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On parallel multisplitting block iterative methods for linear systems arising in the numerical solution of Euler equations

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ABSTRACT

The paper studies the convergence of some parallel multisplitting block iterative methods for the solution of linear systems arising in the numerical solution of Euler equations. Some sufficient conditions for convergence are proposed. As special cases the convergence of the parallel block generalized AOR (BGAOR), the parallel block AOR (BAOR), the parallel block generalized SOR (BGSOR), the parallel block SOR (BSOR), the extrapolated parallel BAOR and the extrapolated parallel BSOR methods are presented. Furthermore, the convergence of the parallel block iterative methods for linear systems with special block tridiagonal matrices arising in the numerical solution of Euler equations are discussed. Finally, some examples are given to demonstrate the convergence results obtained in this paper.

1. Introduction

In this paper we consider the solution methods for the system of km linear equations

$$Ax = b, \tag{1.1}$$

where $A = [A_{ij}] \in \mathbb{C}^{km\times km}$ is an $m \times m$ block matrix with all the blocks $A_{ij} \in \mathbb{C}^{k\times k}$, $b,\ x \in \mathbb{C}^{km\times 1}$. The class of systems arises not only in the numerical solution of 2D and 3D Euler equations in fluid dynamics [1–3], but also in the discretizations of PDEs associated to invariant tori [4,5].

Elsner and Mehrmann in [6,7] gave several convergence results for some block iterative methods such as block Jacobi method, block Gauss–Seidel method and block SOR method for the solution of linear system (1.1) when the coefficient matrix A is either generalized M-matrices (see [6–8]) or consistently ordered p-cyclic matrices (see [9]). Later, Nabben [3,10] established some further results on convergence of block iterative methods for the solution of this class of linear systems with conjugate generalized H-matrices (see [11]). For example, he established convergence of the block Jacobi method, the block Gauss–Seidel method, the block JOR-method and the block SOR-method.

Recently, Zhang et al. [11] further proposed several convergence results for some block iterative methods including the block Jacobi method, the block Gauss–Seidel method, the block SOR method and the block AOR method for the solution of linear systems when the coefficient matrices are generalized H-matrices.

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In what follows we will introduce some iterative methods of the system (1.1). Consider the following splitting of the coefficient matrix A of (1.1),

$$A = D - L - U, \tag{1.2}$$

where D is nonsingular, L and U are not necessarily (block) triangular in general. Assume that $\det(D - \gamma L) \neq 0$. Then the (block) generalized accelerated overrelaxation (GAOR (BGAOR)) method is defined by

$$x^{(i+1)} = \mathcal{L}(\gamma, \omega)x^{(i)} + (D - \gamma L)^{-1}b, \quad i = 1, 2, \ldots, \tag{1.3}$$

where $\mathcal{L}(\gamma, \omega) = (D - \gamma L)^{-1}[(1 - \omega)D + (\omega - \gamma)L + \omega U]$ is the iteration matrix of the method (1.3). For $\omega = \gamma$, the (block) generalized AOR method reduces to the (block) generalized SOR (GSOR (BGSOR)) method. If the splitting (1.2) is standard (block) decomposition (i.e., D is (block) diagonal and nonsingular, L and U are strictly lower and strictly upper (block) triangular, respectively), then the (block) generalized AOR method and the (block) generalized SOR method reduce to the (block) AOR method and the (block) SOR method, respectively. Furthermore, if the method (1.3) is the (block) AOR method and $\gamma = 0$, then we obtain the (block) JOR method.

In this paper, we mainly discuss the convergence of parallel multisplitting block iterative methods of linear system (1.1). The parallel multisplitting iterative methods are investigated in [12–16]. Let us consider the block case.

In order to solve the system (1.1) with parallel multisplitting block iterative methods, the coefficient matrix $A = [A_{ij}] \in \mathbb{C}^{km \times km}$ is split into

$$A = M_s - N_s, \quad s = 1, 2, \ldots, r \tag{1.4}$$

by means of the following block matrices $M_s = [M_{ij}^s]$ with

$$M_{ij}^s = \begin{cases} A_{ij}, & \text{if } (i,j) \in Q_s \text{ and } i = j \in N \\ 0, & \text{if } (i,j) \notin Q_s,\ i \neq j \end{cases} \tag{1.5}$$

and $N_s = [N_{ij}^s]$ with

$$N_{ij}^s = \begin{cases} 0, & \text{if } (i,j) \in Q_s \text{ and } i = j \in N \\ -A_{ij}, & \text{if } (i,j) \notin Q_s,\ i \neq j. \end{cases} \tag{1.6}$$

Here $Q_s \subset P(m) = \{(i,j) \mid i,j \in N = \{1, 2, \ldots, m\},\ i \neq j\}$ and each M_s is nonsingular for $s = 1, 2, \ldots, r$. The splitting (1.4) is called a multisplitting of the matrix A and is denoted by $(M_s, N_s, E_s)_{s=1}^r$. Here, $E_s = \mathrm{diag}(e_s^1 I_k, e_s^2 I_k, \ldots, e_s^m I_k)$ is a $km \times km$ nonnegative diagonal matrix for $s = 1, 2, \ldots, r$ and $\sum_{s=1}^r E_s = I$, the $km \times km$ identity matrix. It follows that a parallel multisplitting block iterative form of (1.1) can be described as follows:

$$x^{(i+1)} = \sum_{s=1}^{r} E_s M_s^{-1} N_s x^{(i)} + \sum_{s=1}^{r} E_s M_s^{-1} b, \quad i = 1, 2, \ldots. \tag{1.7}$$

With $T = \sum_{s=1}^r E_s M_s^{-1} N_s$ and calling T the iteration matrix of the method (1.7), Eq. (1.7) can be changed into the following equations:

$$\begin{aligned} x^{(i+1)} &= \sum_{s=1}^{r} E_s y_s^{(i)}, \quad i = 1, 2, \ldots, \\ y_s^{(i)} &= M_s^{-1} N_s x^{(i)} + M_s^{-1} b \quad s = 1, 2, \ldots, r. \end{aligned} \tag{1.8}$$

Eq. (1.8) shows that this multisplitting method has a natural parallelism, since the calculations of $y_s^{(i)}$ for various values of s are independent and may therefore be performed in parallel. Moreover, the jth component of $y_s^{(i)}$ need not be computed if the corresponding diagonal entry of E_s is zero. This may result in considerable savings of computational time.

