} |f(x,z,\theta)|^2 \mathrm{d}\theta \quad (3)$$

is the total beam intensity. In Eqs. (1)–(3), f represents the so-called coherent density of the coupled incoherent beams and θ is the ratio k_x/k of the transverse to the longitudinal wave number. In the paraxial propagation we consider, this ratio is approximately equal to the angle of the beam with respect to the z axis. In addition, $k = 2\pi n_e/\lambda_0$ (in which λ_0 is the free-space wavelength and n_e is the extraordinary refractive index of the SBN crystal), r_{33} is the electro-optic coefficient involved in the generation of the space-charge field, ρ is the intensity when $x \to \pm\infty$, and $E_0 = \pm V/W$ is the external bias field needed for the development of nonlinearity (here V is the applied bias and W is the transverse width of the SBN crystal). For a biased SBN:75 crystal, the typical values of the parameters mentioned above are $n_e = 2.3$, $r_{33} = 1022$ pm/V, $W = 6$ mm, and $\lambda_0 = 488$ nm [4,9], which we are going to use in our simulations.

Concerning the initial condition—that is, the incident beam—the function $G_N(\theta)$ is the normalized angular power spectrum of the incoherent source, $\phi_0(x)$ is the input complex spatial modulation function of the two incoherent beams, and r_f is the maximum intensity of the incoherent beams. We assume that the normalized angular power spectrum of the incoherent source is Gaussian, that is, $G_N(\theta) = \exp(-\theta^2/\theta_0^2)/(\sqrt{\pi}\theta_0)$ [4,5,9,14], where θ_0 is the angular half-width of the power spectra associated with the incoherent beams. It also a measure of the incoherence—the larger θ_0, the more incoherent the beams. As concerns the modulation function ϕ_0, for bright incoherent beams we define it as

$$\phi_0(x) = \exp\left(-\frac{(x-d)^2}{2x_0^2}\right) + \exp(il\pi)\exp\left(-\frac{(x+d)^2}{2x_0^2}\right), \quad (4)$$

in which $2d$ is the distance between the two Gaussian input beams, l determines the phase difference between the beams,

and x_0 is related to the full width at half-maximum (FWHM) of the Gaussian beam. Corresponding to Eq. (4) we have

$$x_0 = \frac{\text{FWHM}}{2\sqrt{\ln 2}}.$$

For dark incoherent beams, we define ϕ_0 as

$$\phi_0(x) = 1 + \tanh\left(\frac{x-d}{x_0}\right) - \tanh\left(\frac{x+d}{x_0}\right) \quad (5)$$

for the odd-symmetry case. In Eq. (5),

$$x_0 = \frac{\text{FWHM}}{\ln(3+2\sqrt{2})} \quad \text{and} \quad d \geq d_{\text{th}} = \frac{\ln 3}{2}x_0,$$

to guarantee the two solitons are really dark. For the even-symmetry case, we use

$$\phi_0(x) = \sqrt{1 - \frac{S(x)}{\max\{S(x)\}}} \quad (6a)$$

with

$$S(x) = \text{sech}^2\left(\frac{x-d}{x_0}\right) + \text{sech}^2\left(\frac{x+d}{x_0}\right), \quad (6b)$$

in which

$$x_0 = \frac{\text{FWHM}}{\ln(3+2\sqrt{2})}.$$

We should mention at this point that the analytical incoherent soliton solution is hard to obtain, owing to the complexity of the nonlinear dynamical problem. The variational methods [35–40] can be used to solve for the asymptotic soliton solutions; however, as usual with variational methods, they are as good as the presumed ansatz solutions are—optimized but still approximate analytical solutions. Considering that we focus on the interacting dynamics of incoherent bright and dark beams in detail, which cannot be treated analytically, the incident beams given in Eqs. (4,5)–(6) are still similar to the incidents used in previous literature [4,5,9] and are good enough for our purpose. The interactions of asymptotic incoherent solitons obtained by using the variational method are beyond the scope of this paper.

