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Variational multiscale method based on the Crank–Nicolson extrapolation scheme for the non-stationary Navier–Stokes equations

Li Shan[a,b†], Yanren Hou[a,b] and Haibiao Zheng[a,b]*

[a]*School of Mathematics and Statistics, Science, Xi'an Jiaotong University, Xi'an 710049, China;*
[b]*Center of Computational Geoscience, Xi'an Jiaotong University, Xi'an 710049, China*

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In this report, a *variational multiscale* (VMS) method based on the *Crank–Nicolson extrapolation* scheme of time discretization for the turbulent flow is analysed. The flow is modelled by the *fully evolutionary* Navier–Stokes problem. This method has two differences compared to the standard VMS method: (i) For the trilinear term, we use the extrapolation skill to linearize the scheme; (ii) for the projection term, we lag it onto the previous time level to simplify the construction of the projection. These modifications make the algorithm more efficient and feasible. An unconditionally stability and an *a priori* error estimate are given for a case with rather general linear (cellwise constant) viscosity of the turbulent models. Moreover, numerical tests for both linear viscosity and nonlinear Smagorinsky-type viscosity are performed, they confirm the theoretical results and indicate the schemes are effective.

Keywords: turbulent incompressible flows; Crank–Nicolson extrapolation; projection-based variational multicale method; large-eddy simulation; error estimate

2000 AMS Subject Classification: 65N30

1. Introduction

Turbulent flows are characterized by high Reynolds numbers, which occur in many processes in nature as well as in many industrial applications. They posses a richness of scales, that means there are large flow structures and also very small ones. For example, a hurricane has a number of very large eddies but also millions of small eddies. Due to the richness of scales, it is very hard to simulate them by the direct numerical simulation, large eddy simulation (LES) is one of very popular approach for turbulent flow simulation. However, the goal of LES is to compute only the large flow structures accurately. The definition of the large scales by spatial averaging leads to serious problems if the flow is given in a bounded domain, which is the most frequent case

*Corresponding author. Email: hbzheng13@gmail.com
†Current address: College of Science, Liaoning Technical University, Fu Xin 123000, China; School of Mathematics and Statistics, Science, Xi'an Jiaotong University, Xi'an 710049, China; Center of Computational Geoscience, Xi'an Jiaotong University, Xi'an 710049, China.

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in applications. A second serious problem is the definition of appropriate boundary conditions for the large scales, which is unresolved so far. Variational multiscale (VMS) method is another good numerical method for the turbulent flows, see [18]. They consider large scales which are defined by projection into appropriate spaces. It simultaneously discretizes coupled systems of both large and small scales. It has proved that VMS methods are an efficient and simple realization of the idea of introducing eddy viscosity locally into scale space only on the marginally resolved scales and tuned to add dissipation to mimic the loss of energy in the marginally resolved scales caused by breakdown of eddies to unresolved scales. This idea is inspired by Layton [31] and John and Kaya [24]. They start by writing the Navier–Stokes equations as a coupled system of three equations for the three types of scales. Then, the equation for the unresolved scales is neglected and the equation for the resolved small scales will be modelled with a turbulence model. An important feature of the VMS presented in this paper is that it allows the resolved small scales to move across faces of mesh cells. Herein a variationally consistent eddy viscosity turbulence model is introduced acting only on the discrete resolved small scales (fluctuations).

In the last 10 years, there has been an explosion of work on the VMS method for turbulent flow [17–19,21,24,25,28,31,32,43]. It was also introduced in [15,16,20] as a procedure for deriving models and numerical methods capable of dealing with multi-scale phenomena ubiquitous in science and engineering. Similar ideas, such as, subgrid modelling, three-level method, local projection stabilization and so on, are introduced in [3,6,7] and other literatures.

To introduce the idea, suppose the Navier–Stokes equations are written as

$$\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}, \mathbf{u}) + \nu A\mathbf{u} = \mathbf{f}(t). \tag{1}$$

The well-known classical Crank–Nicolson (CN) discretization in time and VMS method in space reads as follows:

$$\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\triangle t} + N\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right) + \nu A\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right) + \nu_T \nabla \cdot P'_H(\nabla \mathbf{u}_h^{n+1}) = \mathbf{f}(t^{n+1/2}),$$

where $P'_H = I - P_H$ is called the fluctuation operator with an L^2-projection defined in Equation (15). Our goal is to solve the discretized nonlinear PDEs efficiently and accurately. Many popular, efficient methods for this purpose are based on multilevel strategies and all require a linearization process somewhere in the algorithm, see [9,11,14,30,33,34,42] and so on. Usually fully implicit schemes are (almost) unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easier in computation. But it suffers a restricted time step size from the stability requirement. A popular approach is based on an implicit scheme for the linear term and a semi-implicit scheme or an explicit scheme for the nonlinear term. In [5], they proposed a first-order semi-implicit scheme using one or two steps to handle the nonlinear term in computations. Obviously, high-order schemes are of more interest since first-order schemes are not efficiently accurate for large time approximations. In this paper, we take the next step in its development by extending the first-order scheme to the second-order CN scheme. Furthermore, we only require to solve one linear system per time step:

$$\begin{aligned} &\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\triangle t} + N\left(\xi_n(\mathbf{u}_h), \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right) + \nu A\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right) + \nu_T \nabla \cdot (\mathbb{D}\mathbf{u}_h^{n+1}) \\ &= \mathbf{f}(t^{n+1/2}) + \nu_T \nabla \cdot P_H(\mathbb{D}\mathbf{u}_h^n), \end{aligned}$$

where $\xi_n(\mathbf{u}_h)$ is the extrapolation of the velocity to $t^{n+1/2}$ from the previous time levels, which is defined in Equation (17), ν_T is called the turbulent viscosity, which may like a cellwise constant or a nonlinear Smagorinsky type. It is a three time levels scheme which always is called the Crank–Nicolson-linearized-extrapolation (CNLE) discretization. Besides, the VMS method is modified by lagging the projection term onto the previous time level, i.e. $\nu_T\nabla\cdot P_H(\mathbb{D}\mathbf{u}_h^n)$. Since treating $\nu_T\nabla\cdot P_H(\mathbb{D}\mathbf{u}_h)$ explicitly is computationally less expensive than treating it implicitly, and in this case, the projection P_H is easy to construct and compute as well. We give a simple proof that the Algorithm 2.1 is unconditionally stable in Theorem 3.2 and then explore the rates of convergence for velocity in Theorem 4.2. We conclude with numerical experiments which verify the theoretical results.

The rest of the paper is organized as follows. Section 2 presents some mathematical preliminariesfor analysis and give the CNLE–VMS scheme for the Navier–Stokes equations. We prove it is unconditionally stable in Section 3. In Section 4, we analyse its error for velocity. Numerical tests are reported in Section 5, followed by conclusions in Section 6.

2. The CNLE–VMS method

The flow in Ω over time interval $[0,T]$ is governed by the time-dependent Navier–Stokes equations:

$$\frac{\partial\mathbf{u}}{\partial t}-2\nu\nabla\cdot\mathbb{D}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in }\Omega\times(0,T],\tag{2}$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in }\Omega\times[0,T],\tag{3}$$

$$\mathbf{u}=0\quad\text{on }\partial\Omega\times(0,T],\tag{4}$$

$$\mathbf{u}(0)=\mathbf{u}_0\quad\text{in }\Omega,\tag{5}$$

with the usual normalization condition that $\int_\Omega p(\mathbf{x},t)\,\mathrm{d}\mathbf{x}=0$ for $0<t\le T$. Here $\mathbf{u}(\mathbf{x},t)$ is the velocity of the fluid, $\mathbb{D}(\mathbf{u})=(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}})/2$ is the velocity deformation tensor (symmetric part of the gradient), $p(\mathbf{x},t)$ is the pressure, $\nu>0$ is the kinematic viscosity which is inversly proportional to the Reynolds number $\mathrm{Re}=\mathcal{O}(\nu^{-1})$, $\mathbf{f}(x,t)$ is the prescribed body force and the initial velocity field $\mathbf{u}_0(x)$.

Define the velocity space $\mathbf{X}$, the pressure space Q and a deformation tensor space $\mathbb{L}$ as follows:

$$\begin{aligned}&\mathbf{X}:=\mathbf{H}_0^1(\Omega)=\{\mathbf{v}:\mathbf{v}\in\mathbf{H}^1(\Omega),\mathbf{v}=0\text{ on }\partial\Omega\},\\&Q:=L_0^2(\Omega):=\{q\in L^2(\Omega),\int_\Omega q\,\mathrm{d}x=0\},\\&\mathbb{L}:=\{\mathbb{S}\in L^2(\Omega)^{d\times d}:\mathbb{S}_{ij}=\mathbb{S}_{ji}\},\end{aligned}$$

and the space $L^2(\Omega)$ is endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot,\cdot)$ and $||\cdot||$. The variational formulation is defined as follows: Find $(\mathbf{u},p)\in(\mathbf{X},Q)$ such that

$$\left(\frac{\partial\mathbf{u}}{\partial t},\mathbf{v}\right)+2\nu(\mathbb{D}\mathbf{u},\mathbb{D}\mathbf{v})+b(\mathbf{u},\mathbf{u},\mathbf{v})-(p,\nabla\cdot\mathbf{v})=(\mathbf{f},\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{X},$$

$$(\nabla\cdot\mathbf{u},q)=0\quad\forall q\in Q.\tag{6}$$

where the skew-symmetric trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=\tfrac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})-\tfrac{1}{2}(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})$$

which satisfies

$$b(\mathbf{u},\mathbf{v},\mathbf{w})=-b(\mathbf{u},\mathbf{w},\mathbf{v}),\tag{7}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{v})=0,\tag{8}$$

and the following lemma.

LEMMA 2.1 (see [30]) *Let* $\Omega \subset \mathbb{R}^2$ *or* $\mathbb{R}^3$. *For all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C||\nabla \mathbf{u}||||\nabla \mathbf{v}||||\nabla \mathbf{w}||, \tag{9}$$

and

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C\sqrt{||\mathbf{u}||||\nabla \mathbf{u}||}||\nabla \mathbf{v}||||\nabla \mathbf{w}||, \tag{10}$$

If, in addition, $\mathbf{v}, \nabla \mathbf{v} \in L^\infty(\Omega)$,

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(||\mathbf{v}||_{L^\infty(\Omega)} + ||\nabla \mathbf{v}||_{L^\infty(\Omega)})||\nabla \mathbf{u}||||\nabla \mathbf{w}||, \tag{11}$$

and

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C(||\mathbf{u}||||\nabla \mathbf{v}||_{L^\infty(\Omega)} + ||\nabla \mathbf{u}||||\mathbf{v}||_{L^\infty(\Omega)})||\mathbf{w}||. \tag{12}$$

We also define the space of divergence free functions

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0, \ \forall \, q \in Q\},$$

then the variational formulation can be simplified as: Find $\mathbf{u} \in \mathbf{V}$ such that

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) + 2\nu(\mathbb{D}\mathbf{u}, \mathbb{D}\mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \, \mathbf{v} \in \mathbf{V}. \tag{13}$$

A standard Galerkin finite-element (FE) discretization of Equation (6) is unstable in the case of the small viscosity (or the large Reynold number). It is necessary to introduce the idea of the turbulence model, which should model the action of the unresolved scales onto the resolved scales.

