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Local and parallel finite element algorithms for time-dependent convection-diffusion equations *

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Abstract Local and parallel finite element algorithms based on two-grid discretization for the time-dependent convection-diffusion equations are presented. These algorithms are motivated by the observation that, for a solution to the convection-diffusion problem, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedures. Hence, these local and parallel algorithms only involve one small original problem on the coarse mesh and some correction problems on the local fine grid. One technical tool for the analysis is the local a priori estimates that are also obtained. Some numerical examples are given to support our theoretical analysis.

Key words local and parallel algorithms, finite element method, convection-diffusion equations

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Introduction

Nowadays, much researches have focused on the parallel finite element computations in modern science and engineering. Recently, a new local and parallel approach has been proposed for a class of linear and nonlinear elliptic boundary value problems by Xu and Zhou[1-3]. Then this local and parallel method was further investigated by He et al.[4-5] and Ma et al.[6] for the steady Stokes equations and Navier-Stokes equations. These local and parallel algorithms are based on the understanding that the global behavior of a solution is mostly governed by the lower frequency components while the local behavior is mostly governed by the higher frequency components. Hence, the main idea is to use a coarse grid to approximate the lower frequency components, and then use a local fine grid to correct the residual by some local and parallel procedures. In fact, this idea is based on the two-grid strategy first introduced by Xu[7-8].

In this article, we extend this local and parallel algorithm to solve the time-dependent convection-diffusion problems. This paper can be considered as a sequel to the papers of Xu and Zhou[1-3], He et al.[4-5], and Ma et al.[6]. To our knowledge, there are few articles applying the local and parallel method to the time-dependent problems. Due to the long time march, the

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parallel algorithms for the time-dependent problems seem promising in enhancing the efficiency of computations.

This paper is organized as follows. Section 1 gives some notations and assumptions. The local and parallel algorithms and error estimates are provided in Section 2. Section 3 presents some numerical examples to support our theoretical analysis.

1 Notations and preliminaries

We consider the time-dependent convection-diffusion equation

$$\begin{cases} u_t - \Delta u + b \cdot \nabla u = f, \quad \forall (x,t) \in \Omega \times [0,T], \\ u(x,0) = u_0, \quad \forall x \in \Omega, \\ u = 0, \quad \forall (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{1}$$

Here, Ω is a bounded domain in R^d $(d = 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$, $u(x,t)$ is a scalar function, the function $b \in L^\infty(0,T; L^\infty(\Omega)^d)$, f is a forcing function, and T represents a finite time.

Throughout this work we use the notation $X = H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$. X is equipped with the inner products $((u,v)) = (\nabla u \cdot \nabla v)$ and the norms $\|u\| = ((u,u))^{\frac{1}{2}}$, where $\forall u, v \in X$ and $(\cdot,\cdot)$ is the usual L^2 inner product.

Let h be a real positive parameter. T_h, which is a partition of Ω into K (a triangle or a quadrilateral), is assumed to be uniformly regular as h tends to 0. The finite element subspace $X^h \subset X$ is characterized by this partition. Let $P_h : X \to X^h$ denote the L^2 orthogonal projection defined by $(P_h v, v_h) = (v, v_h), \forall v \in X, v_h \in X^h$. Throughout this paper, we use $\|\cdot\|_\alpha$ to denote the $H^\alpha(\Omega)$ norm for $\forall \alpha \in \mathbf{R}$. Moreover, in the rest of this paper, we always use c to denote a generic positive constant independent of the partition which may be different at different occurrences.

Let k be the time step length and $N = [T/k]$. Then, the standard Galerkin discretization method for solving (1) reads: $u^{h,0} = P_h u_0$, for all $1 \le n \le \mathbf{N}$. Find $u^{h,n} \in X^h$ such that

$$(u^{h,n}, v) + ka(u^{h,n}, v) = (u^{h,n-1}, v) + k(f^n, v), \quad \forall v \in X^h, \tag{2}$$

where

$$a(u,v) = a_0(u,v) + N(u,v), \quad a_0(u,v) = (\nabla u, \nabla v), \quad N(u,v) = (b \cdot \nabla u, v), \quad \forall u, v \in X.$$

