Supplemental Materials *of*

Observation of π solitons in oscillating waveguide arrays

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S1. Numerical simulations

For numerical simulations of evolution of 1D and 2D π solitons we used dimensionless nonlinear Schrödinger equation (1) in the main text, in which the transverse coordinates x, y are normalized to the characteristic scale $r_0 = 10$ μm, the propagation distance z is scaled to the diffraction length $kr_0^2 \approx 1.14$ mm, $k = 2\pi n/\lambda$ is the wavenumber in the medium with the background refractive index n (for fused silica $n \approx 1.45$) and $\lambda = 800$ nm is the working wavelength. The dimensionless widths of the waveguides are $a_x = 0.25$ and $a_y = 0.75$ (corresponding to 2.5 and 7.5 μm, respectively), and the array depth $p = k^2 r_0^2 \delta n/n$ is proportional to the refractive index contrast δn in the structure. In accordance with experiments, for 1D arrays we set $p = 4.5$ that corresponds to $\delta n \sim 5.0 \times 10^{-4}$ and for 2D array we set $p = 5$ that corresponds to $\delta n \sim 5.6 \times 10^{-4}$.

S2. Fs-laser inscription of oscillating waveguide arrays

SSH-like arrays of oscillating waveguides were inscribed in 10 cm-long fused silica glass samples (JGS1) using focused (with an aspheric lens with NA = 0.3) under the surface at depth 800 μm (in the 1D case) and at depth range $600 - 1000$ μm (in the 2D case) femtosecond laser pulses at the wavelength 515 nm for pulse duration 280 fs, repetition rate 1 MHz, pulse energy 290 nJ (in the 1D case) or 320 nJ (in the 2D case). Translation of the sample during the writing process of each waveguide was performed by the high-precision air-bearing positioner (Aerotech) with identical for all waveguides velocity of 1 mm/s. All such waveguides are elliptical and single-mode, and they exhibit propagation losses not exceeding 0.3 dB/cm at $\lambda = 800$ nm. In all cases for longitudinal periods considered here radiative losses were negligible for oscillation amplitudes $r < 11$ μm, but they become pronounced for $r \sim 13$ μm. After the waveguide arrays had been inscribed, the input/output facets of the sample were optically polished, so the sample length was shortened to 99 mm. Spacings between waveguides in unperturbed arrays (corresponding to $r = 0$) were 30 μm (in the 1D case) and 32 μm (in the 2D case). In the 1D case the waveguides were oscillating along the x axis only (in the plane of array), while in the 2D case they were oscillating along the diagonal of each unit cell. Longer axes of elliptical waveguides in the 2D case were oriented along the diagonal of the entire array to achieve more uniform coupling between waveguides. Longitudinal period of oscillations in the 1D cases was 33 mm (three periods on sample length), while in the 2D case it was 49.5 mm (two periods on sample length). Arrays

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Preprint submitted to Science Bulletin *August 6, 2023*

with fractional lengths were inscribed in the same 10 cm-long sample, so they began inside the sample. To enable excitation the left edge waveguide of the arrays was extended by a straight section at the beginning.

S3. Waveguide arrays excitation

In experiments we employed single-waveguide excitations using 280 fs pulses of variable energy E from 1 kHz fs Ti:sapphire laser at 800 nm central wavelength. The input peak power in the waveguide defined as a ratio of the pulse energy E to the pulse duration τ taking into account the losses for matching with the focusing lens is evaluated as 2.5 kW for each 1 nJ.

S4. Topological characterization of the system

Topological characterization of Floquet systems, like our z-periodic waveguide arrays, requires introduction of special invariants, as shown in [1, 2]. Starting from the periodically varying Hamiltonian $H(z)$ of the system that offers a "stroboscopic" description of the propagation dynamics over a complete longitudinal period Z (here H can be taken in tight-binding approximation for simplicity), one can introduce the evolution operator

$$\mathcal{U}(Z) = \mathcal{Z} e^{-i\int_0^Z H(z)dz} = e^{-iH_{\text{eff}}Z},$$

where $\mathcal{Z}$ is the time-ordering operator, and H_{eff} is the z-independent effective Hamiltonian of the Floquet system [3]. The π gap invariant w_π can be calculated as

$$w_\pi = \frac{i}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}\left[(\mathcal{V}_\pi^+)^{-1}\partial_k \mathcal{V}_\pi^+\right] dk,$$

where $\mathcal{V}(z) = \mathcal{U}(z)e^{iH_{\text{eff}}z}$ and $\mathcal{V}_\pi^+$ is obtained from

$$\mathcal{V}(Z/2) = \begin{bmatrix} \mathcal{V}_\pi^+ & 0 \\ 0 & \mathcal{V}_\pi^- \end{bmatrix}. \tag{S1}$$