Let $\mathcal{T}_H$ denote a coarse FE mesh which is refined (once, twice, ...) to produce the finer mesh $\mathcal{T}_h$, so $h < H$. Let $(\mathbf{X}^h, Q^h) \subset (\mathbf{X}, Q)$ be a pair of conforming velocity-pressure FE spaces satisfying the usual inf-sup condition (see [8]): there exists a constant β independent of h such that

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in \mathbf{X}^h} \frac{(q^h, \nabla \cdot \mathbf{v}^h)}{||q^h||||\nabla \mathbf{v}^h||} \geq \beta > 0. \tag{14}$$

We assume that the spaces $\mathbf{X}^h$ and Q^h contain piecewise continuous polynomials of degree k and $k-1$, respectively, and suppose that the spaces $(\mathbf{X}^h, Q^h)$ satisfy the following approximation properties:

$$\inf_{\mathbf{v}^h \in \mathbf{X}^h} \{||\mathbf{u} - \mathbf{v}^h|| + h||\nabla(\mathbf{u} - \mathbf{v}^h)||\} \leq Ch^{k+1}|\mathbf{u}|_{k+1}.$$

Define the space of discretely divergence-free functions as follows:

$$\mathbf{V}^h = \{\mathbf{v}_h \in \mathbf{X}^h : (q_h, \nabla \cdot \mathbf{v}_h) = 0, \ \forall \, q_h \in Q^h\}.$$

We shall use a space $\mathbb{L}^H$ of 'well resolved' velocity deformations. There are two natural ways to define $\mathbb{L}^H$ (see [24,29]). If $\mathbf{X}^h$ is a higher order FE space on a given mesh, one approach is to define $\mathbb{L}^H$ by using lower order FEs on the same mesh. The second option, and only one for low order elements, is to define $\mathbb{L}^H$ on a coarse mesh leading to a two-level discretization. A semi-discrete version of the VMS model reads: find $\mathbf{u}_h : [0,T] \to X^h, p_h : [0,T] \to Q^h, \mathbb{G}_H : [0,T] \to \mathbb{L}^H$ such that

$$\begin{aligned}
&(\mathbf{u}_{ht}, \mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) + (\nu_T(\mathbb{D}\mathbf{u}_h - \mathbb{G}_H), \mathbb{D}\mathbf{v}_h)\\
&\quad = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}^h,\\
&(q_h, \nabla\cdot\mathbf{u}_h) = 0 \quad \forall\, q_h \in Q^h,\\
&(\mathbb{G}_H - \mathbb{D}\mathbf{u}_h, \mathbb{S}_H) = 0 \quad \forall\, \mathbb{S}_H \in \mathbb{L}^H,
\end{aligned}$$

The third equation implies $\mathbb{G}_H = P_H(\mathbb{D}\mathbf{u}_h)$, where P_H is a L^2-orthogonal projection from $\mathbb{L}$ onto $\mathbb{L}^H$ such that

$$(P_H(\mathbb{D}\mathbf{u}_h) - \mathbb{D}\mathbf{u}_h, \mathbb{S}_H) = 0 \quad \forall \mathbb{S}_H \in \mathbb{L}^H. \tag{15}$$

If we think of a cellwise constant ν_T per cell $K \subset \Omega$, the additional viscosity model term becomes symmetric with

$$\begin{aligned}
(\nu_T(I - P_H)\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) &= \sum_{K\in\mathcal{T}_h} \nu_T^K((I-P_H)\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h)\\
&= \sum_{K\in\mathcal{T}_h} \nu_T^K((I-P_H)\mathbb{D}\mathbf{u}_h, (I-P_H)\mathbb{D}\mathbf{v}_h)\\
&= (\nu_T(I-P_H)\mathbb{D}\mathbf{u}_h, (I-P_H)\mathbb{D}\mathbf{v}_h).
\end{aligned} \tag{16}$$

In this case, one can write the problem as: find $\mathbf{u}_h : [0,T] \to X^h, p_h : [0,T] \to Q^h$, such that

$$\begin{aligned}
&(\mathbf{u}_{ht}, \mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}_h, \mathbb{D}\mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) + (\nu_T(I-P_H)\mathbb{D}\mathbf{u}_h, (I-P_H)\mathbb{D}\mathbf{v}_h)\\
&\quad = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}^h,\\
&(q_h, \nabla\cdot\mathbf{u}_h) = 0 \quad \forall q_h \in Q^h.
\end{aligned}$$

In the analysis part of the paper, we consider a rather simple case for ν_T is a constant independent of the space variable $\mathbf{u}_h$. Later on, we consider a Smagorinsky-type model for $\nu_T = (C_\delta\delta)^2||(I - P_H)\mathbb{D}\mathbf{u}_h||_F$. Numerical analysis of such a method can be found for a constant viscosity parameter in [25], and for the piecewise constant parameter in [36] (which also introduces the so-called grad-div stabilization as a subgrid model for the pressure). John *et al.* [28] considers a Smagorinsky-type parameter, the difference is that the velocity deformation tensor is applied to the fluctuation operator P'_H in [28] first whereas the fluctuation operator is applied to the velocity deformation tensor here. All these works only analyse the semi-discretization of the VMS scheme, in the present paper, we consider a fully discrete version of the method which only need to solve a linear system per time step.

Let $t^n = n\triangle t, n = 0, 1, 2, \ldots, N$ and $T = N\triangle t$. The CN–FE discretization in time and the VMS method in space of Equation (6) reads as follows: find $\{\mathbf{u}_h^{n+1}\}_{n=0}^N \in \mathbf{X}^h, \{p_h^{n+1}\}_{n=0}^N \in Q^h$ such that

$$\begin{aligned}
&\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right), \mathbb{D}\mathbf{v}_h\right) + b\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}, \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}, \mathbf{v}_h\right)\\
&\qquad + (\nu_T(I-P_H)\mathbb{D}\mathbf{u}_h^{n+1}, (I-P_H)\mathbb{D}\mathbf{v}_h) - \left(\frac{p_h^{n+1} + p_h^n}{2}, \nabla\cdot\mathbf{v}_h\right)\\
&\quad = (\mathbf{f}(t^{n+1/2}), \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{X}^h,\\
&(q_h, \nabla\cdot\mathbf{u}_h^{n+1}) = 0 \quad \forall\, q_h \in Q^h.
\end{aligned}$$

with $\mathbf{u}_h(0,\mathbf{x}) = \mathbf{u}_h^0 \in \mathbf{V}^h$.

It is easy to see that CN–FE scheme (17) is nonlinear, we try to linearize it from two respects. Firstly, a variant of CN–FE obtained by extrapolation of the convecting velocity $\mathbf{u}_h$: for example,

$$b\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}, \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h\right) \approx b\left(\xi_n(\mathbf{u}_h), \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h\right),$$

where

$$\xi_n(\mathbf{u}_h) = \begin{cases} \mathbf{u}_h^0 & \text{for } n=0, \\ \frac{3}{2}\mathbf{u}_h^n - \frac{1}{2}\mathbf{u}_h^{n-1} & \text{for } n \geq 1. \end{cases} \tag{17}$$

This method is often called CNLE and was first studied by Baker [1]. The second- and third-order CNLE methods are introduced and analysed in [1,2].

Secondly, the viscosity model term can be written as

$$(\nu_T(I-P_H)\mathbb{D}\mathbf{u}_h^{n+1}, \mathbb{D}\mathbf{v}_h) = (\nu_T\mathbb{D}\mathbf{u}_h^{n+1}, \mathbb{D}\mathbf{v}_h) - (\nu_T P_H\mathbb{D}\mathbf{u}_h^{n+1}, \mathbb{D}\mathbf{v}_h),$$

It is not easy to construct a projection P_H of unknown $\mathbf{u}_h^{n+1}$ on the time level $n+1$, a simple and direct idea to modify it is lagging the second term onto the previous time step n. Thus, CNLE–VMS scheme is presented as follows.

ALGORITHM 2.1 (CNLE–VMS)

Step 1. Let $\mathbf{u}_h^0 = \mathbf{u}_0$*, then at the first time level, we find* $(\mathbf{u}_h^1, p_h^1) \in (\mathbf{X}^h, Q^h)$ *such that*

$$\begin{aligned} &\left(\frac{\mathbf{u}_h^1-\mathbf{u}_h^0}{\triangle t}, \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}\right), \mathbb{D}\mathbf{v}_h\right) + \nu_T(\mathbb{D}\mathbf{u}_h^1, \mathbb{D}\mathbf{v}_h) \\ &\quad + b\left(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1+\mathbf{u}_h^0}{2}, \mathbf{v}_h\right) - \left(\frac{p_h^1+p_h^0}{2}, \nabla\cdot\mathbf{v}_h\right) \\ &\quad = (\mathbf{f}(t^{1/2}), \mathbf{v}_h) + \nu_T(P_H\mathbb{D}\mathbf{u}_h^0, \mathbb{D}\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \\ &(\nabla\cdot\mathbf{u}_h^1, q_h) = 0 \quad \forall q_h \in Q^h. \end{aligned} \tag{18}$$

Step 2. For $n \geq 1$*, given* $(\mathbf{u}_h^n, p_h^n) \in (\mathbf{X}^h, Q^h)$*, find* $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in (\mathbf{X}^h, Q^h)$ *satisfying*

$$\begin{aligned} &\left(\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}\right), \mathbb{D}\mathbf{v}_h\right) + \nu_T(\mathbb{D}\mathbf{u}_h^{n+1}, \mathbb{D}\mathbf{v}_h) \\ &\quad + b\left(\frac{3\mathbf{u}_h^n-\mathbf{u}_h^{n-1}}{2}, \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h\right) - \left(\frac{p_h^{n+1}+p_h^n}{2}, \nabla\cdot\mathbf{v}_h\right) \\ &\quad = (\mathbf{f}(t^{n+1/2}), \mathbf{v}_h) + \nu_T(P_H\mathbb{D}\mathbf{u}_h^n, \nabla\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \\ &(\nabla\cdot\mathbf{u}_h^{n+1}, q_h) = 0 \quad \forall q_h \in Q^h. \end{aligned} \tag{19}$$

3. The stability of the CNLE–VMS method

In this section, we prove the unconditionally stability of Algorithm 2.1. We start it with a useful discrete version of the Gronwall lemma used in [10,39].

LEMMA 3.1 (Discrete Gronwall) *Let $D \geq 0$ and $\kappa_n, A_n, B_n, C_n \geq 0$ for any integer $n \geq 0$ and satisfy*

$$A_N + \triangle t \sum_{n=0}^{N} B_n \leq \triangle t \sum_{n=0}^{N-1} \kappa_n A_n + \triangle t \sum_{n=0}^{N} C_n + D, \quad N \geq 1,$$

then for all $\triangle t > 0$,

$$A_N + \triangle t \sum_{n=0}^{N} B_n \leq \exp\left(\triangle t \sum_{n=0}^{N-1} \kappa_n\right)\left(\triangle t \sum_{n=0}^{N} C_n + D\right), \quad N \geq 1.$$

THEOREM 3.2 *Let $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega))$, we choose cellwise constant turbulent viscosity $\nu_T \geq 0$. Algorithm* 2.1 *is unconditionally stable in the following sense, for any $l \geq 0$*

$$||\mathbf{u}_h^{l+1}||^2 + \nu \triangle t \sum_{n=0}^{l} \left\| \mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)\right\|^2 + \frac{\nu_T \triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^{l+1}||^2 + \nu_T \triangle t ||P_H \mathbb{D}\mathbf{u}_h^{l+1}||^2$$
$$\leq ||\mathbf{u}_h^0||^2 + \frac{\nu_T \triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^0||^2 + \nu_T \triangle t||P_H \mathbb{D}\mathbf{u}_h^0||^2 + \frac{\triangle t}{3\nu}\sum_{n=0}^{l}||\mathbf{f}(t^{n+1/2})||^2_{H^{-1}(\Omega)}. \quad (20)$$

Proof Choosing $\mathbf{v}_h = (\mathbf{u}_h^1 + \mathbf{u}_h^0)/2$ in Equation (18) and using the divergence-free property gives

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^1||^2 - ||\mathbf{u}_h^0||^2) + 2\nu \left\|\mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right\|^2 + \nu_T\left(\mathbb{D}\mathbf{u}_h^1 - P_H \mathbb{D}\mathbf{u}_h^0, \mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right)$$
$$= \left(\mathbf{f}(t^{1/2}), \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right). \quad (21)$$

From the orthogonality of P_H, the identity $2(a+b,a) = |a|^2 - |b|^2 + |a+b|^2$ gives

$$\nu_T\left(\mathbb{D}\mathbf{u}_h^1 - P_H\mathbb{D}\mathbf{u}_h^0, \mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right)$$
$$= \nu_T\left((I - P_H)\mathbb{D}\mathbf{u}_h^1, (I - P_H)\mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right) + \nu_T\left(P_H\mathbb{D}(\mathbf{u}_h^1 - \mathbf{u}_h^0), P_H\mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right)$$
$$= \frac{\nu_T}{4}(||(I - P_H)\mathbb{D}\mathbf{u}_h^1||^2 - ||(I - P_H)\mathbb{D}\mathbf{u}_h^0||^2 + ||(I - P_H)\mathbb{D}(\mathbf{u}_h^1 + \mathbf{u}_h^0)||^2)$$
$$+ \frac{\nu_T}{2}(||P_H\mathbb{D}\mathbf{u}_h^1||^2 - ||P_H\mathbb{D}\mathbf{u}_h^0||^2). \quad (22)$$

By using the Cauchy–Schwarz inequality on the right-hand side (RHS) of Equation (21), and then taking the above equality into it and multiplying it with $2\triangle t$ gives

$$||\mathbf{u}_h^1||^2 + \nu\triangle t\left\|\mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right)\right\|^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^1||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_h^1||^2$$
$$\leq ||\mathbf{u}_0||^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_0||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_0||^2 + \frac{\triangle t}{3\nu}||\mathbf{f}(t^{1/2})||^2_{H^{-1}}. \quad (23)$$

Similarly, for $n \geq 1$, we choose $\mathbf{v}_h = (\mathbf{u}_h^{n+1} + \mathbf{u}_h^n)/2$ in Equation (18) and using the divergence-free property gives

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + 2\nu \left\| \mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)\right\|^2 + \frac{\nu_T}{4}(||(I - P_H)\mathbb{D}\mathbf{u}_h^{n+1}||^2$$
$$- ||(I - P_H)\mathbb{D}\mathbf{u}_h^n)||^2) + \frac{\nu_T}{4}||(I - P_H)\mathbb{D}(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n)||^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}\mathbf{u}_h^{n+1}||^2 - ||P_H\mathbb{D}\mathbf{u}_h^n||^2)$$
$$= \left(\mathbf{f}(t^{n+1/2}), \frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right). \tag{24}$$

