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# Pullback attractor of 2D non-autonomous g-Navier-Stokes equations on some bounded domain*

Jin-ping JIANG (姜金平)[1,2], Yan-ren HOU (侯延仁)[1]

(1. School of Science, Xi'an Jiaotong University, Xi'an 710049, P. R. China;
2. College of Mathematics and Computer Science, Yan'an University, Yan'an 716000, Shaanxi Province, P. R. China)

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**Abstract** The existence of pullback attractors for the 2D non-autonomous g-Navier-Stokes equations on some bounded domains is investigated under the general assumptions of pullback asymptotic compactness. A new method to prove the existence of pullback attractors for the 2D g-Navier-Stokes equations is given.

**Key words** pullback attractor, g-Navier-Stokes equation, pullback asymptotical compact, pullback condition, bounded domain

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## 1 Introduction

Navier-Stokes equations have received increasing attention over the last decades due to their importance in the fluid motion and turbulence[1–7]. The understanding of the asymptotic behaviour of dynamical systems is one of the most important problems of modern mathematical physics. One way to treat this problem for a system with some dissipativity properties is to analyze the existence and structure of the global attractor[1–5,8–10]. At the same time, the theory of pullback attractors has been developed for both the non-autonomous systems and the random dynamical systems[11–19]. It is shown that the theory is very useful in the understanding of the dynamics of non-autonomous dynamical systems. In this paper, we study the existence of the pullback attractor of the g-Navier-Stokes (g-N-S) equations on the bounded domain $\Omega \subset \mathbb{R}^2$, which have the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(x,t) \ \text{ in } \ \Omega \times (0,\infty), \\ \nabla \cdot (gu) = 0 \ \text{ in } \ \Omega \times (0,\infty), \\ u(x,t) = 0 \ \text{ on } \ \partial\Omega, \\ u(x,0) = u_0(x) \ \text{ in } \ \Omega, \end{cases} \tag{1}$$

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Corresponding author Jin-ping JIANG, Associate Professor, Ph. D., E-mail: yadxjjp@163.com

where $u(x,t) \in \mathbb{R}^2$ and $p(x,t) \in \mathbb{R}$ denote the velocity and the pressure, respectively, $\nu > 0$, $f = f(x,t) \in (L^2(\Omega))^2$ is the time-dependent external force, and $0 < m_0 \leqslant g = g(x_1, x_2) \leqslant M_0$. Here, $g = g(x_1, x_2)$ is a suitable real-valued smooth function. When $g = 1$, the equations (1) become the usual 2D Navier-Stokes equations.

Now, some results have been obtained for the research on the global attractor of the autonomous 2D g-N-S equations[20–22]. In [20], Roh studied the existence of the 2D g-N-S equations on some bounded domain using the semiflow theory. In [21], Kwak et al. researched the Hausdorff and Fractal dimension of the global attractor about the 2D g-N-S equations for the periodic and Dirichlet boundary conditions. Moreover, the authors mainly studied the global attractor of the 2D g-N-S equations with linear dampness on $\mathbb{R}^2$ and the fractal dimension in [22]. From the research, we can see that the autonomous 2D g-N-S equations are studied, and the related research about the non-autonomous 2D g-N-S equations is still rare. We would like to use the theory of pullback attractors to study the non-autonomous dynamical system. Therefore, the present research is necessary and has a theoretical basis.

Recently, Caraballo et al. in [12] introduced the notion of pullback $\mathfrak{D}$-attractors for non-autonomous dynamical systems and proved the existence of pullback $\mathfrak{D}$-attractors by using the energy equation method. Obviously, it is hard to prove that a cocycle satisfies the above conditions. Motivated by the ideas in [17–19,23], we present a new equivalent condition (pullback condition) for the pullback $\mathfrak{D}$-asymptotical compact by using the measure of non-compactness. It is easy to be verified for general non-autonomous dynamical systems. As the application of this method, we prove the existence of pullback attractors for the 2D g-N-S equations on some bounded domains.

This paper is organized as follows. In Section 2, we recall some basic notations and results for 2D g-N-S equations and the concept about the pullback asymptotic compactness. In Section 3, using the measure of non-compactness, we prove the existence of the pullback attractor for the 2D g-N-S equations on some bounded domain.

