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A postprocessing mixed finite element method for the Navier–Stokes equations

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A fully discrete postprocessing mixed finite element scheme is considered for solving the time-dependent Navier–Stokes equations. In the PP method, we only consider a non-linear equation in the coarse-level subspace and a linear problem in the fine-level subspace. The analysis shows that the PP scheme can reach the same accuracy as the standard Galerkin method with a very fine mesh size h by an appropriate choice of H. Numerical examples are provided that confirm both the theoretical analysis and the corresponding improvement in computational efficiency.

Keywords: postprocessing; mixed finite element method; Navier–Stokes equations; two-level method; stability and convergence

1. Introduction

This article is to study a fully discrete postprocessing (PP) scheme for the time-dependent Navier–Stokes equations.

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \forall(x,t)\in\Omega\times[0,T], \\ \nabla\cdot u = 0 & \forall(x,t)\in\Omega\times[0,T], \\ u = 0 & \forall t\in[0,T], \forall x\in\partial\Omega, \\ u(x,0) = u_0 & \forall x\in\Omega. \end{cases} \quad (1.1)$$

Here Ω is a bounded domain in R^2 with a Lipschitz continuous boundary, u is the velocity field, u_0 is the initial velocity satisfying $\nabla\cdot u_0 = 0$, p denotes the pressure, f is the density of body forces and $\nu > 0$ is the kinetic viscosity.

For given positive constants h and H with $0 < h < H$, we construct two finite element couples (X_H, M_H) and (X_h, M_h). Based on the usual L^2 orthogonal projection, Ait Ou Ammi and Marion (1994), Marion and Xu (1995) introduced the space splitting as follows

$$X_h = X_H + X_h^H.$$

Note that X_H and X_h^H are orthogonal with respect to the scalar product $(\cdot,\cdot)$. On the basis of this space splitting, the final approximation $u_h \in X_h$ can be naturally decomposed into the large eddy component $v_H \in X_H$ and the small eddy component $w_h \in X_h^H$. According to the theory of the approximate inertial manifold (AIM) for dissipative system initialised by Foias *et al.* (1988), there exists a smooth mapping ϕ from X_H onto X_h^H reflecting the approximate interactive relation between the large and small eddy components such that

$$w_h \approx \phi(v_H),$$

which is frequently expressed via a steady Stokes problem. Thus, a class of two-level schemes called non-linear Galerkin method (NLG) was widely studied by Marion and Temam (1989, 1990), Ait Ou Ammi and Marion (1994), Marion and Xu (1995), in either the finite element or spectral case. The finite element non-linear Galerkin method addressed in Ait Ou Ammi and Marion (1994) is described as: for a finite time $T > 0$, find $v_H \in X_H$, $w_h \in X_h^H$ and $p_h \in M_h$ such that for all $t \in (0,T]$

$$\begin{cases} (v_{Ht}, v) + \nu a(v_H + w_h, v) + b(v_H + w_h, v_H, v) \\ \quad + b(v_H, w_h, v) - d(v, p_h) = (f, v), \quad \forall v \in X_H, \\ a(v_H + w_h, w) + b(v_H, v_H, w) - d(w, p_h) = (f, w), \\ \quad \forall w \in X_h^H, \\ d(v_H + w_h, q) = 0, \quad \forall q \in M_h. \end{cases} \quad (1.2)$$

The detailed definitions of the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and the trilinear form $b(\cdot,\cdot,\cdot)$ will be discussed in section 2.

One of the main advantages of NLG and its variants is that they can present higher convergence rate than the classical Galerkin method as the analysis in Foias *et al.* (1988), Marion and Temam (1989, 1990), Ait Ou Ammi

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and Marion (1994), Marion and Xu (1995). Under the assumptions that the initial data are given by satisfying $u_0 \in (H^2(\Omega))^2$ and $f, f_t \in L^\infty(0,T;(L^2(\Omega))^2)$, then the convergent results of Equation (1.2) are

$$||u(t) - (v_H(t) + w_h(t))||_{H^1} = O(h + H^2),$$
$$||p(t) - p_h(t)||_{L^2} = O(h + H^2), \quad \forall t \in (0, T],$$

where (u, p) is the exact solution of the Navier–Stokes equations.

However, recalling the construction of Equation (1.2), it is easy to find that when computing v_H, we have to use the information of w_h and vice versa. This may inspire doubt whether the better accuracy of NLG is worth its computational cost. Therefore, a class of PP techniques was developed to increase the accuracy and computational efficiency of Galerkin methods for dissipative partial differential equations in a series of work by Titi *et al.* (1998, 1999, 2000, 2003). The PP method in Titi and coworkers (1998, 1999), both its analysis and understanding seemed to depend on the approach of AIM, which was only applicable to the near attractor case. Lately, Novo and coworkers (2005) developed the PP mixed finite element for the Navier–Stokes equations. They first compute a standard Galerkin approximation $(u_H, p_H) \in (X_H, M_H)$, then by solving a steady Stokes problem to obtain a final approximation $(u_h, p_h) \in (X_h, M_h)$. Specifically, find $v_H \in X_H$, $u_h \in X_h$ and $p_h \in M_h$, for all $t \in (0,T]$ such that

$$\begin{cases} (u_{Ht}, v) + \nu a(u_H, v) + b((u_H, u_H, v) - d(v, p_H) = (f, v), \\ \quad \forall v \in X_H, \\ \nu a(u_h, v) - d(v, p_h) = (f, v) - (v_{Ht}(T), v) - b(v_H(T), \\ \quad v_H(T), v), \quad \forall v \in X_h. \end{cases} \tag{1.3}$$

The PP scheme (1.3) is cheaper to be implemented than the finite element NLG (1.2) because the first equation in (1.3) is actually the standard finite element Galerkin method (SGM), which does not use the information on the fine mesh at all. Only at the end of the integration, we have to use the solution v_H in order to refine the solution. Hence, we think the finite element PP method (1.3) is weakly coupled, which can be seen as a very efficient scheme to some extent. Moreover, the PP scheme (1.3) can obtain a similar high convergence rate as the finite element NLG. Novo *et al.* (2005) has shown that for all $t \in (0,T]$

$$||u(t) - u_h(t)||_{L^2} = O(h^2 + L_h^2 H^3),$$
$$||u(t) - u_h(t)||_{H^1} = O(h + L_h^2 H^2),$$
$$||p(t) - p_h(t)||_{L^2} = O(h + L_h^2 H^2),$$

where $L_h \sim |\log(h)|^{1/2}$.

However, both in the NLG scheme (1.2) and PP scheme (1.3), the interaction of the large and small eddies components are reflected by a steady generalised Stokes equations, such schemes are only accepted only for $t > t_0$ when the time derivative of the solution possesses enough regularity. People realised that, in general, such relation should be time-dependent to better describe the interaction. Hence, Titi *et al.* (2003) presented a more general PP scheme called the dynamical postprocessing (DPP) scheme

$$\begin{cases} (u_{Ht}, v) + \nu a(u_H, v) + b((u_H, u_H, v) - d(v, p_H) = (f, v), \\ \quad \forall v \in X_H, \\ (u_{ht}, v) + \nu a(u_h, v) - d(v, p_h) = (f, v) - b(v_H(t), \\ \quad v_H(t), v), \quad \forall v \in X_h^H. \end{cases} \tag{1.4}$$

The omission of all the small eddy components both in PP and DPP leads to a weakly decoupled system. In fact, such approximation can be seen as a first-order linearisation of the non-linear term. This is only valid for the large viscosity cases and our later numerical results also agree with our presumption. To make the algorithms applicable for the small viscosity cases, a second-order linearisation of the non-linear term should be considered. In this article, we are dedicated to construct a fully discrete PP scheme for the time-dependent Navier–Stokes equations, which is weakly coupled like that of Equation (1.3) and can improve the L^2 and H^1 convergence rate about one order compared with the scheme (1.3). In fact, we have obtained the following results

$$||u(t_n) - u_h^n||_{L^2} = O(k + h^2 + L_h H^4),$$
$$||u(t_n) - u_h^n||_{H^1} = O(k + h + L_h H^3),$$
$$||p(t) - p_h^n||_{L^2} = O(k + h + L_h H^3),$$
$$1 \le n \le N,$$

where (u_h^n, p_h^n) is given by our PP scheme, k is the time step length, $t_n = nk$ and $N = [T/k]$.

The remainder of this article is arranged as follows: section 2 gives some mathematical preliminaries. The detailed presentation of the PP scheme and its stability are presented in section 3. The convergent results are investigated in section 4. Finally, section 5 presents some numerical examples to complement our theoretical analysis in previous sections.

2. Mathematical preliminaries

Let us denote

$$X = (H_0^1(\Omega))^2, \quad M = L_0^2(\Omega)$$
$$= \left\{ q \in L^2(\Omega); \int_\Omega q \mathrm{d}x = 0 \right\},$$

$$H = \{v \in (L^2(\Omega))^2; \mathrm{div} v = 0, v \cdot n|_\Omega = 0\},$$

$$V = \{v \in X; \mathrm{div} v = 0\}.$$

The spaces H and V are equipped with the following scalar products and norms

$$(u, v) = \int_\Omega u \cdot v \mathrm{dx}, \quad |u| = (u, u)^{\frac{1}{2}}, \quad \forall u, v \in H,$$

$$((u, v)) = (\nabla u, \nabla v), \quad ||u|| = ((u, u))^{\frac{1}{2}}, \quad \forall u, v \in V.$$

The Stokes operator $A = -P\Delta$ with domain $D(A) = (H^2(\Omega))^2 \cap V$ and the bilinear operator

$$B(u, v) = (u \cdot \nabla)v + \frac{1}{2}(\mathrm{div}\, u)v, \quad \forall u, v \in X,$$

where P is the Leray orthogonal projection of $(L^2(\Omega))^2$ onto H.