Applying Cauchy–Schwarz and Young inequalities leads to

$$\frac{1}{2\triangle t}(||\mathbf{u}_h^{n+1}||^2 - ||\mathbf{u}_h^n||^2) + \frac{\nu}{2}\left\| \mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)\right\|^2 + \frac{\nu_T}{4}(||(I - P_H)\mathbb{D}\mathbf{u}_h^{n+1}||^2$$
$$- ||(I - P_H)\mathbb{D}\mathbf{u}_h^n)||^2) + \frac{\nu_T}{2}(||P_H\mathbb{D}\mathbf{u}_h^{n+1}||^2 - ||P_H\mathbb{D}\mathbf{u}_h^n||^2) \leq \frac{1}{6\nu}||\mathbf{f}(t^{n+1/2})||^2_{H^{-1}(\Omega)}. \tag{25}$$

Multiplying it with $2\triangle t$ and summing over n from $n = 1$ to l gives

$$||\mathbf{u}_h^{l+1}||^2 + \nu\triangle t\sum_{n=1}^{l}\left\| \mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)\right\|^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^{l+1}||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_h^{l+1}||^2$$
$$\leq ||\mathbf{u}_h^1||^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^1||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_h^1||^2 + \frac{\triangle t}{3\nu}\sum_{n=1}^{l}||\mathbf{f}(t^{n+1/2})||^2_{H^{-1}(\Omega)}. \tag{26}$$

By using Equation (23) to bound the RHS of Equation (26), we obtain that for $l \geq 0$

$$||\mathbf{u}_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}\left\| \mathbb{D}\left(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^n}{2}\right)\right\|^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^{l+1}||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_h^{l+1}||^2$$
$$\leq ||\mathbf{u}_h^0||^2 + \frac{\nu_T\triangle t}{2}||(I - P_H)\mathbb{D}\mathbf{u}_h^0||^2 + \nu_T\triangle t||P_H\mathbb{D}\mathbf{u}_h^0||^2 + \frac{\triangle t}{3\nu}\sum_{n=0}^{l}||\mathbf{f}(t^{n+1/2})||^2_{H^{-1}(\Omega)}. \tag{27}$$

■

4. Error estimation

In this section, we prove the error estimate for the velocity. At the beginning, we introduce some notations and definitions,

$$e^n = \mathbf{u}(t^n) - \mathbf{u}_h^n = (\mathbf{u}(t^n) - \tilde{\mathbf{u}}^n) - (\mathbf{u}_h^n - \tilde{\mathbf{u}}^n) =: \tilde{e}^n - e_h^n, \tag{28}$$
$$\phi^n = p(t^n) - p_h^n = (p(t^n) - \tilde{p}^n) - (p_h^n - \tilde{p}^n) =: \tilde{\phi}^n - \phi_h^n. \tag{29}$$

with $(\tilde{\mathbf{u}}^n, \tilde{p}^n)$ is the Stokes projection of $(\mathbf{u}(t^n), p(t^n))$ into $(\mathbf{X}^h, Q^h)$, which defined as below.

DEFINITION 4.1 (Stokes projection) *The Stokes projection operator* $P_s : (\mathbf{X}, Q) \to (\mathbf{X}^h, Q^h)$, $P_s(\mathbf{u}, p) = (\tilde{\mathbf{u}}, \tilde{p})$, *satisfies*

$$\begin{aligned} 2\nu(\mathbb{D}(\mathbf{u} - \tilde{\mathbf{u}}), \nabla \mathbf{v}_h) - (p - \tilde{p}, \nabla \cdot \mathbf{v}_h) &= 0 \quad \forall \mathbf{v}_h \in \mathbf{X}^h, \\ (\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), q_h) &= 0 \quad \forall q_h \in Q^h. \end{aligned} \tag{30}$$

Under the discrete inf-sup condition (14), the Stokes projection is well defined. Its error satisfies (see [30])

$$\nu||\mathbb{D}(\mathbf{u} - \tilde{\mathbf{u}})||^2 \le C[\nu \inf_{\mathbf{v}_h \in \mathbf{X}^h} ||\mathbb{D}(\mathbf{u} - \mathbf{v}_h)||^2 + \nu^{-1} \inf_{q_h \in Q^h} ||p - q_h||^2], \tag{31}$$

where C is a constant independent of h and ν. we choose $\tilde{\mathbf{u}}^0 = \mathbf{u}_h^0$ in the initial error decomposition gives $e_h^0 = 0$.

For simplicity, we will denote time averages by

$$\mathbf{v}^{n+1/2} = \frac{\mathbf{v}^{n+1} + \mathbf{v}^n}{2}.$$

We assume that the exact solution satisfies the following regularity assumptions:

$$\mathbf{u} \in L^2(0, T; H^{k+1}(\Omega)^d) \cap L^\infty(0, T; H^{k+1}(\Omega)^d), \quad \mathbf{u}_t \in L^2(0, T; H^{k+1}(\Omega)^d), \tag{32}$$

$$\mathbf{u}_{tt} \in L^2(0, T; H^1(\Omega)^d), \mathbf{u}_{ttt} \in L^\infty(0, T; L^2(\Omega)^d), \quad p_{tt} \in L^\infty(0, T; L^2(\Omega)). \tag{33}$$

Remark Before providing the error estimate Theorem 4.2, it is important to comment on these regularity assumptions. Turbulent flows, the target of the VMS methods, evolve over a long time period from smooth data and are commonly initialized by either zero initial conditions that are ramped up or by a separate initialization procedure called spin up which provides smooth, compatible, statistically steady initial data. It is unknown if the regularity assumptions hold in general (even for flows that begin as C^∞)for longer time intervals. Thus, error estimates that predict reduced rates of convergence under reduced regularity assumed, while not developed herein, are of great interest and importance. It is also known since the work of Heywood and Rannacher [13] and Rautmann [35] that unless the initial data satisfies a global compatibility condition, the solution can be regular for (small) $t > 0$, but not down to $t = 0$. Like most high-order methods (see [13]), the smoothness assumed as in (32)–(33) implicitly imposes these compatibility conditions on the initial data. Thus, it is also an interesting open question to explore the effect of the regularization term on the initial layer in the error when the conditions fail.

THEOREM 4.2 *Let the FE space* $(\mathbf{X}^h, Q^h)$ *include continuous piecewise polynomials of degree k and* $k - 1$*, respectively,* $(k \ge 2)$*. If* $\triangle t$ *satisfies the condition*

$$C||\mathbf{u}||_{L^\infty(0,T;H^{k+1})} h^{k-3/2} \triangle t \le \tfrac{1}{2}, \tag{34}$$

and we choose $\nu_T = Ch^{k/2}, H = h$ *or* $\nu_T = Ch^{k-1}, H = h^{1/2}$*. There exists a constant* $\hat{C} = \hat{C}(\nu, \Omega, T, \mathbf{u}, p)$ *such that for any* $l \ge 0$

$$\begin{aligned} &\frac{1}{2}\|\mathbf{u}(t^{l+1}) - \mathbf{u}_h^{l+1}\|^2 + \triangle t \nu \sum_{n=0}^{l} \left\| \mathbb{D}\left(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1} + \mathbf{u}(t^n) - \mathbf{u}_h^n}{2} \right) \right\|^2 \\ &\quad + \frac{\nu_T \triangle t}{2} \|(I - P_H)\mathbb{D}(\mathbf{u}(t^{l+1}) - \mathbf{u}_h^{l+1})\|^2 + \nu_T \triangle t \|P_H \mathbb{D}(\mathbf{u}(t^{l+1}) - \mathbf{u}_h^{l+1})\|^2 \\ &\quad \le \hat{C}(h^{2k} + \triangle t^4). \end{aligned} \tag{35}$$

Proof First we prove the error for the first step in Algorithm 2.1. Taking the variational formulation (6) at $t^{1/2}$ gives

$$(\mathbf{u}_t(t^{1/2}), \mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}(t^{1/2}), \mathbb{D}\mathbf{v}_h) + b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), \mathbf{v}_h) - (p(t^{1/2}), \nabla \cdot \mathbf{v}_h) = (\mathbf{f}(t^{1/2}), \mathbf{v}_h), \tag{36}$$

subtracting Equation (18) from Equation (36) gives

$$\begin{aligned}&\left(\mathbf{u}_t(t^{1/2}) - \frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\triangle t}, \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\mathbf{u}(t^{1/2}) - \mathbb{D}\left(\frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}\right), \mathbb{D}\mathbf{v}_h\right)\\ &\quad + b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), \mathbf{v}_h) - b\left(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right)\\ &\quad - \nu_T(\mathbb{D}\mathbf{u}_h^1 - P_H\mathbb{D}\mathbf{u}_h^0, \mathbb{D}\mathbf{v}_h) - \left(p(t^{1/2}) - \frac{p_h^1 + p_h^0}{2}, \nabla \cdot \mathbf{v}_h\right) = 0.\end{aligned} \tag{37}$$

For the trilinear terms, by adding and subtracting $b(\mathbf{u}_h^0 - \mathbf{u}(t^0), (\mathbf{u}(t^1) + \mathbf{u}(t^0))/2, \mathbf{v}_h)$ gives

$$\begin{aligned}&b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), \mathbf{v}_h) - b\left(\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 + \mathbf{u}_h^0}{2}, \mathbf{v}_h\right)\\ &\quad = b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), \mathbf{v}_h) + b(\mathbf{u}_h^0, e^{1/2}, v_h) + b\left(e^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, \mathbf{v}_h\right)\\ &\quad\quad - b\left(\mathbf{u}(t^0), \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, \mathbf{v}_h\right).\end{aligned} \tag{38}$$

For the remaining linear terms in Equation (37), by adding and subtracting

$$\begin{aligned}&\left(\frac{\mathbf{u}(t^1) - \mathbf{u}(t^0)}{\triangle t}, \mathbf{v}_h\right) + 2\nu(\mathbb{D}\mathbf{u}(t^0), \mathbb{D}\mathbf{v}_h) + \left(\frac{p(t^1) + p(t^0)}{2}, \nabla \cdot \mathbf{v}_h\right)\\ &\quad + \nu_T(\mathbb{D}\mathbf{u}(t^1) - P_H\mathbb{D}\mathbf{u}(t^0), \mathbb{D}\mathbf{v}_h)\end{aligned}$$

gives the error equation for the first step as follows:

$$\begin{aligned}&\left(\frac{e^1 - e^0}{\triangle t}, \mathbf{v}_h\right) + 2\nu(\mathbb{D}e^{1/2}, \mathbb{D}\mathbf{v}_h) + \nu_T((I - P_H)\mathbb{D}e^1, (I - P_H)\mathbb{D}\mathbf{v}_h) + \nu_T(P_H\mathbb{D}(e^1 - e^0), P_H\mathbb{D}\mathbf{v}_h)\\ &\quad = (\phi^{1/2}, \nabla \cdot \mathbf{v}_h) - b(\mathbf{u}_h^0, e^{1/2}, \mathbf{v}_h) - b\left(e^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, \mathbf{v}_h\right) + R_1(\mathbf{u}, p; \mathbf{v}_h),\end{aligned} \tag{39}$$

with

$$\begin{aligned}R_1(\mathbf{u}, p; \mathbf{v}_h) &= \left(\frac{\mathbf{u}(t^1) - \mathbf{u}(t^0)}{\triangle t} - \mathbf{u}_t(t^{1/2}), \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2} - \mathbf{u}(t^{1/2})\right), \mathbb{D}\mathbf{v}_h\right)\\ &\quad + \nu_T(\mathbb{D}\mathbf{u}(t^1) - P_H\mathbb{D}\mathbf{u}(t^0), \mathbb{D}\mathbf{v}_h) - \left(\frac{p(t^1) + p(t^0)}{2} - p(t^{1/2}), \nabla \cdot \mathbf{v}_h\right)\\ &\quad - b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), \mathbf{v}_h) + b\left(\mathbf{u}(t^0), \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, \mathbf{v}_h\right).\end{aligned} \tag{40}$$

Using the error decomposition (28)–(29) and setting $\mathbf{v}_h = e_h^{1/2}$ gives

$$\frac{1}{2\triangle t}(||e_h^1||^2 - ||e_h^0||^2) + 2\nu||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T}{4}(||(I-P_H)\mathbb{D}e_h^1||^2 - ||(I-P_H)\mathbb{D}e_h^0||^2)$$
$$+ \nu_T||(I-P_H)\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}e_h^1||^2 - ||P_H\mathbb{D}e_h^0||^2)$$
$$= \left(\frac{\tilde{e}^1 - \tilde{e}^0}{\triangle t}, e_h^{1/2}\right) + 2\nu(\mathbb{D}\tilde{e}^{1/2}, \mathbb{D}e_h^{1/2}) + \nu_T((I-P_H)\mathbb{D}\tilde{e}^1, (I-P_H)\mathbb{D}e_h^{1/2})$$
$$+ \nu_T(P_H\mathbb{D}(\tilde{e}^1 - \tilde{e}^0), \mathbb{D}e_h^{1/2}) - (\phi^{1/2}, \nabla\cdot e_h^{1/2}) - R_1(\mathbf{u}, p; e_h^{1/2})$$
$$+ b(\mathbf{u}_h^0, \tilde{e}^{1/2}, e_h^{1/2}) + b\left(e^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, e_h^{1/2}\right). \tag{41}$$

