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Time-fractional Gardner equation for ion-acoustic waves in negative-ion-beam plasma with negative ions and nonthermal nonextensive electrons

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Nonlinear propagation of ion-acoustic waves is investigated in a one-dimensional, unmagnetized plasma consisting of positive ions, negative ions, and nonthermal electrons featuring Tsallis distribution that is penetrated by a negative-ion-beam. The classical Gardner equation is derived to describe nonlinear behavior of ion-acoustic waves in the considered plasma system via reductive perturbation technique. We convert the classical Gardner equation into the time-fractional Gardner equation by Agrawal's method, where the time-fractional term is under the sense of Riesz fractional derivative. Employing variational iteration method, we construct solitary wave solutions of the time-fractional Gardner equation with initial condition which depends on the nonlinear and dispersion coefficients. The effect of the plasma parameters on the compressive and rarefactive ion-acoustic solitary waves is also discussed in detail.
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I. INTRODUCTION

Recently, the applications of fractional calculus and fractional differential equations have received a great deal of attention in plasma physics. As a generalization of the classical ordinary calculus, fractional calculus was first introduced by Leibniz in 1695.[1] Fractional differential equations have seen an explosive growth because of their non-local property, i.e., the next stage of the system depends not only on its current state but also on all of its historical states.[2] Therefore, fractional differential equations have been successfully applied to model non-conservative real processes in physical world.[2–4] El-Wakil *et al.*[5] used time-fractional KdV equation to study electron-acoustic solitary waves in plasma system with two different electron temperature and stationary ion. It has been found that theoretical analysis with $\alpha \approx 0.78$ (α is the fractional derivative parameter) yields a good agreement with the observation from the Viking satellite in the dayside auroral zone. Nazari-Golshan and Nourazar[6] applied time-fractional modified KdV equation to investigate plasma system with trapped electrons. They found that the soliton amplitude increases as time fractional order increases. In addition, the studies on dust acoustic waves using fractional differential equations have been discussed in Refs. 7–9. For more theoretical studies on fractional differential equations in plasma physics, see Refs. 10–14.

When a beam is penetrated into plasma system involving relative motion, the external energy source makes the plasma unstable and appears in the form of streaming instability.[15] The injection of a beam into plasma system can strongly modify the properties of solitary waves, which has been investigated in Refs. 16–22. However, most of these studies were confined to positive-ion-beam or electron-beam. In order to speed up the progress in negative-ion-beam research, it is highly desirable to investigate the interactions between negative-ion-beam and plasma system. Recently, different methods have been reported to produce sufficiently intense negative-ion-beam by negative ion source in a high vacuum environment.[23] Progress in the negative-ion-beam research indicates that it has broad potential applications in new scientific and technological fields, including inertial confinement fusion and heating warm dense matter.[24,25] Therefore, it is an important work to investigate the effect of negative-ion-beam streaming parameters on the nonlinear structures of solitary waves in plasma system.

The presence of energetic particles, resulting in long-tailed distributions, is an intrinsic element in space and laboratory observations.[26,27] The high energy tail can be characterized by non-Maxwellian distribution function. In the past few decades, different models have been proposed to describe this effect on wave dynamics via phenomenological modification. Cairns *et al.*[28] proposed nonthermal distribution to model the enhanced high energy tails observed by the FREJA satellite. In the nonthermal distribution, the index α characterizes the deviation from the Maxwellian distribution function. Because of the presence of nonthermal electrons, both compressive and rarefactive solitons can be produced in plasma models. In recent years, this nonthermal distribution has been widely applied to investigate nonlinear waves in plasmas.[29–31] A generalization of the Boltzmann-Gibbs-Shannon (BGS) entropy, expressed in terms of a parameter q ($q > -1$), was recognized by Renyi[32] and subsequently proposed by Tsallis.[33] This nonextensive model can extend the standard additivity of the entropies to the nonextensive cases, where q measures the degree of nonextensivity of the plasma system. The q-nonextensive statistics can present a good fit to the experimental data[34,35] and provide a powerful and convenient framework for the analysis of astrophysical phenomena.[36,37]

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A hybrid Cairns-Tsallis distribution function, recently proposed by Tribeche *et al.*,[38] can be useful in fitting to a wider range of plasma system. It is because that the Cairns-Tsallis distribution contains two parameters, which purports to offer enhanced parametric flexibility in modeling nonthermal plasmas.[39] Later, Amour *et al.*[40] used this distribution to investigate the electron acoustic solitary waves in a plasma with nonthermal nonextensive hot electrons. In Ref. 41, Guo and Mei studied the dependence of dust-ion-acoustic rogue waves on the nonthermal index α and the nonextensive index q.

The aim of this paper is to investigate the nonlinear propagation of ion-acoustic waves by using the time-fractional Gardner equation in an unmagnetized plasma consisting of positive ions, negative ions, negative-ion-beam, and nonthermal electrons featuring the Tsallis distribution. The paper is organized as follows. In Sec. II, a set of fluid equations is presented for our theoretical model. Employing reductive perturbation method and Agrawal's method, we derive the time-fractional Gardner equation for ion-acoustic waves. In Sec. III, the time-fractional Gardner equation is solved by variational iteration method. The effect of the plasma parameters on the compressive and rarefactive ion-acoustic solitary waves is discussed in detail. Finally, conclusions are given in Sec. IV.