If $r = 1$, then the multisplitting (1.4) turns into a single splitting

$$A = M_1 - N_1, \tag{1.9}$$

and the corresponding block iterative method is a general block iterative method.

An extrapolated parallel iterative method with a positive extrapolation parameter τ is considered in [15,12]. The following gives the extrapolated parallel block iterative method by the block iteration

$$x^{(i+1)} = \tau \sum_{s=1}^{r} E_s M_s^{-1}(N_s x^{(i)} + b) + (1 - \tau)x^{(i)}, \quad i = 1, 2, \ldots. \tag{1.10}$$

Its iteration matrix is defined by

$$T(\tau) = \tau \sum_{s=1}^{r} E_s M_s^{-1} N_s + (1 - \tau)I.$$

In [15,16], the parallel generalized AOR (GAOR), block AOR (BAOR) and AOR methods are defined. Let

$$A = D_s - L_s - U_s, \quad s = 1, 2, \ldots, r \tag{1.11}$$

where $D_s \in C^{km\times km}$ is a nonsingular block matrix, $L_k \in \mathbb{C}^{km\times km}$ and $U_k \in \mathbb{C}^{km\times km}$ are not necessarily block triangular in general. Assume that $\det(D_s - \gamma_s L_s) \neq 0,\ s = 1, 2, \ldots, r$. Then the parallel block GAOR (BGAOR) method is defined by

$$x^{(i+1)} = \mathcal{L}(\Gamma, \Omega)x^{(i)} + \sum_{s=1}^{r} E_s(D_s - \gamma_s L_s)^{-1}b, \quad i = 1, 2, \ldots, \tag{1.12}$$

where

$$\begin{aligned}\mathcal{L}(\Gamma, \Omega) &= \sum_{s=1}^{r} E_s(D_s - \gamma_s L_s)^{-1}[(1-\omega_s)D_s + (\omega_s - \gamma_s)L_s + \omega_s U_s],\\ \Gamma &= (\gamma_1, \gamma_2, \ldots, \gamma_r), \qquad \Omega = (\omega_1, \omega_2, \ldots, \omega_r).\end{aligned} \tag{1.13}$$

This method may be achieved by the multisplitting (1.4) with

$$\begin{aligned}M_s &= \frac{1}{\omega_s}(D_s - \gamma_s L_s),\\ N_s &= \frac{1}{\omega_s}[(1-\omega_s)D_s + (\omega_s - \gamma_s)L_s + \omega_s U_s], \quad s = 1, 2, \ldots, r.\end{aligned} \tag{1.14}$$

The parallel BGAOR method reduces to the parallel BGSOR (parallel block generalized SOR) method if the parameter pairs (γ_s, ω_s) turn into (ω_s, ω_s) for $s = 1, 2, \ldots, r$ and the parallel BGGS (parallel block generalized Gauss–Seidel) method if the parameter pairs (γ_s, ω_s) turn into (ω_s, ω_s) with $\omega_s = 1$ for $s = 1, 2, \ldots, r$. We denote by $\mathcal{L}(\Omega)$ and $\mathcal{L}_{PBGGS}$ the iteration matrices of the parallel BGSOR and the parallel BGGS methods, respectively.

If the decompositions in (1.11) are the usual block decompositions, i.e., $D_s \in \mathbb{C}^{km\times km}$ is a nonsingular block diagonal part of A, $L_k \in \mathbb{C}^{km\times km}$ and $U_k \in \mathbb{C}^{km\times km}$ are strictly lower and upper block triangular matrices, respectively, then the parallel BGAOR and the parallel BGSOR methods reduce to the parallel block AOR (BAOR) and the parallel block SOR (BSOR) methods, respectively. Lastly, we denote the iteration matrices of the extrapolated BGAOR and BGSOR methods by $\mathcal{L}(\Gamma, \Omega, \tau)$ and $\mathcal{L}(\Omega, \tau)$, respectively.

This paper is organized as follows. Some notations and preliminary results about generalized H-matrices are given in Section 2. The convergence results of parallel block iterative methods for linear systems with generalized H-matrices are established in Section 3. In what follows, the convergence properties of parallel block iterative methods for linear systems with special block tridiagonal matrices arising in special cases from the computations of partial differential equations are discussed in Section 4 and some examples are given in Section 5 to illustrate the convergence results obtained in this paper. Finally, conclusions are given in Section 6.

2. Preliminaries

In this section we give some notions and preliminary results about special matrices that are used in this paper. We denote by $\mathbb{C}^{n\times n}$ ($\mathbb{R}^{n\times n}$) the set of all $n\times n$ complex (real) matrices; $\mathbb{C}^n$ the set of all n-dimensional complex vectors; $\mathbb{R}^n_+$ the set of positive vectors in $\mathbb{R}^n$; A^T the transpose of A; A^H the conjugate transpose of A; $\rho(A)$ the spectral radius of A; $Re(z)$ the real part of z.

Definition 2.1 (*See [17]*)**.** A matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian if $A^H = A$; a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian positive definite if $x^H Ax > 0$ for all $0 \neq x \in \mathbb{C}^n$ and Hermitian semipositive definite if $x^H Ax \geq 0$ for all $x \in \mathbb{C}^n$. A matrix $A \in \mathbb{C}^{n\times n}$ is called positive definite if $Re(x^H Ax) > 0$ for all $0 \neq x \in \mathbb{C}^n$ and semipositive definite if $Re(x^H Ax) \geq 0$ for all $x \in \mathbb{C}^n$.

By $A > 0$ and $A \geq 0$ we denote that A is (Hermitian) positive definite and (Hermitian) semipositive definite. Analogously we write $A < 0$ if $-A > 0$ and $A \leq 0$ if $-A \geq 0$. Furthermore, for $A,\ B \in \mathbb{C}^{n\times n}$, we write $A > B$ and $A \geq B$ if $A - B > 0$ and $A - B \geq 0$.

Definition 2.2. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. If A is Hermitian, then $|A| \in \mathbb{C}^{n\times n}$ is defined as $|A| := \sqrt{AA}$.

