Supplemental Materials *of*
"π-mode solitons in photonic Floquet lattices"

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1. π-mode soliton family at $r = 0.3$

The soliton family presented in Fig. 5 corresponds to $r = 0.25$ in the main text is completely inside a gap [see the spectrum in Fig. 4(d)] and does not overlap with bulk states. In Fig. S1, we present the results obtained for $r = 0.3$. As shown in Fig. S1(a), the linear π-modes that are indicated by red dots overlap with the bulk states, which is different from the case with $r = 0.25$. We find that even in this case the π-mode state does not show notable coupling with bulk states, but light appears in adjacent corners of the structure. Continuous soliton families can still be found in both focusing and defocusing media [see Fig. S1(b)]. Typical π-mode solitons are displayed in Fig. S1(c). The left panel in Fig. S1(c) is for the focusing nonlinearity condition and corresponds to dot 1 in Fig. S1(b), while the right panel in Fig. S1(c) is for the defocusing nonlinearity condition and corresponds to dot 2 in Fig. S1(b). We also find the solutions are stable during propagation over a long distance up to $z = 4000$. Peak amplitudes A versus propagation distance z corresponding to dots 1 and 2 in Fig. S1(b) are exhibited in Fig. S1(d), which do not decay and behave periodically.

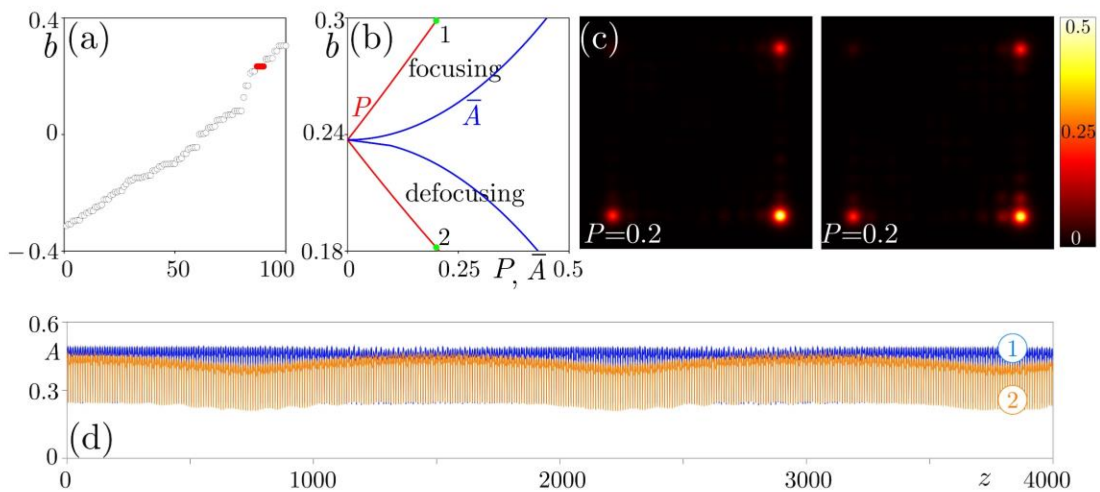

Fig. S1. (a) Linear quasi-propagation constant spectrum at $r = 0.3$, in which red dots correspond to linear π-modes. (b) Nonlinear π-mode family bifurcating from the linear π-mode. (c) Examples of the nonlinear π-mode states corresponding to dots 1 and 2 in (b). (d) Peak amplitude of these states versus distance z in the presence of small perturbations.

2. π-mode solitons survive from defects

We introduce the defect into the modulated lattice by increasing the refractive index change of the second waveguide 10%, and the other parameters are same as those adopted in Fig. 1(d). In Fig. S2(a), defects are highlighted by using thicker waveguides. The corresponding quasi-spectrum of the lattice is shown in Fig. S2(b); in the bandgap there are two pairs of degenerated defect modes in addition to two degenerated π-modes. The modulo profiles of the π-modes and defect modes are shown in Fig. S2(c). The π-mode soliton family in the defocusing condition is shown in Fig. S2(d), in which the dashed region represents unstable π-mode solitons. In Fig. S2(e), we show the modulo profiles of the selected π-mode solitons indicated by green dots in Fig. S2(d). Clearly, the π-mode soliton exists even though the lattice is perturbed.