II. THEORETICAL MODEL AND TIME-FRACTIONAL GARDNER EQUATION

A. Theoretical Model

Consider a one-dimensional, unmagnetized plasma composed of four components: positive ions, negative ions, negative-ion-beam, and nonthermal nonextensive electrons. The former three species (positive ions, negative ions, and negative-ion-beam) are described by fluid equations, while the electrons are assumed to satisfy hybrid Cairns-Tsallis velocity distribution. At equilibrium, the assumption of charge neutrality requires

$$n_{e0} + n_{b0} + n_{n0} = n_{p0},$$

where n_{j0} denotes the equilibrium densities of $j-$ species particles ($j = e, b, n,$ or p for electrons, negative-ion-beam, negative ions, or positive ions, respectively).

For positive ions, the normalized fluid equations are

$$\frac{\partial n_p}{\partial t} + \frac{\partial (n_p u_p)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial x} = -\frac{\partial \Phi}{\partial x}. \tag{2}$$

For negative ions, they are

$$\frac{\partial n_n}{\partial t} + \frac{\partial (n_n u_n)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} = \mu_1 \frac{\partial \Phi}{\partial x}. \tag{4}$$

For negative-ion-beam, they are

$$\frac{\partial n_b}{\partial t} + \frac{\partial (n_b u_b)}{\partial x} = 0, \tag{5}$$

$$\frac{\partial u_b}{\partial t} + u_b \frac{\partial u_b}{\partial x} = \mu_2 \frac{\partial \Phi}{\partial x}. \tag{6}$$

The above six equations are coupled by the Poisson equation

$$\frac{\partial \Phi^2}{\partial x^2} = \delta n_e + \theta n_n + \beta n_b - n_p. \tag{7}$$

In Eqs. (1)–(7), n_j ($j = p, n,$ or b) denotes the number density of j-species particles normalized by the equilibrium values n_{j0}, u_j stands for the fluid velocity of j-species particles normalized by the positive ion sound speed $C_s = (\kappa_B T_e/m_p)^{1/2}$, $\mu_1 = m_p/m_n$, $\mu_2 = m_p/m_e$. The electric potential φ, time variable t, and spatial variable x are normalized by $k_B T_e/e$, positive ion plasma period $\omega_{pi}^{-1} = \sqrt{m_p/(4\pi Z_p^2 e^2 n_{p0})}$, and Debye radius $\lambda_{Di} = \sqrt{k_B T_e/(4\pi Z_p^2 e^2 n_{p0})}$, respectively. Here, k_B is the Boltzmann constant, T_e the electron temperature, m_p the positive ion mass, e the electronic charge. From the overall charge neutrality condition, we obtain $\alpha + \beta + \delta = 1$ with $\theta = n_{n0}/n_{p0}$, $\beta = n_{b0}/n_{p0}$, and $\delta = n_{e0}/n_{p0}$. The normalized electron number density is

$$n_e = \left(1 + \rho_1 \varphi + \rho_2 \varphi^2\right) \times (1 + (q-1)\varphi)^{\frac{(q+1)}{2(q-1)}}. \tag{8}$$

Here, $\rho_1 = -16q\alpha/(3 - 14q + 15q^2 + 12\alpha)$, $\rho_2 = 16(2q-1)q\alpha/(3 - 14q + 15q^2 + 12\alpha)$, the parameter α stands for the number of nonthermal electrons in the distribution, and $q(>-1)$ measures the strength of nonextensivity. If $\alpha = 0$, the distribution (8) reduces to the q-nonextensive distribution $n_e = [1 + (q-1)\varphi]^{\frac{(q+1)}{2(q-1)}}$. If $q \to 1$, Eq. (8) recovers to the well-known Cairns distribution $n_e = \left(1 - \frac{4\alpha}{1+3\alpha}\varphi + \frac{4\alpha}{1+3\alpha}\varphi^2\right)\exp(\varphi)$.

B. Time-Fractional Gardner Equation

To investigate the dynamics of small but finite amplitude ion-acoustic waves in our plasma system, we apply the reductive perturbation method[42] to Eqs. (1)–(8). The independent variables can be stretched as

$$\xi = \epsilon^{1/2}(x - Vt), \ \ \tau = \epsilon^{3/2} t, \tag{9}$$

where ϵ is a small dimensionless expansion parameter and V is the wave speed, which will be determined later. The physical quantities appearing in Eq. (1)–(7) can be expanded as

$$\begin{aligned} n_j(\xi,\tau) &= 1 + \epsilon n_{j1}(\xi,\tau) + \epsilon^2 n_{j2}(\xi,\tau) + \epsilon^3 n_{j3}(\xi,\tau) + \cdots, \\ u_j(\xi,\tau) &= \epsilon u_{j1}(\xi,\tau) + \epsilon^2 u_{j2}(\xi,\tau) + \epsilon^3 u_{j3}(\xi,\tau) + \cdots, \\ \varphi(\xi,\tau) &= \epsilon \varphi_1(\xi,\tau) + \epsilon^2 \varphi_2(\xi,\tau) + \epsilon^3 \varphi_3(\xi,\tau) + \cdots, \end{aligned} \tag{10}$$

where $j = p, n$ or b stands for the positive ions, negative ions, or negative-ion-beam, respectively.

Substituting Eqs. (9) and (10) into Eqs. (1)–(8), we can develop different sets of equations in various powers of ϵ. To the lowest order in ϵ, we obtain

$$\begin{aligned} &u_{p1} - Vn_{p1} = 0, \quad \varphi_1 - Vu_{p1} = 0, \quad u_{n1} - Vn_{n1} = 0, \\ &\mu_1\varphi_1 + Vu_{n1} = 0, \quad u_{b1} - Vn_{b1} = 0, \\ &\mu_2\varphi_1 + Vu_{b1} = 0, \quad -\theta n_{n1} + n_{p1} - \delta c_1\varphi_1 - \beta n_{b1} = 0, \end{aligned} \tag{11}$$

and linear dispersion relationship

$$V = \pm\sqrt{\frac{\theta\mu_1 + \beta\mu_2 + 1}{c_1\delta}}, \tag{12}$$

where $c_1 = \rho_1 + (q+1)/2$. The above expression of the wave velocity indicates that ion-acoustic waves can propagate outward (+) or inward (–). In this paper, we discuss the former case.