Furthermore we introduce the bilinear forms $a(\cdot,\cdot)$ $d(\cdot,\cdot)$ and trilinear form $b(\cdot,\cdot,\cdot)$

$$\begin{aligned} a(u,v) &= \langle Au, v\rangle_{X'} = ((u,v)), \quad \forall u, v \in X, \\ d(v,q) &= -(v, \nabla q) = (q, \mathrm{div} v), \quad \forall v \in X, \quad \forall q \in M, \\ b(u,v,w) &= \langle B(u,v), w\rangle_{X'} \\ &= ((u \cdot \nabla)v, w) + \tfrac{1}{2}((\mathrm{div}\, u)v, w) \\ &= \tfrac{1}{2}((u \cdot \nabla)v, w) - \tfrac{1}{2}((u \cdot \nabla)w, v), \quad \forall u, v, w \in X. \end{aligned}$$

The finite element subspace (X_h, M_h) is characterised by a partition τ_h, assumed to be uniformly regular as h tending to 0. We define the subspace V_h by

$$V_h = \{v_h \in X_h; d(v_h, q_h) = 0, \forall q_h \in M_h\}.$$

Let P_h: $(L^2(\Omega))^2 \to V_h$ be the L^2 orthogonal projection defined by

$$(v - P_h v, v_h) = 0, \qquad \forall v \in (L^2(\Omega))^2, \quad \forall v_h \in V_h.$$

Also, we define ρ_h: $M \to M_h$ by

$$(q - \rho_h q, q_h) = 0, \quad \forall q \in M, \quad \forall q_h \in M_h.$$

Here and after, we will use $|\cdot|_{L^\alpha}$ and $|\cdot|_\alpha$ to denote the $(L^\alpha(\Omega))^2$ and $(H^\alpha(\Omega))^2$ norms, respectively, for all $\alpha \in R$. Furthermore, we always use c to denote a generic positive constant depending only on the data (v, Ω, f) and use κ to denote a constant which may depend additionally on u_0 and on time t, assumed to be continuous with respect to t.

We assume that the couple (X_h, M_h) satisfies the following approximate properties

(A1) For each $v \in (H^2(\Omega))^2 \cap X$ and $q \in H^1(\Omega) \cap M$, there exists approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\begin{cases} d(v - I_h v, q_h) = 0 \quad \forall q_h \in M_h, \\ |v - I_h v| + h||v - I_h v|| \le ch^2||v||_{H^2}, \\ |q - J_h q| \le ch||q||, \end{cases}$$

(A2) Inverse inequality in the finite element subspace

$$||v_h|| \le ch^{-1}|v_h|, \quad \forall v_h \in X_h,$$

(A3) The so-called inf-sup condition: for each $q_h \in M_h$, there exists $v_h \in X_h$ and $v_h \ne 0$ such that

$$d(v_h, q_h) \ge \beta|q_h|gt||v_h||,$$

where β is a constant independent of h.

The following properties are classical consequences of above assumptions (A1)–(A3) (see Girault and Raviart 1986).

$$\begin{aligned} &||P_h v|| \le c||v||, \quad \forall v \in X, \\ &|v - P_h v| + h||v - P_h v|| \le ch^2|Av|, \quad \forall v \in D(A), \\ &|v - P_h v| \le ch||v - P_h v||, \quad \forall v \in X. \end{aligned} \tag{2.1}$$

The semi-discrete Galerkin approximation of (1.1) based on (X_h, M_h) reads: find $(U_h, p_h) \in (X_h, M_h)$, such that for $\forall t \in (0,T]$

$$\begin{aligned} &(U_{ht}, v) + va(U_h, v) + b(U_h, U_h, v) - d(v, p_h) + d(U_h, q) \\ &\quad = (f, v), \forall (v, q) \in (X_h, M_h) U_h(0) = P_h u_0. \end{aligned} \tag{2.2}$$

Above semi-discrete Galerkin approximation based on V_h reads: find $U_h \in V_h$, such that for $\forall t \in (0,T]$

$$\begin{aligned} &(U_{ht}, v) + va(U_h, v) + b(U_h, U_h, v) = (f, v), \quad \forall v \in V_h, \\ &U_h(0) = P_h u_0. \end{aligned} \tag{2.3}$$

In Heywood and Rannacher (1982), it has shown the following theorem.

Theorem 2.1: *Under the assumptions (A1)–(A3), let* $u_0 \in D(A)$ and $f, f_t \in L^\infty(0,T;(L^2(\Omega))^2) \cap L^2(0,T;X)$ *be given, then the solution of Equation* (2.2) *satisfies the following estimates*

$$|u(t) - U_h(t)| + h||u(t) - U_h(t)|| + h|$$
$$p(t) - p_h(t)| \le C(t)h^2, \quad 0 \le t \le T,$$

where $C(t)$ is a continuous function with respect to t.

We define a discrete analogue $A_h \in L(X_h, X_h)$ of Stokes operator A defined by $A_h = -P_h\Delta_h$ through the condition that $(-\Delta_h u_h, v_h) = ((u_h, v_h))$, $\forall u_h, v_h \in X_h$. Thanks to Heywood and Rannacher (1990), we know the restriction of A_h to V_h is invertible, with inverse denoted by A_h^{-1}. Since A_h^{-1} is self-adjoint and positive definite, we define discrete Sobolev norms on V_h that $|v_h|_r = |A_h^{r/2} v_h|, \forall r \in R, \forall v_h \in V_h$. These discrete norms will be assumed to have various properties similar to their continuous counterparts.

Furthermore, some regularity results hold for the semi-discrete solution U_h proven in Heywood and Rannacher (1982).

Theorem 2.2: *Under the assumptions of Theorem* 2.1, *there exists a time $t_0 > 0$ such that the semi-discrete solution U_h satisfies the following estimates*

$$|A_h U_h(t)| \le \kappa, \quad ||U_h^{(j)}(t)|| \le \kappa, \quad 0 \le j \le 3, \quad t \ge t_0,$$

where $U_h^{(j)}(t)$ represents the j-th partial derivative with respect to t.

To end this section, we recall some useful inequalities. The trilinear form satisfies the following properties used in Heywood and Rannacher (1990), Olshanskii (1999), He (2003).

$$b(u_h, v_h, w_h) = -b(u_h, w_h, v_h), \tag{2.4}$$

$$|b(u_h, v_h, w_h)| \le c(|u_h|_{L^\infty}||v_h|| + ||u_h||\ ||v_h|_{L^\infty})|w_h|, \tag{2.5}$$

$$|b(u_h, v_h, w_h)| \le c|u_h|(||v_h||\ |w|_{L^\infty} + |v_h|_{L^\infty}||w_h||), \tag{2.6}$$

$$|b(u_h, v_h, w_h)| \le c(|u_h|\ ||v_h|| + ||u_h||\ |v_h|)|w_h|_{L^\infty}, \tag{2.7}$$

$$|b(u_h, v_h, w_h)| \le c(|u_h|_{L^\infty}||v_h|| + ||u_h||_{L^4}|v_h|_{L^4})|w_h|, \tag{2.8}$$

$$\begin{aligned} &|b(u_h, v_h, w_h)| \le c|u_h|_{1+s}||v_h||\ |w_h|, \\ &c||u_h||\ |v_h|_{1+s}|w_h|, \\ &c|u_h||v_h|_{1+s}||w_h||, \quad \forall s \in (0, 1], \end{aligned} \tag{2.9}$$

$$|b(u_h, v_h, w_h)| \le c(|u_h|_{L^4}||v_h||_{L^4} + ||u_h||_{L^4}\ |v_h|_{L^4})|w_h|, \tag{2.10}$$

$$|b(u_h, v_h, w_h)| \le c|A_h u_h||A_h v_h||w_h|_{-1}, \tag{2.11}$$

for $\forall u_h,\ v_h,\ w_h \in X_h$.

We have the Brezis-Gallouet inequality and Galiardo-Nireberg estimates provided in Brezis and Gallout (1980), Hill and Suli (2000)

$$|v_h|_{L^\infty} \le cL_h|A_h^{\frac{1}{2}} v_h|, \quad \forall v_h \in V_h, \tag{2.12}$$

$$|v_h|_{L^4} \le c|v_h|^{\frac{1}{2}}||v_h||^{\frac{1}{2}}, \quad |\nabla v_h|_{L^4} \le ||v_h||^{\frac{1}{2}}|A_h v_h|^{\frac{1}{2}}, \forall v_h \in V_h. \tag{2.13}$$

3. Fully discrete postprocessing scheme and its stability

We consider the fully discrete approximation of Equation (1.1) by defining the two sequences

$$\begin{aligned} &(u_H^n, p_H^n) \in (X_H, M_H) \quad \text{and} \\ &(u_h^n, p_h^n) \in (X_h, M_h), \quad 0 \le n \le N. \end{aligned}$$

They are given by $u_h^0 = P_h u_o$, $u_H^O = P_H u_o, p_h^o = \rho_h p_o$ $p_H^o = \rho_H p_o$ and

$$\begin{aligned} &(u_H^{n+1}, v) + kva(u_H^{n+1}, v) + kb(u_H^{n+1}, u_H^{n+1}, v) \\ &\quad - kd(v, p_H^{n+1}) + kd(u_H^{n+1}, q) = k(f^{n+1}, v) \\ &\quad + (u_H^n, v), \quad \forall (v, q) \in (X_H, M_H), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &(d_t u_H^{n+1}, v) + va(u_h^{n+1}, v) + b(u_H^{n+1}, u_h^{n+1}, v) \\ &\quad - d(v, p_h^{n+1}) + d(u_h^{n+1}, q) = (f_{n+1}, v) \\ &\quad + b(u_H^{n+1}, u_h^{n+1}, v), \quad \forall (v, q) \in (X_h, M_h) \end{aligned} \tag{3.2}$$