Assume that $f(t) \in X' = H^{-1}(\Omega)$ and $u(t) \in H^2(\Omega)$ for all $t \in [0,T]$. Then we have

$$\|u(t_n) - u^{h,n}\|_{0,\Omega} \le c(k + h^2), \quad 1 \le n \le N. \tag{3}$$

Now we define a discrete analogue A_h of the Laplace operator by $A_h = -P_h \Delta_h$ through the condition that $(-\Delta_h u_h, v_h) = ((u_h, v_h))$, $\forall u_h, v_h \in X^h$. In fact, the restriction of A_h to X^h is invertible, with the inverse denoted by A_h^{-1}. Since A_h^{-1} is self-adjoint and positive definite, we define the "discrete" Sobolev norms on X^h of any order $r \in \mathbf{R}$ by $\|v_h\|_r = \|A_h^{r/2} v_h\|_{L^2}$, $\forall v_h \in X^h$. These discrete norms are assumed to have various properties similar to their continuous counterparts. In particular,

$$\|v_h\|_0 = \|v_h\|_{L^2}, \quad \|v_h\|_1 = \|v_h\|_{H_0^1}, \quad \|v_h\|_2 = \|A_h v_h\|_{L^2}, \quad \forall v_h \in X^h.$$

The mesh size $h(x)$ is the diameter h_τ of the element τ containing x. A basic assumption on the mesh is that it is not exceedingly over-refined locally, namely,

A.0 There exists $\gamma \ge 1$ such that

$$h_\Omega^\gamma \le h(x), \quad \forall x \in \Omega,$$

where $h_\Omega = \max_{x\in\Omega} h(x)$. We will drop the subscript in h_Ω, and write h for the mesh size on a domain that is clear in the context.

Associated with a mesh partition $T_h(\Omega)$, $X^h(\Omega) \subset H^1(\Omega)$ is assumed to be a finite dimensional subspace on Ω and $X_0^h(\Omega) = X^h(\Omega) \cap H_0^1(\Omega)$. Given $\Omega_0 \subset \Omega$, we define $X^h(\Omega_0)$ and $T_h(\Omega_0)$ to be the restriction of $X^h(\Omega)$ and $T_h(\Omega)$ to Ω_0, and $X_0^h(\Omega_0) = \{v \in X^h(\Omega), \text{supp } v \subset\subset \Omega_0\}$.

We now state some basic assumptions on the finite element spaces.

A.1 Approximation

There exists $r \geq 1$ such that, for all $w \in H_0^1(\Omega)$,

$$\inf_{v\in X_0^h(\Omega)} (\|w - v\|_{0,\Omega} + h\|w - v\|_{1,\Omega}) \leq ch^{s+1}\|w\|_{1+s,\Omega}, \quad 0 \leq s \leq r.$$

A.2 Inverse estimates

For any $v \in X^h(\Omega_0)$, there holds

$$\|v\|_{1,\Omega_0} \leq ch^{-1}\|v\|_{0,\Omega_0}, \quad \|v\|_{0,\Omega_0} \leq ch^{-1}\|v\|_{-1,\Omega_0}.$$

A.3 Super approximation

For $G \subset\subset \Omega$, let $\omega \in C_0^\infty(\Omega)$ with supp $\omega \subset\subset G$. Then for any $w \in X^h(G)$, there exists $v \in X_0^h(G)$ such that

$$\|\omega w - v\|_{1,G} \leq ch\|w\|_{1,G}.$$

2 Local and parallel algorithms

In this section, we first present a new local algorithm and local a priori error estimates for the time-dependent convection-diffusion equations, which will play a crucial role in our analysis. Based on the local algorithm, we construct a parallel algorithm and derive the error estimates.

There are two positive mesh grid scales H and h with $0 \leq h \leq H \leq 1$. The local algorithm, which obtains an approximate solution on a given subdomain mostly by local computations, can be described as follows.

2.1 Local algorithm

(i) Find a global coarse grid solution $u_H^n \in X_0^H(\Omega)$ such that

$$(u_H^n, v) + ka(u_H^n, v) = (u_H^{n-1}, v) + k(f^n, v), \quad \forall v \in X_0^H(\Omega). \tag{4}$$

(ii) Find a local fine grid correction $e_h^n \in X_0^h(\Omega_0)$ such that

$$(e_h^n, v) + ka(e_h^n, v) = (e_h^{n-1}, v) + k(f^n, v) - ((u_H^n - u_H^{n-1}), v) - ka(u_H^n, v), \quad \forall v \in X_0^h(\Omega_0). \tag{5}$$

(iii) Correction is completed as

$$u_h^n|_D = u_H^n + e_h^n,$$

where $1 \leq n \leq N$, $D \subset\subset \Omega_0$ (see Fig. 1).

First, let us show a discrete Gronwall lemma provided by Heywood and Rannacher[9].