Considering Eq. (11) and the coefficients of $O(\epsilon^2)$, and eliminating the second order perturbed quantities n_{j2}, u_{j2} ($j = p, n$, or b), and φ_2, we obtain the following KdV equation

$$\frac{\partial\Phi}{\partial\tau} + A_1\Phi\frac{\partial\Phi}{\partial\xi} + B\frac{\partial^3\Phi}{\partial\xi^3} = 0. \tag{13}$$

Here, $\Phi = \varphi_1$, the nonlinear coefficient A_1 and dispersion coefficient B are

$$\begin{aligned} A_1 &= \frac{-3\theta\mu_1^2 - 3\beta\mu_2^2 + 1 - 2V^4c_2 + 2V^4c_2\beta + 2V^4c_2\theta}{2V(\theta\mu_1 + \beta\mu_2 + 1)}, \\ B &= \frac{V^3}{2(\theta\mu_1 + \beta\mu_2 + 1)}, \end{aligned} \tag{14}$$

where $c_2 = ((3-q)(q+1) + 4\rho_1(q+1) + 8\rho_2)/8$.

However, there is a certain value that the concentration of negative ions reaches to the so-called critical concentration value, which makes the nonlinear coefficient A_1 in a small neighborhood of zero ($A_1 \to 0$). In this case, the above KdV equation (13) is not appropriate to describe the properties of ion-acoustic waves. To overcome this problem, we assume $A_1 = O(\epsilon)$ and introduce the following new stretched coordinates:

$$\xi = \epsilon(x - Vt), \ \ \tau = \epsilon^3 t. \tag{15}$$

Substitution of Eqs. (15) and (10) into Eqs. (1)–(8) yields sets of equations in various powers of ϵ. To the lowest order in ϵ, we find the same equations for n_{j1}, u_{j1}, φ_1, and V as in Eqs. (11) and (12), respectively.

To the next order in ϵ, we obtain

$$\begin{aligned} &u_{p2} - Vn_{p2} + \frac{\varphi_1^2}{V^3} = 0, \quad u_{p2} = \frac{\varphi_2}{V} + \frac{\varphi_1^2}{V^3}, \\ &u_{n2} - Vn_{n2} + \frac{\mu_1^2\varphi_1^2}{V^3} = 0, \quad u_{p2} = \frac{-\mu_1\varphi_2}{V} + \frac{\mu_1^2\varphi_1^2}{V^3}, \\ &u_{b2} - Vn_{b2} + \frac{\mu_2^2\varphi_1^2}{V^3} = 0, \quad u_{b2} = -\frac{\mu_2\varphi_2}{V} + \frac{\mu_2^2\varphi_1^2}{V^3}. \end{aligned} \tag{16}$$

Proceeding to the order of ϵ^3, and considering Eqs. (11) and (16) and the assumption $A_1 = O(\epsilon)$, we obtain the following Gardner equation:

$$\frac{\partial\Phi}{\partial\tau} + A_1\Phi\frac{\partial\Phi}{\partial\xi} + A_2\Phi^2\frac{\partial\Phi}{\partial\xi} + B\frac{\partial^3\Phi}{\partial\xi^3} = 0, \tag{17}$$

where $\Phi = \varphi_1$, the coefficients A_1 and B are the same as in Eq. (14), and the coefficient A_2 is given by

$$A_2 = \frac{15\left(1 + \beta\mu_2^3 + \theta\right)}{4V^3(\beta\mu_2 + \theta\mu_1 + 1)}.$$

Next, we convert the classical Gardner equation (17) to the time-fractional Gardner equation.

Using the potential function $\Psi(\xi,\tau)$, where $\Phi(\xi,\tau) = \Psi_\xi(\xi,\tau)$, we can obtain the following potential form of Gardner equation (17)

$$\Psi_{\tau\xi} + A_1\Psi_\xi\Psi_{\xi\xi} + A_2\Psi_\xi^2\Psi_{\xi\xi} + B\Psi_{\xi\xi\xi\xi} = 0, \tag{18}$$

where the subscripts stand for the partial derivative of Ψ with respect to the parameters. The functional of Eq. (18) can be given as

$$\begin{aligned} J(\Psi) = \int_R d\xi\int_T d\tau\Psi(\xi,\tau)[&d_1\Psi_{\tau\xi} + d_2A_1\Psi_\xi\Psi_{\xi\xi} \\ &+ d_3A_2\Psi_\xi^2\Psi_{\xi\xi} + d_4B\Psi_{\xi\xi\xi\xi}]. \end{aligned} \tag{19}$$

Here, R denotes the boundary of space and T stands for the limits of time. The constants d_1, d_2, d_3, and d_4 are to be determined later. Integrating Eq. (19) by parts and taking $\Psi_\tau|_R = \Psi_\xi|_R = \Psi_\tau|_T = 0$, we obtain

$$J(\Psi) = \int_R d\xi\int_T d\tau H(\Psi_\tau, \Psi_\xi, \Psi_{\xi\xi}), \tag{20}$$

where $H(\Psi_\tau, \Psi_\xi, \Psi_{\xi\xi}) = -d_1\Psi_\tau\Psi_\xi - \frac{d_2A_1}{2}\Psi_\xi^3 - \frac{d_3A_2}{3}\Psi_\xi^4 + d_4B\Psi_{\xi\xi}$. Taking the variation of the above functional, we obtain

$$\begin{aligned} &\delta J(\Psi) \\ &= \int_R d\xi\int_T d\tau\left[\left(\frac{\partial H}{\partial\Psi_\tau}\right)\delta\Psi_\tau + \left(\frac{\partial H}{\partial\Psi_\xi}\right)\delta\Psi_\xi + \left(\frac{\partial H}{\partial\Psi_{\xi\xi}}\right)\delta\Psi_{\xi\xi}\right]. \end{aligned} \tag{21}$$