Similarly, considering above schemes (3.1)–(3.2) in the subspaces V_H and V_h respectively implies the following PP scheme

$$\begin{aligned} &(u_H^{n+1}, v) + kva(u_H^{n+1}, v) + kb(u_H^{n+1}, u_H^{n+1}, v) \\ &\quad = k(f^{n+1}, v) + (u_H^n, v), \quad \forall v \in V_H, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &(d_t u_h^{n+1}, v) + va(u_h^{n+1}, v) + b(u_H^{n+1}, u_h^{n+1}, v) \\ &\quad + b(u_h^{n+1}, u_H^{n+1}, v) = (f^{n+1}, v) \\ &\quad + b(u_H^{n+1}, u_H^{n+1}, v), \quad \forall v \in V_h. \end{aligned} \tag{3.4}$$

We decompose u_h^{n+1} and u_h^{n+1} as

$$u_h^{n+1} = u_H^{n+1} + \hat{u}_h^{n+1}, \quad p_h^{n+1} = p_H^{n+1} + \hat{p}_h^{n+1}.$$

Here, $u_H^{n+1} = P_H u_h^{n+1}$, $\hat{u}_h^{n+1} = Q_H u_h^{n+1}$, $Q_H = I - P_H$. Furthermore, the first Equation (3.1) is to get the large eddy component (u_H^{n+1}, p_H^{n+1}) which is nothing but the SGM equation. Regarding $(u_H^{n+1}, p_H^{n+1}) \in (X_H, M_H)$ as an initial guess, the second equation is to obtain the final approximation $(u_h^{n+1}, p_h^{n+1}) \in (X_h, M_h)$ which is a one-step Newton iteration in fact. Thus, we call our PP scheme (3.1)–(3.2) the DPP scheme of Newton type (DPPN).

We recall two Gronwall lemmas provided in Heywood and Rannacher 1990, He 2003.

Lemma 3.1: *Let k, B and a_n, b_n, c_n, d_n for integer $k_0 \geq 0$ be non-negative number such that*

$$a_m + k\sum_{n=1}^{m} b_n \leq k\sum_{n=0}^{m-1} d_n a_n + k\sum_{n=0}^{m-1} c_n + B, \quad \forall m \geq 1.$$

Then,

$$a_m + k\sum_{n=1}^{m} b_n \leq \exp\left(k\sum_{n=0}^{m-1} d_n\right)\left(k\sum_{n=0}^{m-1} c_n + B\right), \quad \forall m \geq 1.$$

Lemma 3.2: *Let k, B* and *a_n, b_n, c_n, d_n for integer $n \geq 1$ be non-negative number such that*

$$a_m + k\sum_{n=1}^{m} b_n \leq k\sum_{n=1}^{m} d_n a_n + k\sum_{n=1}^{m} c_n + B, \quad \forall m \geq 1.$$

Suppose that $kd_n < 1$, for all n, and set $\gamma_n = (1-kd_n)^{-1}$, then,

$$a_m + k\sum_{n=1}^{m} b_n \leq \exp\left(k\sum_{n=1}^{m} \gamma_n d_n\right)\left(k\sum_{n=1}^{m} c_n + B\right), \quad \forall m \geq 1.$$

Lemma 3.3: *Suppose that the assumptions (A1)–(A3) are valid and the time step length $k \leq \tau_T$, where τ_T is a constant which can take different values at its different occurrences. Then there exists a positive constant κ such that*

$$||u_H^n||^2 + k\nu\sum_{i=1}^{n} |A_H u_H^i| \leq \kappa, \quad 1 \leq n \leq N.$$

Proof The proof will be similar to the procedure of Theorem 3.3 provided in He and Liu (2006). Here we omit it. □

Lemma 3.4: *Under the assumptions of Lemma 3.3, there exists a positive constant κ_0 such that*

$$|u_h^n|^2 + k\nu\sum_{i=1}^{n} ||u_h^i||^2 \leq \kappa_0^2, \quad 1 \leq n \leq N.$$

Proof First, we rewrite the scheme (3.4) as

$$(d_t u_h^{i+1}, v) + \nu a(u_h^{i+1}, v) + b(u_h^{i+1}, u_h^{i+1}, v) - b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, v) = (f^{i+1}, v), \quad \forall v \in V_h.$$

Taking $v = 2ku_h^{i+1}$ in above equation and using Equation (2.4), there holds

$$|u_h^{i+1}|^2 + |u_h^{i+1} - u_h^i|^2 - |u_h^i|^2 + 2k\nu||u_h^{i+1}||^2 - 2kb(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, u_H^{i+1}) = 2k(f^{i+1}, u_h^{i+1}). \quad (3.5)$$

Using Equation (2.9), we have

$$2k|(f^{i+1}, u_h^{i+1})| \leq 2k|f|_{-1}\,||u_h^{i+1}|| \leq \frac{k\nu}{2}||u_h^{i+1}||^2 + \frac{ck}{\nu}|f|_{-1}^2, 2k|b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, u_H^{i+1})| \leq ck|\hat{u}_h^{i+1}|\,||\hat{u}_h^{i+1}||\;|A_H u_H^{i+1}| \leq \frac{k\nu}{2}||u_h^{i+1}||^2 + \frac{ck}{\nu}|A_H u_H^{i+1}|^2|u_h^{i+1}|^2$$

where $|f|_{-1} = ||f|_{L^\infty(0,T;X')}$.

Summing Equation (3.5) from $i = 0$ to $i = n-1$ and noticing above estimates implies

$$|u_h^n|^2 + k\nu\sum_{i=1}^{n} ||u_h^{i+1}||^2 \leq |u_0|^2 + \frac{ck}{\nu}\sum_{i=1}^{n}(|f|_{-1}^2 + |A_H u_H^i|^2|u_h^i|^2). \quad (3.6)$$

Finally, applying Gronwall Lemma 3.2 to Equation (3.6) and noticing Lemma 3.3 yields the prescribed result of this lemma. □

Lemma 3.5: *Under the assumptions of Lemma* 3.3, *there exists a positive constant κ_1 such that*

$$||u_h^n||^2 + k\nu\sum_{i=1}^{n} |A_h u_h^i|^2 \leq \kappa_1^2, \quad 1 \leq n \leq N,$$

provided that $ckL_h^2\kappa^2 \leq \frac{\nu}{2}$.

Proof First, the scheme (3.4) can be rewritten as

$$(d_t u_h^{i+1}, v) + \nu a(u_h^{i+1}, v) + b(u_H^{i+1}, u_h^{i+1}, v) + b(\hat{u}_h^{i+1}, u_H^{i+1}, v) = (f^{i+1}, v), \quad \forall v \in V_h.$$

Taking $v = 2A_h u_h^{i+1}$ in the above equation leads to

$$|u_h^{i+1}||^2 + ||u_h^{i+1} - u_h^i||^2 - ||u_h^i||^2 + 2k\nu|A_h u_h^{i+1}|^2 + 2kb(u_H^{i+1}, u_h^{i+1}, A_h u_h^{i+1}) + 2kb(\hat{u}_h^{i+1}, u_H^{i+1}, A_h u_h^{i+1}) = 2k(f^{i+1}, A_h u_h^{i+1}).$$

With the help of Equation (2.5), (2.8) and (2.12), we have

$$\begin{aligned}
&2k|(f^{i+1}, A_h u_h^{i+1})| \\
&\leq 2k|f|\,|A_h u_h^{i+1}| \leq \frac{k\nu}{2}|A_h u_h^{i+1}|^2 \\
&\quad + \frac{ck}{\nu}|f|^2, 2k|b(u_H^{i+1}, u_h^{i+1}, A_h u_h^{i+1})| \\
&\quad + 2k|b(\hat{u}_h^{i+1}, u_H^{i+1}, A_h u_h^{i+1})| \\
&\leq ck|u_H^{i+1}|_{L^\infty}||u_h^{i+1} - u_h^i||\,|A_h u_h^{i+1}| \\
&\quad + ck||u_H^{i+1}||\,|u_h^{i+1} - u_h^i|_{L^\infty}|A_h u_h^{i+1}| \\
&\quad + ck|u_H^{i+1}|_{L^\infty}||u_h^i||\,|A_h u_h^{i+1}| \\
&\quad + ck||u_H^{i+1}||_{L^4}|u_h^i|_{L^4}|A_h u_h^{i+1}| \\
&\leq ckL_H\tilde{\kappa}_1||u_h^{i+1} - u_h^i||\,|A_h u_h^{i+1}| + ckL_h\tilde{\kappa}_1||u_h^{i+1} - u_h^i|| \\
&\quad \times |A_h u_h^{i+1}| + ck|u_h^{i+1}|^{\frac{1}{2}}||u_h^i||\,|A_h u_h^{i+1}|^{\frac{3}{2}} \\
&\quad + ck||u_H^{i+1}||^{\frac{1}{2}}||u_h^i||\,|A_h u_h^{i+1}|^{\frac{3}{2}} \\
&\leq \frac{ckL_H^2\tilde{\kappa}_1^2}{\nu}||u_h^{i+1} - u_h^i||^2 + \frac{ckL_h^2\tilde{\kappa}_1^2}{\nu}||u_h^{i+1} - u_h^i||^2 \\
&\quad + \frac{ck}{\nu^3}|u_h^{i+1}|^2||u_h^i||^4 + \frac{ck}{\nu^3}||u_H^{i+1}||^2||u_h^i||^4 + \frac{k\nu}{2}|A_h u_h^{i+1}|^2.
\end{aligned}$$

Hence, summing Equation (3.7) from $i = 0$ to $i = n-1$ and noticing above estimates, we have

$$\begin{aligned}
||u_h^n||^2 + k\nu\sum_{i=1}^{n}|A_h u_h^i|^2 &\leq ||u_0||^2 + k\sum_{i=0}^{n-1} d_i||u_h^i||^2 \\
&\quad + c\nu^{-1}T|f|^2, \qquad (3.7)
\end{aligned}$$

where $d_i = c\nu^{-3}(|u_h^{i+1}|^2||u_h^i||^2 + ||u_h^{i+1}||^2||u_h^i||^2)$ for $0 \leq i \leq n-1$.