From the definition of the Stokes projection (30), we know that

$$2\nu(\mathbb{D}\tilde{e}^{1/2}, \mathbb{D}e_h^{1/2}) - (\phi^{1/2}, \nabla\cdot e_h^{1/2}) = 0. \tag{42}$$

Applying the Cauchy–Schwarz and Young inequalities to the linear terms on the RHS of Equation (41) gives

$$\left(\frac{\tilde{e}^1 - \tilde{e}^0}{\triangle t}, e_h^{1/2}\right) + \nu_T((I-P_H)\mathbb{D}\tilde{e}^1, (I-P_H)\mathbb{D}e_h^{1/2}) + \nu_T(P_H\mathbb{D}(\tilde{e}^1 - \tilde{e}^0), \mathbb{D}e_h^{1/2})$$
$$\leq \frac{\nu}{4}\|\mathbb{D}e_h^{1/2}\|^2 + \frac{C}{\nu}\left\|\frac{\tilde{e}^1 - \tilde{e}^0}{\triangle t}\right\|^2 + \frac{2\nu_T^2}{\nu}\|P_H\mathbb{D}(\tilde{e}^1 - \tilde{e}^0)\|^2$$
$$+ \frac{\nu_T}{2}\|(I-P_H)\mathbb{D}\tilde{e}^1\|^2 + \frac{\nu_T}{2}\|(I-P_H)\mathbb{D}e_h^{1/2}\|^2. \tag{43}$$

Taking Equation (43) into Equation (41) yields

$$\frac{1}{2\triangle t}(||e_h^1||^2 - ||e_h^0||^2) + \frac{7\nu}{4}||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T}{4}(||(I-P_H)\mathbb{D}e_h^1||^2 - ||(I-P_H)\mathbb{D}e_h^0||^2)$$
$$+ \frac{\nu_T}{2}\|(I-P_H)\mathbb{D}e_h^{1/2}\|^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}e_h^1||^2 - ||P_H\mathbb{D}e_h^0||^2)$$
$$\leq \frac{C}{\nu}\left\|\frac{\tilde{e}^1 - \tilde{e}^0}{\triangle t}\right\|^2 + \frac{\nu_T}{2}\left\|(I-P_H)\mathbb{D}\tilde{e}^1\right\|^2 + \frac{2\nu_T^2}{\nu}\left\|P_H\mathbb{D}(\tilde{e}^1 - \tilde{e}^0)\right\|^2$$
$$+ |b(\mathbf{u}_h^0, \tilde{e}^{1/2}, e_h^{1/2})| + \left|b\left(e^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, e_h^{1/2}\right)\right| + |R_1(\mathbf{u}, p; e_h^{1/2})|, \tag{44}$$

For the trilinear terms on the RHS of Equation (44) is bounded by using Equation (9) and the regularity assumptions on $\mathbf{u}$,

$$|b(\mathbf{u}_h^0, \tilde{e}^{1/2}, e_h^{1/2})| \leq |b(\tilde{e}^0, \tilde{e}^{1/2}, e_h^{1/2})| + |b(\mathbf{u}(t^0), \tilde{e}^{1/2}, e_h^{1/2})|$$
$$\leq \frac{5\nu}{24}||\mathbb{D}e_h^{1/2}||^2 + \frac{C}{\nu}||\nabla\tilde{e}^{1/2}||^2 + \frac{C}{\nu}||\nabla\tilde{e}^0||^2, \tag{45}$$
$$\left|b(e^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, e_h^{1/2})\right| = \left|b(\tilde{e}^0, \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, e_h^{1/2})\right|$$
$$\leq C||\nabla\tilde{e}^0||||\mathbb{D}e_h^{1/2}|| \leq \frac{5\nu}{24}||\mathbb{D}e_h^{1/2}||^2 + \frac{C}{\nu}||\nabla\tilde{e}^0||^2. \tag{46}$$

We bound $R_1(\mathbf{u}, p; e_h^{1/2})$ as follows:

$$\left|\left(\frac{\mathbf{u}(t^1) - \mathbf{u}(t^0)}{\triangle t} - \mathbf{u}_t(t^{1/2}), e_h^{1/2}\right)\right| \leq \frac{\nu}{15}||\mathbb{D}e_h^{1/2}||^2 + C\nu^{-1}\triangle t^4 \max_{t^0 \leq t \leq t^1} ||\mathbf{u}_{ttt}(t)||^2, \quad (47)$$

$$2\nu|(\mathbb{D}(\mathbf{u}(t^1) + \mathbf{u}(t^0)2 - \mathbf{u}(t^{1/2})), \mathbb{D}e_h^{1/2})| \leq \frac{\nu}{15}||\mathbb{D}e_h^{1/2}||^2 + C\nu\triangle t^4 \max_{t^0 \leq t \leq t^1} ||\mathbb{D}\mathbf{u}_{tt}(t)||^2, \quad (48)$$

and

$$\begin{aligned}
&\nu_T|(\mathbb{D}\mathbf{u}(t^1) - P_H\mathbb{D}\mathbf{u}(t^0), \mathbb{D}e_h^{1/2})| \\
&\quad = \nu_T|((I - P_H)\mathbb{D}\mathbf{u}(t^1), (I - P_H)\mathbb{D}e_h^{1/2})| + \nu_T|(P_H\mathbb{D}(\mathbf{u}(t^1) - \mathbf{u}(t^0)), \mathbb{D}e_h^{1/2})| \\
&\quad \leq \frac{\nu}{15}||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T}{2}\|(I - P_H)\mathbb{D}e_h^{1/2}\|^2 + \frac{C\nu_T^2\triangle t}{\nu}\int_{t^0}^{t^1} ||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t + C\nu_T H^{2k+2}||\mathbf{u}(t^1)||_{k+2}^2,
\end{aligned}$$

$$\left|\left(\frac{p(t^1) + p(t^0)}{2} - p(t^{1/2}), \nabla\cdot e_h^{1/2}\right)\right| \leq \frac{\nu}{15}||\mathbb{D}e_h^{1/2}||^2 + C\nu^{-1}\triangle t^4 \max_{t^0 \leq t \leq t^1} ||p_{tt}(t)||^2.$$

For the trilinear term in R_1, Lemma 2.1 and the regularity assumption on $\mathbf{u}$ gives

$$\begin{aligned}
&\left|-b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), e_h^{1/2}) + b\left(\mathbf{u}(t^0), \frac{\mathbf{u}(t^1) + \mathbf{u}(t^0)}{2}, e_h^{1/2}\right)\right| \\
&\quad = |b(\mathbf{u}(t^0), \mathbf{u}(t^{1/2}) + C\triangle t^2\mathbf{u}_{tt}(t^\theta), e_h^{1/2}) + b(\mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), e_h^{1/2})| \\
&\quad \leq |b(\mathbf{u}(t^0) - \mathbf{u}(t^{1/2}), \mathbf{u}(t^{1/2}), e_h^{1/2})| + C\triangle t^2|b(\mathbf{u}(t^0), \mathbf{u}_{tt}(t^\theta), e_h^{1/2})| \\
&\quad \leq \triangle t|b(\mathbf{u}_t(t^\theta), \mathbf{u}(t^{1/2}), e_h^{1/2})| + C\triangle t^2|b(\mathbf{u}(t^0), \mathbf{u}_{tt}(t^\theta), e_h^{1/2})|, \qquad (49)
\end{aligned}$$

with any $t^\theta \in (0, \triangle t)$. The first term in Equation (49) can be bounded by Equation (12)

$$\begin{aligned}
&\triangle t|b(\mathbf{u}_t(t^\theta), \mathbf{u}(t^{1/2}), e_h^{1/2})| \\
&\quad \leq C\triangle t(||\mathbf{u}_t(t^\theta)||||\nabla\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)} + ||\nabla\mathbf{u}_t(t^\theta)||||\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)})||e_h^{1/2}|| \\
&\quad \leq \frac{1}{4\triangle t}||e_h^1||^2 + C\triangle t^3(||\mathbf{u}_t(t^\theta)||||\nabla\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)} + ||\nabla\mathbf{u}_t(t^\theta)||||\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)})^2.
\end{aligned}$$

The second term in Equation (49) can be bounded by

$$C\triangle t^2|b(\mathbf{u}(t^0), \mathbf{u}_{tt}(t^\theta), e_h^{1/2})| \leq \frac{\nu}{15}||\mathbb{D}e_h^{1/2}||^2 + C\nu^{-1}\triangle t^4||\nabla\mathbf{u}(t^0)||^2||\mathbf{u}_{tt}(t^\theta)||^2. \qquad (50)$$

Combining Equation (47) with Equation (50) gives

$$\begin{aligned}
|R_1(\mathbf{u}, p; e_h^{1/2})| \leq{}& \frac{\nu}{3}||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T}{2}\|(I - P_H)\mathbb{D}e_h^{1/2}\|^2 + \frac{1}{4\triangle t}||e_h^1||^2 \\
&+ C\nu^{-1}\triangle t^4 \max_{t^0 \leq t \leq t^1} ||\mathbf{u}_{ttt}(t)||^2 + C\nu\triangle t^4 \max_{t^0 \leq t \leq t^1} ||\mathbb{D}\mathbf{u}_{tt}(t)||^2 \\
&+ \frac{C\nu_T^2\triangle t}{\nu}\int_{t^0}^{t^1} ||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t + C\nu_T H^{2k+2}||\mathbf{u}(t^1)||_{k+2}^2 \\
&+ C\triangle t^3(||\mathbf{u}_t(t^\theta)||||\nabla\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)} + ||\nabla\mathbf{u}_t(t^\theta)||||\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)})^2 \\
&+ C\nu^{-1}\triangle t^4||\nabla\mathbf{u}(t^0)||^2||\mathbf{u}_{tt}(t^\theta)||^2. \qquad (51)
\end{aligned}$$

Multiplying Equation (44) with $2\triangle t$ and combining all the estimates above, we have

$$\begin{aligned}
&\frac{1}{2}||e_h^1||^2 + \nu\triangle t||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T\triangle t}{2}||(I-P_H)\mathbb{D}e_h^1||^2 + \nu_T\triangle t||P_H\mathbb{D}e_h^1||^2 \\
&\quad \leq \frac{C\triangle t}{\nu}\left\|\frac{\tilde{e}^1-\tilde{e}^0}{\triangle t}\right\|^2 + \frac{\nu_T\triangle t}{2}\|(I-P_H)\mathbb{D}\tilde{e}^1\|^2 + \frac{2\nu_T^2\triangle t}{\nu}\|P_H\mathbb{D}(\tilde{e}^1-\tilde{e}^0)\|^2 + \frac{C\triangle t}{\nu}||\nabla\tilde{e}^{1/2}||^2 \\
&\quad + \frac{C\triangle t}{\nu}||\nabla\tilde{e}^0||^2 + C\nu^{-1}\triangle t^5\max_{t^0\leq t\leq t^1}||\mathbf{u}_{ttt}(t)||^2 + C\nu\triangle t^5\max_{t^0\leq t\leq t^1}||\nabla\mathbf{u}_{tt}(t)||^2 \\
&\quad + \frac{C\nu_T^2\triangle t^2}{\nu}\int_{t^0}^{t^1}||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t + C\nu_T H^{2k+2}\triangle t||\mathbf{u}(t^1)||_{k+2}^2 \\
&\quad + C\triangle t^4(||\mathbf{u}_t(t^\theta)||||\nabla\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)} + ||\nabla\mathbf{u}_t(t^\theta)||||\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)})^2 \\
&\quad + C\nu^{-1}\triangle t^5||\nabla\mathbf{u}(t^0)||^2||\mathbf{u}_{tt}(t^\theta)||^2.
\end{aligned} \tag{52}$$

By using the interpolation inequality and the property of P_H and the approximation error, we have

$$\begin{aligned}
&\frac{1}{2}||e_h^1||^2 + \nu\triangle t||\mathbb{D}e_h^{1/2}||^2 + \frac{\nu_T\triangle t}{2}||(I-P_H)\mathbb{D}e_h^1||^2 + \nu_T\triangle t||P_H\mathbb{D}e_h^1||^2 \\
&\quad \leq \frac{Ch^{2k+2}}{\nu\triangle t}\int_{t^0}^{t^1}||\mathbf{u}_t(t)||_{k+1}^2\,\mathrm{d}t + C\nu_T\triangle t h^{2k}||\mathbf{u}(t^1)||_{k+1}^2 \\
&\quad + \frac{C\nu_T^2h^{2k}\triangle t}{\nu}\int_{t^0}^{t^1}||\mathbf{u}_t(t)||_{k+1}^2\,\mathrm{d}t + C\nu^{-1}\triangle t h^{2k}(||\mathbf{u}(t^1)||_{k+1}^2 + ||\mathbf{u}(t^0)||_{k+1}^2) \\
&\quad + C\nu^{-1}\triangle t^5\max_{t^0\leq t\leq t^1}||\mathbf{u}_{ttt}(t)||^2 + C\nu\triangle t^5\max_{t^0\leq t\leq t^1}||\mathbb{D}\mathbf{u}_{tt}(t)||^2 \\
&\quad + \frac{C\nu_T^2\triangle t^2}{\nu}\int_{t^0}^{t^1}||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t + C\nu_T H^{2k+2}\triangle t||\mathbf{u}(t^1)||_{k+2}^2 \\
&\quad + C\triangle t^4(||\mathbf{u}_t(t^\theta)||||\nabla\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)} + ||\nabla\mathbf{u}_t(t^\theta)||||\mathbf{u}(t^{1/2})||_{L^\infty(\Omega)})^2 \\
&\quad + C\nu^{-1}\triangle t^5||\nabla\mathbf{u}(t^0)||^2||\mathbf{u}_{tt}(t^\theta)||^2.
\end{aligned} \tag{53}$$