Optimizing Eq. (21), i.e., $\delta J(\Psi) = 0$, we obtain

$$\begin{aligned} &2d_1\Psi_{\tau\xi} + 3d_2A_1\Psi_\xi\Psi_{\xi\xi} + 4d_3A_2\Psi_\xi^2\Psi_{\xi\xi} \\ &+ 2d_4B\Psi_{\xi\xi\xi\xi} = 0. \end{aligned} \tag{22}$$

Because the above equation must be equal to Eq. (18), we obtain

$$d_1 = \frac{1}{2}, \quad d_2 = \frac{1}{3}, \quad d_3 = \frac{1}{4}, \quad d_4 = \frac{1}{2}.$$

From the functional expressed in Eq. (20), we can see that the Lagrangian of the classical Gardner equation is given by

$$H(\Psi_\tau, \Psi_\xi, \Psi_{\xi\xi}) = -\frac{1}{2}\Psi_\tau\Psi_\xi - \frac{A_1}{6}\Psi_\xi^3 - \frac{A_2}{12}\Psi_\xi^4 + \frac{1}{2}B\Psi_{\xi\xi}.$$

Similar to this form, the Lagrangian of time-fractional Gardner equation can be expressed in the form

$$\hat{H}\left({}_0D_\tau^\gamma\Psi, \Psi_\xi, \Psi_{\xi\xi}\right) = -\frac{1}{2}\left({}_0D_\tau^\gamma\Psi\right)\Psi_\xi - \frac{A_1}{6}\Psi_\xi^3 - \frac{A_2}{12}\Psi_\xi^4 + \frac{1}{2}B\Psi_{\xi\xi}, \quad 0 < \gamma \le 1. \tag{23}$$

In the above equation, ${}_0D_\tau^\gamma$ is the left Riemann-Liouville fractional derivative operator defined as

$${}_aD_t^\gamma f(t) = \frac{1}{\Gamma(k-\gamma)}\frac{d^k}{dt^k}\left[\int_a^t d\tau(t-\tau)^{k-\gamma-1}f(\tau)\right],$$
$$k-1 < \gamma \le k, \ t \in [a,b].$$

where $k \ge 1$ is a positive integer. The functional corresponding to the time-fractional Lagrangian (23) is defined as

$$\hat{J}(\Psi) = \int_R d\xi \int_T d\tau \hat{H}({}_0D_\tau^\gamma\Psi, \Psi_\xi, \Psi_{\xi\xi}). \tag{24}$$

where $\hat{H}({}_0D_\tau^\gamma\Psi, \Psi_\xi, \Psi_{\xi\xi})$ is expressed in Eq. (23). The variation of functional (24) with respect to $\Psi(\xi,\tau)$ yields

$$\delta\hat{J}(\Psi) = \int_R d\xi \int_T d\tau \left[\left(\frac{\partial H}{\partial {}_0D_\tau^\gamma\Psi}\right)\delta {}_0D_\tau^\gamma\Psi + \left(\frac{\partial H}{\partial \Psi_\xi}\right)\delta\Psi_\xi + \left(\frac{\partial H}{\partial \Psi_{\xi\xi}}\right)\delta\Psi_{\xi\xi}\right]. \tag{25}$$

Assuming $\delta\Psi|_T = \delta\Psi|_R = \delta\Psi_\xi|_R = 0$, integrating Eq. (25) by parts, and using the following formula:[2]

$$\int_a^b dt f(t)\,{}_aD_t^\gamma g(t) = \int_a^b dt g(t)\,{}_tD_b^\gamma f(t), \tag{26}$$

we obtain

$$\delta\hat{J}(\Psi) = \int_R d\xi \int_T d\tau \left[{}_\tau D_{T_0}^\gamma\left(\frac{\partial H}{\partial {}_0D_\tau^\gamma\Psi}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial H}{\partial\Psi_\xi}\right) + \frac{\partial^2}{\partial\xi^2}\left(\frac{\partial H}{\partial\Psi_{\xi\xi}}\right)\right]\delta\Psi_\xi. \tag{27}$$

Here, Eq. (26) is the rule for fractional integration by parts, and ${}_tD_b^\gamma$ is the right Riemann-Liouville fractional derivative operator defined as

$${}_tD_b^\gamma f(t) = \frac{(-1)^k}{\Gamma(k-\gamma)}\frac{d^k}{dt^k}\left[\int_a^t d\tau(\tau-t)^{k-\gamma-1}f(\tau)\right],$$
$$k-1 < \gamma \le k, \ t \in [a,b].$$

Optimizing the variation of the functional $\hat{J}(\Psi)$, i.e., $\delta\hat{J}(\Psi) = 0$, we obtain the following Euler-Lagrange equation of time-fractional Gardner equation:

$${}_\tau D_{T_0}^\gamma\left(\frac{\partial\hat{H}}{\partial {}_0D_\tau^\gamma\Psi}\right) - \frac{\partial}{\partial\xi}\left(\frac{\partial\hat{H}}{\partial\Psi_\xi}\right) + \frac{\partial^2}{\partial\xi^2}\left(\frac{\partial\hat{H}}{\partial\Psi_{\xi\xi}}\right) = 0.$$