Finally, applying Gronwall Lemma 3.1 to (3.7) and taking into account Lemma 3.3, Lemma 3.4, we obtain the prescribed result of this lemma. □

Lemma 3.6: *Under the assumptions of Lemma* 3.5, *there exist constants* $\kappa'_r > 0$ *such that*

$$|d_t u_h^n|_r^2 + k\nu\sum_{i=1}^{n}|d_t u_h^i|_{r+1}^2 \leq \kappa'_r, \quad r = 0, 1, \quad 1 \leq n \leq N.$$

Proof From the PP scheme (3.4), we derive

$$\begin{aligned}
&(d_{tt}u_h^{i+1}, v) + \nu a(d_t u_h^{i+1}, v) + b(d_t u_h^{i+1}, u_h^{i+1}, v) \\
&\quad + b(u_h^i, d_t u_h^{i+1}, v) - b(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, v) \\
&\quad - b(\hat{u}_h^i, d_t\hat{u}_h^{i+1}, v) = (d_t f^{i+1}, v), \quad \forall v \in V_h.
\end{aligned}$$

Taking $v = 2kA_h^r d_t u_h^{i+1}$ in Equation (3.8) yields

$$\begin{aligned}
&|d_t u_h^{i+1}|_r^2 + |d_t u_h^{i+1} - d_t u_h^i|_r^2 - |d_t u_h^i|_r^2 + 2k\nu|d_t u_h^{i+1}|_{r+1}^2 \\
&\quad + 2kb(d_t u_h^{i+1}, u_h^{i+1}, A_h^r d_t u_h^{i+1}) \\
&\quad + 2kb(u_h^i, d_t u_h^{i+1}, A_h^r d_t u_h^{i+1}) \\
&\quad - 2kb(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h^r d_t u_h^{i+1}) - 2kb(\hat{u}_h^i, d_t\hat{u}_h^{i+1}, A_h^r d_t u_h^{i+1}) \\
&\quad = 2k(d_t f^{i+1}, A_h^r d_t u_h^{i+1}). \qquad (3.8)
\end{aligned}$$

With the help of Equation (2.5), (2.6), (2.9) and (2.12), we find

$$\begin{aligned}
&2k|b(d_t u_h^{i+1}, u_h^{i+1}, A_h^r d_t u_h^{i+1})| \\
&\quad + 2k|b(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h^r d_t u_h^{i+1})| \\
&\qquad \leq ckL_h|d_t u_h^{i+1}|_{r+1}||u_h^{i+1}||\,|d_t u_h^{i+1} - d_t u_h^i|_r \\
&\qquad\quad + ck|d_t u_h^i|_r\,|A_h u_h^{i+1}|\,|d_t u_h^i|_{r+1} \\
&\qquad \leq \frac{k\nu}{3}|d_t u_h^{i+1}|_{r+1}^2 + \frac{ckL_h^2}{\nu}||u_h^{i+1}||^2|d_t u_h^{i+1} - d_t u_h^i|_r^2 \\
&\qquad\quad + \frac{ck}{\nu}|A_h u_h^{i+1}|^2|d_t u_h^i|_r^2,
\end{aligned}$$

$$\begin{aligned}
&2k|b(u_h^i, d_t u_h^{i+1}, A_h^r d_t u_h^{i+1})| + 2k|b(\hat{u}_h^i, d_t\hat{u}_h^{i+1}, A_h^r d_t u_h^{i+1})| \\
&\quad \leq ckL_h|d_t u_h^{i+1} - d_t u_h^i|_r|\,|u_h^i||\,|d_t u_h^{i+1}|_{r+1} \\
&\qquad + ck|A_h u_h^i|\,|d_t u_h^{i+1}|_{r+1}|d_t u_h^i|_r \\
&\quad \leq \frac{k\nu}{3}|d_t u_h^{i+1}|_{r+1}^2 + \frac{ckL_h^2}{\nu}||u_h^i||^2|d_t u_h^{i+1} - d_t u_h^i|_r^2 \\
&\qquad + \frac{ck}{\nu}|A_h u_h^i|^2|d_t u_h^i|_r^2,
\end{aligned}$$

$$\begin{aligned}
2k|(d_t f^{i+1}, A_h^r d_t u_h^{i+1})| &\leq 2k|d_t f^{i+1}|_{r-1}\,|d_t u_h^{i+1}|_{r+1} \\
&\leq \frac{k\nu}{3}|d_t u_h^{i+1}|_{r+1}^2 + \frac{ck}{\nu}|f_t|_{r-1}^2,
\end{aligned}$$

where $|f_t| = |f_t|_{L^\infty(0,T;(L^2(\Omega))^2)}$.

Hence, summing Equation (3.8) from $i = 0$ to $i = n-1$ and using above estimates, we have

$$\begin{aligned}
|d_t u_h^n|_r^2 + k\nu\sum_{i=1}^{n}|d_t u_h^i|_{r+1}^2 &\leq |d_t u_h^0|_r^2 + k\sum_{i=0}^{n-1} d_i|d_t u_h^i|_r^2 \\
&\quad + c\nu^{-1}T|f_t|_{r-1}^2, \qquad (3.9)
\end{aligned}$$

where $d_i = c\nu^{-1}(|A_h u_h^{i+1}|^2 + |A_h u_h^i|^2)$ for $0 \leq i \leq n-1$.

Finally, using Gronwall Lemma 3.1 to Equation (3.9) together with Lemma 3.5 suggests the results of this lemma. □

Lemma 3.7: *Under the assumptions of Lemma 3.5, there exists a positive constant* κ_2 *such that*

$$|A_h u_h^n| \leq \kappa_2, \quad 1 \leq n \leq N.$$

Proof We rewrite (3.2) as

$$\nu P_h A_h u_h^n = -d_t u_h^n - P_h B(u_H^n, u_h^n) - P_h B(\hat{u}_h^n, u_H^n) + P_h f^n.$$

As pointed in Heywood and Rannacher (1982), we have

$$|A_h u_h^n| \le c|P_h A_h u_h^n|.$$

Therefore, the following estimate holds

$$\nu|A_h u_h^n| \le c|d_t u_h^n| + c|B(u_H^n, u_h^n)| + c|B(\hat{u}_h^n, u_H^n)| + c|f^n|. \quad (3.10)$$

By using Equations (2.10) and (2.13), we have

$$\begin{aligned} &|B(u_H^n, u_h^n)| \\ &\le c|u_H^n|^{\frac{1}{2}}||u_H^n||^{\frac{1}{2}}||u_h^n||^{\frac{1}{2}}|A_h u_h^n|^{\frac{1}{2}} + c||u_H^n||^{\frac{1}{2}}|u_h^n|^{\frac{1}{2}}||u_h^n||^{\frac{1}{2}}|A_h u_h^n|^{\frac{1}{2}} \\ &\le \frac{\nu}{4}|A_h u_h^n| + \frac{c}{\nu}||u_H^n||\ |u_h^n|\ ||u_h^n||, \\ &|B(\hat{u}_h^n, u_H^n)| \\ &\le c|\hat{u}_h^n|^{\frac{1}{2}}||\hat{u}_h^n||^{\frac{1}{2}}||u_H^n||^{\frac{1}{2}}|A_h u_h^n|^{\frac{1}{2}} + c||\hat{u}_h^n||^{\frac{1}{2}}|u_H^n|^{\frac{1}{2}}||u_H^n||^{\frac{1}{2}}|A_h u_h^n|^{\frac{1}{2}} \\ &\le \frac{\nu}{4}|A_h u_h^n| + \frac{c}{\nu}||u_H^n||\ |u_h^n|\ ||u_h^n||. \end{aligned}$$

Finally, combining above estimates with Equation (3.10) and noticing Lemma 3.5, we obtain the result of this lemma. □

We derive from Equation (3.2) that for $\forall \upsilon \in X_h$

$$\begin{aligned} d(v, p_h^n) = (d_t u_h^n, v) + \nu a(u_h^n, v) + b(u_H^n, u_h^n, v) \\ + b(\hat{u}_h^n, u_H^n, v) - (f^n, v), \quad 0 \le n \le N. \end{aligned}$$

Due to the inf–sup condition (A3) and (2.9), there holds

$$|p_h^n| \le \beta^{-1}(|d_t u_h^n|_{-1} + \nu||u_h^n|| + c||u_H^n||\ ||u_h^n|| + |f^n|_{-1}).$$

The combination of Lemma 3.4, Lemma 3.5 and Lemma 3.6, Lemma 3.7 permits us to conclude the following theorem. □

Theorem 3.8: *Under the assumptions of Theorem* 2.1 *and Lemma* 3.3, *furthermore, if k satisfies the following stability condition*

$$ckL_h^2\kappa_1^2 \le \frac{\nu}{2} \quad (3.11)$$

then there exist positive constants κ_r, κ_r', κ_p *and* κ_2 *for all* $1 \le n \le N$ *such that*

$$|u_h^n|_r^2 + k\nu\sum_{i=1}^{n}|u_h^i|2_{r+1} \le \kappa_r^2, \quad r = 0, 1,$$

$$|d_t u_h^n|_r^2 + k\nu\sum_{i=1}^{n}|d_t u_h^i|_{r+1}^2 \le \kappa_r', \quad r = 0, 1,$$

$$|p_h^n| \le \kappa_p, \quad |A_h u_h^n| \le \kappa_2.$$

Remark 3.1. The stability condition (3.11) implies that the restriction on the time step of DPPN is similar to the fully implicit one-level method or the cheaper semi-implicit one-level method. However, the fully implicit one-level method is a time-consuming procedure. The linearized nonlinear term in the semi-implicit scheme is similar to the first *b*–term from Equation (3.2). It is the first-order linearisation of the non-linear term, which is only valid for the large viscosity case. In spite of the easy implementation of the semi-implicit one-level method, it will lose effectiveness for solving the Navier–Stokes equations with large Reynolds number. Some detailed comparisons of these algorithms will be provided in section 5.