Definition 2.3 (*See [6,3]*)**.**

1. $Z^k_m = \{A = [A_{ij}] \in \mathbb{C}^{km\times km} \mid A_{ij} \in \mathbb{C}^{k\times k}$ is Hermitian for all $i, j \in N = \{1, 2, \ldots, m\}$ and $A_{ij} \leq 0$ for all $i \neq j,\ i, j \in N\}$;
2. $\widehat{Z}^k_m = \{A = [A_{ij}] \in Z^k_m \mid A_{ii} > 0,\ i \in N\}$;
3. $M^k_m = \{A \in \widehat{Z}^k_m \mid$ there exists $u \in \mathbb{R}^m_+$ such that $\sum_{j=1}^m u_j A_{ij} > 0$ for all $i \in N\}$, where $\mathbb{R}^m_+$ denotes all positive vectors in $\mathbb{R}^m$, and A matrix $A \in \widehat{Z}^k_m$ is called a generalized M-matrix if $A \in M^k_m$;
4. $D^k_m = \{A = [A_{ij}] \in \mathbb{C}^{km\times km} \mid A_{ij} \in \mathbb{C}^{k\times k}$ is Hermitian for all $i, j \in N$ and $A_{ii} > 0$ for all $i \in N\}$;

5. $H_m^k = \{A \in D_m^k \mid \mu(A) \in M_m^k\}$, where $\mu(A) = [M_{ij}] \in \mathbb{C}^{mk\times mk}$ is the block comparison matrix of A and is defined as

$$M_{ij} := \begin{cases} |A_{ii}|, & \text{if } i = j \\ -|A_{ij}|, & \text{if } i \neq j, \end{cases}$$

and A matrix $A \in D_m^k$ is called a generalized H-matrix if $A \in H_m^k$.

3. Main results

In this section we discuss the convergence of parallel multisplitting block iterative methods when the coefficient matrices are generalized H-matrices. The following lemmas will be used in this section.

Lemma 3.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ with a multisplitting $(M_s, N_s, E_s)_{s=1}^r$, and let $T = \sum_{s=1}^r E_s M_s^{-1} N_s$ and $\hat{A} = \hat{M} - \hat{N}$, where*

$$\hat{M} = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_r \end{bmatrix}, \qquad \hat{N} = \begin{bmatrix} N_1E_1 & N_1E_2 & \cdots & N_1E_r \\ N_2E_1 & N_2E_2 & \cdots & N_2E_r \\ \vdots & \vdots & \ddots & \vdots \\ N_rE_1 & N_rE_2 & \cdots & N_rE_r \end{bmatrix}. \tag{3.1}$$

Then $\rho(T) = \rho(\hat{M}^{-1}\hat{N})$, where $\rho(T)$ denotes the spectral radius of the matrix T.

Proof.

$$\begin{aligned}
\rho(T) &= \rho\left(\sum_{s=1}^r E_s M_s^{-1} N_s\right) \\
&= \rho\left(\begin{bmatrix} E_1 & E_2 & \cdots & E_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} M_1^{-1}N_1 & 0 & \cdots & 0 \\ M_2^{-1}N_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_r^{-1}N_r & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \rho\left(\begin{bmatrix} M_1^{-1}N_1 & 0 & \cdots & 0 \\ M_2^{-1}N_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_r^{-1}N_r & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} E_1 & E_2 & \cdots & E_r \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \\
&= \rho\left(\begin{bmatrix} M_1^{-1}N_1E_1 & M_1^{-1}N_1E_2 & \cdots & M_1^{-1}N_1E_r \\ M_2^{-1}N_2E_1 & M_2^{-1}N_2E_2 & \cdots & M_2^{-1}N_2E_r \\ \vdots & \vdots & \ddots & \vdots \\ M_r^{-1}N_rE_1 & M_r^{-1}N_rE_2 & \cdots & M_r^{-1}N_rE_r \end{bmatrix}\right) \\
&= \rho(\hat{M}^{-1}\hat{N}),
\end{aligned} \tag{3.2}$$

where $\hat{M}$ and $\hat{N}$ are defined as in (3.1). This completes the proof. □

Lemma 3.2 (See [11]). *Let $A = [A_{ij}] \in H_m^k$ with a splitting $A = M_1 - N_1$ as in (1.9). Then $\rho(M_1^{-1}N_1) < 1$.*

Theorem 3.3. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting $(M_s, N_s, E_s)_{s=1}^r$. Then the parallel multisplitting block iterative method (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Proof. We only prove that $\rho(T) < 1$. Lemma 3.1 shows that $\rho(T) = \rho(\hat{M}^{-1}\hat{N})$, where $\hat{M}$ and $\hat{N}$ are defined as in (3.1). Since $A \in H_m^k$ indicates $\mu(A) \in M_m^k$, it follows from Definition 2.3 that there exists a positive diagonal matrix $F = \mathrm{diag}(f_1 I_k, f_2 I_k, \ldots, f_m I_k)$, where I_k is the $k \times k$ identity matrix, such that AF satisfies

$$f_i|A_{ii}| - \sum_{j=1, j\neq i}^{m} |A_{ij}| f_j > 0, \tag{3.3}$$

for all $i \in N$. Note that $(M_s, N_s, E_s)_{s=1}^r$ is a multisplitting of A, $E_s = \mathrm{diag}(e_s^1 I_k, \ldots, e_s^m I_k)$ is a $km \times km$ nonnegative diagonal matrix for $s = 1, 2, \ldots, r$ and $\sum_{s=1}^r E_s = I$, the $km \times km$ identity matrix. Then we have

$$\sum_{s=1}^r e_s^i = 1, \quad i = 1, 2, \ldots, m \text{ and } e_s^i \geq 0,\ s = 1, 2, \ldots, r. \tag{3.4}$$

As a result, $A = M_s - N_s = M_s - N_s \sum_{s=1}^{r} E_s \in H_m^k$ satisfying (3.3) for all $s = 1, 2, \ldots, r$. Following (3.3) and (3.4), we have that for $s = 1, 2, \ldots, r$,

$$\begin{aligned}\left[f_i|A_{ii}| - \sum_{(i,j)\in Q_s} |A_{ij}|f_j\right] - \sum_{s=1}^{r}\left[\sum_{(i,j)\in Q_s; j\neq i} |A_{ij}|f_j\right] e_s^i &= f_i|A_{ii}| - \sum_{(i,j)\in Q_s} |A_{ij}|f_j - \sum_{(i,j)\in Q_s; j\neq i}\left(\sum_{s=1}^{r} |A_{ij}|e_s^i\right) f_j \\ &= f_i|A_{ii}| - \left[\sum_{(i,j)\in Q_s} |A_{ij}|f_j + \sum_{(i,j)\in Q_s; j\neq i} |A_{ij}|f_j\right] \\ &> 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{3.5}$$