3. NUMERICAL SIMULATIONS AND DISCUSSION

A. Bright Incoherent Beams

We first investigate the interactions of two in-phase [viz., $l = 0$ in Eq. (4)] incoherent beams; the numerical results are shown in Figs. 1(a1)–1(e1). It is seen that two incoherent solitons attract each other during propagation and form a bound breather. With the interval between beams increasing, the interaction strength decreases, and the two incoherent beams need a longer distance to form a breather.

If $l = 1/2$, the phase difference between the interacting incoherent beams is $\pi/2$; the corresponding results are displayed in Figs. 1(a2)–1(e2). It is interesting to note that the interacting beams still attract each other to form a deflected
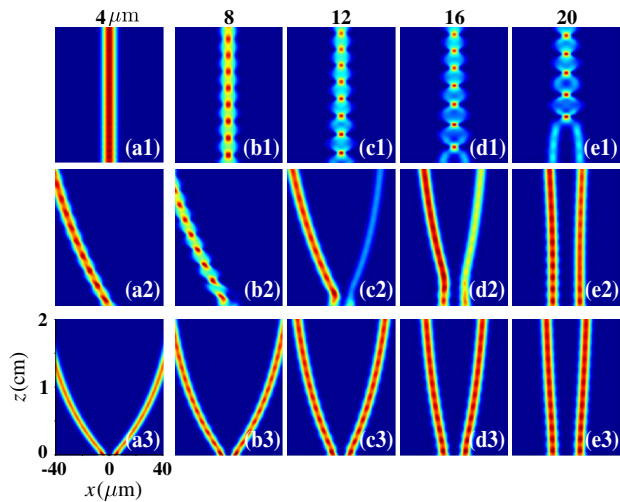
Fig. 1.   Interaction of two incoherent bright beams with different intervals $2d$ between them (the numbers displayed on the top of the panels). Other parameters are $l = 0, 0.5$, and 1 for (a1)–(e1), (a2)–(e2), and (a3)–(e3), respectively; FWMH = 7 μm; $V = 550$ V; and $\theta_0 = 1$ mrad.

breather if the interval is not big, as shown in Figs. 1(a2) and 1(b2). When the interval is big enough, the interacting beams repel each other, as shown in Figs. 1(c2)–1(e2). The reason is that there exist attraction and repulsion simultaneously between the two interacting beams; when the interval is small, the attraction prevails, while if the interval is big, the repulsion prevails.

The reason why the breather is deflected is also not difficult to explain. From Eq. (4) we can find the transverse group velocity v_x of the wave packet, which can be written as

$$v_x = -\frac{x - d}{x_0^2} \exp\left(-\frac{(x - d)^2}{2x_0^2}\right) - \exp(il\pi)\frac{x + d}{x_0^2} \exp\left(-\frac{(x + d)^2}{2x_0^2}\right). \quad (7)$$

When d is not big, the two beams behave as one wave packet, so that we can check the speed at $x = 0$ to obtain $v_x|_{x=0} \neq 0$. Hence, the breather should deflect. Furthermore, the beams accelerate transversely, as there are forces acting on them.

In the out-of-phase case [viz., $l = 1$ in Eq. (4)], exhibited in Figs. 1(a3)–1(e3), the interaction is predominantly repulsion. And the smaller the interval, the stronger the repulsion. Also note that in the interactions of bright incoherent beams, practically no radiation is observed—the solitons form fast from the incident beams and continue to propagate quite stably.