4. Error estimates

Now, we begin to establish the convergence theorem of DPPN, which gives us the mathematical guidance on the configuration of H and h. For simplicity of analysis, from now on, we always assume that the results in Theorem 2.2 are valid for $t_0 = 0$.

Lemma 4.1: *Under the assumptions of Theorem* 3.8, *there holds*

$$|\hat{u}_h^n|^2 + k\nu\sum_{i=1}^{n}||\hat{u}_h^i||^2 \le \kappa H^4, \quad 1 \le n \le N.$$

Proof First, from Equation (3.4), we derive

$$\begin{aligned} (d_t u_h^{i+1}, v) + \nu a(u_h^{i+1}, v) + b(u_h^{i+1}, u_h^{i+1}, v) \\ - b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, v) = (f^{i+1}, v), \quad \forall v \in Q_H V_h. \quad (4.1) \end{aligned}$$

Taking $\upsilon = 2k\hat{u}_h^{i+1}$ in Equation (4.1) and noticing Equation (2.4) leads to

$$\begin{aligned} |\hat{u}_h^{i+1}|^2 + |\hat{u}_h^{i+1} - \hat{u}_h^i|^2 - |\hat{u}_h^i|^2 + 2k\nu||\hat{u}_h^{i+1}||^2 \\ + 2kb(u_h^{i+1}, u_h^{i+1}, \hat{u}_h^{i+1}) = 2k(f^{i+1}, \hat{u}_h^{i+1}). \quad (4.2) \end{aligned}$$

With the help of Equation (2.11), we obtain

$$\begin{aligned} &2k|b(u_h^{i+1}, u_h^{i+1}, \hat{u}_h^{i+1})| \\ &\le ck|A_h u_h^{i+1}|^2|\hat{u}_h^{i+1}|_{-1} \\ &\le \frac{k\nu}{3}||\hat{u}_h^{i+1}||^2 + \frac{ck}{\nu}|A_h u_h^{i+1}|^4 H^4, 2k|(f^{i+1}, \hat{u}_h^{i+1})| \\ &\le 2k||f^{i+1}||\ |\hat{u}_h^{i+1}|_{-1} \\ &\le \frac{k\nu}{3}||\hat{u}_h^{i+1}||^2 + \frac{ck}{\nu}||f^{i+1}||^2 H^4. \end{aligned}$$

Summing Equation (4.2) from $i = 0$ to $i = n-1$ and considering above inequalities, we arrive at

$$|\hat{u}_h^n|^2 + kv\sum_{i=1}^{n}||\hat{u}_h^i||^2 \le |q^0|^2 + \frac{ck}{v}\sum_{i=1}^{n}\left(|A_h u_h^i|^4 H^4 + ||f^i||^2 H^4\right).$$

Finally, noticing Theorem 3.8 implies the prescribed result of this lemma. □

For given time step length $k > 0$ and space discretisation scales h,H with $0 < h < H$, we denote

$$t_n = nk, \quad U_h^n = U_h(t_n),$$
$$e_h^n = U_h^n - u_h^n, \quad 0 \le n \le N,$$

Lemma 4.2: *Under the assumptions of Theorem 3.8, there holds*

$$|e_h^n|_r^2 + kv\sum_{i=1}^{n}|e_h^i|_{r+1}^2 \le \kappa(L_h^2 H^{8-2r} + k^2),$$
$$r = 0, 1, \quad 1 \le n \le N.$$

Proof Subtracting Equation (3.4) from Equation (2.3) at $t = t_{i+1}$, for $\forall v \in V_h$, we have

$$(d_t e_h^{i+1}, v) + va(e_h^{i+1}, v) + b(e_h^{i+1}, U_h^{i+1}, v) + b(u_h^{i+1}, e_h^{i+1}, v) + b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, v) = (h^{i+1}, v) \tag{4.3}$$

where $h^{i+1} = \frac{1}{k}\int_{t_i}^{t_{i+1}}(U_{ht}(s) - U_{ht}(t_{i+1}))\mathrm{d}s$.

First, taking $v = 2ke_h^{i+1}$ in Equation (4.3) and noticing Equation (2.4), we obtain

$$|e_h^{i+1}|^2 + |e_h^{i+1} - e_h^i|^2 - |e_h^i|^2 + 2kv||e_h^{i+1}||^2 + 2kb(e_h^{i+1}, U_h^{i+1}, e_h^{i+1}) + 2kb(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, e_h^{i+1}) = 2k(h^{i+1}, e_h^{i+1}). \tag{4.4}$$

By virtue of Equations (2.5), (2.9) and (2.12), there holds

$$2k|b(e_h^{i+1}, U_h^{i+1}, e_h^{i+1})| \le ck||e_h^{i+1}||\,|A_h U_h^{i+1}||e_h^{i+1}| \le \frac{kv}{3}||e_h^{i+1}||^2 + \frac{ck}{v}|A_h U_h^{i+1}|^2|e_h^{i+1}|^2, 2k|b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, e_h^{i+1})| \le ckL_h|\hat{u}_h^{i+1}|\,||\hat{u}_h^{i+1}||\,||e_h^{i+1}|| \le \frac{kv}{3}||e_h^{i+1}||^2 + \frac{ckL_h^2}{v}|\hat{u}_h^{i+1}|^2||\hat{u}_h^{i+1}||^2, 2k|(h^{i+1}, e_h^{i+1})| \le 2k|h^{i+1}|_{-1}||e_h^{i+1}|| \le \frac{kv}{3}||e_h^{i+1}||^2 + \frac{ck}{v}|h^{i+1}|_{-1}^2.$$

Summing Equation (4.4) from $i = 0$ to $i = n-1$ along with above estimates gives

$$e_h^n|^2 + kv\sum_{i=1}^{n}||e_h^i||^2 \le \frac{ck}{v}\sum_{i=1}^{n}(|A_h U_h^i|^2|e_h^i|^2 + L_h|\hat{u}_h^i|^2||\hat{u}_h^i||^2 + |h^i|_{-1}^2). \tag{4.5}$$

Then, applying Gronwall Lemma 3.2 to Equation (4.5) along with Lemma 4.1 and Theorem 2.2 yields the inequality of this lemma with $r = 0$.

Next, taking $v = 2kA_h e_h^{i+1}$ in Equation (4.3), we have

$$||e_h^{i+1}||^2 + ||e_h^{i+1} - e_h^i||^2 - ||e_h^i||^2 + 2kv|A_h e_h^{i+1}|^2 + 2kb(e_h^{i+1}, U_h^{i+1}, A_h e_h^{i+1}) + b(u_h^{i+1}, e_h^{i+1}, A_h e_h^{i+1}) + b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h e_h^{i+1}) = 2k(h^{i+1}, A_h e_h^{i+1}). \tag{4.6}$$

The usage of Equations (2.5), (2.9) and (2.12) suggests

$$2k|b(e_h^{i+1}, U_h^{i+1}, A_h e_h^{i+1})| \le ck||e_h^{i+1}||\,|A_h U_h^{i+1}||A_h e_h^{i+1}| \le \frac{kv}{4}|A_h e_h^{i+1}|^2 + \frac{ck}{v}|A_h U_h^{i+1}|^2||e_h^{i+1}||^2, 2k|b(u_h^{i+1}, e_h^{i+1}, A_h e_h^{i+1})| \le ck|A_h u_h^{i+1}|\,||e_h^{i+1}||\,|A_h e_h^{i+1}| \le \frac{kv}{4}|A_h e_h^{i+1}|^2 + \frac{ck}{v}|A_h u_h^{i+1}|^2||e_h^{i+1}||^2, 2k|b(\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h e_h^{i+1})| \le ckL_h||\hat{u}_h^{i+1}||^2|A_h e_h^{i+1}| \le \frac{kv}{4}|A_h e_h^{i+1}|^2 + \frac{ckL_h^2}{v}||\hat{u}_h^{i+1}||^4, 2k|(h^{i+1}, A_h e_h^{i+1})| \le 2k|h^{i+1}||A_h e_h^{i+1}| \le \frac{kv}{4}|A_h e_h^{i+1}|^2 + \frac{ck}{v}|h^{i+1}|^2.$$