Substitution of Eq. (23) (the Lagrangian of time-fractional Gardner equation) into the above Euler-Lagrange equation yields

$$-\frac{1}{2}{}_\tau D_{T_0}^\gamma\Psi_\xi + \frac{1}{2}{}_0D_\tau^\gamma\Psi_\xi + A_1\Psi_\xi\Psi_{\xi\xi} + A_2\Psi_\xi^2\Psi_{\xi\xi} + B\Psi_{\xi\xi\xi\xi} = 0. \tag{28}$$

Substituting the potential function $\Psi_\xi(\xi,\tau) = \Phi(\xi,\tau)$ into Eq. (28), we obtain the time-fractional Gardner equation in the form

$${}_0^RD_\tau^\gamma\Phi + A_1\Phi\frac{\partial\Phi}{\partial\xi} + A_2\Phi^2\frac{\partial\Phi}{\partial\xi} + B\frac{\partial^3\Phi}{\partial\xi^3} = 0,$$
$$0 < \gamma \le 1, \tau \in [0, T_0]. \tag{29}$$

where ${}_0^RD_\tau^\gamma$ is Riesz fractional derivative operator defined as

$${}_0^RD_t^\gamma f(t) = \frac{1}{2}\left[{}_aD_t^\gamma f(t) + (-1)^k\,{}_tD_b^\gamma f(t)\right] = \frac{1}{2}\frac{1}{\Gamma(k-\gamma)}\frac{d^k}{dt^k}\left[\int_b^a d\tau|t-\tau|^{k-\gamma-1}f(\tau)\right],$$
$$k-1 < \gamma \le k, \ t \in [a,b]. \tag{30}$$

III. RESULTS AND DISCUSSIONS

In this section, we will construct the approximate compressive and rarefactive solitary wave solutions of time-fractional Gardner equation (29) by variational iteration method[43,44] (see the Appendix), and discuss the features of these two kinds of ion-acoustic solitary waves related to the plasma parameters.

Because we discuss the ion-acoustic waves propagating outward ($V > 0$), the compressive solitary wave solution of time-fractional Gardner equation (29) can be obtained by taking the initial value of the classical Gardner equation as the zeroth order approximation

$$\Phi_0(\xi,\tau) = \frac{24B\,\mathrm{sech}^2(\xi)}{(A_1 - \sqrt{\sigma}) - (A_1 + \sqrt{\sigma})\tanh^2(\xi)}, \tag{31}$$

where $\sigma = A_1^2 + 24A_2B$. Substituting this zeroth order approximation into iteration formula (A3) and using the definition of Riesz fractional derivative (30), we get the first order approximate solution with the aid of Maple software

$$\Phi_1(\xi,\tau) = \Phi_0(\xi,\tau) - \frac{f_1(\xi,\tau)}{f_2(\xi,\tau)}, \tag{32}$$

where the functions $f_1(\xi,\tau)$ and $f_2(\xi,\tau)$ are

$$
\begin{aligned}
f_1(\xi,\tau) &= 96\sinh(\xi)\cosh(\xi)B^2\frac{\tau^\gamma}{\Gamma(1+\gamma)}[2\cosh^2(\xi)A_1^3 + 48B\cosh^2(\xi)A_1A_2 - 2\sqrt{\sigma}\cosh^2(\xi)A_1^2 + 12B\sqrt{\sigma}A_2 \\
&\quad +2\sqrt{\sigma}\cosh^4(\xi)A_1^2 + 48B\sqrt{\sigma}\cosh^4(\xi)A_2 - 48B\sqrt{\sigma}\cosh^2(\xi)A_2 + A_1^2\sqrt{\sigma} - A_1^3 - 24BA_1A_2], \\
f_2(\xi,\tau) &= 4\cosh^2(\xi)A_1^3\sqrt{\sigma} + A_2^2B^2\left[-576\cosh^2(\xi) + 1728\cosh^4(\xi) - 2304\cosh^6(\xi) + 1152\cosh^8(\xi) + 72\right] \\
&\quad +A_1^3\sqrt{\sigma}\left[4\cosh^6(\xi) - 6\cosh^4(\xi) - 1\right] + A_2^4 + A_1^2A_2B[96\cosh^8(\xi) - 120\cosh^2(\xi) - 192\cosh^6(\xi) \\
&\quad +216\cosh^4(\xi) - 12\sqrt{\sigma} + 24] + A_1\sqrt{\sigma}A_2B\left[72\cosh^2(\xi) + 96\cosh^6(\xi) - 144\cosh^4(\xi)\right] \\
&\quad + A_1^4\left[2\cosh^8(\xi) - 4\cosh^2(\xi) + 6\cosh^4(\xi) - 4\cosh^6(\xi)\right].
\end{aligned}
$$

The higher order approximations $\Phi_2, \Phi_3, \ldots$ which converge to the exact solution of Eq. (29) can be calculated by the similar procedure. For simplicity, we do not list them there.

Similarly, we can obtain the approximate rarefactive solitary wave solution of time-fractional Gardner equation (29) by substituting the following initial value of the classical Gardner equation into iteration formula (A3)

$$
\Phi_0(\xi,\tau) = \frac{24B\,\mathrm{sech}^2(\xi)}{(A_1+\sqrt{\sigma}) - (A_1-\sqrt{\sigma})\tanh^2(\xi)}.
$$

Here, we omit the higher order approximations for simplicity.

For a better understanding of time-fractional Gardner equation (29), the compressive and rarefactive solitary wave solutions are depicted to show the time evolutions of electrostatic potential $\Phi(\xi,\tau)$ in Fig. 1.