To check the influence of the degree of incoherence, which is estimated by the width of the power spectrum, we change θ_0 from 1 to 3 mrad and redo the simulation; the corresponding results are displayed in Fig. 2. In comparison with those shown in Fig. 1, we find that the interactions weaken. Especially, by comparing Fig. 2(c2) with Fig. 1(c2), we see that the competition between attraction and repulsion is also affected by θ_0, the degree of incoherence. It can be predicted that bigger θ_0 and smaller interval make the attraction more dominant.
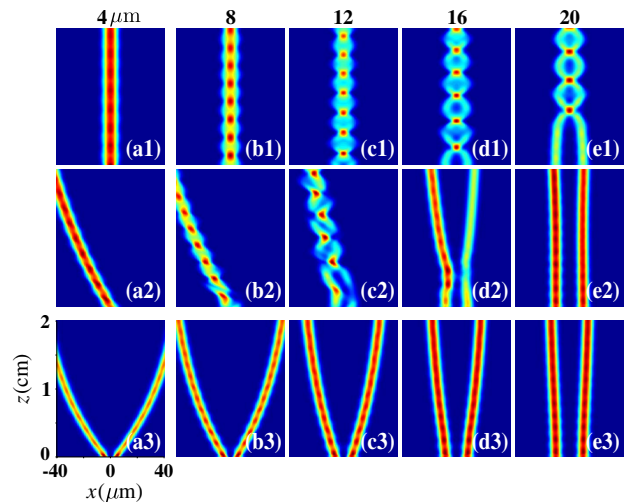

Fig. 2.   Same as Fig. 1, but with $\theta_0 = 3$ mrad.

B. Dark Incoherent Beams

As established previously [9,23,24], dark incoherent beams tend split into a doublet under the even conditions, or into a triplet or evolve into a gray soliton under the odd conditions (depending whether FWHMs are large or small, respectively). We discuss the interactions of dark incoherent beams under the odd conditions first, and then under the even conditions. To these ends, we use the modulation functions displayed in Eqs. (5) and (6), respectively.

We assume first that the incident beams are narrow, FWMH = 10 μm; the results are shown in Fig. 3. In Figs. 3(f)–3(j), in which the intervals are bigger than the FWHM, one can see that two gray solitons are formed, between which there is little or no interaction. When $d - d_{\mathrm{th}}$ is small, as shown in Figs. 3(a)–3(e), there is attraction first and then the repulsion between the two interacting dark incoherent beams. The reason is that the two dips in the input incoherent beams are so close that they will fuse into one at the beginning, strongly radiate, and then separate into two gray solitons. In addition, the function displayed in Eq. (5) looks like an even function for small intervals, so the interaction is then in analogy with the even case, which leads to the formation of a doublet [9]. One should note that, generally, the dark beams breathe and initially strongly radiate but exchange little energy as they propagate.
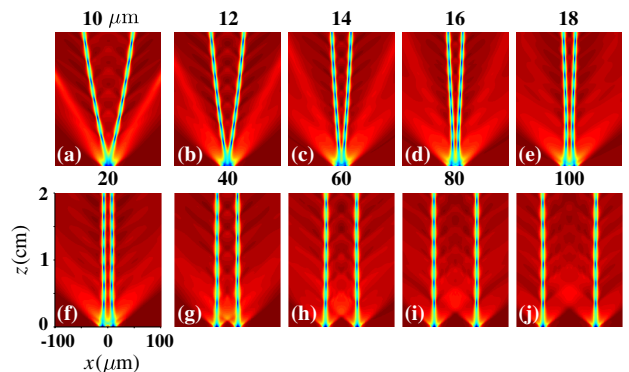

Fig. 3.   Interaction of two dark incoherent beams with FWMH = 10 μm, for changing interval $2d$ under odd conditions. Other parameters: $V = -550$ V and $\theta_0 = 9.6$ mrad.
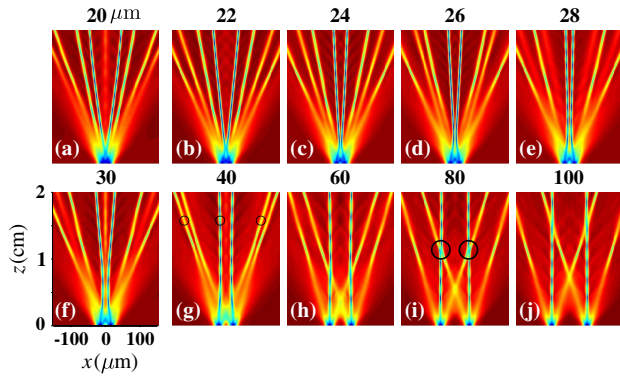
Fig. 4.   Same as Fig. 3 but with FWMH = 20 μm.

Fig. 5.   Figure setup is as Fig. 4 but under even conditions.