For $n \geq 1$. Taking the variational formulation (6) at $t^{n+1/2}$ gives

$$\begin{aligned}
&(\mathbf{u}_t(t^{n+1/2}), \mathbf{v}_h) + 2\nu(\mathbb{D}\mathbf{u}(t^{n+1/2}), \mathbb{D}\mathbf{v}_h) + b(\mathbf{u}(t^{n+1/2}), \mathbf{u}(t^{n+1/2}), \mathbf{v}_h) \\
&\quad - (p(t^{n+1/2}), \nabla\cdot\mathbf{v}_h) = (\mathbf{f}(t^{n+1/2}), \mathbf{v}_h),
\end{aligned} \tag{54}$$

Subtracting Equation (18) from Equation (54) gives

$$\begin{aligned}
&\left(\mathbf{u}_t(t^{n+1/2}) - \frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\triangle t}, \mathbf{v}_h\right) + 2\nu\left(\mathbb{D}\left(\mathbf{u}(t^{n+1/2}) - \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}\right), \mathbb{D}\mathbf{v}_h\right) \\
&\quad + b(\mathbf{u}(t^{n+1/2}), \mathbf{u}(t^{n+1/2}), \mathbf{v}_h) - b\left(\xi_n(\mathbf{u}_h), \frac{\mathbf{u}_h^{n+1}+\mathbf{u}_h^n}{2}, \mathbf{v}_h\right) \\
&\quad - \nu_T(\mathbb{D}\mathbf{u}_h^{n+1} - P_H\mathbb{D}\mathbf{u}_h^n, \mathbb{D}\mathbf{v}_h) - \left(p(t^{n+1/2}) - \frac{p_h^{n+1}+p_h^n}{2}, \nabla\cdot\mathbf{v}_h\right) = 0,
\end{aligned} \tag{55}$$

Adding and subtracting

$$\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\triangle t},\mathbf{v}_h\right)+2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{2}\right),\mathbb{D}\mathbf{v}_h\right)+\nu_T(\mathbb{D}\mathbf{u}(t^{n+1})-P_H\mathbb{D}\mathbf{u}(t^n),\mathbb{D}\mathbf{v}_h)$$
$$+\left(\frac{p(t^{n+1})+p(t^n)}{2},\nabla\cdot\mathbf{v}_h\right)+b\left(\mathbf{u}(t^{n+1/2}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},\mathbf{v}_h\right)$$
$$+b\left(\xi_n(\mathbf{u}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},\mathbf{v}_h\right)+b\left(\xi_n(\mathbf{u}_h),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},\mathbf{v}_h\right) \tag{56}$$

to Equation (55) to derive the error equation, where $\xi_n(\mathbf{u})=\frac{3}{2}\mathbf{u}(t^n)-\frac{1}{2}\mathbf{u}(t^{n-1})$,

$$\left(\frac{e^{n+1}-e^n}{\triangle t},\mathbf{v}_h\right)+2\nu(\mathbb{D}e^{n+1/2},\mathbb{D}\mathbf{v}_h)+\nu_T((I-P_H)\mathbb{D}e^{n+1},(I-P_H)\mathbb{D}\mathbf{v}_h)$$
$$+\nu_T(P_H\mathbb{D}(e^{n+1}-e^n),P_H\mathbb{D}\mathbf{v}_h)$$
$$=(\phi^{1/2},\nabla\cdot\mathbf{v}_h)-b(\xi_n(\mathbf{u}_h),e^{n+1/2},\mathbf{v}_h)$$
$$-b\left(\xi_n(e),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},\mathbf{v}_h\right)+R_n(\mathbf{u},p;\mathbf{v}_h). \tag{57}$$

where

$$R_n(\mathbf{u},p;\mathbf{v}_h)=\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\triangle t}-\mathbf{u}_t(t^{n+1/2}),\mathbf{v}_h\right)$$
$$+2\nu\left(\mathbb{D}\left(\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}-\mathbf{u}(t^{n+1/2})\right),\mathbb{D}\mathbf{v}_h\right)$$
$$+\nu_T(\mathbb{D}\mathbf{u}(t^{n+1})-P_H\mathbb{D}\mathbf{u}(t^n),\mathbb{D}\mathbf{v}_h)-\left(\frac{p(t^{n+1})+p(t^n)}{2}-p(t^{n+1/2}),\nabla\cdot\mathbf{v}_h\right)$$
$$+b\left(\mathbf{u}(t^{n+1/2}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}-\mathbf{u}(t^{n+1/2}),\mathbf{v}_h\right)$$
$$+b\left(\xi_n(\mathbf{u})-\mathbf{u}(t^{n+1/2}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},\mathbf{v}_h\right). \tag{58}$$

Using the error decomposition and setting $\mathbf{v}_h=e_h^{n+1/2}$ in Equation (57) gives

$$\frac{1}{2\triangle t}(||e_h^{n+1}||^2-||e_h^n||^2)+2\nu||\mathbb{D}e_h^{n+1/2}||^2+\frac{\nu_T}{4}(||(I-P_H)\mathbb{D}e_h^{n+1}||^2-||(I-P_H)\mathbb{D}e_h^n||^2)$$
$$+\nu_T||(I-P_H)\mathbb{D}e_h^{n+1/2}||^2+\frac{\nu_T}{2}(||P_H\mathbb{D}e_h^{n+1}||^2-||P_H\mathbb{D}e_h^n||^2)$$
$$=\left(\frac{\tilde{e}^{n+1}-\tilde{e}^n}{\triangle t},e_h^{n+1/2}\right)+2\nu(\mathbb{D}\tilde{e}^{n+1/2},\mathbb{D}e_h^{n+1/2})+\nu_T((I-P_H)\mathbb{D}\tilde{e}^{n+1},(I-P_H)\mathbb{D}e_h^{n+1/2})$$
$$+(P_H\mathbb{D}(\tilde{e}^{n+1}-\tilde{e}^n),\mathbb{D}e_h^{n+1/2})-(\phi^{n+1/2},\nabla\cdot e_h^{n+1/2})$$
$$+b(\xi_n(\mathbf{u}_h),\tilde{e}^{n+1/2},e_h^{n+1/2})+b\left(\xi_n(\tilde{e}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},e_h^{n+1/2}\right)$$
$$+b\left(\xi_n(e_h),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},e_h^{n+1/2}\right)-R_n(\mathbf{u},p;e_h^{n+1/2}). \tag{59}$$

From the definition of the Stokes projection, we know that

$$2\nu(\mathbb{D}\tilde{e}^{n+1/2}, \mathbb{D}e_h^{n+1/2}) - (\phi^{n+1/2}, \nabla \cdot e_h^{n+1/2}) = 0. \quad (60)$$

Applying the Cauchy–Schwarz and Young inequalities to the linear terms on the RHS of Equation (59) gives

$$\left(\frac{\tilde{e}^{n+1} - \tilde{e}^n}{\triangle t}, e_h^{n+1/2}\right) \leq \frac{\nu}{4}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}\left\|\frac{\tilde{e}^{n+1} - \tilde{e}^n}{\triangle t}\right\|^2,$$

$$\nu_T((I - P_H)\mathbb{D}\tilde{e}^{n+1}, (I - P_H)\mathbb{D}e_h^{n+1/2}) \leq \frac{\nu_T}{2}||(I - P_H)\nabla e_h^{n+1/2}||^2 + \frac{\nu_T}{2}||(I - P_H)\mathbb{D}\tilde{e}^{n+1}||^2,$$

$$(P_H\mathbb{D}(\tilde{e}^{n+1} - \tilde{e}^n), \mathbb{D}e_h^{n+1/2}) \leq \frac{\nu}{4}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C\nu_T^2}{\nu}||P_H\mathbb{D}(\tilde{e}^{n+1} - \tilde{e}^n)||^2.$$

Taking them into Equation (59) again yields

$$\begin{aligned}
&\frac{1}{2\triangle t}(||e_h^{n+1}||^2 - ||e_h^n||^2) + \frac{3\nu}{2}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T}{4}(||(I - P_H)\mathbb{D}e_h^{n+1}||^2 - ||(I - P_H)\mathbb{D}e_h^n||^2) \\
&\quad + \frac{\nu_T}{2}||(I - P_H)\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}e_h^{n+1}||^2 - ||P_H\mathbb{D}e_h^n||^2) \\
&\quad \leq \frac{C}{\nu}\left\|\frac{\tilde{e}^{n+1} - \tilde{e}^n}{\triangle t}\right\|^2 + \frac{\nu_T}{2}||(I - P_H)\mathbb{D}\tilde{e}^{n+1}||^2 + \frac{C\nu_T^2}{\nu}||P_H\mathbb{D}(\tilde{e}^{n+1} - \tilde{e}^n)||^2 \\
&\quad + |b(\xi_n(\mathbf{u}_h), \tilde{e}^{n+1/2}, e_h^{n+1/2})| + \left|b\left(\xi_n(\tilde{e}), \frac{\mathbf{u}(t^{n+1}) + \mathbf{u}(t^n)}{2}, e_h^{n+1/2}\right)\right| \\
&\quad + \left|b\left(\xi_n(e_h), \frac{\mathbf{u}(t^{n+1}) + \mathbf{u}(t^n)}{2}, e_h^{n+1/2}\right)\right| + |R_n(\mathbf{u}, p; e_h^{n+1/2})|. \quad (61)
\end{aligned}$$

For the trilinear terms on the RHS of Equation (61), we analyse them individually. For the first one, by using Lemma 2.1, we get

$$\begin{aligned}
&|b(\xi_n(\mathbf{u}_h), \tilde{e}^{n+1/2}, e_h^{n+1/2})| \\
&\quad \leq |b(\xi_n(e), \tilde{e}^{n+1/2}, e_h^{n+1/2})| + |b(\xi_n(\mathbf{u}), \tilde{e}^{n+1/2}, e_h^{n+1/2})| \\
&\quad \leq |b(\xi_n(\tilde{e}), \tilde{e}^{n+1/2}, e_h^{n+1/2})| + |b(\xi_n(e_h), \tilde{e}^{n+1/2}, e_h^{n+1/2})| + |b(\xi_n(\mathbf{u}), \tilde{e}^{n+1/2}, e_h^{n+1/2})|. \quad (62)
\end{aligned}$$

For the first one in Equation (62), from Equation (9), we have

$$\begin{aligned}
|b(\xi_n(\tilde{e}), \tilde{e}^{n+1/2}, e_h^{n+1/2})| &\leq C||\nabla\xi_n(\tilde{e})||||\nabla\tilde{e}^{n+1/2}||||\nabla e_h^{n+1/2}|| \\
&\leq \frac{\nu}{18}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}||\nabla\xi_n(\tilde{e})||^2||\nabla\tilde{e}^{n+1/2}||^2 \\
&\leq \frac{\nu}{18}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)||\nabla\tilde{e}^{n+1/2}||^2. \quad (63)
\end{aligned}$$

For the second one in Equation (62), from Equation (10), we have

$$\begin{aligned}
|b(\xi_n(e_h), \tilde{e}^{n+1/2}, e_h^{n+1/2})| &\leq C||e_h^n||^{1/2}||e_h^{n-1}||^{1/2}||\nabla\tilde{e}^{n+1/2}||||\nabla e_h^{n+1/2}|| \\
&\leq CC_Ih^{-3/2}||e_h^n||^{1/2}||e_h^{n-1}||^{1/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^{n+1}|| + ||e_h^n||) \quad (64) \\
&\leq CC_Ih^{-3/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^n|| + ||e_h^{n-1}||)(||e_h^{n+1}|| + ||e_h^n||).
\end{aligned}$$

For the third one in Equation (62), from Equation (9) and the regularity of $\mathbf{u}$, we have

$$|b(\xi_n(\mathbf{u}), \tilde{e}^{n+1/2}, e_h^{n+1/2})| \le C||\nabla\xi_n(\mathbf{u})||||\nabla\tilde{e}^{n+1/2}||||\nabla e_h^{n+1/2}|| \\ \le \frac{\nu}{18}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}||\nabla\tilde{e}^{n+1/2}||^2 \tag{65}$$

Taking Equations (63)–(65) into Equation (62), we have

$$|b(\xi_n(\mathbf{u}_h), \tilde{e}^{n+1/2}, e_h^{n+1/2})| \le \frac{\nu}{6}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}||\nabla\tilde{e}^{n+1/2}||^2 \\ + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)||\nabla\tilde{e}^{n+1/2}||^2 \\ + Ch^{-3/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^n|| + ||e_h^{n-1}||)(||e_h^{n+1}|| + ||e_h^n||). \tag{66}$$