## 2 Preliminaries

Now, we assume that the Poincaré inequality holds on $\Omega$, i.e., there exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g dx \leqslant \frac{1}{\lambda_1} \int_\Omega |\nabla \phi|^2 g dx, \quad \forall \phi \in H_0^1(\Omega). \tag{2}$$

The mathematical frameworks of (1) are as follows. Let $L^2(g) = (L^2(\Omega))^2$ with the inner products

$$(u, v) = \int_\Omega u \cdot \nu g dx$$

and the norms

$$|\cdot| = (\cdot, \cdot)^{\frac{1}{2}}, \quad u, v \in L^2(g).$$

Let $H_0^1(g) = (H_0^1(\Omega))^2$, which is endowed with the inner products

$$((u, v)) = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j g dx$$

and the norms

$$||\cdot|| = ((\cdot, \cdot))^{\frac{1}{2}}, \quad u = (u_1, u_2), \quad v = (v_1, v_2) \in H_0^1(g).$$

From (2), the norm $||\cdot||$ is equivalent to the usual one in $H_0^1(\Omega)$. Let $D(\Omega)$ be the space of $\mathcal{C}^\infty$ functions with the compact support contained in $\Omega$, and let

$$\begin{aligned}\aleph &= \{v \in (D(\Omega))^2 : \nabla \cdot gv = 0 \ \text{ in } \Omega\},\\ H_g &= \text{closure of } \ \aleph \ \text{ in } L^2(g),\\ V_g &= \text{closure of } \ \aleph \ \text{ in } H_0^1(g).\end{aligned}$$

With $H_g$ and $V_g$ endowed with the inner product and norm of $L^2(g)$ and $H_0^1(g)$, respectively, it follows from (2) that

$$|u|^2 \leqslant \frac{1}{\lambda_1}||u||^2, \quad \forall u \in V_g. \tag{3}$$

Now, we define a g-Laplacian operator as follows:

$$-\Delta_g u = -\frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - \frac{1}{g}\nabla g \cdot \nabla u.$$

Using the g-Laplacian operator, we rewrite the first equation of (1) as follows:

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{\nabla g}{g} \cdot \nabla u + (u, \nabla)u + \nabla p = f. \tag{4}$$

We define a g-orthogonal projection

$$P_g : L^2(g) \to H_g$$

and a g-Stokes operator

$$A_g u = -P_g\Big(\frac{1}{g}(\nabla \cdot (g\nabla u))\Big),$$

which satisfies the following proposition.

**Proposition 1**[20] *For the linear operator $A_g$, the following results hold*:

(i) $A_g$ *is a positive self-adjoint operator with compact inverse, where the domain of* $A_g$ *is* $D(A_g) = V_g \cap H^2(\Omega)$.

(ii) *There exist countable eigenvalues of* $A_g$ *satisfying* $0 < \lambda_g \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ , *where* $\lambda_g = \frac{4\pi^2 m_0}{M_0}$, *and* $\lambda_1$ *is the smallest eigenvalue of* $A_g$. *In addition, there exists the corresponding collection of eigenfunctions* $\{e_1, e_2, e_3, \cdots\}$, *which forms an orthonormal basis for* $H_g$.

When we apply the projection $P_g$ into (4), we can obtain the following weak formulation of (1). Let $f \in V_g$ and $u_0 \in H_g$. Then, we find that

$$u \in L^\infty(0, T; H_g) \cap L^2(0, T; V_g), \quad T > 0 \tag{5}$$

such that

$$\frac{d}{dt}(u, v) + \nu((u, v)) + b_g(u, u, v) + \nu(Ru, v) = \langle f, v\rangle, \quad \forall v \in V_g, \quad \forall t > 0, \tag{6}$$

$$u(0) = u_0, \tag{7}$$

where $b_g : V_g \times V_g \times V_g \to \mathbb{R}$ is given by

$$b_g(u, v, w) = \sum_{i,j=1}^{2} \int u_i \frac{\partial v_j}{\partial x} w_j g dx \tag{8}$$

and

$$Ru = P_g\Big(\frac{1}{g}(\nabla g\cdot\nabla)u\Big), \quad \forall u\in V_g.$$

Then, the weak formulation of (6) and (7) is equivalent to the functional equations

$$\frac{du}{dt}+\nu A_g u+Bu+\nu Ru=f, \tag{9}$$
$$u(0)=u_0, \tag{10}$$

where $A_g : V_g \to V_g'$ is the g-Stokes operator defined by

$$\langle A_g u, v\rangle = ((u,v)), \quad \forall u,v\in V_g, \tag{11}$$

$B(u)=B(u,u)=P_g(u\cdot\nabla)u$ is a bilinear operator, and $B : V_g\times V_g \to V_g'$ is defined by

$$\langle B(u,v), w\rangle = b_g(u,v,w), \quad \forall u,v,w\in V_g.$$

Now, we recall some well-known inequalities[24] that we will be using in what follows.

For every $u,v\in D(A_g)$,

$$|B(u,v)|\leqslant C|u|^{\frac12}|A_g u|^{\frac12}||v||. \tag{12}$$

Here, $C$ denotes the positive constant, which may be different from line to line and even in the same line.

$$|\varphi|_{L^\infty(\Omega)^2}\leqslant C||\varphi||\Big(1+\log\frac{|A_g\varphi|^2}{\lambda_1||\varphi||^2}\Big)^{\frac12}, \quad \forall\varphi\in D(A_g), \tag{13}$$

from which we can deduce

$$|B(u,v)|\leqslant |(u\cdot\nabla)v|\leqslant |u|_{L^\infty(\Omega)}|\nabla v|. \tag{14}$$

Using (13), we obtain

$$|B(u,v)|\leqslant C||u||||v||\Big(1+\log\frac{|A_g u|^2}{\lambda_1||u||^2}\Big)^{\frac12}. \tag{15}$$

The g-Stokes operator $A_g$ is an isomorphism from $V_g$ into $V_g'$, while $B$ and $R$ satisfy the following inequalities[20,25]:

$$||B(u)||_{V'}\leqslant c|u|||u||, \quad ||Ru||_{V'}\leqslant \frac{|\nabla g|_\infty}{m_0\lambda_1^{\frac12}}||u||, \quad \forall u\in V. \tag{16}$$

We have the following concept and result[26,6].