When the FWHM is increased to 20 μm, the results are displayed in Fig. 4. For d close to d_{th}, as shown in Figs. 4(a)–4(d), the dynamics is complex. There are many channels generated during propagation, and the two main channels in the middle first attract and then repel each other. The reason is similar to that of the case in Fig. 3—the input is similar to an even case with a much wider FWHM, which leads to additional channels [23,24].

However, when the interval is large, the input cannot be in analogy to the even case and has to revert to the odd case. This means that only six channels can form during propagation—three beam components for each of the dark incoherent beams—and the ensuing interaction is between the two triplets. This is clearly seen in Figs. 4(e)–4(j). In Fig. 4(g), the three channels marked by the three black circles originate from the same dark incoherent beam source: the main channel close to the center and the two secondary channels. One can note that the secondary channels are not symmetrically distributed about the main channel. For this case, the interaction is relatively strong, and we can also find that the secondary channels from different sources attract each other. Note the strong radiation formed in the central region, as well as the strong radiation emanating sideways in the beginning.

In Fig. 4(i), with the interval increasing further, the interaction between the two main channels becomes weak. In addition, we see that the secondary channels collide with the main channels. As marked by the two circles, the collision does not lead to the energy exchange between the secondary and main channels but leads to some momentum exchange. The reason for this elastic collision is that the intensities of the two participating beams are almost of the same size at the collision point. However, in the case of Fig. 4(j), there are also collisions between the secondary and main channels, but these collisions seem to be transparent, because the intensity of the secondary channel is much smaller than that of the main channel. We can also observe such transparent collisions in Figs. 4(g) and 4(h). These phenomena obey the usual soliton conservation laws, but with the caveat of considerable initial energy loss in the form of shed radiation.

Finally, we turn to the cases of even conditions, which are exhibited in Fig. 5. In Figs. 5(a) and 5(b), the intervals between the input beams are not big, so the total incident beams can be viewed as hyperbolic-like wave packets with FWMH = 30 μm and FWMH = 40 μm, respectively. Thus, we can classify the case as a splitting of one dark incoherent beam with large
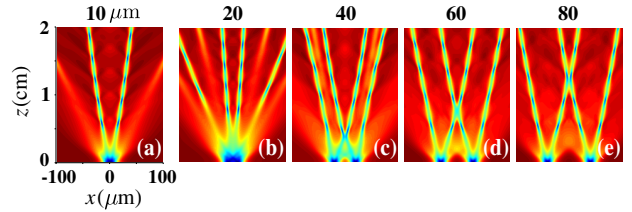
FWHM into a doublet rather than the interaction of two dark incoherent beams [24]. Indeed, we observe that more channels begin to appear at longer propagation distances. In Figs. 5(c)–5(e), the doublets from different incidences begin to interact. Due to the symmetry of the intensity distribution, the intensities of the two interacting participants are the same, and the collisions are always elastic. All the formed dark solitary beams breathe and radiate.

4. CONCLUSION

In summary, we have investigated the interactions of bright and dark incoherent localized beams in a PR medium. For the bright incoherent beams, the interaction is attraction if the interacting beams are in-phase, and repulsion if they are out-of-phase. If the phase difference is $\pi/2$, the interaction will lead to either a deflected bound breather or two repulsive solitons, depending on d and θ_0. The bright beams may breathe or propagate steadily over large distances, without visible radiation.

For the dark incoherent beams, we have discussed the cases of both odd and even symmetry conditions. Under the odd conditions, the collisions may be elastic or transparent, while under even conditions, the collisions are always elastic. Quite complex beam interaction scenarios may exist but still be consistent with the theory developed. These interactions comply with the soliton-like conservation laws, even though the beams breathe and radiate.