Thus, there exists a positive diagonal matrix $\hat{F} = \mathrm{diag}(F, F, \ldots, F)$ such that $\hat{A}\hat{F}$ satisfies (3.5) for $i = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, r$, which shows that $\hat{A} \in H_{rm}^k$. From (3.1), we know that $\hat{A} = \hat{M} - \hat{N}$ is a splitting as in (1.9). It then follows from Lemma 3.2 that $\rho(T) = \rho({M_Q}^{-1}N_Q) < 1$ which completes the proof. □

Theorem 3.4. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting $(M_s, N_s, E_s)_{s=1}^{r}$. Then the extrapolated parallel multisplitting block iterative method (1.10) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$, provided $\tau \in (0, 2/(1+\rho))$, where $\rho = \rho(T)$ and T is the iteration matrix of the method (1.7).*

Proof. Since the iteration matrix of the extrapolated parallel multisplitting block iterative method is

$$T(\tau) = \tau \sum_{s=1}^{r} E_s M_s^{-1} N_s + (1-\tau)I = \tau T + (1-\tau)I,$$

we have $\rho(T(\tau)) = \rho(\tau T + (1-\tau)I) \le \tau\rho(T) + |1-\tau|$. Theorem 3.3 implies that $\rho(T) < 1$. As a result, $\rho(T(\tau)) \le \tau\rho(T) + |1-\tau| < 1$ for all $\tau \in (0, 2/(1+\rho))$. Thus, the extrapolated parallel multisplitting block iterative method converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. This completes the proof. □

In what follows, we consider convergence of the parallel BGAOR iterative method of the system (1.1).

Theorem 3.5. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting (1.11). If $0 \le \gamma_s \le \omega_s \le 1$ and $0 < \omega_s$ for $s = 1, 2, \ldots, r$, then the parallel BGAOR iterative method (1.12) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Proof. Since the parallel BGAOR iterative method (1.12) is induced by the multisplitting $(M_s, N_s, E_s)_{s=1}^{r}$ defined in (1.4) with

$$\begin{aligned} M_s &= \frac{1}{\omega_s}(D_s - \gamma_s L_s), \\ N_s &= \frac{1}{\omega_s}[(1-\omega_s)D_s + (\omega_s - \gamma_s)L_s + \omega_s U_s], \quad s = 1, 2, \ldots, r, \end{aligned} \tag{3.6}$$

it follows from Lemma 3.1 that $\rho(\mathcal{L}(\Gamma, \Omega)) = \rho(\sum_{s=1}^{r} E_s M_s^{-1} N_s) = \rho(\hat{M}^{-1}\hat{N})$, where $\hat{M}$ and $\hat{N}$ are defined as (3.1). Following, we will prove that $\hat{A} = \hat{M} - \hat{N}$ is a generalized H-matrix. Let $R_s, S_s, T_s \subset P(m) = \{(i,j) \mid i, j \in N = \{1, 2, \ldots, m\},\ i \neq j\}$, $R_s \cap S_s = R_s \cap T_s = T_s \cap S_s = \emptyset$ and $R_s \cup S_s \cup T_s = P(m)$. Then for $s = 1, 2, \ldots, r$, $D_s = [D_{ij}] \in \mathbb{C}^{km\times km}$, $L_s = [L_{ij}] \in \mathbb{C}^{km\times km}$ and $U_s = [U_{ij}] \in \mathbb{C}^{km\times km}$ in (3.6) are defined by

$$\begin{aligned} D_{ij} &= \begin{cases} A_{ij}, & (i,j) \in R_s \text{ and } i = j \in N \\ 0, & (i,j) \overline{\in} R_s,\ i \neq j \end{cases} \\ L_{ij} &= \begin{cases} A_{ij}, & (i,j) \in S_s \\ 0, & (i,j) \overline{\in} S_s, \end{cases} \\ U_{ij} &= \begin{cases} A_{ij}, & (i,j) \in T_s \\ 0, & (i,j) \overline{\in} T_s. \end{cases} \end{aligned} \tag{3.7}$$

Since $A \in H_m^k$ indicates $\mu(A) \in M_m^k$, Definition 2.3 shows that there exists a positive diagonal matrix $F = \mathrm{diag}(f_1 I_k, f_2 I_k, \ldots, f_m I_k)$, where I_k is the $k \times k$ identity matrix, such that AF satisfies

$$f_i|A_{ii}| - \sum_{j=1, j\neq i}^{m} |A_{ij}|f_j > 0, \tag{3.8}$$

for all $i \in N$. Note that $(M_s, N_s, E_s)_{s=1}^{r}$ is a multisplitting of A, $E_s = \mathrm{diag}(e_s^1 I_k, \ldots, e_s^m I_k)$ is a $km \times km$ nonnegative diagonal matrix for $s = 1, 2, \ldots, r$ and $\sum_{s=1}^{r} E_s = I$, the $km \times km$ identity matrix. Then we have

$$\sum_{s=1}^{r} e_s^i = 1, \quad i = 1, 2, \ldots, m \text{ and } e_s^i \ge 0,\ s = 1, 2, \ldots, r. \tag{3.9}$$

As a result, $A = M_s - N_s = M_s - N_s \sum_{s=1}^{r} E_s \in H_m^k$ satisfying (3.8) for all $s = 1, 2, \ldots, r$. Let $\hat{A} = [\hat{A}_{ij}] \in C_{rm}^k$. Since $0 \le \gamma_s \le \omega_s \le 1$ and $0 < \omega_s$ for $s = 1, 2, \ldots, r$, it follows from (3.8) and (3.9) that