For the 1D case using tight-binding description of z-periodic array, one finds that when $w_\pi = 1$ a topological π mode appears in the spectrum, while when $w_\pi = 0$, there are no such modes and the system is topologically trivial [2]. Similar approach was used for characterization of the 2D Floquet system using dimensionality reduction to the 1D case by considering equal Bloch momenta $k_x = k_y = k$ in the tight-binding description of 2D lattice (see details in [4] reporting on realization of acoustic higher-order Floquet insulator).

S5. Linear spectra of arrays with oscillating waveguides

Linear spectrum of the the Floquet structures considered here can be obtained numerically from Eq. (2) in the main text with omitted nonlinear term using "propagation and projection" method. At the first step we calculate the eigenvalues $b_{j\in[1,N]}$ and eigenmodes $\psi_{j\in[1,N]}^{\text{in}}$ of the optical potential $\mathcal{R}(x,y,z=0)$ using plane-wave expansion method or the finite-difference method (here N is the total number of single-mode waveguides in the structure). Next, we propagate all such "static" eigenmodes $\psi_{j\in[1,N]}^{\text{in}}$ over one period Z in the oscillating array $\mathcal{R}(x,y,z)$ to obtain output distributions $\psi_{j\in[1,N]}^{\text{out}}$ at $z = Z$. We calculate the matrix of projections with elements $P_{mn} = \langle \psi_m^{\text{in}}, \psi_n^{\text{out}} \rangle$, whose eigenvalues are the Floquet exponents e^{iZb_n} allowing to extract b_n – quasi-propagation constants of the n-th mode (all such constants form the spectrum, that includes π mode). For each b_n, we find the index ℓ_n of the maximum element of the corresponding eigenvector V_n of P_{mn}. The Floquet eigenmode of $\mathcal{R}(x,y,z)$ with any index n can then be constructed as $\psi_n^{\mathcal{R}} = \sum_{j=1}^{N} \psi_j^{\text{in}} V_j(\ell_n)$.

S6. Iterative calculation of the π solitons

The π solitons bifurcating from the linear π modes, can be obtained by using modification of the iterative method proposed in [5]. We first propagate the linear π mode ψ_π^{in} with a given power U (that will determine eventually quasi-propagation constant of the nonlinear Floquet state) according to Eq. (1) in the main text to obtain the dynamical lattice modified by the nonlinearity, i.e. $\mathcal{R}_\pi(x,y,z) = \mathcal{R}(x,y,z) + |\psi_\pi^{\text{in}}(x,y,z)|^2$. Next, we propagate all linear eigenstates $\psi_{n\in[1,N]}^{\text{in}}$ of $\mathcal{R}$ that include ψ_π^{in} in the modified dynamical lattice $\mathcal{R}_\pi$ for a one period Z and obtain corresponding output distributions $\psi_{n\in[1,N]}^{\text{out}}$. Then, we again calculate the projection $P_{mn} = \langle \psi_m^{\text{in}}, \psi_n^{\text{out}} \rangle$, whose eigenvalues are Floquet exponents e^{iZb_n}. For each b_n, we seek for the index ℓ_n of the maximum element of the corresponding eigenvector V_n of P_{mn} that allows us to construct the the Floquet eigenmodes $\psi_{n\in[1,N]}^{\text{re}}$ of modified optical structure $\mathcal{R}_\pi$: $\psi_n^{\text{re}} = \sum_{j=1}^{N} \psi_j^{\text{in}} V_j(\ell_n)$. Further, one can pick out the modified π state ψ_π^{re} from $\psi_{n\in[1,N]}^{\text{re}}$ and normalize it to the given power U. The iterations are continued until the difference between ψ_π^{re} and ψ_π^{in} reduces below the required small level.

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