REFERENCES

1. M. Mitchell, Z. Chen, M.-F. Shih, and M. Segev, "Self-trapping of partially spatially incoherent light," Phys. Rev. Lett. **77**, 490–493 (1996).
2. M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, "Theory of self-trapped spatially incoherent light beams," Phys. Rev. Lett. **79**, 4990–4993 (1997).
3. M. Mitchell and M. Segev, "Self-trapping of incoherent white light," Nature **387**, 880–883 (1997).
4. D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, "Theory of incoherent self-focusing in biased photorefractive media," Phys. Rev. Lett. **78**, 646–649 (1997).
5. D. N. Christodoulides, T. H. Coskun, and R. I. Joseph, "Incoherent spatial solitons in saturable nonlinear media," Opt. Lett. **22**, 1080–1082 (1997).
6. V. P. Nayyar, "Propagation of partially coherent Gaussian Schell-model sources in nonlinear media," J. Opt. Soc. Am. B **14**, 2248–2253 (1997).

7. D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, "Multimode incoherent spatial solitons in logarithmically saturable nonlinear media," Phys. Rev. Lett. **80**, 2310–2313 (1998).

8. D. N. Christodoulides, T. H. Coskun, M. Mitchell, Z. Chen, and M. Segev, "Theory of incoherent dark solitons," Phys. Rev. Lett. **80**, 5113–5116 (1998).

9. T. H. Coskun, D. N. Christodoulides, M. Mitchell, Z. Chen, and M. Segev, "Dynamics of incoherent bright and dark self-trapped beams and their coherence properties in photorefractive crystals," Opt. Lett. **23**, 418–420 (1998).

10. Z. Chen, M. Mitchell, M. Segev, T. H. Coskun, and D. N. Christodoulides, "Self-trapping of dark incoherent light beams," Science **280**, 889–892 (1998).

11. N. Akhmediev, W. Królikowski, and A. W. Snyder, "Partially coherent solitons of variable shape," Phys. Rev. Lett. **81**, 4632–4635 (1998).

12. V. V. Shkunov and D. Z. Anderson, "Radiation transfer model of self-trapping spatially incoherent radiation by nonlinear media," Phys. Rev. Lett. **81**, 2683–2686 (1998).

13. M. I. Carvalho, T. H. Coskun, D. N. Christodoulides, M. Mitchell, and M. Segev, "Coherence properties of multimode incoherent spatial solitons in noninstantaneous Kerr media," Phys. Rev. E **59**, 1193–1199 (1999).

14. T. H. Coskun, D. N. Christodoulides, Z. Chen, and M. Segev, "Dark incoherent soliton splitting and 'phase-memory' effects: theory and experiment," Phys. Rev. E **59**, R4777–R4780 (1999).

15. A. A. Sukhorukov and N. N. Akhmediev, "Coherent and incoherent contributions to multisoliton complexes," Phys. Rev. Lett. **83**, 4736–4739 (1999).

16. W. Królikowski, N. Akhmediev, and B. Luther-Davies, "Collision-induced shape transformations of partially coherent solitons," Phys. Rev. E **59**, 4654–4658 (1999).

17. D. Kip, M. Soljacic, M. Segev, E. Eugenieva, and D. N. Christodoulides, "Modulation instability and pattern formation in spatially incoherent light beams," Science **290**, 495–498 (2000).

18. T. H. Coskun, D. N. Christodoulides, Y.-R. Kim, Z. Chen, M. Soljacic, and M. Segev, "Bright spatial solitons on a partially incoherent background," Phys. Rev. Lett. **84**, 2374–2377 (2000).

19. W. Królikowski, D. Edmundson, and O. Bang, "Unified model for partially coherent solitons in logarithmically nonlinear media," Phys. Rev. E **61**, 3122–3126 (2000).

20. T. H. Coskun, A. G. Grandpierre, D. N. Christodoulides, and M. Segev, "Coherence enhancement of spatially incoherent light beams through soliton interactions," Opt. Lett. **25**, 826–828 (2000).

21. D. N. Christodoulides, E. D. Eugenieva, T. H. Coskun, M. Segev, and M. Mitchell, "Equivalence of three approaches describing partially incoherent wave propagation in inertial nonlinear media," Phys. Rev. E **63**, 035601 (2001).

22. C.-F. Huang, R. Guo, S.-M. Liu, and Z.-H. Liu, "Interaction of incoherent white-light solitons," Opt. Commun. **269**, 174–178 (2007).