**Proposition 2** *Given $f\in L^2(g)$ and $u_0(x)\in H_g$, there exists a unique solution*

$$u(x,t)\in L^\infty(\mathbb{R}^+;H_g)\cap L^2(0,T;V_g)\cap C(\mathbb{R}^+;H_g), \quad \forall T>0$$

*such that* (6) *and* (7) *hold.*

Let $\Gamma$ be a nonempty set. We define a family $\{\theta_t\}_{t\in\mathbb{R}}$ of mappings $\theta_t:\Gamma\to\Gamma$ satisfying

(i) $\theta_0\gamma=\gamma$ for all $\gamma\in\Gamma$;

(ii) $\theta_t(\theta_\tau\gamma)=\theta_{t+\tau}\gamma$ for all $\gamma\in\Gamma$, $t,\tau\in\mathbb{R}$.

Here, the operators $\theta_t$ are called the shift operators.

Let $X$ be a metric space with distance $d(\cdot,\cdot)$, and $\phi$ be a $\theta$-cocycle on $X$, i.e., a mapping $\phi:\mathbb{R}_+\times\Gamma\times X\to X$ satisfying

(i) $\phi(0,\gamma,x)=x$ for all $(\gamma,x)\in\Gamma\times X$;

(ii) $\phi(t+\tau,\gamma,x)=\phi(t,\theta_\tau\gamma,\phi(\tau,\gamma,x))$ for all $t,\tau\in\mathbb{R}_+$ and $(\gamma,x)\in\Gamma\times X$.

The $\theta$-cocycle $\phi$ is said to be continuous if for all $(t,\gamma)\in\mathbb{R}_+\times\Gamma$, the mapping $\phi(t,\gamma,\cdot):X\to X$ is continuous. Let $\mathcal{P}(x)$ be the family of all nonempty subsets of $X$, and $\varphi$ the class of all families $\widetilde{D}=\{D(\gamma):\gamma\in\Gamma\}\subset\mathcal{P}(X)$. Let a nonempty subclass $\mathcal{D}\subset\varphi$.

**Definition 1** *The $\theta$-cocycle $\phi$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $\gamma\in\Gamma$, any $\widetilde{D}\in\mathcal{D}$, and any sequences $t_n\to+\infty$ and $x_n\in D(\theta_{-t_n}\gamma)$, the sequence $\phi(t_n,\theta_{-t_n}\gamma,x_n)$ possesses a convergent subsequence.*

**Definition 2** *A family $\widetilde{B}=\{B(\gamma);\gamma\in\Gamma\}\in\varphi$ is said to be pullback $\mathcal{D}$-absorbing if for each $\gamma\in\Gamma$ and $\widetilde{D}\in\mathcal{D}$, there exists $t_0(\gamma,\widetilde{D})\geqslant 0$ such that*

$$\phi(t,\theta_{-t}\gamma,D(\theta_{-t}\gamma))\subset B(\gamma)\quad \textit{for all}\quad t\geqslant t_0(\gamma,\widetilde{D}).$$

We define the Hausdorff semi-distance between $C_1$ and $C_2$ as

$$\operatorname{dist}(C_1,C_2)=\sup_{x\in C_1}\inf_{y\in C_2}d(x,y)\quad\text{for } C_1,C_2\subset X.$$

**Definition 3** *A family $\widetilde{A}=\{A(\gamma);\gamma\in\Gamma\}\in\varphi$ is said to be a pullback $\mathcal{D}$-attractor if it satisfies the following conditions*:

(i) $A(\gamma)$ *is compact for any* $\gamma\in\Gamma$.

(ii) $\widetilde{A}$ *is pullback $\mathcal{D}$-attracting, i.e.,*

$$\lim_{t\to+\infty}\operatorname{dist}(\phi(t,\theta_{-t}\gamma,D(\theta_{-t}\gamma)),A(\gamma))=0\quad\textit{for all}\ \widetilde{D}\in\mathcal{D},\quad \gamma\in\Gamma.$$

(iii) $\widetilde{A}$ *is invariant, i.e.,*

$$\phi(t,\gamma,A(\gamma))=A(\theta_t\gamma)\quad\textit{for any}\ (t,\gamma)\in\mathbb{R}_+\times\Gamma.$$

## 3 Existence of pullback attractor for 2D g-N-S equations on some bounded domains

In this section, we present the measure of non-compactness to prove the existence of pullback attractors of 2D g-N-S equations on bounded domains. First, we recall some basic notions about the measure of non-compactness[23].