In Ref. 45, the propagation of ion-acoustic solitary wave was investigated in negative ion plasma with nonextensive electrons. It has been found that the amplitudes of slow compressive and rarefactive ion-acoustic solitary waves decrease by increasing the values of nonextensive index q and electron to positive ion number density ratio δ. To compare the results presented in Ref. 45, we show the variation of electrostatic potential $\Phi(\xi,\tau)$ versus ξ for different values of q and δ in Fig. 2. From plot (a), we can see that the amplitude of compressive ion-acoustic solitary wave decreases with an increase of nonextensive index q. The amplitude of rarefactive ion-acoustic solitary wave decreases by increasing δ value, which is presented in plot (b). These results are in

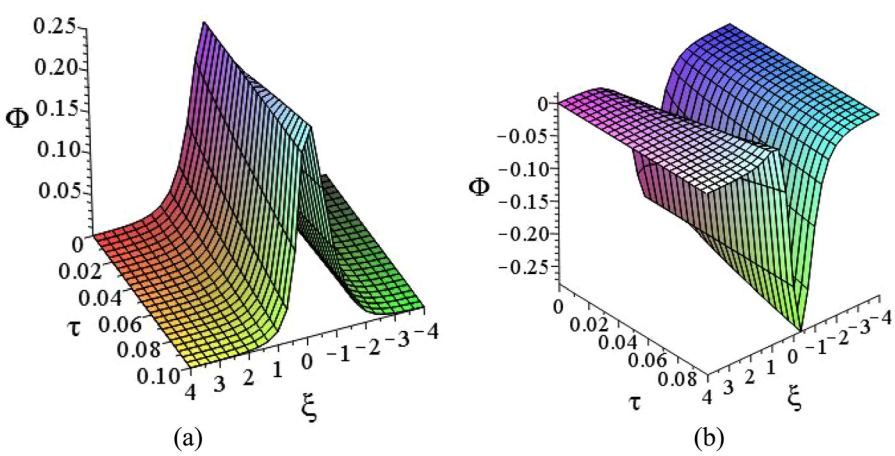

FIG. 1. Time evolutions of (a) compressive solitary wave solution and (b) rarefactive solitary wave solution of time-fractional Gardner equation (29). The parameters are $\theta = 0.1$, $\beta = 0.1$, $\mu_1 = 0.5$, $\mu_2 = 0.8$, $q = 1.8$, $\alpha = 0.2$, $\gamma = 0.8$.

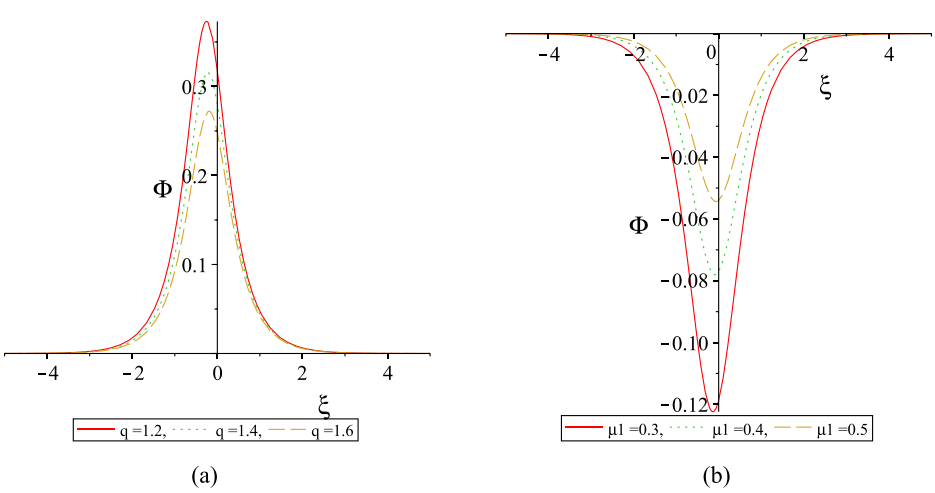

FIG. 2. The electrostatic potential distribution $\Phi(\xi,\tau)$ vs. ξ at $\tau = 0.1$ for different values of (a) q and (b) δ. The parameters are $\beta = 0$, $\mu_1 = 0.89$, $\alpha = 0$, $\gamma = 1$, (a) $\delta = 0.2$, (b) $q = 1.3$.