$$\begin{aligned}
f_i|\hat{A}_{ii}| - \sum_{s=1}^{r}\sum_{j=1,j\neq i}^{m} |\hat{A}_{i,(s-1)m+j}|f_j \ge{} & \left[\left(f_i|A_{ii}| - \sum_{(i,j)\in R_s; j\neq i} |A_{ij}|f_j\right) - \gamma_s \sum_{(i,j)\in S_s} |A_{ij}|f_j\right] \\
& - \sum_{s=1}^{r}\left[(1-\omega_s)\left(f_i|A_{ii}| - \sum_{(i,j)\in R_s; j\neq i} |A_{ij}|f_j\right)\right. \\
& \left. + (\omega_s - \gamma_s)\sum_{(i,j)\in S_s} |A_{ij}|f_j + \omega_s \sum_{(i,j)\in T_s} |A_{ij}|f_j\right] e_s^i \\
={} & \left[\left(f_i|A_{ii}| - \sum_{(i,j)\in R_s; j\neq i} |A_{ij}|f_j\right) - \gamma_s \sum_{(i,j)\in S_s} |A_{ij}|f_j\right] \\
& - \left[(1-\omega_s)\left(f_i|A_{ii}| - \sum_{(i,j)\in R_s; j\neq i} |A_{ij}|f_j\right)\right. \\
& \left. + (\omega_s - \gamma_s)\sum_{(i,j)\in S_s} |A_{ij}|f_j + \omega_s \sum_{(i,j)\in T_s} |A_{ij}|f_j\right] \\
={} & \omega_s\left[f_i|A_{ii}| - \sum_{(i,j)\in R_s; j\neq i} |A_{ij}|f_j - \sum_{(i,j)\in S_s} |A_{ij}|f_j - \sum_{(i,j)\in T_s} |A_{ij}|f_j\right] \\
={} & f_i|A_{ii}| - \sum_{j=1,j\neq i}^{m} |A_{ij}|f_j \\
>{} & 0, \quad i = 1, 2, \ldots, m; \; s = 1, 2, \ldots, r.
\end{aligned} \tag{3.10}$$

Therefore, there exists a positive diagonal matrix $\hat{F} = \mathrm{diag}(F, F, \ldots, F)$ such that $\hat{A}\hat{F}$ satisfies (3.10) for $i = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, r$, which shows that $\hat{A} \in H_{rm}^k$. (3.1) shows that $\hat{A} = \hat{M} - \hat{N}$ is a splitting as in (1.9). It then follows from Lemma 3.2 that $\rho(\mathcal{L}(\Gamma, \Omega)) = \rho(\sum_{s=1}^{r} E_s M_s^{-1} N_s) = \rho(\hat{M}^{-1}\hat{N}) < 1$ which completes the proof. □

It is easy to obtain immediately the following corollaries from Theorem 3.5.

Corollary 3.6. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting (1.11). If $0 \le \gamma_s \le \omega_s \le 1$ and $0 < \omega_s$ for $s = 1, 2, \ldots, r$, then the parallel BAOR iterative method converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Corollary 3.7. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting (1.11). If $0 < \omega_s \le 1$ for $s = 1, 2, \ldots, r$, then the parallel BGSOR and BSOR iterative method converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Theorem 3.8. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting (1.11). If $0 \le \gamma_s \le \omega_s \le 1$ and $0 < \omega_s$ for $s = 1, 2, \ldots, r$, then the extrapolated parallel BGAOR iterative method converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Proof. Similar to the proof of Theorem 3.4, it is easy to obtain the proof coming from Theorem 3.5. □

Corollary 3.9. *Let $A = [A_{ij}] \in H_m^k$ with a multisplitting (1.11). If $0 < \omega_s \le 1$ for $s = 1, 2, \ldots, r$, then the extrapolated parallel BGSOR iterative method converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

4. Applications to special cases from the solution of partial differential equations

In this section, we will discuss the convergence of matrices arising in the numerical solution of some special partial differential equations such as the Euler equation [2], the Navier–Stokes equation [1], elliptic equations [9] and so on. These matrices have the following form

$$M := \begin{bmatrix} T & S_1 & & \\ S_2 & T & \ddots & \\ & \ddots & \ddots & S_1 \\ & & S_2 & T \end{bmatrix} \in \mathbb{C}^{prk\times prk}, \tag{4.1}$$

where $T_1,\ S_1,\ S_2 \in \mathbb{C}^{rk\times rk}$ are defined by

$$T = \begin{bmatrix} C & -A^- & & \\ -A^+ & C & \ddots & \\ & \ddots & \ddots & -A^- \\ & & -A^+ & C \end{bmatrix}, \tag{4.2}$$

$$S_1 = \begin{bmatrix} -B^- & & \\ & \ddots & \\ & & -B^- \end{bmatrix}, \qquad S_2 = \begin{bmatrix} -B^+ & & \\ & \ddots & \\ & & -B^+ \end{bmatrix}. \tag{4.3}$$

Here $A = A^+ - A^- \in \mathbb{C}^{k\times k}$ and $B = B^+ - B^- \in \mathbb{C}^{k\times k}$ are decompositions of Hermitian (indefinite) matrices A, B into positive semidefinite parts A^+, B^+ and negative semidefinite parts $-A^-, -B^-$, while $C = A^+ + A^- + B^+ + B^-$. Furthermore, $N(A) \cap N(B) = \emptyset$, where $N(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$ is the right null space of the matrix A.

With $T = M_s - N_s,\ s = 1, 2, \dots, t$, where M_s and N_s are defined by (1.5) and (1.6), one has the splitting

$$M = P_s - Q_s,\ s = 1, 2, \dots, t, \tag{4.4}$$

where

$$P_s = \mathrm{diag}(M_s, M_s, \dots, M_s) \in \mathbb{C}^{prk\times prk}, \tag{4.5}$$

and

$$Q_s = \begin{bmatrix} N_s & -S_1 & & \\ -S_2 & N_s & \ddots & \\ & \ddots & \ddots & -S_1 \\ & & -S_2 & N_s \end{bmatrix} \in \mathbb{C}^{prk\times prk}. \tag{4.6}$$

Let

$$T = D_s' - L_s' - U_s', \quad s = 1, 2, \dots, t \tag{4.7}$$

be as in (1.11). Then the matrix M can be written as

$$M = D_s - L_s - U_s, \quad s = 1, 2, \dots, t, \tag{4.8}$$

where

$$D_s = \mathrm{diag}(D_s', D_s', \dots, D_s') \in \mathbb{C}^{prk\times prk},$$

$$L_s = \begin{bmatrix} L_s' & & & \\ -S_2 & L_s' & & \\ & \ddots & \ddots & \\ & & -S_2 & L_s' \end{bmatrix} \in \mathbb{C}^{prk\times prk}, \tag{4.9}$$

and

$$U_k = \begin{bmatrix} U_s' & -S_1 & & \\ & U_s' & \ddots & \\ & & \ddots & -S_1 \\ & & & U_s' \end{bmatrix} \in \mathbb{C}^{prk\times prk}. \tag{4.10}$$

Based on the splittings (4.4) and (4.8), this section will establish some convergence results for the parallel multisplitting block iterative method and the parallel multisplitting block GAOR (AOR) method, respectively.