Let $B(X)$ be the set of all bounded subsets of $X$ and $B\in B(X)$. Its Kuratowski measure of non-compactness $\alpha(B)$ is defined by

$$\alpha(B)=\inf\{\delta|B \text{ admits a finite cover by the set of diameter}\leqslant\delta\}.$$

It has the following properties[25,27].

**Lemma 1** *Let $B,B_1,B_2\in B(X)$. Then,*

(i) $\alpha(B)=0\Leftrightarrow\alpha(N(B,\varepsilon))\leqslant 2\varepsilon\Leftrightarrow\overline{B}$ *is compact.*

(ii) $\alpha(B_1+B_2)\leqslant\alpha(B_1)+\alpha(B_2)$.

(iii) $\alpha(B_1)\leqslant\alpha(B_2)$ *whenever* $B_1\subset B_2$.

(iv) $\alpha(B_1\cup B_2)\leqslant\max\{\alpha(B_1),\alpha(B_2)\}$.

(v) $\alpha(\overline{B})=\alpha(B)$.

(vi) *If $B$ is a ball of radius $\varepsilon$, then $\alpha(B)\leqslant 2\varepsilon$.*

**Lemma 2** *Let $\cdots \supset F_n \supset F_{n+1} \supset \cdots$ be a sequence of non-empty closed subsets of $X$ such that $\alpha(F_n) \to 0$ as $n \to \infty$. Then, $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.*

We have some results[17].

**Definition 4** *Let $\phi$ be a $\theta$-cocycle on $X$. A set $B_0 \subset X$ is said to be a uniformly absorbing set for $\phi$ if for any $B \in B(X)$, there exists $T_0 = T_0(B) \in \mathbb{R}^+$ such that*

$$\phi(t, \gamma, B) \subset B_0 \quad \textit{for all} \quad t \geqslant T_0, \quad \gamma \in \Gamma.$$

**Definition 5** *Let $\phi$ be a $\theta$-cocycle on $X$. $\phi$ is said to be pullback $\omega$-limit compact if for any $B \in B(X)$ and $\gamma \in \Gamma$,*

$$\lim_{t \to +\infty} \alpha\Big(\bigcup \phi(t, \theta_{-t}(\gamma), B)\Big) = 0.$$

**Definition 6** *Let $\phi$ be a $\theta$-cocycle on $X$. Define the pullback $\omega$-limit set $\Lambda_\gamma(B)$ of $B$ by the following form:*

$$\Lambda_\gamma(B) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \phi(t, \theta_{-t}(\gamma), B)}.$$

**Theorem 1** *Let $\phi$ be a $\theta$-cocycle on $X$. If $\phi$ is continuous and possesses a uniformly absorbing set $B_0$. Then, $\phi$ possesses a pullback attractor $\mathcal{A} = \{A_\gamma\}_{\gamma \in \Gamma}$ satisfying*

$$A_\gamma = \Lambda_\gamma(B_0), \quad \forall \gamma \in \Gamma$$

*if and only if it is pullback $\omega$-limit compact.*

**Definition 7** *Let $\phi$ be a $\theta$-cocycle on $X$. A cocycle $\phi$ is said to satisfy the pullback condition $(PC)$ if for any $\gamma \in \Gamma, B \in B(X)$, and $\varepsilon > 0$, there exist $t_0 = t_0(\gamma, B, \varepsilon) \geqslant 0$ and a finite dimensional subspace $X_1$ of $X$ such that*

(i) $P\Big(\bigcup_{t \geqslant t_0} \phi(t, \theta_{-t}(\gamma), B)\Big)$ *is bounded.*

(ii) $\Big\|(I - P)\Big(\bigcup_{t \geqslant t_0} \phi(t, \theta_{-t}(\gamma), x)\Big)\Big\| \leqslant \varepsilon, \ \forall x \in B.$

Here, $P : X \to X_1$ is a bounded projector.

**Theorem 2** *Let $X$ be a Banach space and $\phi$ be a $\theta$-cocycle on $X$. If $\phi$ satisfies the PC, then $\phi$ is pullback $\omega$-limit compact. Moreover, let $X$ be a uniformly convex Banach space. Then, $\phi$ is pullback $\omega$-limit compact if and only if the PC holds.*

Denote by $L^2_{\rm loc}(\mathbb{R}, X)$ the metrizable space of function $f(s) \in X$ with $s \in \mathbb{R}$, where $X$ is locally two-power integrable in the Bochner sense. It is equipped with the local two-power mean convergence topology. Now, we apply the new method to prove the existence of pullback attractors for 2D g-N-S equations.