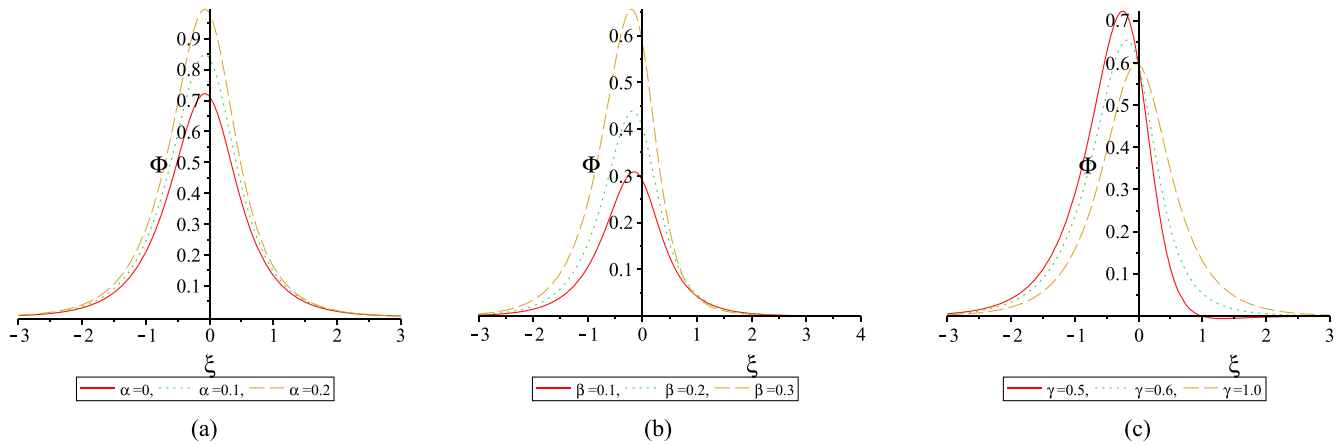

FIG. 3. Variation of compressive solitary waves for different values of (a) α, (b) β, and (c) γ. The parameters are $\theta = 0.2$, $\mu_1 = 0.6$, $\mu_2 = 0.7$, (a) $\beta = 0.3$, $q = 1.0$, $\gamma = 0.85$, (b) $q = 1.8$, $\alpha = 0.2$, $\gamma = 0.85$, and (d) $\beta = 0.3$, $q = 1.7$, $\alpha = 0.15$.

agreement with the corresponding results obtained in Ref. 45.

In Fig. 3, we explore the effect of nonthermal parameter α, negative-ion-beam to positive ion density ratio β, and time fractional order γ on the nonlinear structures of compressive ion-acoustic solitary waves. We can see that the profiles of the waves are pronounced affected by these plasma parameters, and the corresponding physical interpretations can be given as follows:

(i) From plot (a), it can be seen that the amplitude of the compressive solitary wave increases by increasing α value. We speculate that this phenomenon can be explained as follows: The nonlinearity of the plasma with nonthermal electrons ($\alpha > 0$) is higher than that of the plasma with isothermal electrons ($\alpha = 0$). In addition, an increase in nonthermal electron population can further enhance the nonlinearity of the plasma system, which makes the solitary pulses taller.

(ii) Plot (b) reveals that an increase in β value can lead to an increase of the amplitude of compressive solitary wave. This behavior can be explained as follows: Increasing the negative-ion-beam concentration can enhance the nonlinearity of our plasma system and concentrate a significant amount of energy, so the amplitude of the compressive solitary wave has an increasing trend and the pulses become taller.

(iii) Plot (c) indicates that when the value of time fractional order γ increases, the amplitude of the solitary pulses decreases. This feature can be interpreted as follows: The time fractional parameter can be used to modify the profile of the solitary wave instead of adding higher order nonlinearity or dispersion terms to the equation governing the medium.

The effect of nonthermal parameter α, negative-ion-beam to positive ion density ratio β, and time fractional order γ on the rarefactive solitary waves is studied in Fig. 4, which provides a better understanding the nature of ion-acoustic waves. It is easy to see that these plasma parameters have the same effects on the rarefactive solitary wave as on the compressive solitary wave. The physical interpretations of these qualitative behaviors can be given as follows: Increasing the nonthermal electron population or negative-ion-beam concentration can enhance the nonlinearity of the system, which makes the solitary pulses taller. However, the nonlinearity of the system can be reduced with increasing γ value, so the pulses of the rarefactive solitary waves become shorter.

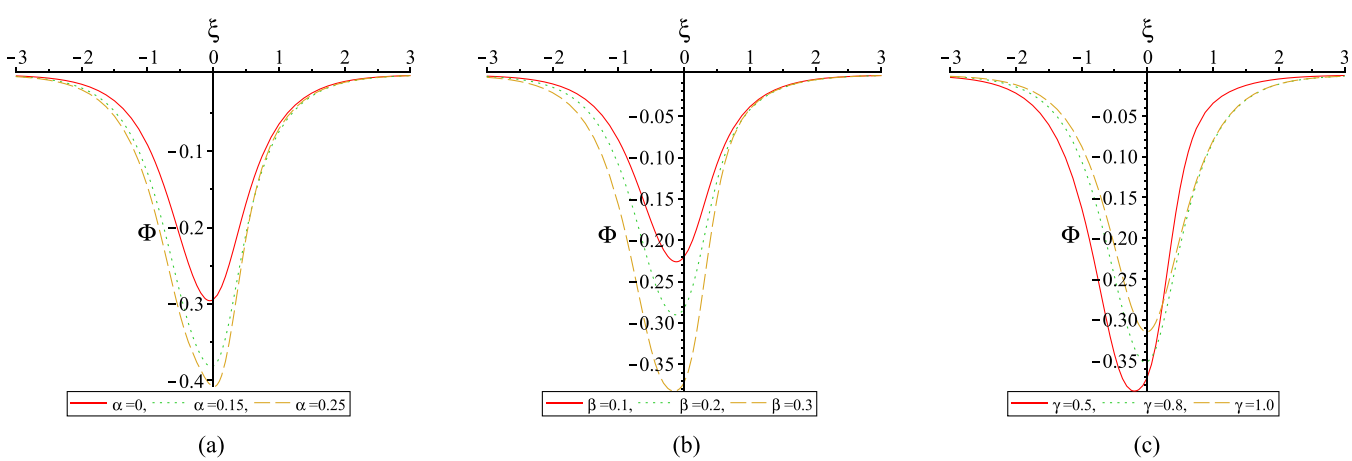

FIG. 4. Variation of rarefactive solitary waves for different values of (a) α, (b) β, and (c) γ. The parameters are $\theta = 0.1$, $\mu_1 = \mu_2 = 0.9$, (a) $\beta = 0.2$, $q = 1.0$, $\gamma = 0.7$, (b) $q = 1.6$, $\alpha = 0.1$, $\gamma = 0.7$, and (c) $\beta = 0.2$, $q = 1.5$, $\alpha = 0.05$.