For the second trilinear term in Equation (61), using Lemma 2.1 and the regularity assumption of $\mathbf{u}$, we have

$$\left|b\left(\xi_n(\tilde{e}), \frac{\mathbf{u}(t^{n+1}) + \mathbf{u}(t^n)}{2}, e_h^{n+1/2}\right)\right| \le C||\nabla\xi_n(\tilde{e})||||\mathbb{D}e_h^{n+1/2}|| \\ \le \frac{\nu}{6}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2). \tag{67}$$

The last trilinear term in Equation (61) is bounded by the third inequality in Lemma 2.1,

$$\left|b\left(\xi_n(e_h), \frac{\mathbf{u}(t^{n+1}) + \mathbf{u}(t^n)}{2}, e_h^{n+1/2}\right)\right| \le C||\xi_n(e_h)||||\mathbb{D}e_h^{n+1/2}|| \\ \le \frac{\nu}{6}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{C}{\nu}(||e_h^n||^2 + ||e_h^{n-1}||^2). \tag{68}$$

Combing Equations (66)–(68) with Equation (61) gives

$$\frac{1}{2\triangle t}(||e_h^{n+1}||^2 - ||e_h^n||^2) + \nu||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T}{4}(||(I-P_H)\mathbb{D}e_h^{n+1}||^2 - ||(I-P_H)\mathbb{D}e_h^n||^2) \\ + \frac{\nu_T}{2}||(I-P_H)\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}e_h^{n+1}||^2 - ||P_H\mathbb{D}e_h^n||^2) \\ \le \frac{C}{\nu}\left\|\frac{\tilde{e}^{n+1} - \tilde{e}^n}{\triangle t}\right\|^2 + \frac{\nu_T}{2}\|(I-P_H)\mathbb{D}\tilde{e}^{n+1}\|^2 + \frac{C\nu_T^2}{\nu}\|P_H\mathbb{D}(\tilde{e}^{n+1} - \tilde{e}^n)\|^2 \\ + \frac{C}{\nu}||\nabla\tilde{e}^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)||\nabla\tilde{e}^{n+1/2}||^2 \\ + Ch^{-3/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^n|| + ||e_h^{n-1}||)(||e_h^{n+1}|| + ||e_h^n||) \\ + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2) + \frac{C}{\nu}(||e_h^n||^2 + ||e_h^{n-1}||^2) + |R_n(\mathbf{u}, p; e_h^{n+1/2})|. \tag{69}$$

Now we estimate the last term $|R_n(\mathbf{u},p;e_h^{n+1/2})|$, each of its linear terms can be bounded by the Cauchy–Schwarz and Young's inequalities,

$$\left|\left(\frac{\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)}{\triangle t}-\mathbf{u}_t(t^{n+1/2}),e_h^{n+1/2}\right)\right|$$
$$\leq \frac{\nu}{10}||\mathbb{D}e_h^{n+1/2}||^2+C\nu^{-1}\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||^2,$$
$$\nu\left|\left(\mathbb{D}\left(\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}-\mathbf{u}(t^{n+1/2})\right),\mathbb{D}e_h^{n+1/2}\right)\right|$$
$$\leq \frac{\nu}{10}||\mathbb{D}e_h^{n+1/2}||^2+C\nu\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2,$$
$$\left|\left(\frac{p(t^{n+1})+p(t^n)}{2}-p(t^{n+1/2}),\nabla\cdot e_h^{n+1/2}\right)\right|$$
$$\leq \frac{\nu}{10}||\mathbb{D}e_h^{n+1/2}||^2+C\nu^{-1}\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||p_{tt}(t)||^2$$

as well as

$$\nu_T|(\mathbb{D}\mathbf{u}(t^{n+1})-P_H\mathbb{D}\mathbf{u}(t^n),\nabla e_h^{n+1/2})|$$
$$=\nu_T|((I-P_H)\mathbb{D}\mathbf{u}(t^{n+1}),(I-P_H)\mathbb{D}e_h^{n+1/2})|+\nu_T|(P_H\mathbb{D}(\mathbf{u}(t^{n+1})-\mathbf{u}(t^n)),\mathbb{D}e_h^{n+1/2})|$$
$$\leq \frac{\nu}{10}||\mathbb{D}e_h^{n+1/2}||^2+\frac{\nu_T}{2}||(I-P_H)\mathbb{D}e_h^{n+1/2}||^2+\frac{C\nu_T^2\triangle t}{\nu}\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t$$
$$+C\nu_T H^{2k+2}||\mathbf{u}(t^{n+1})||_{k+2}^2.$$

From Lemma 2.1 and the regularity assumption of $\mathbf{u}$, the trilinear term in $R_n(\cdot,\cdot;\cdot)$ can be bounded by

$$\left|b\left(\mathbf{u}(t^{n+1/2}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}-\mathbf{u}(t^{n+1/2}),e_h^{n+1/2}\right)\right|$$
$$\leq C\left\|\nabla\left(\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}-\mathbf{u}(t^{n+1/2})\right)\right\|\|\nabla\mathbf{u}(t^{n+1/2})\|\|\nabla e_h^{n+1/2}\|$$
$$\leq \frac{\nu}{20}\|\mathbb{D}e_h^{n+1/2}\|^2+C\nu^{-1}\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2.$$

and

$$\left|b\left(\xi_n(\mathbf{u})-\mathbf{u}(t^{n+1/2}),\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2},e_h^{n+1/2}\right)\right|$$
$$\leq C\|\nabla(\xi_n(\mathbf{u})-\mathbf{u}(t^{n+1/2}))\|\left\|\nabla\left(\frac{\mathbf{u}(t^{n+1})+\mathbf{u}(t^n)}{2}\right)\right\|\|\nabla e_h^{n+1/2}\|$$
$$\leq \frac{\nu}{20}\|\mathbb{D}e_h^{n+1/2}\|^2+C\nu^{-1}\triangle t^4\max_{t^{n-1}\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2.$$

Combining these estimations above, we have

$$|R_n(\mathbf{u},p;e_h^{n+1/2})|\leq\frac{\nu}{2}||\mathbb{D}e_h^{n+1/2}||^2+\frac{\nu_T}{2}||(I-P_H)\mathbb{D}e_h^{n+1/2}||^2+C\nu^{-1}\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\mathbf{u}_{ttt}(t)||^2$$
$$+C\nu\triangle t^4\max_{t^n\leq t\leq t^{n+1}}||\nabla\mathbf{u}_{tt}(t)||^2+C\nu^{-1}\nu_T^2\triangle t\int_{t^n}^{t^{n+1}}||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t$$

$$+ C\nu_T H^{2k+2}||\mathbf{u}(t^{n+1})||_{k+2}^2 + C\nu^{-1}\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||p_{tt}(t)||^2$$
$$+ C\nu^{-1}\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||\nabla \mathbf{u}_{tt}(t)||^2 + C\nu^{-1}\triangle t^4 \max_{t^{n-1} \le t \le t^{n+1}} ||\nabla \mathbf{u}_{tt}(t)||^2. \quad (70)$$

Combining Equation (70) with the error equation (69) gives

$$\frac{1}{2\triangle t}(||e_h^{n+1}||^2 - ||e_h^n||^2) + \frac{\nu}{2}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T}{2}(||P_H\mathbb{D}e_h^{n+1}||^2 - ||P_H\mathbb{D}e_h^n||^2)$$
$$+ \frac{\nu_T}{4}(||(I-P_H)\mathbb{D}e_h^{n+1}||^2 - ||(I-P_H)\mathbb{D}e_h^n||^2)$$
$$\le \frac{C}{\nu}\left\|\frac{\tilde{e}^{n+1}-\tilde{e}^n}{\triangle t}\right\|^2 + \frac{\nu_T}{2}\|(I-P_H)\mathbb{D}\tilde{e}^{n+1}\|^2 + \frac{C\nu_T^2}{\nu}\|P_H\mathbb{D}(\tilde{e}^{n+1}-\tilde{e}^n)\|^2$$
$$+ \frac{C}{\nu}||\nabla\tilde{e}^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)||\nabla\tilde{e}^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)$$
$$+ Ch^{-3/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^n|| + ||e_h^{n-1}||)(||e_h^{n+1}|| + ||e_h^n||)$$
$$+ \frac{C}{\nu}(||e_h^n||^2 + ||e_h^{n-1}||^2) + C\nu^{-1}\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||\mathbf{u}_{ttt}(t)||^2$$
$$+ C\nu\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||\nabla\mathbf{u}_{tt}(t)||^2 + C\nu^{-1}\nu_T^2\triangle t\int_{t^n}^{t^{n+1}} ||\nabla\mathbf{u}_t(t)||^2\,\mathrm{d}t$$
$$+ C\nu_T H^{2k+2}||\mathbf{u}(t^{n+1})||_{k+2}^2 + C\nu^{-1}\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||p_{tt}(t)||^2$$
$$+ C\nu^{-1}\triangle t^4 \max_{t^n \le t \le t^{n+1}} ||\nabla\mathbf{u}_{tt}(t)||^2 + C\nu^{-1}\triangle t^4 \max_{t^{n-1} \le t \le t^{n+1}} ||\nabla\mathbf{u}_{tt}(t)||^2. \quad (71)$$

From the interpolation property, we bound the terms including $\tilde{e}^n, \tilde{e}^{n+1/2}$ or $\tilde{e}^{n+1}$ in Equation (71) below,

$$\frac{C}{\nu}\left\|\frac{\tilde{e}^{n+1}-\tilde{e}^n}{\triangle t}\right\|^2 \le \frac{C}{\nu\triangle t}\int_{t^n}^{t^{n+1}} ||\tilde{e}_t(t)||^2\,\mathrm{d}t \le \frac{Ch^{2k+2}}{\nu\triangle t}\int_{t^n}^{t^{n+1}} ||\mathbf{u}_t(t)||_{k+1}^2\,\mathrm{d}t, \quad (72)$$
$$\frac{\nu_T}{2}||(I-P_H)\nabla\tilde{e}^{n+1}||^2 \le C\nu_T||\nabla\tilde{e}^{n+1}||^2 \le C\nu_T h^{2k}||\mathbf{u}(t^{n+1})||_{k+1}^2, \quad (73)$$
$$\frac{C\nu_T^2}{\nu}||P_H\nabla(\tilde{e}^{n+1}-\tilde{e}^n)||^2 \le \frac{C\nu_T^2 h^{2k}\triangle t}{\nu}\int_{t^n}^{t^{n+1}} ||\mathbf{u}_t(t)||_{k+1}^2\,\mathrm{d}t, \quad (74)$$
$$\frac{C}{\nu}||\nabla\tilde{e}^{n+1/2}||^2 + \frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)$$
$$\le \frac{Ch^{2k}}{\nu}(||\mathbf{u}(t^{n+1})||_{k+1}^2 + ||\mathbf{u}(t^n)||_{k+1}^2 + ||\mathbf{u}(t^{n-1})||_{k+1}^2), \quad (75)$$
$$\frac{C}{\nu}(||\nabla\tilde{e}^n||^2 + ||\nabla\tilde{e}^{n-1}||^2)||\nabla\tilde{e}^{n+1/2}||^2$$
$$\le \frac{Ch^{4k}}{\nu}(||\mathbf{u}(t^{n+1})||_{k+1}^2 + ||\mathbf{u}(t^n)||_{k+1}^2)(||\mathbf{u}(t^n)||_{k+1}^2 + ||\mathbf{u}(t^{n-1})||_{k+1}^2), \quad (76)$$
$$Ch^{-3/2}||\nabla\tilde{e}^{n+1/2}||(||e_h^n|| + ||e_h^{n-1}||)(||e_h^{n+1}|| + ||e_h^n||)$$
$$\le Ch^{k-3/2}(||\mathbf{u}(t^{n+1})||_{k+1} + ||\mathbf{u}(t^n)||_{k+1})(||e_h^{n+1}||^2 + ||e_h^n||^2 + ||e_h^{n-1}||^2). \quad (77)$$

Multiplying both side of Equation (69) by $2\triangle t$ and summing it over n from 1 to l, combined with the first time step error (53), we have

$$
\begin{aligned}
&||e_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T\triangle t}{2}||(I-P_H)\mathbb{D}e_h^{l+1}||^2 + \nu_T\triangle t||P_H\mathbb{D}e_h^{l+1}||^2\\
&\le \frac{Ch^{2k+2}}{\nu}||\mathbf{u}_t||^2_{L^2(0,T;H^{k+1}(\Omega))} + C\nu_T h^{2k}||\mathbf{u}||^2_{L^2(0,T;H^{k+1}(\Omega))} + \frac{C\nu_T^2h^{2k}\triangle t^2}{\nu}||\mathbf{u}_t||^2_{L^2(0,T;H^{k+1}(\Omega))}\\
&\quad + \frac{Ch^{2k}}{\nu}||\mathbf{u}||^2_{L^2(0,T;H^{k+1}(\Omega))} + \frac{C\triangle t^5}{\nu}(||\mathbf{u}_{ttt}||^2_{L^\infty(0,T;L^2(\Omega))} + ||p_{tt}||^2_{L^\infty(0,T;L^2(\Omega))}\\
&\quad + ||\nabla\mathbf{u}_{tt}||^2_{L^\infty(0,T;L^2(\Omega))}) + C\nu\triangle t^4||\mathbf{u}_{ttt}||^2_{L^\infty(0,T;L^2(\Omega))} + C\nu^{-1}\nu_T^2\triangle t^2||\nabla\mathbf{u}_t||^2_{L^2(0,T;L^2(\Omega))}\\
&\quad + C\nu_T H^{2k+2}||\mathbf{u}||^2_{L^2(0,T;H^{k+2}(\Omega))} + C(\nu^{-1} + h^{k-3/2}||\mathbf{u}||_{L^\infty(0,T;H^{k+1})})\triangle t\sum_{n=0}^{l}||e_h^n||^2\\
&\quad + C||\mathbf{u}||_{L^\infty(0,T;H^{k+1})}h^{k-3/2}\triangle t||e_h^{l+1}||^2.
\end{aligned}
$$