**Lemma 3** *Suppose $f \in L^2_{\rm loc}(\mathbb{R}, H_g)$ such that*

$$|f|_b^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} |f(s)|^2 ds < \infty,$$

*and $u_0(x) \in H_g$. Let $u(x,t) \in L^\infty(\mathbb{R}^+, H_g) \cap L^2_{\rm loc}(0, T, V_g) \cap \mathcal{C}(\mathbb{R}^+, H_g)$ $(\forall t > 0)$ be a weak solution of (1). Then, for all $t \geqslant \tau$ and $\sigma = \nu\lambda_1$, the following estimates hold:*

$$|u(t)|^2 \leqslant |u_0|^2 \mathrm{e}^{-\sigma\gamma_0(t-\tau)} + R_1^2, \tag{17}$$

*where $R_1^2 = \sigma^{-1}(1 - \mathrm{e}^{-\sigma\gamma_0})^{-1}|f|_b^2$ and $\gamma_0 = 1 - 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}$ for sufficiently small $|\nabla g|_\infty$.*

**Proof** Let $u(x,t)$ be a solution of (1). Since $u\in L^2(0,T;V_g)$ and $u'\in L^2(0,T;V_g')$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2&=\langle u',u\rangle\\&=\langle f-\nu A_gu-Bu-\nu Ru,u\rangle\\&=\langle f,u\rangle-\nu||u||^2-b_g(u,u,u)-\nu\Big(\Big(\frac{1}{g}\nabla g\cdot\nabla\Big)u,u\Big).\end{aligned}$$

Taking into account that $b_g(u,u,u)=0$, we have

$$\frac{d}{dt}|u|^2+2\nu||u||^2=2\langle f,u\rangle-2\nu\Big(\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u,u\Big).$$

Applying the Poincaré inequality, we obtain

$$\begin{aligned}\frac{d}{dt}|u|^2+2\nu||u||^2&\leqslant\frac{|f|^2}{\nu\lambda_1}+\nu\lambda_1|u|^2+2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u||^2\\&\leqslant\frac{|f|^2}{\nu\lambda_1}+\nu||u||^2+2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u||^2.\end{aligned}$$

Then,

$$\frac{d}{dt}|u|^2+\nu||u||^2\leqslant\frac{|f|^2}{\nu\lambda_1}+2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u||^2.$$

We have

$$\frac{d}{dt}|u|^2+\nu\gamma_0||u||^2\leqslant\frac{|f|^2}{\nu\lambda_1},$$

where $\gamma_0=1-2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}$ for sufficiently small $|\nabla g|_\infty$.

$$\frac{d}{dt}|u|^2+\nu\lambda_1\gamma_0|u|^2\leqslant\frac{|f|^2}{\nu\lambda_1}.$$

Let $\sigma=\nu\lambda_1$. Using Gronwall's lemma, we have

$$\begin{aligned}|u(t)|^2&\leqslant|u_0|^2\mathrm{e}^{-\sigma\gamma_0(t-\tau)}+\frac{1}{\sigma}\int_\tau^t\mathrm{e}^{-\sigma\gamma_0(t-s)}|f(s)|^2ds\\&\leqslant|u_0|^2\mathrm{e}^{-\sigma\gamma_0(t-\tau)}+\frac{1}{\sigma}\Big(\int_{t-1}^t\mathrm{e}^{-\sigma\gamma_0(t-s)}|f(s)|^2ds+\int_{t-2}^{t-1}\mathrm{e}^{-\sigma\gamma_0(t-s)}|f(s)|^2ds+\cdots\Big)\\&\leqslant|u_0|^2\mathrm{e}^{-\sigma\gamma_0(t-\tau)}+\frac{1}{\sigma}(1+\mathrm{e}^{-\sigma\gamma_0}+\mathrm{e}^{-2\sigma\gamma_0}+\cdots)\sup_{t\in\mathbb{R}}\int_t^{t+1}|f(s)|^2ds\\&\leqslant|u_0|^2\mathrm{e}^{-\sigma\gamma_0(t-\tau)}+R_1^2,\end{aligned}$$

where $R_1^2=\sigma^{-1}(1-\mathrm{e}^{-\sigma\gamma_0})^{-1}|f|_b^2$.

For any $f\in\Gamma$ and $|f|_b^2=|f_0|_b^2$, using (17), we obtain that

$$B_0=\{u\in H_g||u|\leqslant 2R_1^2\triangleq\rho_0^2\}$$

is the uniformly absorbing set in $H_g$.

**Lemma 4** *Suppose* $f \in L^2_{\rm loc}(\mathbb{R}, H_g)$ *such that*

$$|f|_b^2 = \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(s)|^2 ds < \infty,$$

*and* $u_0(x) \in H_g$. *Let*

$$u(x,t) \in L^\infty(\mathbb{R}^+, V_g) \cap L^2_{\rm loc}(0,T,D(A_g)) \cap \mathcal{C}(\mathbb{R}^+, V_g), \quad u'(x,t) \in L^2_{\rm loc}(\mathbb{R}_\tau; H_g), \quad \forall t > 0$$

*be a strong solution of* (1). *Then, for all* $t \geqslant \tau$, *the following estimates hold*:

$$||u(t)||^2 \leqslant ||u(\tau)||^2 {\rm e}^{-\beta(t-\tau)} + (1-{\rm e}^{-\beta})^{-1} |f|_b^2, \tag{18}$$

*where* $\beta = \lambda\big(2\nu - 1 - \frac{2C\rho_0}{\lambda_0^{1/2}} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{1/2}}\big)$ *for sufficiently small* $|\nabla g|_\infty$.

