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ERROR ESTIMATES OF SPLITTING GALERKIN METHODS FOR HEAT AND SWEAT TRANSPORT IN TEXTILE MATERIALS*

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Abstract. This paper is concerned with a single-component model of heat and vapor (sweat) transport through three-dimensional porous textile materials with phase change, which is described by a nonlinear, degenerate, and strongly coupled parabolic system. An uncoupled (splitting) Galerkin method with semi-implicit Euler scheme in time direction is proposed for the system. In this method, a linearized scheme is applied for the approximation to Darcy's velocity simultaneously in the mass and energy equations, which leads to physical conservation of the method in the flow convection. The existence and uniqueness of solution of the finite element system is proved and the optimal error estimate in an energy norm is obtained. Numerical results are presented to confirm our theoretical analysis and are compared with experimental data.

Key words. heat and sweat transfer, fibrous porous media, error estimates, strongly coupled, splitting FEM

AMS subject classifications. 35K61, 65M60, 65N30

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1. Introduction.

Mathematical modeling and numerical simulations on heat and moisture transfer in porous media have been attractive in recent decades. Applications can be found in many areas of engineering and science, such as the food industry [13], building materials [12], fuel cells [21, 22], and recently the textile industry [2, 5, 10, 16]. Here we focus on heat and sweat transport through a porous textile assembly; see Figure 1 for the schematic diagram. The physical process can be viewed as a single component (vapor) and multiphase flow, described with the conservation of mass and energy by [10, 16, 17, 27]

$$(\epsilon C)_t + \nabla \cdot (u\epsilon C) = -\Gamma_{\mathrm{ce}}, \tag{1.1}$$

$$\left(\epsilon C_g CT + (1-\epsilon)C_f T\right)_t + \nabla \cdot (\epsilon u C_g CT) - \nabla \cdot (\kappa_c \nabla T) = \lambda \Gamma_{\mathrm{ce}}, \tag{1.2}$$

where C is the vapor density (mol/m^3), T is the absolute temperature (K), ϵ is the porosity of the medium, κ is the thermal conductivity, λ is the latent heat of evaporation/condensation, and C_g and C_f are the molar heat capacity of vapor and volumetric heat capacity of fiber, respectively.

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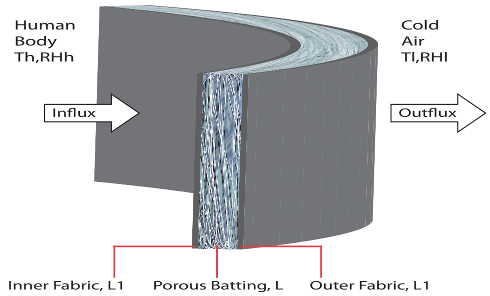

FIG. 1. *Schematic diagram of the porous textile assembly.*

The gas velocity (volumetric discharge) is given by Darcy's law,

$$Cu = -\frac{k}{\mu_g}\nabla P, \tag{1.3}$$

where k and μ_g are the permeability and the viscosity of the vapor, respectively. The molar rate of phase change per unit volume Γ_{ce} is defined by the Hertz–Knudsen equation [15]

$$\Gamma_{\mathrm{ce}} = \beta_\Gamma \left(\frac{P - P_{\mathrm{sat}}(T)}{\sqrt{T}} \right), \tag{1.4}$$

where β_Γ is a positive constant. The vapor pressure is given by $P = RCT$ due to the ideal gas assumption, and the saturation pressure P_{sat} is determined from experimental measurements [9].

With the Darcy velocity and the ideal gas law, (1.1)–(1.2) define a degenerate, nonlinear, and strongly coupled parabolic system. Mathematical analysis has been studied recently in [27] for the existence and uniqueness of a classical solution of a steady state model and in [17] for the existence of weak solutions of the vapor-temperature system (1.1)–(1.2). The proof in [17] was based on the physical conservation of gas convection. The positivity of temperature and nonnegativity of vapor density were also proved. Some more general cases were studied in [1, 18].