We assume that

$$C||\mathbf{u}||_{L^\infty(0,T;H^{k+1})}h^{k-3/2}\triangle t \le \tfrac{1}{2}, \tag{78}$$

then by using the regularity of $\mathbf{u}$ and p, Equation (78) becomes

$$
\begin{aligned}
&\frac{1}{2}||e_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T\triangle t}{2}||(I-P_H)\mathbb{D}e_h^{l+1}||^2 + \nu_T\triangle t||P_H\mathbb{D}e_h^{l+1}||^2\\
&\quad \le C(\nu^{-1} + h^{k-3/2}||\mathbf{u}||_{L^\infty(0,T;H^{k+1})})\triangle t\sum_{n=0}^{l}||e_h^n||^2\\
&\qquad + C(h^{2k} + \nu_T h^{2k} + \nu_T^2h^{2k}\triangle t^2 + \nu_T^2\triangle t^2 + \nu_T H^{2k+2}).
\end{aligned}
$$

Finally, it follows from the discrete Gronwall lemma, that there exists $\hat{C} = \hat{C}(\nu, \Omega, T, \mathbf{u}, p)$ such that for any $l \ge 0$

$$
\begin{aligned}
&\frac{1}{2}||e_h^{l+1}||^2 + \nu\triangle t\sum_{n=0}^{l}||\mathbb{D}e_h^{n+1/2}||^2 + \frac{\nu_T\triangle t}{2}||(I-P_H)\mathbb{D}e_h^{l+1}||^2 + \nu_T\triangle t||P_H\mathbb{D}e_h^{l+1}||^2\\
&\quad \le \hat{C}(h^{2k} + \nu_T h^{2k} + \nu_T^2h^{2k}\triangle t^2 + \nu_T^2\triangle t^2 + \nu_T H^{2k+2}).
\end{aligned} \tag{79}
$$

The final result is easily obtained by using the triangle inequality and choosing $\nu_T = Ch^{k/2}, H = h$ or $\nu_T = Ch^{k-1}, H = h^{1/2}$. ■

Remark We prove that our scheme is conditionally convergent, but condition (34) is not necessary theoretically, it might be removed. However, we could not make it herein. We would like to leave it as an open question.

COROLLARY 4.3 *Let* $(\mathbf{X}^h, Q^h)$ *is a Taylor–Hood FE pair, under the assumption* (*34*), *we choose* $\nu_T = Ch, H = h$ *or* $h^{1/2}$. *There exists a constant* $\hat{C} = \hat{C}(\nu, \Omega, T, \mathbf{u}, p)$ *such that for any* $l \ge 0$

$$
\begin{aligned}
&\frac{1}{2}\|\mathbf{u}(t^{l+1}) - \mathbf{u}_h^{l+1}\|^2 + \triangle t\nu\sum_{n=0}^{l}\left\|\mathbb{D}\left(\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}_h^{n+1} + \mathbf{u}(t^n) - \mathbf{u}_h^n}{2}\right)\right\|^2\\
&\quad + \frac{\nu_T\triangle t}{2}\|(I-P_H)\mathbb{D}(\mathbf{u}(t^{l+1}) - \mathbf{u}_h^{l+1})\|^2 \le \hat{C}(h^4 + \triangle t^4).
\end{aligned} \tag{80}
$$

5. Numerical results

In all experiments, algorithms are implemented by using the public domain FE software Freefem++ [12]. The Taylor–Hood elements are chosen for the velocity-pressure FE spaces $(\mathbf{X}^h, Q^h)$, the well-resolved space $\mathbb{L}^H$ is the piecewise constant space on the given mesh.

5.1 *Rates of convergence study*

Let us consider Ω as the unit square in $\mathbb{R}^2$. The uniform mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square in the same direction. Here,according to the numeral analysis, we set $\nu_T = 0.1h$.

We choose the true solution $(\mathbf{u} = (u_1, u_2), p)$ as follows:

$$u_1 = -\cos(\pi x)\sin(\pi y)\exp\left(\frac{-2\pi^2 t}{\mathrm{Re}}\right),$$

$$u_2 = \sin(\pi x)\cos(\pi y)\exp\left(\frac{-2\pi^2 t}{\mathrm{Re}}\right),$$

$$p = -0.25(\cos(2\pi x) + \cos(2\pi y))\exp\left(\frac{-4\pi^2 t}{\mathrm{Re}}\right),$$

which is the solution of an interesting test problem of simulating decay of the Green–Taylor vortex. It was used as a numerical test in Chorin [4], Tafti [41] and John and Layton [26].

To demonstrate the convergence with respect to time step $\triangle t$, we choose $h = \triangle t/10$, which means that h is much smaller than $\triangle t$. If we denote the approximation errors by $O(\triangle t^\gamma) + O(h^\mu)$, then for the errors by using the CNLE–VMS method and CN–VMS method (CN time discretization and full implicit scheme) for $\mathbf{u}_h^{n+1}$ in L^2 norm, $\gamma < \mu$. This implies $\triangle t^\gamma \gg h^\mu$. In this case, the approximation errors are not dominated by the h-term $O(h^\mu)$ when h varies in the given range. Thus, we list the errors for $\mathbf{u}_h^{n+1}$ in L^2 norm with both spacing h and time step $\triangle t$ decreasing for both CN–VMS and CNLE–VMS schemes in Table 1, we also compare their CPU costs. The errors listed in the second and fourth column in Table 1 are similar, which means that the CNLE–VMS method is comparable to the CN–VMS method. However, CPU cost by CNLE–VMS method is relatively less than CN–VMS method. As expected, since CNLE–VMS is the linearized version of the CN–VMS method, which does not include any iterations in computing. Moreover, from the orders listed in third and sixth column in Table 1, we can see that with the relatively smaller h, both the CN–VMS method and the CNLE–VMS method are second-order convergent in time.

Table 1. Errors and CPU costs by using both the CN–VMS method and the CNLE–VMS method, with Re = 1000.

h	Δt	$\Vert\mathbf{u}-\mathbf{u}_h\Vert^{CN}_{L^2(0,T;L^2)}$	Order	CPU^{CN}	$\Vert\mathbf{u}-\mathbf{u}_h\Vert^{CNLE}_{L^2(0,T;L^2)}$	Order	CPU^{CNLE}
$\frac{1}{40}$	$\frac{1}{4}$	0.000201355	–	16.25	0.000200775	–	12.328
$\frac{1}{80}$	$\frac{1}{8}$	3.78435e−005	2.41162	131.765	3.78089e−005	2.40878	98.656
$\frac{1}{120}$	$\frac{1}{12}$	1.30953e−005	2.61726	469.437	1.30879e−005	2.61639	348.407
$\frac{1}{160}$	$\frac{1}{16}$	6.02309e−006	2.69969	1176.7	6.02073e−006	2.69909	827

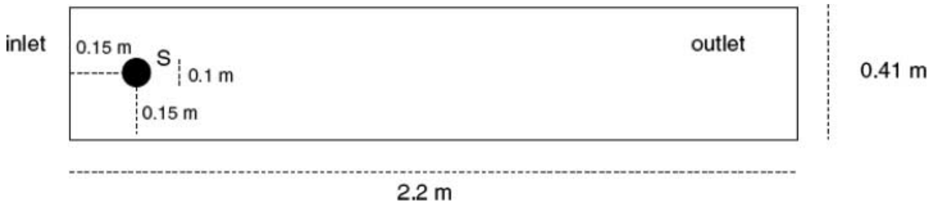

Figure 1. Domain Ω of the test problem.

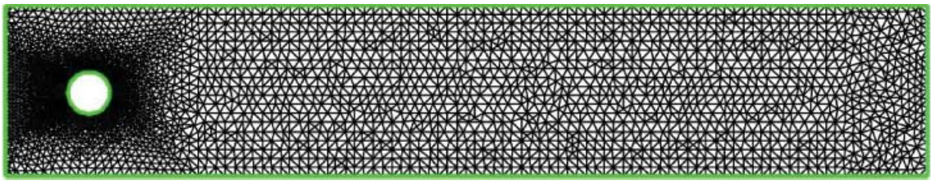

Figure 2. The triangulation of the computational domain for the CNLE–VMS method.

5.2 *Flow around a cylinder*

The second example is the 'flow around a cylinder' which is a popular benchmark problem for testing numerical schemes. The Ω is the channel with the cylinder presented in Figure 1. This is a well-known benchmark problem taken from Shäfer and Turek [38] and John [23]. The domain Ω with 7510 triangles is presented in Figure 2. The time-dependent inflow and outflow profile are showed as follows

$$u_1(0,y,t) = u_1(2.2,y,t) = \frac{6}{0.41^2}\sin\left(\frac{\pi t}{8}\right)y(0.41-y),$$
$$u_2(0,y,t) = u_2(2.2,y,t) = 0.$$

No-slip conditions are prescribed at the other boundaries. Computations are performed for the Reynolds number corresponding to $\nu = 10^{-3}$, and the external force $\mathbf{f} = 0$. Different meshes (1864, 4236, 7516 triangles) with different time step sizes (0.02, 0.01, 0.005) are used, and linear viscosity $\nu_T = 0.1h$ and nonlinear Smagorinsky viscosity (81) are compared, with $C_\delta = 0.1, \delta = h, h = \min_{T\in\tau_h}\{\text{diam}(T)\}$.

In this section, the nonlinear Smagorinsky model [37,40], which is a simple and popular LES model with

$$\nu_T = (C_\delta \delta)^2 ||(I - P_H)\mathbb{D}\mathbf{u}_h||_F \tag{81}$$

is also used for the turbulent viscosity. Here, C_δ is a constant, δ is the filter width of LES, which is related to the mesh width h, and $||\cdot||_F$ denotes the Frobenius norm of a tensor.

The parameters of interest are the drag coefficient $c_d(t)$ at the cylinder, the lift coefficient $c_l(t)$ and the difference of the pressure between the front and the back of the cylinder

$$\triangle p(t) = p(t; 0.15, 0.2) - p(t; 0.25, 0.2).$$

The definitions of $c_d(t)$ and $c_l(t)$ in reference [38] are as follows:

$$c_d(t) = \frac{2}{\rho L U_{\max}^2}\int_S \left(\rho\nu \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_y - p(t) n_x\right) \mathrm{d}S, \tag{82}$$

$$c_l(t) = -\frac{2}{\rho L U_{\max}^2}\int_S \left(\rho\nu \frac{\partial \mathbf{u}_{t_S}(t)}{\partial n} n_x + p(t) n_y\right) \mathrm{d}S. \tag{83}$$

Here $n = (n_x, n_y)^{\mathrm{T}}$ is the normal vector on S directing into Ω, $t_S = (n_y, -n_x)^{\mathrm{T}}$ the tangential vector and $\mathbf{u}_{t_S}$ the tangential velocity. A straightforward calculation gives

$$c_d(t) = -20 \int_{\Omega} [\nu \nabla \mathbf{u}(t) : \nabla \mathbf{v}_d + (\mathbf{u}(t)\cdot)\mathbf{u}(t) \cdot \mathbf{v}_d - p(t)(\nabla \cdot \mathbf{v}_d)] \,\mathrm{d}x\,\mathrm{d}y \tag{84}$$

for all functions $\mathbf{v}_d \in (H^2(\Omega))^2$ with $(\mathbf{v}_d)|_S = (1, 0)^{\mathrm{T}}$ and $\mathbf{v}_d$ vanishes on all other boundaries. Similarly, one obtains

$$c_l(t) = -20 \int_{\Omega} [\nu \nabla \mathbf{u}(t) : \nabla \mathbf{v}_l + (\mathbf{u}(t)\cdot)\mathbf{u}(t) \cdot \mathbf{v}_l - p(t)(\nabla \cdot \mathbf{v}_l)] \,\mathrm{d}x\,\mathrm{d}y \tag{85}$$

for all test functions $\mathbf{v}_l \in (H^2(\Omega))^2$ with $(\mathbf{v}_l)|_S = (0, 1)^{\mathrm{T}}$ and $\mathbf{v}_l$ vanishes on all other boundaries. We have the experience that the volume integral formulations (84), (85) are more accurate and less sensitive to the approximation of the circular boundary S than the line integral (82), (83), see [22]. The actual choice of $\mathbf{v}_d$ and $\mathbf{v}_l$ in our computations is the same as in the steady-state problem investigated in [27].