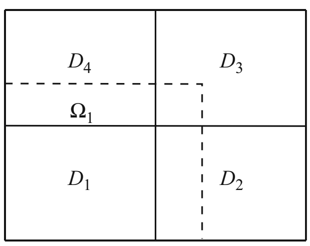

Fig. 1 Domain decomposition

Lemma 2.1 *Let k, B, a_n, b_n, c_n, and d_n for the integer $n \ge 1$ be nonnegative numbers such that*

$$a_m + k\sum_{n=1}^{m} b_n \le k\sum_{n=1}^{m} d_n a_n + k\sum_{n=1}^{m} c_n + B, \quad \forall m \ge 1.$$

Suppose that $kd_n < 1$ for all n and set $\gamma_n = (1 - kd_n)^{-1}$. Then,

$$a_m + k\sum_{n=1}^{m} b_n \le \exp\left(k\sum_{n=1}^{m}\gamma_n d_n\right)\left(k\sum_{n=1}^{m} c_n + B\right), \quad \forall m \ge 1.$$

Before presenting the local a priori error estimates, we recall the following technical results presented by Xu and Zhou[1].

Lemma 2.2 *Let $D \subset\subset \Omega_0$ and $\omega \in C_0^\infty(\Omega)$ such that* supp $\omega \subset\subset \Omega_0$. *Then we have*

$$a_0(\omega w, \omega w) \le 2a(w, \omega^2 w) + c\|w\|_{0,\Omega_0}^2, \quad \forall v \in H_0^1(\Omega). \tag{6}$$

Lemma 2.3 *Suppose that $f \in L^\infty(0,T;X')$ and $D \subset\subset \Omega_0 \subset\subset \Omega$. Furthermore, if Assumptions* A.1–A.3 *hold, $k \le ch_{\Omega_0}^2$, and $w \in X^h(\Omega_0)$ satisfies*

$$(w, v) + ka(w, v) = (f, v), \quad \forall v \in X_0^h(\Omega_0), \tag{7}$$

then

$$\|w\|_{0,D} + k^{1/2}\|w\|_{1,D} \le c(\|w\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}), \tag{8}$$

where $\|f\|_{-1,\Omega_0} = \|f\|_{L^\infty(0,T;X')}$.

Proof Let p be an integer such that $p \ge 2\gamma - 1$ and $\Omega_j (j = 1, 2, \cdots, p)$ satisfy

$$D \subset\subset \Omega_p \subset\subset \Omega_{p-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Then we can choose a $D_1 \subset\subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_p$ and $\omega \in C_0^\infty(\Omega)$ such that $\omega \equiv 1$ on D_1 and supp $\omega \subset\subset \Omega_p$. From Assumption A.3, there exists $v \in X_0^h(\Omega_p)$ such that

$$\|\omega^2 w - v\|_{1,\Omega_p} \le ch_{\Omega_0}\|w\|_{1,\Omega_p},$$

which implies

$$a(w, \omega^2 w - v) \le ch_{\Omega_0}\|w\|_{1,\Omega_p}^2$$

and

$$|(f, v)| \le \|f|_{-1,\Omega_0}\|v\|_{1,\Omega_p} \le c\|f\|_{-1,\Omega_0}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega_p}).$$

Since $v \in X_0^h(\Omega_p) \subset X_0^h(\Omega_0)$, (7) implies

$$(w, \omega^2 w) + ka(w, \omega^2 w) = (w, \omega^2 w - v) + ka(w, \omega^2 w - v) + (f, v).$$

Hence, combining (6) with the above equality, we have

$$\begin{aligned}\|\omega w\|_{0,\Omega}^2 + k\|\omega w\|_{1,\Omega}^2 \le\; & ch_{\Omega_0}\|w\|_{0,\Omega_p}^2 + ckh_{\Omega_0}\|w\|_{1,\Omega_p}^2 + ck\|w\|_{0,\Omega_0}^2 \\ & + c\|f\|_{-1,\Omega_p}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega_p})\end{aligned}$$

or

$$\|w\|_{0,D} + k^{1/2}\|w\|_{1,D} \le ch_{\Omega_0}^{1/2}\|w\|_{0,\Omega_p}^2 + ck^{1/2}h_{\Omega_0}^{1/2}\|w\|_{1,\Omega_p} + ck^{1/2}\|w\|_{0,\Omega_0} + c\|f\|_{-1,\Omega_0}. \tag{9}$$