**Proof** We multiply (9) by $-\Delta u(t)$ and obtain

$$\frac{1}{2}\frac{d}{dt}||u||^2 + \nu|\Delta u|^2 = (f, -\Delta u) - (Bu, -\Delta u) - \nu(Ru, -\Delta u).$$

Using Young's inequality,

$$\begin{aligned}
\frac{d}{dt}||u||^2 + 2\nu|\Delta u|^2 &= 2(f, -\Delta u) - 2(Bu, -\Delta u) - 2\nu(Ru, -\Delta u) \\
&\leqslant |f|^2 + |\Delta u|^2 + 2|(Bu, -\Delta u)| + 2\nu|(Ru, -\Delta u)| \\
&\leqslant |f|^2 + |\Delta u|^2 + 2|Bu||\Delta u| + 2\nu|Ru||\Delta u| \\
&\leqslant |f|^2 + |\Delta u|^2 + \frac{2C}{\lambda_0^{\frac{1}{2}}}|u|^{\frac{1}{2}}|\Delta u|^2 + \frac{2\nu|\nabla g|_\infty}{m_0}||u|||\Delta u| \\
&\leqslant |f|^2 + |\Delta u|^2 + \frac{2C\rho_0}{\lambda_0^{\frac{1}{2}}}|\Delta u|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{\frac{1}{2}}}|\Delta u|^2,
\end{aligned}$$

we have

$$\frac{d}{dt}||u||^2 + \left(2\nu - 1 - \frac{2C\rho_0}{\lambda_0^{\frac{1}{2}}} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{\frac{1}{2}}}\right)|\Delta u|^2 \leqslant |f|^2.$$

Using the Poincaré inequality, we obtain

$$\frac{d}{dt}||u||^2 + \lambda\left(2\nu - 1 - \frac{2C\rho_0}{\lambda_0^{\frac{1}{2}}} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{\frac{1}{2}}}\right)||u||^2 \leqslant |f|^2.$$

Let

$$\beta = \lambda\left(2\nu - 1 - \frac{2C\rho_0}{\lambda_0^{\frac{1}{2}}} - \frac{2\nu|\nabla g|_\infty}{m_0\lambda_0^{\frac{1}{2}}}\right).$$

Then, we have

$$\frac{d}{dt}||u||^2 + \beta||u||^2 \leqslant |f|^2.$$

Applying Gronwall's lemma, we obtain

$$||u||^2 \leqslant ||u(\tau)||^2 \mathrm{e}^{-\beta(t-\tau)} + \int_\tau^t \mathrm{e}^{-\beta(t-s)}|f|^2 ds$$

$$||u||^2 \leqslant ||u(\tau)||^2 \mathrm{e}^{-\beta(t-\tau)} + \int_{t-1}^t \mathrm{e}^{-\beta(t-s)}|f|^2 ds + \int_{t-2}^{t-1} \mathrm{e}^{-\beta(t-s)}|f|^2 ds + \cdots$$

$$||u||^2 \leqslant ||u(\tau)||^2 \mathrm{e}^{-\beta(t-\tau)} + (1 + \mathrm{e}^{-\beta} + \mathrm{e}^{-2\beta} + \cdots) \sup_{t\in\mathbb{R}} \int_t^{t+1} |f|^2 ds$$

$$||u(\tau)||^2 \mathrm{e}^{-\beta(t-\tau)} + (1 - \mathrm{e}^{-\beta})^{-1}|f|_b^2.$$

Let

$$B_1 = \bigcup_{f\in\Gamma} \bigcup_{t>t_0+1} \phi(t_0+1, f, B_0).$$

Using (18), we know that $B_1$ is bound, $||u||^2 \leqslant \rho_1^2$ for all $u \in B_1$, and $B_1$ is the uniformly absorbing set in $V_g$.

**Lemma 5** *Suppose that $H_g$ is a Hilbert space, and $\{\omega_i\}_{i\in\mathbb{N}}$ is orthonormal in $H_g$. Let $f(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; H_g)$, and suppose that there exists a $\sigma > 0$ such that for any $t \in \mathbb{R}$, $\int_{-\infty}^t \mathrm{e}^{\sigma s}||f(x,s)||^2_{H_g} ds < \infty$. Then,*

$$\lim_{n\to\infty} \int_{-\infty}^t \mathrm{e}^{\sigma s}||(I - P_m)f(x,s)||^2_{H_g} ds = 0, \quad \forall t \in \mathbb{R}, \tag{19}$$