Since it is not positive to give the complete data of $c_d(t)$, $c_l(t)$ and $\triangle p(t)$ as reference values, we will concentrate on one special value for each parameter. As proposed in [38], we take the maximal drag and lift coefficients together with the corresponding times and the final pressure difference $\triangle p(8s)$. The values for the maximal drag $c_{d,\max}$, maximal lift $c_{l,\max}$ and $\triangle p(8s)$ (here $\triangle p(t) = p(t; 0.15, 0.2) - p(t; 0.25, 0.2)$) for both CNLE–VMS methods with constant and Smagorinsky viscosities are presented in Tables 2 and 3, respectively. The following reference intervals are

Table 2. Results maximal drag $c_{d,\max}$, maximal lift $c_{l,\max}$ and $\triangle p(8s)$ by CNLE–VMS with linear viscosity.

Mesh	$\triangle t$	$t(c_{d,\max})$	$c_{d,\max}$	$t(c_{l,\max})$	$c_{l,\max}$	$\triangle p(8s)$
1864	0.02	3.92	2.86688	5.46	0.665305	−0.104658
4236	0.01	3.94	2.91683	5.72	0.543051	−0.108992
7516	0.005	3.93	2.93339	5.715	0.490709	−0.110030

Table 3. Results maximal drag $c_{d,\max}$, maximal lift $c_{l,\max}$ and $\triangle p(8s)$ by CNLE–VMS with the Smagorinsky viscosity.

Mesh	$\triangle t$	$t(c_{d,\max})$	$c_{d,\max}$	$t(c_{l,\max})$	$c_{l,\max}$	$\triangle p(8s)$
1864	0.02	3.92	2.91882	5.42	0.615220	−0.094712
4236	0.01	3.94	2.93886	5.7	0.528322	−0.107860
7516	0.005	3.93	2.94649	5.705	0.481234	−0.109300

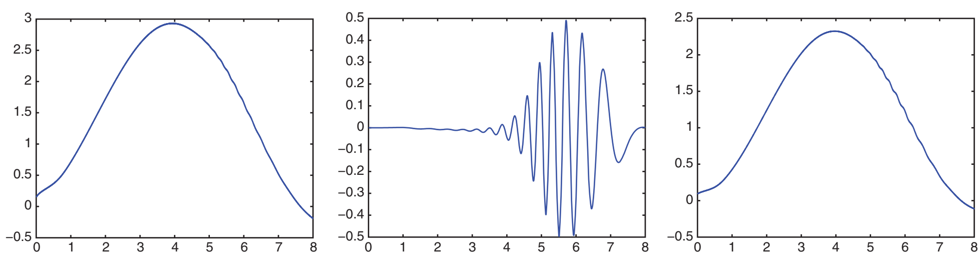

Figure 3. The evolutions of $c_{d,\max}$, $c_{l,\max}$ and $\triangle p$ by the CNLE–VMS method with constant viscosity and 7516 triangles and $\triangle t = 0.005$.

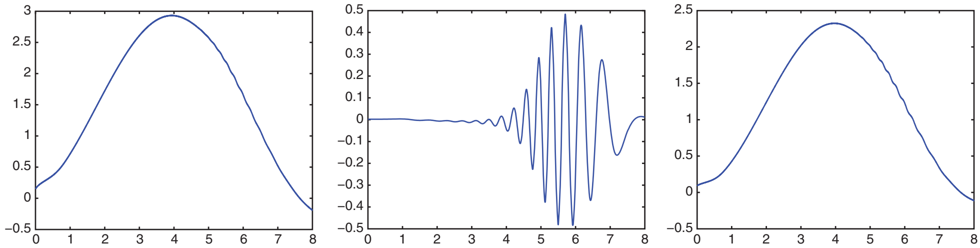

Figure 4. The evolutions of $c_{d,\max}$, $c_{l,\max}$ and Δp by the CNLE–VMS method with the Smagorinsky viscosity and 7516 triangles and $\triangle t = 0.005$.

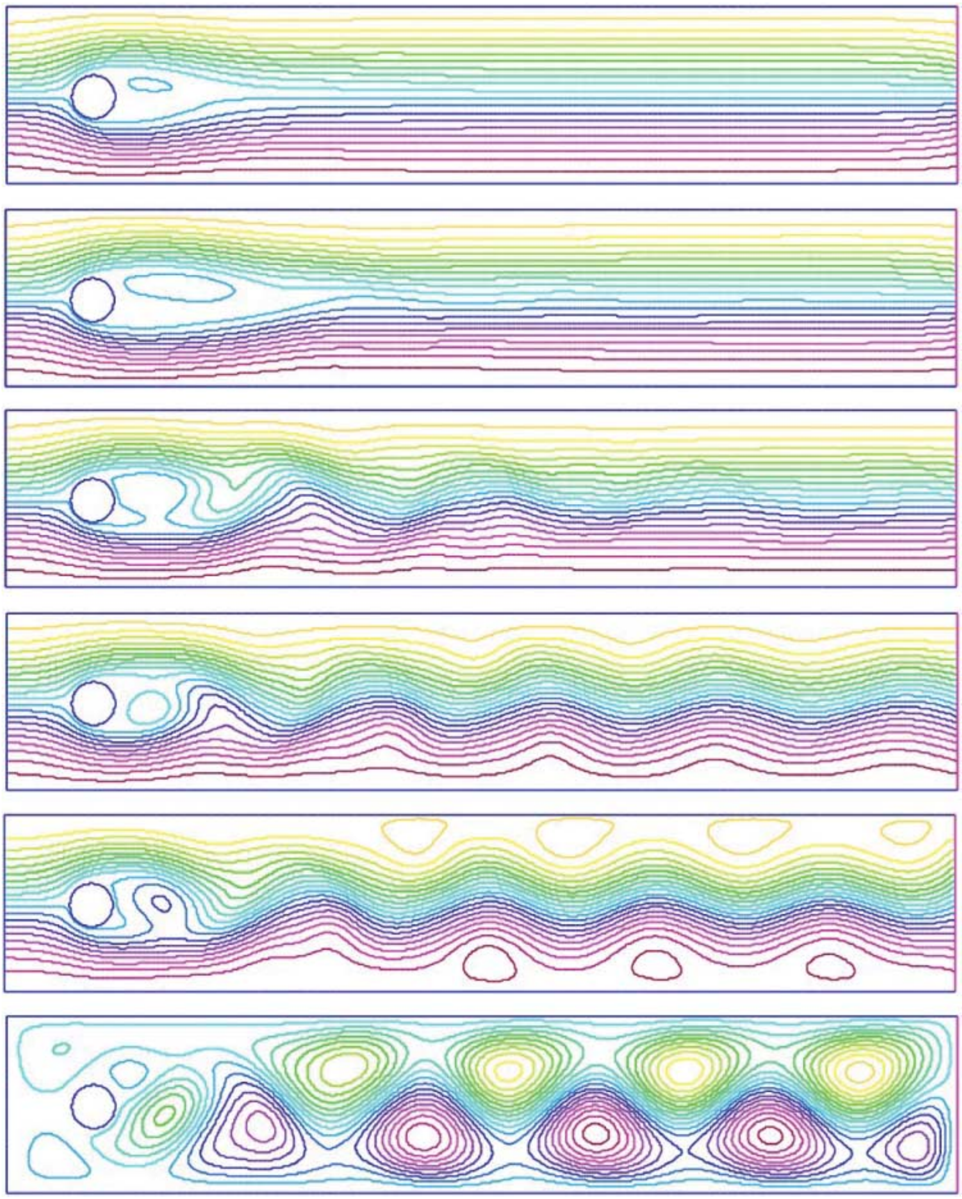

Figure 5. The streamline at $t = 2, 4, 5, 6, 7, 8$ by the CNLE–VMS method with linear viscosity and with 7516 triangles and $\triangle t = 0.005$.

given in [38],

$$c_{d,\max}^{\text{ref}} \in [2.93, 297], \quad c_{l,\max}^{\text{ref}} \in [0.47, 0.49], \quad \Delta p(8s)^{\text{ref}} \in [-0.115, -0.105].$$

The results list in both tables show that as the time step size and mesh size decreases, all coefficients approach the reference results as we expected.

The evolutions of $c_{d,\max}$, $c_{l,\max}$ and Δp with 7516 triangles and $\Delta t = 0.005$ for the CNLE–VMS method with constant and Smagorinsky viscosities are presented in Figures 3 and 4, respectively. It is easy to see that the results for both the CNLE–VMS methods are almost the same. They also coincide with the results provided in [23].

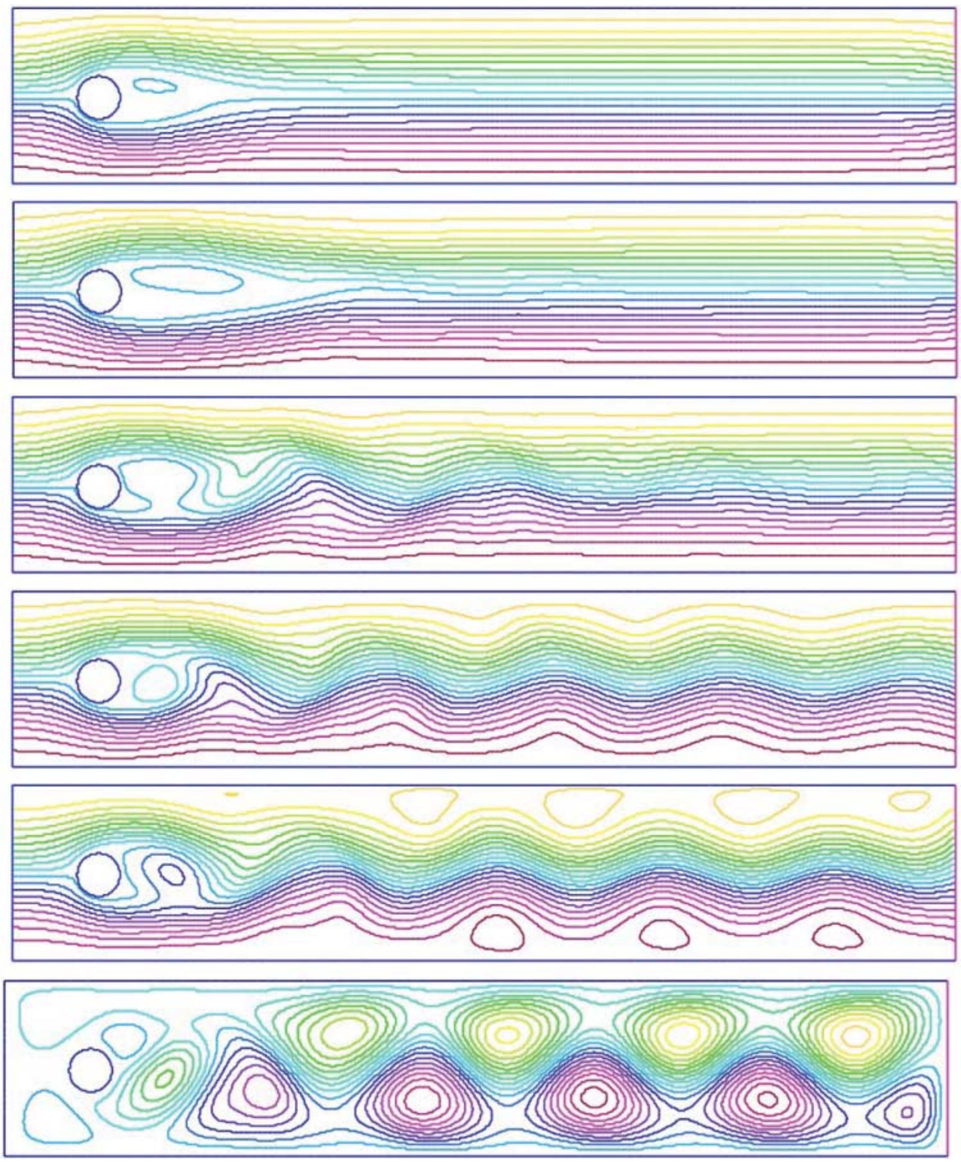

Figure 6. The streamline at $t = 2, 4, 5, 6, 7, 8$ by CNLE–VMS method with Smagorinsky viscosity and with 7516 triangles and $\triangle t = 0.005$.

Furthermore, the development of the flows by both the CNLE–VMS methods with 7510 triangles and $\triangle t = 0.05$ are depicted in Figures 5 and 6, respectively. From both figures, we notice that from $t = 2$ to $t = 4$, as time progresses, two vortices start to develop behind the cylinder. Then, the vortices separate from the cylinder between $t = 4$ and $t = 5$, and a vortex street develops, and they continue to be visible through the final time $t = 8$, which agrees with the results of [23,38].

6. Conclusions

In this report, a VMS method based on the CN extrapolation scheme for the non-stationary Naiver–Stokes problem is considered. This method is unconditionally stable, an *a priori* error estimate is given, but it is sub-optimal in space for velocity with respect to the L^2-norm. We leave it as our future work to provide an optimal error analysis. In the numerical studies, both constant viscosity and the nonlinear Smagorinsky type viscosity are performed, the results show that these schemes are efficient and feasible for turbulent flow simulations.

Acknowledgements

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