Theorem 4.1. *Let M be as in (4.1)–(4.3). For the splitting (4.4) of M, the parallel multisplitting block iterative method (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Proof. According to Theorem 6.1 in [3], we have $M + M^H \in M_{pr}^k$. It is easy to obtain $M \in H_{pr}^k$ from Lemma 3.1 in [18]. It follows from Theorem 3.3 that $\rho(T) < 1$, where $T = \sum_{s=1}^{r} E_s M_s^{-1} N_s$, i.e., the parallel multisplitting block iterative method (1.7) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$. $\square$

Theorem 4.2. *Let M be as in (4.1)–(4.3). For the splitting (4.8) of M, if $0 \le \gamma_s \le \omega_s \le 1$ and $0 < \omega_s$ for $s = 1, 2, \dots, t$, then the parallel BGAOR iterative method (1.12) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$.*

Proof. The proof is similar to that for Theorem 4.1 and is easy to obtain from Theorem 3.5. □

5. Numerical examples

In this section some examples are given to illustrate the results obtained in Sections 3 and 4.

Example 5.1. Let the coefficient matrix A of linear system (1.1) be given by

$$A = \begin{bmatrix} 3 & -2 & 2 & -1 & 1 & -1 \\ -2 & 3 & -1 & 2 & -1 & 1 \\ 40 & -35 & 100 & -80 & -50 & 40 \\ -35 & 40 & -80 & 90 & 40 & -40 \\ 3 & -3 & -6 & 4 & 10 & -8 \\ -3 & 3 & 4 & -5 & -8 & 9 \end{bmatrix}. \tag{5.1}$$

It is easy to see that $A \in H_3^2$. Now we verify the convergence results of some block iterative methods for linear systems with given matrix $A \in H_3^2$ in Section 3.

We choose

$$M_1 = \begin{bmatrix} 3 & -2 & 2 & -1 & 1 & -1 \\ -2 & 3 & -1 & 2 & -1 & 1 \\ 0 & 0 & 100 & -80 & -50 & 40 \\ 0 & 0 & -80 & 90 & 40 & -40 \\ 0 & 0 & 0 & 0 & 10 & -8 \\ 0 & 0 & 0 & 0 & -8 & 9 \end{bmatrix}, \tag{5.2}$$

$$M_2 = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 0 \\ 40 & -35 & 100 & -80 & 0 & 0 \\ -35 & 40 & -80 & 90 & 0 & 0 \\ 3 & -3 & -6 & 4 & 10 & -8 \\ -3 & 3 & 4 & -5 & -8 & 9 \end{bmatrix} \tag{5.3}$$

and

$$M_3 = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & -80 & 0 & 0 \\ 0 & 0 & -80 & 90 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & -8 \\ 0 & 0 & 0 & 0 & -8 & 9 \end{bmatrix}. \tag{5.4}$$

Then, $N_s = M_s - A$ for $s = 1, 2, 3$. Set $E_1 = \mathrm{diag}(1/2, 1/2, 1/6, 1/6, 1/3, 1/3)$, $E_2 = \mathrm{diag}(1/3, 1/3, 1/2, 1/2, 1/6, 1/6)$ and $E_3 = \mathrm{diag}(1/6, 1/6, 1/3, 1/3, 1/2, 1/2)$. Then, we have $\sum_{s=1}^{3} E_s = I$, and consequently, $(M_s, N_s, E_s)_{s=1}^{3}$ is a multisplitting of the matrix A and $T = \sum_{s=1}^{3} E_s M_s^{-1} N_s$ is the iteration matrix. Direct computation yields $\rho(T) = 0.8987 < 1$, which shows that the parallel multisplitting block iterative method (1.7) is convergent.

Example 5.2. Consider the following linear system arising in the numerical solution of the Euler equation [2]:

$$Mx = b, \tag{5.5}$$

where $M \in \mathbb{C}^{(4\times3\times2)\times(4\times3\times2)}$ is as in (4.1)–(4.3) and $b = [1, 3, 1, 2, 5, 3, 2, 1, 7, 5, 9, 0, 2, 0, 1, 2, 1, 0, 1, 3, 1.2, 4, 6, 8]^T$. Here $A^+ = A^- = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $B^+ = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B^- = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $C = A^+ + A^- + B^+ + B^- = \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}$. Then $A = A^+ - A^- = 0$ and $B = B^+ - B^- = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ and hence $N(A) \cap N(B) = \emptyset$. Then

$$M := \begin{bmatrix} T & S_1 & & \\ S_2 & T & S_1 & \\ & S_2 & T & S_1 \\ & & S_2 & T \end{bmatrix} \in \mathbb{C}^{(4\times3\times2)\times(4\times3\times2)}, \tag{5.6}$$

Table 5.1
The comparison of convergence speed with different r and $E_s = \frac{1}{r}I_{4\times3\times2}$.

r	1	2	3	4	5	6
$\rho(\mathbb{T}_r)$	0.1801	0.2901	0.2844	0.2959	0.2894	0.2796
Number of iterations	11	13	13	13	13	12

Table 5.2
The comparison of convergence speed with different r and E_s.

r	1	2	3	4	5	6
$\rho(\mathbb{T}_r)$	0.1801	0.1801	0.1801	0.1801	0.2719	0.2719
Number of iterations	11	12	12	12	12	12

Note that in Table 5.2 the weighting matrices E_s are chosen as follows: $E_1 = \mathrm{diag}(I_6, 0, I_6, 0)$ and $E_2 = \mathrm{diag}(0, I_6, 0, I_6)$ when $r = 2$; $E_1 = \mathrm{diag}(I_6, 0, I_6, 0)$, $E_2 = \mathrm{diag}(0, I_6, 0, 0)$ and $E_3 = \mathrm{diag}(0, 0, 0, I_6)$ when $r = 3$; $E_1 = \mathrm{diag}(I_6, 0, 0, 0)$, $E_2 = \mathrm{diag}(0, I_6, 0, 0)$, $E_3 = \mathrm{diag}(0, 0, I_6, 0)$ and $E_4 = \mathrm{diag}(0, 0, 0, I_6)$ when $r = 4$; $E_1 = \mathrm{diag}(I_6, 0, 0, 0)$, $E_2 = \mathrm{diag}(0, I_6, 0, 0)$, $E_3 = \mathrm{diag}(0, 0, I_6, 0)$ and $E_4 = E_5 = \mathrm{diag}(0, 0, 0, \frac{1}{2}I_6)$ when $r = 5$; $E_1 = \mathrm{diag}(I_6, 0, 0, 0)$, $E_2 = \mathrm{diag}(0, I_6, 0, 0)$, $E_3 = E_6 = \mathrm{diag}(0, 0, \frac{1}{2}I_6, 0)$ and $E_4 = E_5 = \mathrm{diag}(0, 0, 0, \frac{1}{2}I_6)$ when $r = 6$, where I_6 is the 6×6 identity matrix.