*where $P_m : H_g \to \mathrm{span}\{\omega_1, \cdots, \omega_n\}$ is an orthogonal projector.*

**Proof** Let $\xi_i(t) = (f(x,t), \omega_i)_{H_g}$. Then,

$$f(x,t) = \frac{1}{g} \sum_{i=1}^\infty \xi_i(t)\omega_i.$$

For any $t \in \mathbb{R}, \varepsilon > 0$, since

$$\int_{-\infty}^t \mathrm{e}^{\sigma s}||f(x,s)||^2_{H_g} ds = \sum_{i=1}^\infty \int_{-\infty}^t \mathrm{e}^{\sigma s}||\xi_i(s)||^2_{H_g} ds < \infty.$$

we have

$$\int_{-\infty}^t \mathrm{e}^{\sigma s}||(I - P_m)f(x,s)||^2_{H_g} ds = \sum_{i=N_0}^\infty \int_{-\infty}^t \mathrm{e}^{\sigma s}||\xi_i(s)||^2_{H_g} ds < \varepsilon$$

for any $n \geqslant N_0$ with the sufficiently large $N_0$.

**Theorem 3** *If $f(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; H_g)$, then the cocycle $\{\phi(t,\gamma,x)\}$ corresponding to* (1) *possesses a compact pullback attractor*

$$\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma} = \{\Lambda_\gamma(B_1)\}_{\gamma\in\Gamma},$$

*where $B_1$ is the uniformly* (w.r.t. $\gamma \in \Gamma$) *absorbing set in $V_g$.*

**Proof** From Theorem 2, we only need to verify that the family of cocycles $\{\phi(t,\gamma,x)\}$ satisfies the PC in $V_g$.

Since $(-\Delta)^{-1}$ is a continuous compact operator in $H_g$, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^\infty$, where

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \leqslant \lambda_j \to \infty \text{ as } j \to \infty, \tag{20}$$

and a family of elements $\{\omega_j\}_{j=1}^{\infty}$ of $D(-\Delta)$ that are orthonormal in $H_g$ such that

$$-\Delta\omega_j = \lambda_j\omega_j, \quad \forall j \in \mathbb{N}.$$

Let $V_m = \text{span}\{\omega_1, \omega_2, \cdots, \omega_m\}$ in $V_g$ and $P_m : V_g \to V_m$ be an orthogonal projector.

For any $u \in D(-\Delta)$, we write

$$u = P_m u + (I - P_m)u = u_1 + u_2.$$

Taking the inner product of (9) with $-\Delta u_2$ in $H_g$, we have

$$\frac{1}{2}\frac{d}{dt}||u_2||^2 + \nu|\Delta u_2|^2 + (B(u), -\Delta u_2) + \nu(Ru, -\Delta u_2) = (f, -\Delta u_2).$$

Using Young's inequality, together with (12) and (15), we have

$$\begin{aligned}|(B(u), -\Delta u_2)| &\leqslant |(B(u_1, u_1 + u_2), -\Delta u_2)| + |(B(u_2, u_1 + u_2), -\Delta u_2)|\\ &\leqslant cL^{\frac{1}{2}}||u_1|||\Delta u_2|(||u_1|| + ||u_2||) + c|u_2|^{\frac{1}{2}}|\Delta u_2|^{\frac{3}{2}}(||u_1|| + ||u_2||)\\ &\leqslant \frac{\nu}{4}|\Delta u_2|^2 + \frac{c}{\nu}\rho_1^4 L + \frac{c}{\nu^3}\rho_0^2\rho_1^4, \qquad t \geqslant t_0 + 1,\end{aligned}$$

where $|\Delta u_1|^2 \leqslant \lambda_m||u_1||^2$, and $L = 1 + \log\frac{\lambda_{m+1}}{\lambda_1}$.

$$\begin{aligned}|(Ru, -\Delta u_2)| &\leqslant |Ru| \cdot |\Delta u_2|\\ &\leqslant \frac{|\nabla g|_\infty}{m_0}||u|| \cdot |\Delta u_2|\\ &\leqslant \frac{|\nabla g|_\infty}{m_0}\Big(\frac{|\Delta u_2|^2}{2} + 2||u||^2\Big)\\ &\leqslant \frac{|\nabla g|_\infty}{m_0}\Big(\frac{|\Delta u_2|^2}{2} + 2\rho_1^2\Big),\end{aligned}$$

and

$$\begin{aligned}&(f, -\Delta u_2) \leqslant \frac{|f|^2}{\nu} + \frac{\nu|\Delta u_2|^2}{4},\\ &\frac{d}{dt}||u_2||^2 + 2\nu|\Delta u_2|^2\\ &\leqslant 2(f, -\Delta u_2) - 2(B(u), -\Delta u_2) - 2\nu(Ru, -\Delta u_2)\\ &\leqslant \frac{2|f|^2}{\nu} + \frac{\nu|\Delta u_2|^2}{2} + \frac{\nu}{2}|\Delta u_2|^2 + \frac{2c}{\nu}\rho_1^4 L + \frac{2c}{\nu^3}\rho_0^2\rho_1^4 + \frac{2|\nabla g|_\infty}{m_0}\Big(\frac{\nu|\Delta u_2|^2}{2} + \frac{2\rho_1^2}{\nu}\Big)\\ &\leqslant \frac{2|f|^2}{\nu} + \nu|\Delta u_2|^2 + \frac{\nu|\nabla g|_\infty}{m_0}|\Delta u_2|^2 + \frac{2c}{\nu}\rho_1^4 L + \frac{2c}{\nu^3}\rho_0^2\rho_1^4 + \frac{4|\nabla g|_\infty}{\nu m_0}\rho_1^2.\end{aligned}$$