23. Y.-Q. Zhang, K.-Q. Lu, M.-Z. Zhang, K.-H. Li, S. Liu, and Y.-P. Zhang, "Dynamics of incoherent photovoltaic spatial solitons," Chin. Phys. Lett. **26**, 034212 (2009).

24. Y.-Q. Zhang, K.-Q. Lu, M.-Z. Zhang, K.-H. Li, S. Liu, and Y.-P. Zhang, "Dark incoherent spatial solitons in logarithmically saturable nonlinear media," Chin. Phys. B **18**, 5385–5394 (2009).

25. T.-S. Ku, M.-F. Shih, A. A. Sukhorukov, and Y. S. Kivshar, "Coherence controlled soliton interactions," Phys. Rev. Lett. **94**, 063904 (2005).

26. G. I. Stegeman and M. Segev, "Optical spatial solitons and their interactions: universality and diversity," Science **286**, 1518–1523 (1999).

27. R. N. Thurston and A. M. Weiner, "Collisions of dark solitons in optical fibers," J. Opt. Soc. Am. B **8**, 471–477 (1991).

28. G. E. Torres-Cisneros, R. J. Romero-Troncoso, J. J. Sónchez-Mondragón, and E. Alvarado-Méndez, "Collision of two real dark spatial solitons," J. Mod. Opt. **42**, 2323–2328 (1995).

29. Y. Ohta, D.-S. Wang, and J. Yang, "General *N*-dark-dark solitons in the coupled nonlinear Schrödinger equations," Stud. Appl. Math. **127**, 345–371 (2011).

30. Z. Chen, M.-f. Shih, M. Segev, D. W. Wilson, R. E. Muller, and P. D. Maker, "Steady-state vortex-screening solitons formed in biased photorefractive media," Opt. Lett. **22**, 1751–1753 (1997).

31. W. Królikowski, M. Saffman, B. Luther-Davies, and C. Denz, "Anomalous interaction of spatial solitons in photorefractive media," Phys. Rev. Lett. **80**, 3240–3243 (1998).

32. W. Królikowski, B. Luther-Davies, C. Denz, J. Petter, C. Weilnau, A. Stepken, and M. Belic, "Interaction of two-dimensional spatial incoherent solitons in photorefractive medium," Appl. Phys. B **68**, 975–982 (1999).

33. W. Krolikowski, E. A. Ostrovskaya, C. Weilnau, M. Geisser, G. McCarthy, Y. S. Kivshar, C. Denz, and B. Luther-Davies, "Observation of dipole-mode vector solitons," Phys. Rev. Lett. **85**, 1424–1427 (2000).

34. B. Terhalle, A. S. Desyatnikov, D. N. Neshev, W. Krolikowski, C. Denz, and Y. S. Kivshar, "Effect of nonlinearity on dynamic diffraction and interband coupling in two-dimensional hexagonal photonic lattices," Phys. Rev. A **86**, 013821 (2012).

35. B. A. Malomed, "Variational methods in nonlinear fiber optics and related fields," Prog. Opt. **43**, 71–193 (2002).

36. D. Buccoliero, A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, "Laguerre and Hermite soliton clusters in nonlocal nonlinear media," Phys. Rev. Lett. **98**, 053901 (2007).

37. M. Shen, X. Chen, J. Shi, Q. Wang, and W. Krolikowski, "Incoherently coupled vector dipole soliton pairs in nonlocal media," Opt. Commun. **282**, 4805–4809 (2009).

38. Q. Kong, Q. Wang, O. Bang, and W. Krolikowski, "Analytical theory of dark nonlocal solitons," Opt. Lett. **35**, 2152–2154 (2010).

39. M. Shen, H. Ding, Q. Kong, L. Ruan, S. Pang, J. Shi, and Q. Wang, "Self-trapping of two-dimensional vector dipole solitons in nonlocal media," Phys. Rev. A **82**, 043815 (2010).

40. S. Pu, C. Hou, K. Zhan, C. Yuan, and Y. Du, "Lagrangian approach for dark soliton in nonlocal nonlinear media," Opt. Commun. **285**, 3631–3635 (2012).