where $T,\ S_1,\ S_2 \in \mathbb{C}^{(3\times2)\times(3\times2)}$ are defined by

$$T = \begin{bmatrix} C & -A^- & \\ -A^+ & C & -A^- \\ & -A^+ & C \end{bmatrix},$$
$$S_1 = \begin{bmatrix} -B^- & & \\ & -B^- & \\ & & -B^- \end{bmatrix}, \qquad S_2 = \begin{bmatrix} -B^+ & & \\ & -B^+ & \\ & & -B^+ \end{bmatrix}. \tag{5.7}$$

Writing $T = M_s - N_s$, where M_s and N_s are defined by

$$M_1 = \begin{bmatrix} C & 0 & \\ -A^+ & C & 0 \\ & -A^+ & C \end{bmatrix}, \qquad M_2 = \begin{bmatrix} C & -A^- & \\ 0 & C & -A^- \\ & 0 & C \end{bmatrix}$$
$$M_3 = \begin{bmatrix} C & 0 & \\ -A^+ & C & -A^- \\ & 0 & C \end{bmatrix}, \qquad M_4 = \begin{bmatrix} C & -A^- & \\ 0 & C & 0 \\ & -A^+ & C \end{bmatrix} \tag{5.8}$$
$$M_5 = \begin{bmatrix} C & -A^- & \\ -A^+ & C & 0 \\ & 0 & C \end{bmatrix}, \qquad M_6 = \begin{bmatrix} C & 0 & \\ 0 & C & -A^- \\ & -A^+ & C \end{bmatrix}$$

and $N_s = M_s - T$ for $s = 1, 2, 3, 4, 5, 6$, then we have a multisplitting $(P_s, Q_s, E_s)_{s=1}^r$ of the matrix M with $1 \le r \le 6$, where P_s and Q_s are defined by (4.4)–(4.6), and $E_s = \frac{1}{r}I_{4\times3\times2}$, where $I_{4\times3\times2}$ is the $(4\times3\times2)\times(4\times3\times2)$ identity matrix for $s = 1, 2, \ldots, r$. Furthermore, the iteration matrix is $\mathbb{T}_r = \sum_{s=1}^r E_s P_s^{-1} Q_s$. By direct computation, one obtains $\rho(\mathbb{T}_2) = 0.2901$, $\rho(\mathbb{T}_4) = 0.2959$, $\rho(\mathbb{T}_5) = 0.2894$ and $\rho(\mathbb{T}_6) = 0.2796$. This shows that the parallel multisplitting block iterative method (1.7) for linear system (5.5) converges to the unique solution of (5.5) for any choice of the initial guess $x^{(0)}$.

In what follows we consider the convergence speed (i.e., quantity of spectral radius of iteration matrix and number of iterations required for given accuracy ϵ) of the parallel multisplitting method for different values of r. As is shown in [14,19], for a given linear system, the convergence speed of the parallel multisplitting method depends not only on the choice of the parallel multisplitting of the coefficient matrix and the weighting matrix but also on the number r of splittings in such a parallel multisplitting.

Tables 5.1–5.2 indicate the changing on both the quantity of spectral radius of iteration matrix and the number M of iterations required for given accuracy $\epsilon = \|x^{(M)} - x^{(M-1)}\|_2 < 10^{-4}$ for different r and different choice of weighting matrices E_s, where $\|x\|_2$ denotes 2-norm of the vector x. The initial guess was taken to be the vector of all one's.

Finally, we test the convergence of the parallel BGAOR iterative method (1.12) for linear system (5.5). Assume that (5.6) and (5.7) hold. Let M_s be defined as in (5.8) and $N_s = M_s - T$ for $s = 1, 2, 3, 4$. Let $T = D_s' - L_s' - U_s'$, where $D_s' = M_s$, $L_s' = 0$ and $U_s' = N_s$ for $s = 1, 2, 3, 4$. Then $M = D_s - L_s - U_s$, where $D_s,\ L_s$ and U_s are defined in (4.9) and (4.10), and thus, $(P_s, Q_s, E_s)_{s=1}^4$ is a multisplitting of the matrix M, where $P_s = \omega^{-1}(D_s - \gamma L_s),\ Q_s = \omega^{-1}[(1-\omega)D_s + (\omega-\gamma)L_s + \omega U_s]$, $0 \le \gamma \le \omega \le 1,\ 0 < \omega$ and $E_s = 0.25I_{4\times3\times2}$ with $I_{4\times3\times2}$ the $(4\times3\times2)\times(4\times3\times2)$ identity matrix for $s = 1, 2, 3, 4$. As a consequence, $\mathcal{L}(\gamma, \omega) = \sum_{s=1}^r E_s P_s^{-1} Q_s$ is the iteration matrix of the parallel BGAOR iterative method (1.12). Let $\rho(\mathcal{L}(\gamma, \omega))$ denote the spectral radius of $\mathcal{L}(\gamma, \omega)$. The comparison results of $\rho(\mathcal{L}(\gamma, \omega))$ with different parameter pairs (γ, ω) are shown in Table 5.3 to show that the change of the convergence of the parallel BGAOR iterative method with parameter pair (γ, ω) changing.

Table 5.3
The comparison results of $\rho(\mathcal{L}(\gamma, \omega))$ with different parameter pairs (γ, ω).

(γ, ω)	(0.1, 0.2)	(0.3, 0.4)	(0.5, 0.6)	(0.7, 0.8)	(0.8, 0.9)	(0.9, 1)
$\rho(\mathcal{L}(\gamma, \omega))$	0.8592	0.7184	0.5776	0.4367	0.3663	0.2959
(γ, ω)	(0.8, 0.8)	(0.9, 0.9)	(0.9, 0.95)	(0.95, 0.99)	(0.99, 0.99)	(1, 1)
$\rho(\mathcal{L}(\gamma, \omega))$	0.4367	0.3663	0.3561	0.3030	0.3005	0.2959

The table shows that the change in the convergence of the parallel BGAOR iterative method with change in the parameter pair (γ, ω).