Summing Equation (4.6) from $i = 0$ to $i = n-1$ together with above inequalities yields

$$||e_h^n||^2 + kv\sum_{i=1}^{n}|A_h e_h^i|^2 \le \frac{ck}{v}\sum_{i=1}^{n}((|A_h U_h^i|^2 + |A_h u_h^i|^2)||e_h^i||^2 + L_h^2||\hat{u}_h^i||^4 + |h^i|^2). \tag{4.7}$$

Hence, using Gronwall Lemma 3.2 to Equation (4.7), noticing Lemma 4.1 and Theorem 2.2, we obtain the inequality of this lemma with $r = 1$. □

Lemma 4.3: *Under the assumptions of Lemma* 4.2, *we obtain*

$$|d_t e_h^n|_{-1}^2 + kv\sum_{i=1}^{n}|d_t e_h^i|^2 \le \kappa(L_h^2 H^6 + k^2), \quad 1 \le n \le N.$$

Proof First, we deduce from Equation (4.3) that

$$\begin{aligned}&(d_{tt}e_h^{i+1}, v) + va(d_t e_h^{i+1}, v) + b(d_t e_h^{i+1}, U_h^{i+1}, v)\\&\quad + b(e_h^i, d_t U_h^{i+1}, v) + b(d_t u_h^{i+1}, e_h^{i+1}, v) + b(u_h^i, d_t e_h^{i+1}, v)\\&\quad + b(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, v) + b(\hat{u}_h^i, d_t\hat{u}_h^{i+1}, v)\\&\qquad = (d_t h^{i+1}, v), \quad \forall v \in V_h.\end{aligned} \tag{4.8}$$

Taking $v = 2kA_h^{-1}\, d_t e_h{}^{i+1}$ in Equation (4.8) leads to

$$\begin{aligned}&|d_t e_h^{i+1}|_{-1}^2 + |d_t e_h^{i+1} - d_t e_h^i|_{-1}^2 - |d_t e_h^i|_{-1}^2\\&\quad + 2kv|d_t e_h^{i+1}|^2 + 2kb(d_t e_h^{i+1}, U_h^{i+1}, A_h^{-1}d_t e_h^{i+1})\\&\quad + 2kb(e_h^i, d_t U_h^{i+1}, A_h^{-1}d_t e_h^{i+1})\\&\quad + 2kb(d_t u_h^{i+1}, e_h^{i+1}, A_h^{-1}d_t e_h^{i+1})\\&\quad + 2kb(u_h^i, d_t e_h^{i+1}, A_h^{-1}d_t e_h^{i+1})\\&\quad + 2kb(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h^{-1}d_t e_h^{i+1})\\&\quad + 2kb(\hat{u}_h^i, d_t\hat{u}_h^{i+1}, A_h^{-1}d_t e_h^{i+1}) = (d_t h^{i+1}, A_h^{-1}d_t e_h^{i+1}).\end{aligned}$$

The application of Equations (2.4) and (2.9) gives

$$\begin{aligned}&2k|b(d_t e_h^{i+1}, U_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le ck|d_t e_h^{i+1}||A_h U_h^{i+1}||d_t e_h^{i+1}|_{-1} \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}|A_h U_h^{i+1}|^2|d_t e_h^{i+1}|_{-1}^2, 2k|b(e_h^i, d_t U_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le ck||e_h^i|||d_t e_h^{i+1}||d_t U_h^{i+1}| \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}|d_t U_h^{i+1}|^2||e_h^i||^2, 2k|b(d_t u_h^{i+1}, e_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le ck|d_t u_h^{i+1}||\,|e_h^{i+1}||\,|d_t e_h^{i+1}| \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}|d_t u_h^{i+1}|^2||e_h^{i+1}||^2, 2k|b(u_h^i, d_t e_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le ck|A_h u_h^i||d_t e_h^{i+1}|_{-1}|d_t e_h^{i+1}| \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}|A_h u_h^i|^2|d_t e_h^{i+1}|_{-1}, 2k|b(d_t\hat{u}_h^{i+1}, \hat{u}_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le ck|d_t\hat{u}_h^{i+1}|||u_h^{i+1}||\,|d_t e_h^{i+1}| \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}||\hat{u}_h^{i+1}||^2|d_t\hat{u}_h^{i+1}|^2, 2k|b(\hat{u}^i, d_t\hat{u}_h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le c||\hat{u}_h^i||\,|d_t e_h^{i+1}||d_t\hat{u}_h^{i+1}| \le \frac{kv}{7}|d_t e_h^{i+1}|^2\\&\quad + \frac{ck}{v}||\hat{u}_h^i||^2|d_t\hat{u}_h^{i+1}|^2, 2k|(d_t h^{i+1}, A_h^{-1}d_t e_h^{i+1})|\\&\quad \le 2k|d_t e_h^{i+1}||d_t h^{i+1}|_{-2} \le \frac{kv}{7}|d_t e_h^{i+1}|^2 + \frac{ck}{v}|d_t h^{i+1}|_{-2}^2.\end{aligned}$$

Therefore, summing Equation (4.9) from $i = 0$ to $i = n-1$ and using above estimates, we obtain

$$\begin{aligned}&|d_t e_h^n|_{-1}^2 + kv\sum_{i=1}^{n}|d_t e_h^i|^2 \le |d_t e_h^0|_{-1}^2 + k\sum_{i=1}^{n}d_i|d_t e_h^i|_{-1}^2\\&\quad + \frac{ck}{v}\sum_{i=1}^{n}(||e_h^{n-1}||^2|d_t U_h^i|^2 + ||e_h^i||^2|d_t u_h^i|^2)\\&\quad + \frac{ck}{v}\sum_{i=1}^{n}(|d_t\hat{u}_h^i|^2(||\hat{u}_h^i||^2 + ||\hat{u}_h^{n-1}||^2) + |d_t h^i|_{-2}^2),\end{aligned} \tag{4.9}$$

where $d_i = cv^{-1}(|A_h U_h^i|^2 + |A_h^i u_h^i|^2$.

Finally, applying Gronwall Lemma 3.2 to Equation (4.9) together with Lemma 4.1, Lemma 4.2, Theorem 3.8 and Theorem 2.2 suggests the results of this lemma. □

We subtract Equation (3.2) from (2.2) at $t = t_n$ for $0 \le n \le N$, obtaining

$$\begin{aligned}&d(v, p_h(t_n) - p_h^n) = (d_t e_h^n, v) + va(e_h^n, v)\\&\quad + b(e_h^{n+1}, U_h^{n+1}, v) + b(u_h^{n+1}, e_h^{n+1}, v) + b(\hat{u}_h^{n+1}, \hat{u}_h^{n+1}, v)\\&\quad - (h^n, v_h), \quad \forall v \in X_h.\end{aligned}$$

Also, thanks to the inf–sup condition (A3) and (2.7), one finds

$$\begin{aligned}&|p_h(t_n) - p_h^n| \le \beta^{-1}(|d_t e_h^n|_{-1} + v||e_h^n|| + c||e_h^n||(||U_h^n||\\&\quad + ||u_h^n||) + cL_h|\hat{u}_h^n|\;||\hat{u}_h^n|| + |h^n|_{-1}).\end{aligned}$$

Therefore, applying Lemma 4.2, Lemma 4.3 in above inequality together with Theorem 2.1 allows us to conclude the following estimates.

Theorem 4.4: *Under the assumptions of Theorem* 3.8, *we have*

$$\begin{aligned}&|u(t_n) - u_h^n| \le \kappa(L_h H^4 + k) + C(t_n)h^2, \quad 1 \le n \le N,\\&||u(t_n) - u_h^n|| + |p(t_n) - p_h^n| \le \kappa(L_h H^3 + k)\\&\quad + C(t_n)h, \quad 1 \le n \le N.\end{aligned}$$

Remark 4.1. The results of Theorem 4.4 implies that, to make the DPPN reach the same accuracy as the SGM with a fine mesh size h, we should choose

$$L_h H^4 \sim D(v)h^2, \quad L_h H^3 \sim D(v)h,$$

where $D(v) = C(t_n)/\kappa$ can be seen as the functions of v. A careful observation will allow us to find that κ has an exponential factor with respect to $1/v$ owing to the applications of Gronwall Lemma in above error estimates. Therefore, $D(v)$ will increase rapidly when v decreases, we should choose a smaller H to keep above configurations for a fixed h.

5. Numerical examples

This section gives some numerical examples to confirm the theoretical analysis in previous sections. All the numerical experiments are carried out by using the Taylor-Hood element. That is, the Lagrange quadratic elements are used to approximate the velocity and line elements to approximate the pressure. For all the implicit time stepping, we use Newton iterative method with tolerance 10^{-6}. And we use GMRES iterative method to solve the linear algebraic equations arising at every time step with tolerance 10^{-9}.

Example 1. The first example to be considered here is to confirm the error estimates, given in previous sections, are also obtained numerically. In this example, we choose domain $\Omega = [0, 1]^2$ and a time interval $[0, T]$ with $T = 1$. In the following calculations, we divide the domain into triangles, which are induced by the set of nodes $(i/M, j/M)$, $0 \le i, j \le M$, where $M = |\Omega|/h$ is a positive integer. For convenience of computing the error, we give an exact solution, then obtain the forcing term f. We choose the exact solution as follows

$$\begin{cases} u_1(x,y,t) = 10(1-x)^2x^2(1-2y)(1-y)y\tanh(t), \\ u_2(x,y,t) = -10(1-2x)(1-x)x(1-y)^2y^2\tanh(t), \\ p(x,y) = 10(1-2x)(1-2y). \end{cases}$$

A simple calculation can verify that $\nabla\cdot u = 0$ and $p \in L_0^2(\Omega)$.