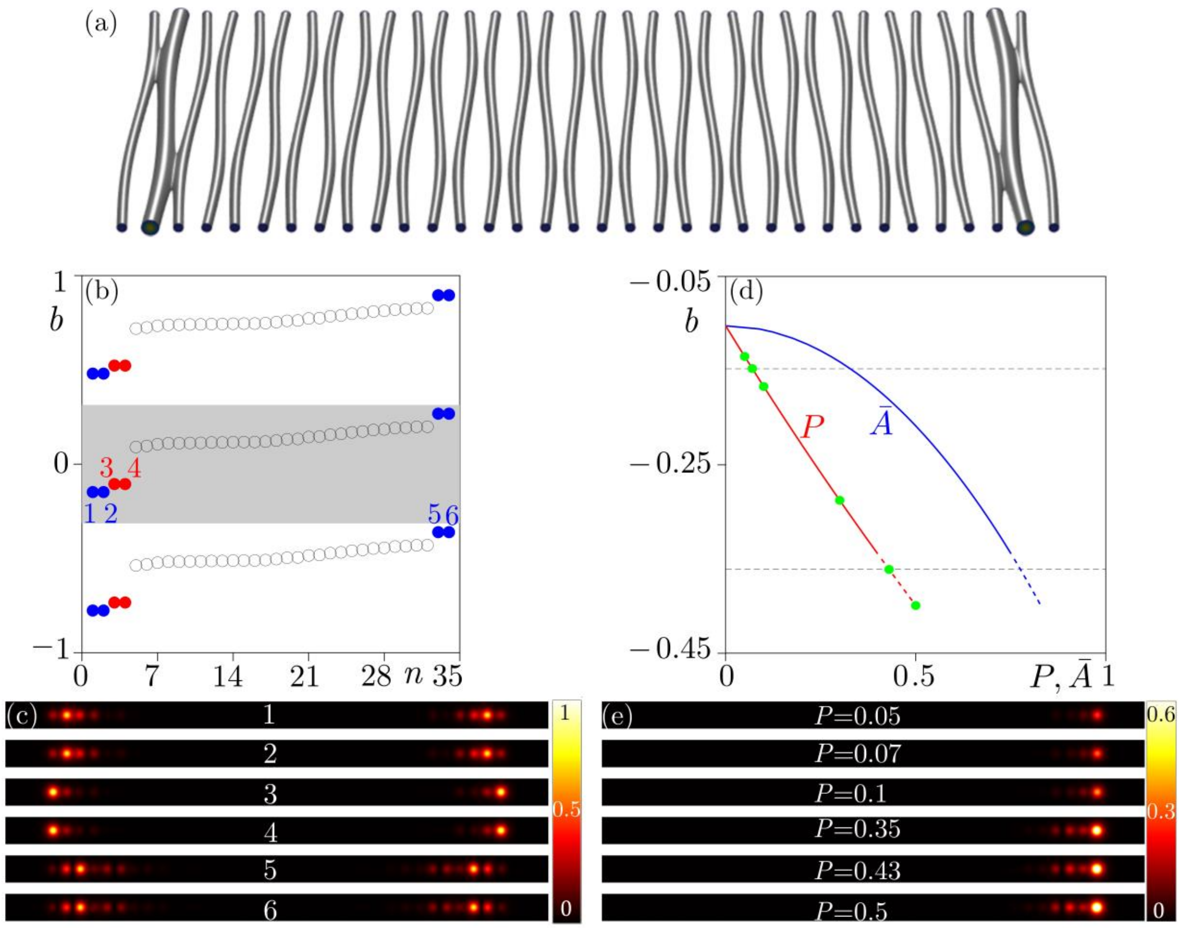

Fig. S2. (a) The Floquet lattice with defect that is highlighted with thicker waveguides. (b) The corresponding quasi-spectrum, in which π-modes are indicated by red dots (numbered 3 and 4) and defect modes by blue dots (numbered 1, 2, 5 and 6). The grey region1 shows the first Brillouin zone of the longitudinal periodic modulation. (c) Modulo profiles of the π-modes and defect modes. (d) π-mode soliton family bifurcating from the linear π-mode numbered 3 in (b) under the defocusing condition. Solid and dashed curves represent stable and unstable solitons. Horizontal dashed lines mark the energy of the linear defect-modes. (e) Modulo profiles of the 6 selected π-mode solitons in (d).