IV. CONCLUSIONS

By applying the fractional calculus, we investigate nonlinear propagation of ion-acoustic waves in an unmagnetized plasma consisting of positive ions, negative ions, negative-ion-beam, and nonthermal nonextensive electrons. Via reductive perturbation method and Agrawal's method, we reduce the basic set of fluid equations to time-fractional Gardner equation governing the nonlinear behavior of small but finite amplitude ion-acoustic waves. The approximate compressive and rarefactive solitary wave solutions of the time-fractional Gardner equation are obtained by the variational iteration method.

It is found that the amplitudes of compressive and rarefactive ion-acoustic solitary wave decrease by increasing the value of nonextensive index q or electron to positive ion number density ratio δ. These results are in agreement with the corresponding results obtained previously.[45] Moreover, increasing the value of nonthermal parameter α or negative-ion-beam to positive ion density ratio β can enhance the nonlinearity of the system, and so the pulses of the solitary waves become taller. However, an increase of time fractional order γ value can shrink off nonlinearity of the system and reduce the amplitudes of the solitary pulses. It is also concluded that time fractional order introduces higher order nonlinearity or dispersion relationship into the plasma system, which plays an important role in varying the amplitude of solitary waves.

Our theoretical investigations can provide a better understanding of negative-ion-beam plasma in laboratory experiments and spatial observations.

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APPENDIX: VARIATIONAL ITERATION METHOD FOR THE TIME-FRACTIONAL GARDNER EQUATION

The approximate solutions of time-fractional Gardner equation (29) can be obtained by variational iteration method[43,44] as follows:

Considering the following property of Riesz fractional derivative[1,2]

$$ {}_a^RD_b^{\gamma}\left[{}_a^RD_b^{\rho}f(t)\right] = {}_a^RD_b^{\gamma+\rho}f(t) - \sum_{j=1}^{k} {}_a^RD_b^{\rho-j}f(t)\Big|_{t=a}\frac{(t-a)^{-\gamma-j}}{\Gamma(1-\gamma-j)}, $$
$$ 0<\gamma\le 1,\ k-1<\rho\le k. $$

and acting from left by the fractional operator ${}_0^RD_\tau^{1-\gamma}$ on Eq. (29), we obtain

$$ \frac{\partial\Phi}{\partial\tau} = {}_0^RD_\tau^{\gamma-1}\Phi\Big|_{\tau=0}\frac{\tau^{\gamma-2}}{\Gamma(\alpha-1)} - {}_0^RD_\tau^{1-\gamma} \times\left[A_1\Phi\frac{\partial\Phi}{\partial\xi} + A_2\Phi^2\frac{\partial\Phi}{\partial\xi} + B\frac{\partial^3\Phi}{\partial\xi^3}\right], \quad 0<\gamma\le 1,\ \tau\in[0,T_0]. \tag{A1} $$

Then, the correction functional of Eq. (29) can be constructed as

$$ \Phi_{n+1}(\xi,\tau) = \Phi_n(\xi,\tau) + \int_0^\tau d\tau'\lambda(\tau')\left\{\frac{\partial\Phi_n(\xi,\tau')}{\partial\tau'} - {}_0^RD_\tau^{\gamma-1}\Phi_n(\xi,\tau')\Big|_{\tau=0}\frac{\tau'^{\gamma-2}}{\Gamma(\gamma-1)} + {}_0^RD_{\tau'}^{1-\gamma}\left[A_1\tilde{\Phi}_n(\xi,\tau')\frac{\partial\tilde{\Phi}_n(\xi,\tau')}{\partial\xi} + A_2\tilde{\Phi}_n^2(\xi,\tau')\frac{\partial\tilde{\Phi}_n(\xi,\tau')}{\partial\xi} + B\frac{\partial^3\tilde{\Phi}_n(\xi,\tau')}{\partial\xi^3}\right]\right\}, \tag{A2} $$

where $\lambda(\tau')$ is the Lagrange multiplier, the subscript $n(>0)$ denotes the nth-order approximation, and $\tilde{\Phi}_n(\xi,\tau')$ is the restricted variation, i.e., $\delta\tilde{\Phi}_n(\xi,\tau')=0$. Making the above correction functional stationary, we obtain

$$ \frac{d\lambda(\tau')}{d\tau'} = 0,\ \ 1+\lambda(\tau') = 0. $$

Therefore, the Lagrange multiplier can be identified as $\lambda(\tau')=-1$. Substitution of the Lagrange multiplier $\lambda(\tau')=-1$ into the correction functional (A2) yields the following iteration formula:

$$ \Phi_{n+1}(\xi,\tau) = \Phi_n(\xi,\tau) - \int_0^\tau d\tau'\left\{\frac{\partial\Phi_n(\xi,\tau')}{\partial\tau'} - {}_0^RI_{\tau'}^{1-\gamma}\Phi_n(\xi,\tau')|_{\tau=0}\frac{\tau^{\gamma-2}}{\Gamma(\gamma-1)} + {}_0^RD_{\tau'}^{1-\gamma}\left[A_1\Phi_n(\xi,\tau')\frac{\partial\Phi_n(\xi,\tau')}{\partial\xi} + A_2\Phi_n^2(\xi,\tau')\frac{\partial\Phi_n(\xi,\tau')}{\partial\xi} + B\frac{\partial^3\Phi_n(\xi,\tau')}{\partial\xi^3}\right]\right\}. \tag{A3} $$

As $\gamma-1<0$, the Riesz fractional derivative operator ${}_0^RD_{\tau'}^{\gamma-1}$ is reduced to the Riesz fractional integral operator ${}_0^RI_{\tau'}^{1-\gamma}$. Because the right Riemann-Liouville fractional derivative is interpreted as a future state of the process,[2] we set the right-derivative to zero in the calculations.

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