Table 1 shows the relative errors, convergence rates and CPU time of SGM (the fully implicit one-level method) with different space discretisation scale h. Table 2 presents the performances of DPPN with various h and H. The numerical results presented in these two tables are obtained when $\nu = 0.01$. Moreover, we are careful to choose the time step length $k = 10^{-4}$ which ensures the dominant error in the computations is the spatial discretisation error and satisfy the stability condition (3.11). DPPN first solves a non-linear equation on a coarse-level subspace, then to get the final approximate solution by solving a linear equation in the fine-level subspace which will involve much less work compared with SGM with the fine mesh scale h.

In Figures 1–3, we compare the errors of the fully implicit one-level method, semi-implicit one-level method, PP, DPP and DPPN with a larger time step length $k = 0.05$ when $\nu = 10^{-2}, 10^{-3}, 10^{-4}$ at the final time instant $t = 1$. For fully implicit one-level method and semi-implicit one-level method, we alter the mesh size $h = 1/M$, $M = 2, 4, 8, 16, 24, 32$. For PP, DPP and DPPN, we fix the finest scale $h = 1/32$, then to vary the coarse mesh size $H = 1/M$, $M = 2, 4, 8, 16, 24, 32$. We find that the fully implicit one-level method presents very good approximations whereas the semi-implicit one-level method can't reach so good approximations compared with other schemes. On the other hand, PP, DPP and DPPN can reach the same accuracy as the fully implicit one-level method with a fine mesh size h for an appropriate coarse mesh scale H for large viscosity cases. However, for further small viscosity cases, for example $\nu \le 10^{-3}$, the PP and DPP can't get the optimal accuracy except $H = h$. These results verify our presumption in the introduction that PP, DPP are only efficient for the large viscosity cases. On the contrary, the DPPN can keep the good

Table 1. Convergent rates of standard Galerkin method (SGM).

h	L^2 error	rate	H^1 error	rate	L^2 pressure	rate	CPU time
1/4	0.038172		0.164771		0.048414		378.2s
1/8	0.004840	2.979	0.044217	1.898	0.012103	2.000	1403.8s
1/16	0.000603	3.005	0.011316	1.966	0.003026	2.000	6062.3s
1/32	7.6332e-05	2.982	0.002849	1.990	0.000756	2.001	32347.7s

Table 2. Convergent rates of the dynamical postprocessing method of Newton type (DPPN).

H	h	L^2 error	rate	H^1 error	rate	L^2 pressure	rate	CPU time
1/2	1/4	0.038207		0.164771		0.0484136		327.9s
1/4	1/8	0.004841	2.981	0.044217	1.898	0.0121031	2.000	1041.3s
1/8	1/16	0.000604	3.003	0.011316	1.966	0.003026	2.000	4312.4s
1/16	1/32	7.6319e-05	2.984	0.002848	1.990	0.000756442	2.000	17211.4s

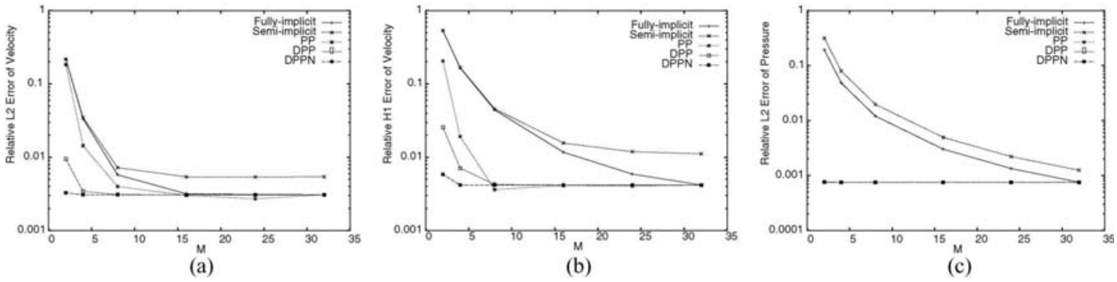

Figure 1. (a) L^2 error of velocity, (b) H^1 of velocity, (c) L^2 error of pressure $v = 10^{-2}$.

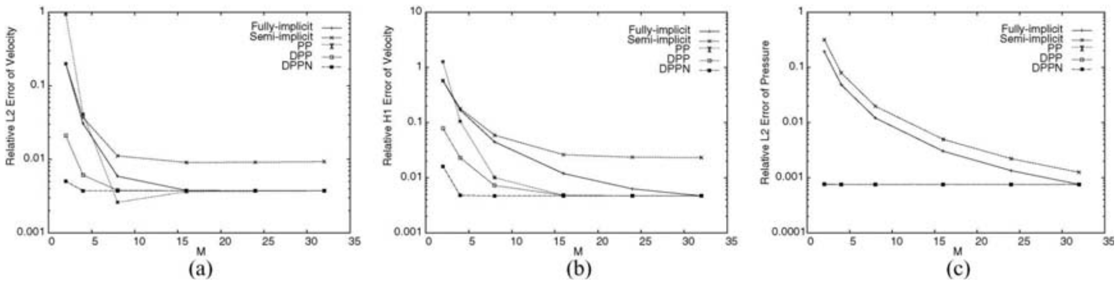

Figure 2. (a) L^2 error of velocity, (b) H^1 of velocity, (c) L^2 error of pressure $v = 10^{-3}$.

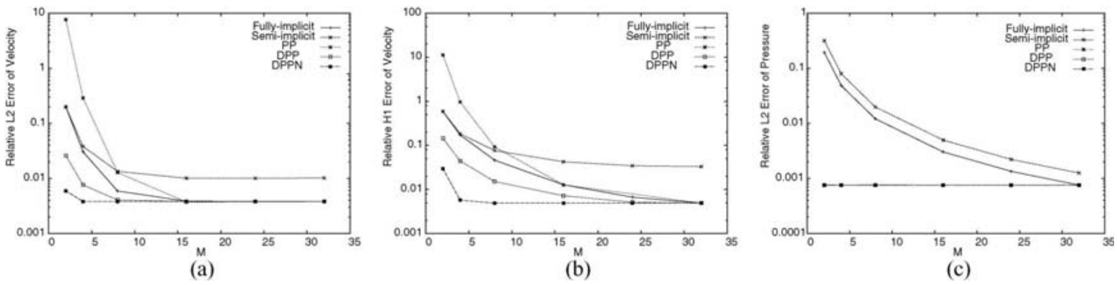

Figure 3. (a) L^2 error of velocity, (b) H^1 of velocity, (c) L^2 error of pressure $v = 10^{-4}$.

accuracy for the small viscosity cases. Moreover, we choose a finer H to reach the prescribed accuracy with the decrease of viscosity coefficient v, which is in good agreement with the analysis in *Remark* 4.1.

Example 2. The second example to be considered here is the flow around a circular cylinder. The flow around a cylinder has been widely studied from the theoretical, experimental and numerical point of view, e.g., see Van Dyke (1982) for the experimental results, Franca and Nesliturk (2001) employed the finite-dimensional spaces consisting of piecewise polynomials enriched with residual-free bubble function, Ding (2003) for a dynamic mesh method, Wang *et al*. (2007) for the computation of leading eigenvalues and eigenvectors. At upper and lower computational boundaries and at the inflow section, a uniform free-stream velocity boundary condition is imposed. The traction-free condition is imposed at the outflow boundary. The geometry and the boundary conditions are shown in Figure 4. Figure 5 states the coarse and fine meshes distributions employed in the simulations. We choose the time step length $k = 0.05$ in this example.

The cylinder problem can be characterised by the Reynolds number which is based on the free-stream velocity and the cylinder diameter d, i.e. $Re = (u_\infty d)/v$. For Reynolds numbers less than or equal to 40, the flow is steady, causing a wake behind the cylinder to develop. For $Re = 0.01$ shown in Figure 6(a), the flow is symmetrical upstream and downstream, the right

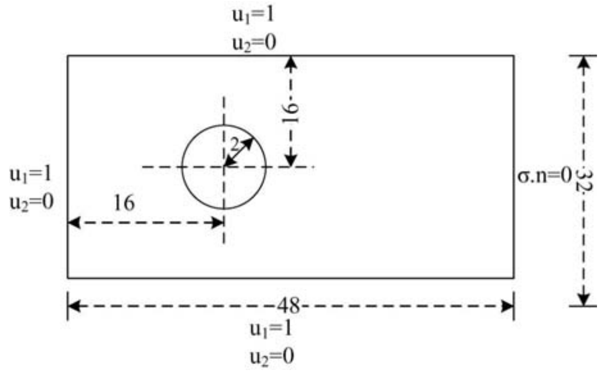

Figure 4. The statement of the flow around a cylinder.

hand of Figure 6(a) is the mirror image of the left hand. As Reynolds number increases, the upstream–downstream symmetry disappears. When Reynolds number exceeds about 4, two attached eddies appear behind the cylinder which will become bigger with increasing Reynolds number. Figure 6 shows these features and Figure 7 presents the pressure contours when flow reaches its final steady state at $Re = 0.01$, 10, 20, 40. Moreover, some comparisons of the wake length L which is from the rearmost point of the cylinder to the end of the wake and the separation angle θ with some previous simulations are listed in Table 3.