The similar arguments may be repeated for $\|w\|_{0,\Omega_p} + k^{1/2}\|w\|_{1,\Omega_p}$ on the right to yield

$$\begin{aligned}&\|w\|_{0,\Omega_j} + k^{1/2}\|w\|_{1,\Omega_j}\\ \le\,& ch_{\Omega_0}^{1/2}\|w\|_{0,\Omega_{j-1}} + ck^{1/2}h_{\Omega_0}^{1/2}\|w\|_{1,\Omega_{j-1}} + ck^{1/2}\|w\|_{0,\Omega_0} + c\|f\|_{-1,\Omega_0}, \quad j = 1, 2\cdots, p. \quad (10)\end{aligned}$$

Combining (9) with (10) and noticing Assumption A.2 and the condition of k in this lemma, we have

$$\begin{aligned}\|w\|_{0,D} + k^{1/2}\|w\|_{1,D} &\le ch_{\Omega_0}^{(p+1)/2}\|w\|_{0,\Omega_0} + ck^{1/2}h_{\Omega_0}^{(p+1)/2}\|w\|_{1,\Omega_0} + ck^{1/2}\|w\|_{0,\Omega_0} + c\|f\|_{-1,\Omega_0}\\ &\le c(\|w\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}).\end{aligned}$$

This ends the proof of the lemma.

Thanks to the classical two-grid method, we have the following estimates which are derived similarly to the procedure in [7–8].

Lemma 2.4 *Assume that $u^{h,n}$ and u_H^n are obtained by* (2) *and* (4), *respectively. We have*

$$\|u^{h,m} - u_H^m\|_{-1,\Omega}^2 + k\sum_{i=1}^{m}\|u^{h,n} - u_H^n\|_{0,\Omega}^2 + k\sum_{i=1}^{m}\|d_t(u^{h,n} - u_H^n)\|_{-2,\Omega}^2 \le cH^6, \quad 1 \le m \le N,$$

where $d_t(u^{h,n} - u_H^n) = [u^{h,n} - u_H^n - (u^{h,n-1} - u_H^{n-1})]/k$.

Theorem 2.1 *Under the conditions of Lemma* 2.3, *assume that $u_h^n \in X^h(\Omega_0)$ is obtained by the local algorithm* (4)–(5). *Then*

$$\|u^{h,n} - u_h^n\|_{0,D} \le cH^3, \quad 1 \le n \le N. \quad (11)$$

Proof We derive from (2) and the local algorithm (4)–(5) that

$$(u^{h,n} - u_h^n, v) + ka(u^{h,n} - u_h^n, v) = (u^{h,n-1} - u_h^{n-1}, v), \quad \forall v \in X^h(\Omega_0).$$

From Lemma 2.3, we obtain

$$\begin{aligned}&\|u^{h,n} - u_h^n\|_{0,D} + k^{1/2}\|u^{h,n} - u_h^n\|_{1,D}\\ \le\,& c(\|u^{h,n} - u_h^n\|_{-1,\Omega_0} + \|u^{h,n-1} - u_h^{n-1}\|_{-1,\Omega_0})\\ \le\,& c(\|u^{h,n} - u_H^n\|_{-1,\Omega_0} + \|u^{h,n-1} - u_H^{n-1}\|_{-1,\Omega_0} + \|e_h^n\|_{-1,\Omega_0} + \|e_h^{n-1}\|_{-1,\Omega_0}). \quad (12)\end{aligned}$$

Hence, the following work is to estimate $\|e_h^n\|_{-1,\Omega_0}$ for all $1 \le n \le N$.

Taking $v = 2A_h^{-1}e_h^n$ in (5), we have

$$\begin{aligned}&\|e_h^n\|_{-1,\Omega_0}^2 + \|e_h^n - e_h^{n-1}\|_{-1,\Omega_0}^2 - \|e_h^{n-1}\|_{-1,\Omega_0}^2 + 2k\|e_h^n\|_{0,\Omega_0}^2\\ =& -2kN(e_h^n, A_h^{-1}e_h^n) + 2((u^{h,n} - u_H^n) - (u^{h,n-1} - u_H^{n-1}), A_h^{-1}e_h^n)\\ &+ 2ka(u^{h,n} - u_H^n, A_h^{-1}e_h^n). \quad (13)\end{aligned}$$