In the following, we will discuss the convergence of the parallel BGAOR iterative method (1.12) for linear system (5.5). It is easy to see from Table 5.3 that $\rho(\mathcal{L}(\gamma, \omega))$ decreases gradually when r and ω increase from 0.1 and 0.2, respectively, to 1. Furthermore, we have

$$\min_{\gamma,\omega\in(0,1],\gamma\le\omega} \rho(\mathcal{L}(\gamma, \omega)) = \rho(\mathcal{L}(1, 1)) = \rho(\mathcal{L}_{PBGGS}), \tag{5.9}$$

where $\mathcal{L}_{PBGGS}$ denotes the iteration matrix of the parallel BGGS methods.

In addition, since the parallel BGSOR, the parallel BAOR and the parallel BSOR methods are special cases of the parallel BGAOR-method, the same results for the parallel BGSOR, the parallel BAOR and the parallel BSOR methods can also obtained.

Example 5.3. Consider a large sparse linear system arising in the numerical solution of the elliptic equations [9]:

$$Ax = b, \tag{5.10}$$

where

$$A = \begin{bmatrix} B & -I & & \\ -I & B & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & B \end{bmatrix} \in \mathbb{C}^{mn\times mn} \tag{5.11}$$

where I is the $m \times m$ identity matrix and $B \in \mathbb{C}^{m\times m}$ are defined by

$$B = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix} \in \mathbb{C}^{m\times m}. \tag{5.12}$$

For $r = 2$ and two positive integers $m_1,\ m_2$ with $1 \le m_2 < m_1 \le n$, we define a multisplitting $A = D - L_s - U_s$ of the block matrix A, where

$$\begin{aligned} &D = \mathrm{diag}[B, B, \dots, B] \in \mathbb{C}^{mn\times mn};\\ &L_s = [L_{ij}^{(s)}] \in \mathbb{C}^{mn\times mn}, \quad s = 1, 2;\\ &U_s = [U_{ij}^{(s)}] \in \mathbb{C}^{mn\times mn}, \quad s = 1, 2 \end{aligned} \tag{5.13}$$

with

$$\begin{aligned} &L_{ij}^{(1)} = \begin{cases} I, & j=i-1,\ 2\le i\le m_1,\\ 0, & \text{otherwise}, \end{cases}\\ &L_{ij}^{(2)} = \begin{cases} I, & j=i-1,\ m_2\le i\le n,\\ 0, & \text{otherwise}, \end{cases}\\ &U_{ij}^{(1)} = \begin{cases} I, & j=i-1,\ m_1+1\le i\le n,\\ I, & j=i+1,\ 1\le i\le n-1,\\ 0, & \text{otherwise}, \end{cases}\\ &U_{ij}^{(2)} = \begin{cases} I, & j=i-1,\ 2\le i\le m_2-1,\\ I, & j=i+1,\ 1\le i\le n-1,\\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{5.14}$$

and two weighted matrices

$$E_s = \mathrm{diag}[E_{11}^{(s)}, \dots, E_{nn}^{(s)}] \in \mathbb{C}^{mn\times mn}, \quad s = 1, 2 \tag{5.15}$$

Table 5.4
Multisplitting BGAOR method with $n = m$.

m	5	7	11	13	15	20
(i)						
Time	0.0483	0.785	0.892	0.7120	1.9663	20.2959
Iter	19	30	56	75	93	148
(ii)						
Time	0.0613	0.0837	0.0880	0.7052	1.9551	20.3108
Iter	19	30	56	75	93	148

Table 5.5
Multisplitting BGAOR method when the cases (i) and (ii) for $n = m = 10$.

(γ, ω)	(0.9,1)	(0.7,1)	(0.5,1)	(0.7,1.1)	(1.1,1)	(1,1)
(i)						
Time	0.0753	0.0815	0.0819	0.0895	0.0884	0.0726
Iter	42	51	52	105	84	39
(ii)						
Time	0.1130	0.0737	0.0810	0.103	0.0923	0.0731
Iter	44	51	56	115	83	41

where

$$E_{ii}^{(1)} = \begin{cases} I, & 1 \le i \le m_2, \\ I/2, & m_2 + 1 \le i \le m_1 - 1 \\ 0, & m_1 \le i \le n \end{cases}$$
$$E_{ii}^{(2)} = \begin{cases} 0, & 1 \le i \le m_2, \\ I/2, & m_2 + 1 \le i \le m_1 - 1, \\ I, & m_1 \le i \le n. \end{cases} \tag{5.16}$$

We let (i) $m_1 = [\frac{3n}{4}],\ m_2 = [\frac{n}{4}]$; (ii) $m_1 = [\frac{5n}{6}],\ m_2 = [\frac{n}{6}]$, where [ ] denotes the integer part of corresponding real number. Then we get two weighted matrices E_1 and E_2. The initial guess of x_0 is taken as a zero vector. Here $\|x^{k+1} - x^k\| / \|x^{k+1}\| \le 10^{-6}$ is used as the stopping criterion. All experiments were executed on a PC using MATLAB programming package.

In Table 5.4, $\gamma = \gamma_1 = \gamma_2 = 0.7$ and $\omega = \omega_1 = \omega_2 = 1$, we report the CPU time (Time) and the number of iterations (Iter) for the multisplitting block GAOR iterative method. In Tables 5.5, let $m = 10$, we report the CPU time (Time) and the number of iterations (Iter) for the multisplitting block GAOR iterative method for different γ and ω. Following from Tables 5.5, for $(\gamma, \omega) = (1, 1)$ it can be seen that the convergence rate of the multisplitting block GAOR iterative method is faster than the other parameterized iterative method for generalized H-matrices.

6. Conclusions

The paper is devoted to the study of the convergence properties of some parallel multisplitting block iterative methods for the solution of linear systems arising in the numerical solution of the Euler equation. We give sufficient conditions for the convergence of parallel multisplitting block iterative methods including the parallel block generalized AOR (BGAOR), the parallel block AOR (BAOR), the parallel block generalized SOR (BGSOR), the parallel block SOR (BSOR), the extrapolated parallel BAOR and the extrapolated parallel BSOR methods. Furthermore, we present the convergence of the parallel block iterative methods for linear systems with special block tridiagonal matrices arising in the numerical solution of the Euler equation. Finally, we have given some examples to demonstrate the convergence results obtained in this paper.

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