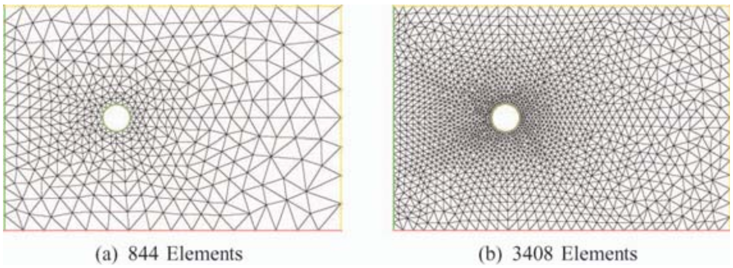

Figure 5. The coarse and fine meshes tested.

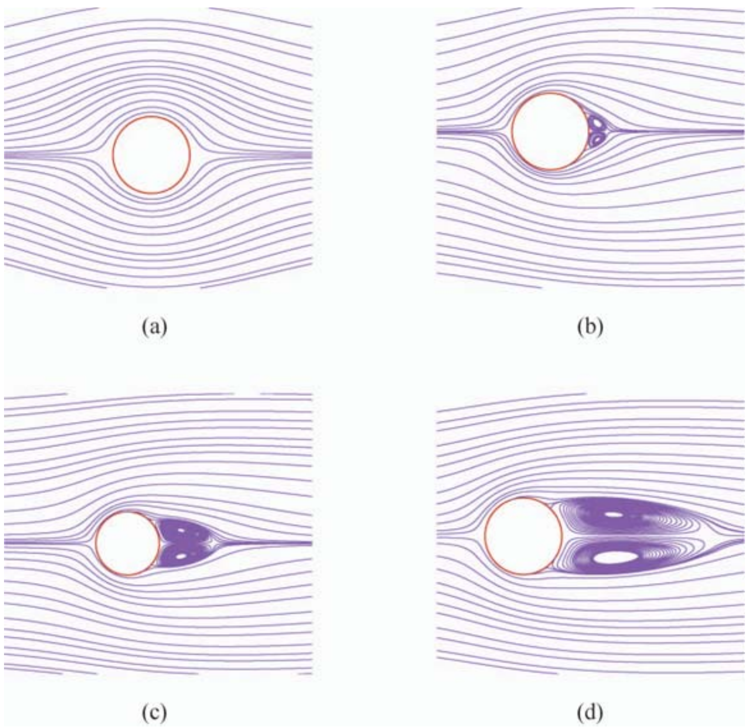

Figure 6. Streamlines. (a) $Re = 0.01$, (b) $Re = 10$, (c) $Re = 20$, (d) $Re = 40$.

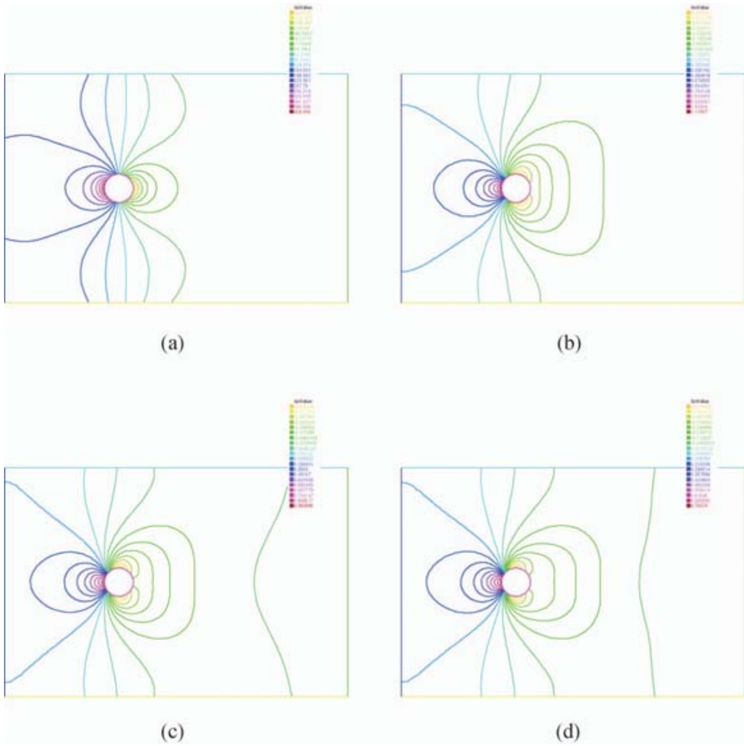

Figure 7. Pressure contours. (a) $Re = 0.01$, (b) $Re = 10$, (c) $Re = 20$, (d) $Re = 40$.

Table 3. Recirculation length and detachment angle.

Re	Method	L/r	θ
	Present work	0.52	25.8
	Dennis and Chang (1970)	0.53	29.6
	Nieuwstadt and Keller (1973)	0.434	27.96
	Coutanceau and Bouard (1977)	0.68	32.5
	He and Doolen (1997)	0.474	26.89
10	Rome *et al.* (2007)	0.549	25.47
	Present work	1.945	43.63
	Dennis and Chang (1970)	1.88	43.7
	Nieuwstadt and Keller (1973)	1.786	43.37
	Coutanceau and Bouard (1977)	1.86	44.8
	He and Doolen (1997)	1.842	42.96
20	Rome *et al.* (2007)	2.16	41.98
	Present work	4.55	53.13
	Dennis and Chang (1970)	4.69	53.8
	Nieuwstadt and Keller (1973)	4.357	53.34
	Coutanceau and Bouard (1977)	4.26	53.5
	He and Doolen (1997)	4.49	52.84
40	Rome *et al.* (2007)	4.94	50.99

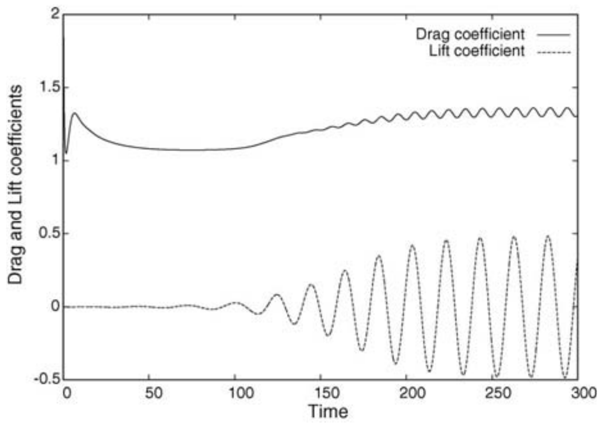

Figure 8. Drag and lift coefficients at $Re = 160$.

We also consider another frame so that the flow is unsteady for $40 < Re < 190$, leading to periodic vortex shedding, known as the Karman vortex street. The eddy on one side is being shed while on the other side it is reforming. Figure 8 shows the temporal development of the drag and lift coefficients. Figure 9 shows the streamlines around the cylinder during a period for $Re = 160$. From Figure 8, we get the shedding period $T_p = 20.5$ and obtain the Strouhal number $St = d/(u_\infty T_p)$. Table 4 states the Strouhal number comparisons obtained by DPPN with some other results.

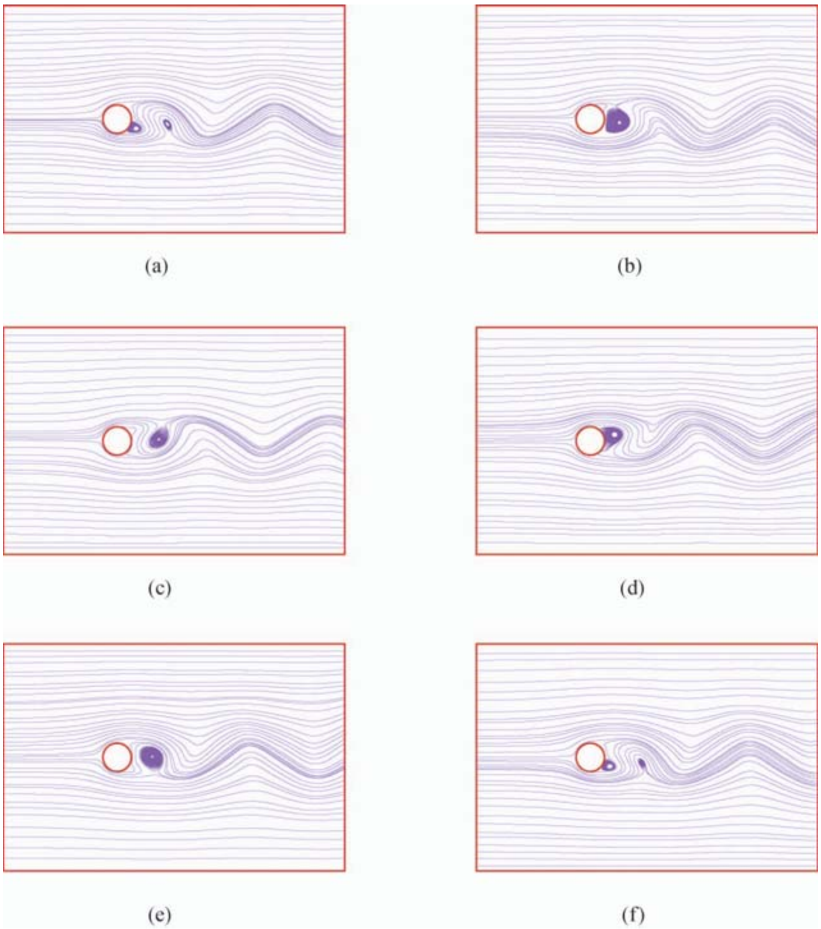

Figure 9. Streamline of the flow around a cylinder during a period at $Re = 160$.

Table 4. Some Strouhal number comparisons at $Re = 160$.

Method	St
Present work	0.195
Williamson and Brown (1998)	0.188
Rome *et al*. (2007)	0.192

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