Thanks to the Holder inequality, we obtain

$$2k|N(e_h^n, A_h^{-1}e_h^n)| \le ck\|e_h^n\|_{0,\Omega_0}\|e_h^n\|_{-1,\Omega_0} \le \frac{k}{3}\|e_h^n\|_{0,\Omega_0}^2 + ck\|e_h^n\|_{-1,\Omega_0}^2,$$

$$\begin{aligned}2|((u^{h,n} - u_H^n) - (u^{h,n-1} - u_H^{n-1}), A_h^{-1}e_h^n)| &\le 2k\|d_t(u^{h,n} - u_H^n)\|_{-2,\Omega_0}\|e_h^n\|_{0,\Omega_0}\\ &\le \frac{k}{3}\|e_h^n\|_{0,\Omega_0}^2 + ck\|d_t(u^{h,n} - u_H^n)\|_{-2,\Omega_0}^2,\end{aligned}$$

$$\begin{aligned}2k|a(u^{h,n} - u_H^n, A_h^{-1}e_h^n)| &\le 2\|u^{h,n} - u_H^n\|_{0,\Omega_0}\|e_h^n\|_{0,\Omega_0}\\ &\le \frac{k}{3}\|e_h^n\|_{0,\Omega_0}^2 + ck\|u^{h,n} - u_H^n\|_{0,\Omega_0}^2.\end{aligned}$$

Then (13) combining with the above estimates gives

$$\|e_h^n\|_{-1,\Omega_0}^2 + k\|e_h^n\|_{0,\Omega_0}^2 \le \|e_h^{n-1}\|_{-1,\Omega_0}^2 + ck\|e_h^n\|_{-1,\Omega_0}^2 + ck\|d_t(u^{h,n} - u_H^n)\|_{-2,\Omega_0}^2 + ck\|u^{h,n} - u_H^n\|_{0,\Omega_0}^2. \quad (14)$$

Finally, summing (14) from $n = 1$ to $n = m$ for all $1 \le m \le N$ leads to

$$\|e_h^m\|_{-1,\Omega_0}^2 + k\sum_{n=1}^{m}\|e_h^n\|_{0,\Omega_0}^2 \le ck\sum_{n=1}^{m}(\|e_h\|_{-1,\Omega_0}^2 + \|d_t(u^{h,n} - u_H^n)\|_{-2,\Omega_0}^2 + \|u^{h,n} - u_H^n\|_{0,\Omega_0}^2).$$

The application of Lemma 2.1 to the above inequality, along with (12) and Lemma 2.4, implies (11).

2.2 Parallel algorithm

The basic idea of the parallel algorithm is that we just apply the local algorithm (4)–(5) in parallel in all Ω_j for $j = 1, 2, \cdots, J$. Based on the local algorithm, we can easily construct the parallel algorithm.

(i) Find a global coarse grid solution $u_H^n \in X_0^H(\Omega)$ such that

$$(u_H^n, v) + ka(u_H^n, v) = (u_H^{n-1}, v) + k(f^n, v), \quad \forall v \in X_0^H(\Omega). \quad (15)$$

(ii) Find a local fine grid correction $e_{h,j}^n \in X_0^h(\Omega_j)$ such that

$$(e_{h,j}^n, v) + ka(e_{h,j}^n, v) = (e_{h,j}^{n-1}, v) + k(f^n, v) - ((u_H^n - u_H^{n-1}), v) - ka(u_H^n, v), \quad \forall v \in X_0^h(\Omega_j). \quad (16)$$

(iii) Correction is completed as

$$u_h^n|_{D_j} = u_H^n + e_{h,j}^n,$$

where $1 \le n \le N$, $D_j \subset\subset \Omega_j \subset\subset \Omega$, $\cup D_j = \Omega$, and $D_i \cap D_j = \emptyset$ $(i \ne j)$ for $i, j = 1, 2, \cdots, J$.

Theorem 2.2 *Under the conditions of Theorem* 2.1, *assume that* $u_h^n \in X^h(\Omega)$ *is obtained by the parallel algorithm* (15)–(16), *Then, for all* $1 \le n \le N$,

$$\|u^{h,n} - u_h^n\|_{0,\Omega} \le cH^3, \quad (17)$$

$$\|u(t_n) - u_h^n\|_{0,\Omega} \le c(k + h^2 + H^3). \quad (18)$$

Proof From Theorem 2.1, we have

$$\|u^{h,n} - u_h^n\|_{0,D_j} \le cH^3, \quad j = 1, 2, \cdots, J,$$

which leads to (17). The combination of (3) and (17) yields the inequality (18).