3. Numerical method on the quasi-spectrum

The beam propagation equation in the Floquet modulated wave guide array in our system can be written as:

$$i\frac{\partial\psi}{\partial z} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi - \mathcal{R}(x,y,z)\psi - \delta|\psi|^2\psi. \tag{1}$$

To calculate for its quasi-spectrum, one takes the following procedures:

(1) By dropping the nonlinear term in Eq. (1), we firstly calculate for the spectrum of the straight array with the transverse distribution same at the dynamic lattice at $z = 0$, i.e., $\mathcal{R}(x,y,z=0)$. The corresponding solution can be written as

$$\psi = u(x,y)e^{ibz}. \tag{2}$$

Plug solution (2) into Eq. (1), then we can get

$$bu = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u + \mathcal{R}(x, y, z = 0)u, \tag{3}$$

Which is an eigenvalue problem that can be solved by using the plane-wave expansion method or the finite-difference method. One obtains $b_{j\in[1,N]}$ and $\psi_{j\in[1,N]}^{\text{in}}$ for $\mathcal{R}(x, y, z = 0)$ with N being the lattice number.

(2) Propagate u over one period Z in the modulated array $\mathcal{R}(x, y, z)$ successively, and one obtains $\psi_{j\in[1,N]}^{\text{out}}$ at $z = Z$.

(3) Calculate for the projection:

$$U_{mn} = \langle \psi_m^{\text{in}}, \psi_n^{\text{out}} \rangle, \tag{4}$$

whose eigenvalues are Floquet exponents e^{iZb_n}. Here b_n is the quasi-spectrum.

(4) For each b_n, one finds the index ℓ_n of the maximum element of the corresponding eigenvector V_n of U_{mn}. The eigenstate of $\mathcal{R}(x, y, z)$ can be constructed as: $\psi_n^{\mathcal{R}} = \sum_{j=1}^{N} \psi_j^{\text{in}} V_j(\ell_n)$, which includes the π mode.

4. Numerical method on π state solitons

Now, as shown in Section 3 in this Supplemental Materials, one obtains linear states $\psi_{j\in[1,N]}^{\text{in}}$ (i.e. $\psi_{j\in[1,N]}^{\mathcal{R}}$ in Section 3) of $\mathcal{R}(x, y, z)$, and here we label the π state as ψ_π^{in}. The procedure of calculation of the π state soliton can be divided into several steps:

(1) We propagate the linear π state ψ_π^{in} with a given power P (that will determine eventually quasi-propagation constant of the nonlinear Floquet state) according to Eq. (1) to obtain the dynamical lattice modified by the nonlinearity, i.e. $\mathcal{R}_\pi = \mathcal{R} + |\psi_\pi^{\text{in}}|^2$.

(2) After this we propagate all linear eigenstates $\psi_{n\in[1,N]}^{\text{in}}$ of $\mathcal{R}$ that include ψ_π^{in} in the modified dynamical lattice $\mathcal{R}_\pi$ for a whole period Z and obtain corresponding output distributions $\psi_{n\in[1,N]}^{\text{out}}$.

(3) One calculates the projection $U_{mn} = \langle \psi_m^{\text{in}}, \psi_n^{\text{out}} \rangle$, whose eigenvalues are Floquet exponents e^{iZb_n}.

(4) For each b_n, one finds the index ℓ_n of the maximum element of the corresponding eigenvector V_n of U_{mn}. The eigenstate $\psi_{n\in[1,N]}^{\text{re}}$ of $\mathcal{R}_\pi$ can be constructed as: $\psi_n^{\text{re}} = \sum_{j=1}^{N} \psi_j^{\text{in}} V_j(\ell_n)$.

(5) One picks out the modified π state ψ_π^{re} from $\psi_{n\in[1,N]}^{\text{re}}$ and normalizes it to the given power P.

(6) The steps (1)-(5) are repeated until the difference between ψ_π^{re} and ψ_π^{in} reduces below required small level.