We obtain

$$\begin{aligned}&\frac{d}{dt}||u_2||^2 + \nu\Big(1 - \frac{|\nabla g|_\infty}{m_0}\Big)|\Delta u_2|^2 \leqslant \frac{2|f|^2}{\nu} + \frac{2c}{\nu}\rho_1^4 L + \frac{2c}{\nu^3}\rho_0^2\rho_1^4 + \frac{4|\nabla g|_\infty}{\nu m_0}\rho_1^2,\\ &\frac{d}{dt}||u_2||^2 + \nu\Big(1 - \frac{|\nabla g|_\infty}{m_0}\Big)|\Delta u_2|^2 \leqslant 2c\Big(\frac{1}{c\nu}|(I - P_m)f|^2 + \frac{1}{\nu}\rho_1^4 L + \frac{1}{\nu^3}\rho_0^2\rho_1^4 + \frac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2\Big).\end{aligned}$$

Let $\alpha = (1 - \frac{|\nabla g|_\infty}{m_0})$. Then, we have

$$\frac{d}{dt}||u_2||^2 + \nu\lambda_{m+1}\alpha||u_2||^2 \leqslant 2c\Big(\frac{1}{c\nu}|(I - P_m)f|^2 + \frac{1}{\nu}\rho_1^4 L + \frac{1}{\nu^3}\rho_0^2\rho_1^4 + \frac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2\Big).$$

Applying Gronwall's lemma, we deduce

$$\begin{aligned}||u_2||^2 &\leqslant ||u_2(t_0+1)||^2 \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-(t_0+1))} + \int_{t_0+1}^{t} \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-s)} \Big(2c\Big(\frac{1}{c\nu}|(I-P_m)f|^2 \\ &\quad + \frac{1}{\nu}\rho_1^4 L + \frac{1}{\nu^3}\rho_0^2\rho_1^4 + \frac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2\Big)\Big)ds \\ &= ||u_2(t_0+1)||^2 \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-(t_0+1))} + 2c\Big(\frac{1}{\nu}\rho_1^4 L + \frac{1}{\nu^3}\rho_0^2\rho_1^4 + \frac{2|\nabla g|_\infty}{c\nu m_0}\rho_1^2\Big) \\ &\quad \cdot \int_{t_0+1}^{t} \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-s)} ds + \frac{2}{\nu}\int_{t_0+1}^{t} \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-s)} |(I-P_m)f|^2 ds \\ &= ||u_2(t_0+1)||^2 \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-(t_0+1))} + \frac{2c}{\nu^2\lambda_{m+1}\alpha}\Big(\rho_1^4 L + \frac{\rho_0^2\rho_1^4}{\nu^2} + \frac{2|\nabla g|_\infty}{c m_0}\rho_1^2\Big) \\ &\quad + \frac{2}{\nu}\int_{t_0+1}^{t} \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-s)} |(I-P_m)f|^2 ds.\end{aligned}$$

By (17) and Lemma 4, for any $\varepsilon > 0$, we can take $m+1$ large enough such that

$$\frac{2}{\nu}\int_{t_0+1}^{t} \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-s)} |(I-P_m)f|^2 ds \leqslant \frac{\varepsilon}{3},$$
$$\frac{2c}{\nu^2\lambda_{m+1}\alpha}\Big(\rho_1^4 L + \frac{\rho_0^2\rho_1^4}{\nu^2} + \frac{2|\nabla g|_\infty}{c m_0}\rho_1^2\Big) \leqslant \frac{\varepsilon}{3}.$$

Let $t_2 = t_0 + 1 + \frac{1}{\nu\lambda_{m+1}\alpha \ln \frac{3\rho_1^2}{\varepsilon}}$. Then, $t \geqslant t_2$. We have

$$||u_2(t_0+1)||^2 \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-(t_0+1))} \leqslant \rho_1^2 \mathrm{e}^{-\nu\lambda_{m+1}\alpha(t-(t_0+1))} \leqslant \frac{\varepsilon}{3}.$$

Hence, we have

$$||u_2(t)||^2 \leqslant \varepsilon, \quad \forall t \geqslant t_2,$$

which indicates that the family $\{\phi(t,\gamma,x)\}$ in $V_g$ satisfies the PC in $V_g$. Applying Theorem 2, the proof is completed.

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