3 Numerical examples

This section gives some numerical examples to confirm the theoretical analysis in the previous sections. We divide the unit square domain $\Omega = [0,1]^2$ into triangles (see Fig.2), and all the numerical experiments are conducted by using the linear elements (i.e., the P_1 finite element). Especially, we choose the time step length $k = 0.001$ which is small enough such that the entire error will not improve when k becomes much smaller. We divide the domain Ω into four subdomains (see Fig. 1),

$$D_1 = (0, 1/2) \times (0, 1/2), \quad D_2 = (1/2, 1) \times (0, 1/2),$$
$$D_3 = (1/2, 1) \times (1/2, 1), \quad D_4 = (0, 1/2) \times (1/2, 1),$$

and

$$\Omega_1 = (0, 5/8) \times (0, 5/8), \quad \Omega_2 = (3/8, 1) \times (0, 5/8),$$
$$\Omega_3 = (3/8, 1) \times (3/8, 1), \quad \Omega_4 = (0, 5/8) \times (3/8, 1).$$

We remark that our main interest here is to check the feasibility of the local and parallel algorithm (15)–(16) for the time-dependent convection-diffusion equation in the simplest form. Hence, we compute the finite element solutions in subdomains Ω_i $(i = 1, 2, 3, 4)$ independently on a single chip computer. When the local and parallel algorithms are applied to the more complex equations involving scientific computing on a large scale, such as the time-dependent Navier-Stokes equations, it is really necessary to implement the algorithms on the parallel computing platform. The application of the local and parallel algorithms to the time-dependent Naiver-Stokes equations will be discussed elsewhere. Furthermore, for simplicity, we choose an exact solution in advance and then f is determined by this exact solution. The exact solution is $u(x, t) = t^2 \cos(x_1 x_2^2)$ with $x = (x_1, x_2)$, $b = (2, -1)$, $\Omega = (0, 1)^2$, and $T = 1$. In the following tables, the errors $\|e_h\|$ denote the relative error. That is, $\|e_h\|_{0,D} = \max_{n=1,2,\cdots,N} \frac{\|u(t_n)-u_h^n\|_{0,D}}{\|u(t_n)\|_{0,D}}$.

Table 1 presents the numerical results achieved by the local and parallel algorithm (15)–(16) for different H and h. From Table 1, we see that the local and parallel algorithm can reach the convergent rate with respect to h obtained in Theorem 2.2. Since we only do some corrections on the local domains in parallel while the usual two-grid methods do the corrections on the whole domain Ω, the local and parallel methods (LPM) can save much computational time compared with the usual two-grid methods (TGM). The CPU time comparisons of the TGM and the LPM are shown in Table 2. The CPU time presented in Table 2 is the maximum of

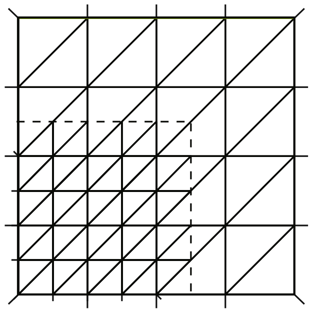

Fig. 2 Mesh

Table 1 Errors and convergent rates of the LPM

H	h	$\|e_h\|_{0,\Omega}$	Rate	$\|\nabla e_h\|_{0,\Omega}$	Rate
1/2	1/4	0.034 603		0.207 558	
1/4	1/8	0.006 582	2.39	0.091 588	1.18
1/8	1/16	0.001 507	2.13	0.043 097	1.09
1/16	1/32	0.000 378	2.00	0.020 903	1.04

Table 2 CPU time comparisons

H	h	TGM/s	LPM/s
1/2	1/4	14.23	4.12
1/4	1/8	26.29	10.99
1/8	1/16	77.45	39.42
1/16	1/32	265.41	137.25

the CPU time over the four subdomains. Finally, Table 3 shows that the relative error in $\Omega_1 \backslash D$ is much larger than that in D, which is the reason that we abandon the approximate solution in $\Omega_1 \backslash D$ to heighten the approximate accuracy.

Table 3 Relative errors in different domains

H	h	$\|e_h\|_{0,D}$	$\|\nabla e_h\|_{0,D}$	$\|e_h\|_{0,\Omega_1\backslash D}$	$\|\nabla e_h\|_{0,\Omega_1\backslash D}$
1/2	1/4	0.011 091	0.115 844	0.031 049	0.164 929
1/4	1/8	0.000 355	0.007 277	0.002 701	0.063 243
1/8	1/16	0.000 277	0.003 278	0.000 482	0.012 866
1/16	1/32	9.799 8E−05	0.001 570	0.000 161	0.006 124

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