Numerical methods and simulations for heat and moisture transport in porous textile materials have been studied by many authors [4, 10, 12, 14, 16, 28]. In [10], a classical finite difference method is applied for solving a single-component multi-phase model with a thermal radiation in one-dimensional space, where the thermal radiation equation was solved analytically. A semi-implicit finite volume method was presented in [14] for a multicomponent heat and moisture model. Several practical cases of clothing assemblies were investigated in a comparison of experimental data. However, due to the degeneracy, the strong nonlinearity and the coupling of the system, error analysis of these existing numerical methods is limited. A Galerkin FEM was proposed in [4] for a simplified one-dimensional model, in which only vapor diffusion and heat conductive processes are included and the vapor bulk motion is neglected ($u = 0$). The optimal-order error estimate of the FEM in energy norm was presented. Error analysis of a semi-implicit finite difference method was given in a recent work [24] for the vapor-temperature system in one-dimensional space. In this paper, we present a splitting (uncoupled) Galerkin method and its analysis for the nonlinear vapor-temperature system in d-dimensional space, $d = 1, 2, 3$, with a class of commonly used flux type boundary conditions. In this method, a backward semi-implicit Euler scheme is applied in the time direction and Galerkin approximation is used in the spatial direction. The difficulty lies in the strong nonlinearity and coupling. The equations in this nonlinear parabolic system are coupled mainly via Darcy's velocity, which induces the vapor flow and the heat convective transfer. In our scheme, a linearized approximation to the mass flux is used in both the mass and energy equations so that the energy flux due to convection equals the mass flux multiplied by temperature. This maintains the physical conservation to a certain degree, as noted in previous theoretical analysis of the nonlinear parabolic system [17]. Based on the physical conservation, existence, uniqueness, and optimal error estimates in an energy norm are obtained.

A related model is the fluid flow in petroleum engineering and groundwater hydrology (see [7] and references therein), which usually is described by an elliptic pressure (or density) equation coupled with parabolic concentration (or saturation) equations for incompressible case and a system of parabolic equations for compressible case. The corresponding mathematical analysis can be found in [11, 19]. Numerical solutions for both the elliptic-parabolic and parabolic-parabolic systems have been investigated by many authors with a variety of numerical methods, e.g., see [3, 6, 7, 8, 20, 23, 25] and references therein. Among these methods, mixed FEMs, a modified method of characteristics, and an Euler–Lagrangian localized adjoint method are competitive to stabilize the numerical approximations. An optimal error estimate for a family of Ellam approximations to the incompressible porous medium flow was studied in [25, 26] under certain restrictions on both the time step and the spatial step. In all these works, the temperature was ignored and the phase change (condensation/evaporation) does not occur due to the nature of these applications, while both temperature and phase change in the vapor-temperature system play important roles and result in stronger nonlinearity and coupling.

Under the above assumptions, the system (1.1)–(1.2) with nondimensionalization reduces to the equations

$$c_t - \nabla \cdot \nabla(c\theta) = -\Gamma(c,\theta), \tag{1.5}$$

$$(c\theta + \sigma\theta)_t - \nabla \cdot (\theta \nabla(c\theta)) - \nabla \cdot (\kappa(c)\nabla\theta) = \lambda \Gamma(c,\theta) \tag{1.6}$$

for $x \in \Omega$ and $t > 0$, where $\kappa(c) = \kappa_1 + \kappa_2 c$, $\Gamma(c,\theta) = c\sqrt{\theta} - p_s(\theta)$, $p_s(\theta) \sim P_{\text{sat}}(\theta)/\sqrt{\theta}$ is a smooth and increasing function satisfying $p_s(0) = 0$, and Ω is a smooth and bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$.

Here we consider a class of commonly used flux type boundary conditions [10, 12, 14]

$$\nabla(c\theta) \cdot \vec{n} = \alpha(\mu - c) \quad \text{on } \partial\Omega, \quad 0 < t \le \mathcal{T}, \tag{1.7}$$

$$\kappa(c)\nabla\theta \cdot \vec{n} = \beta(\nu - \theta) \quad \text{on } \partial\Omega, \quad 0 < t \le \mathcal{T}, \tag{1.8}$$

and the initial conditions

$$c(x,0) = c_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \Omega, \tag{1.9}$$

where α represents the mass transfer coefficient, β the heat transfer coefficient, μ the vapor density in the background, and ν the background temperatures. Physically, $\alpha, \beta \ge 0$ and all other parameters are positive constants and $c_0(x) \ge \underline{c}$, $\theta_0(x) \ge \underline{\theta}$ with $\underline{c}$ and $\underline{\theta}$ being positive constants.

The rest of the paper is organized as follows. In section 2, we present a splitting Galerkin method for the nonlinear vapor-temperature system and our main results. In section 3, we prove the existence and uniqueness of the discrete Galerkin system and provide optimal error estimates of a numerical solution in an energy norm. Numerical examples will be given in section 4 to illustrate our theoretical analysis and compare with experimental data.

We suppose that the system (1.5)–(1.9) has a unique solution $\{c(x,t), \theta(x,t)\}$ and there exist positive constants $\theta_{\min}, c_{\max}, \theta_{\max}$ such that

$$0 \le c \le c_{\max}, \qquad \theta_{\min} \le \theta \le \theta_{\max}\,. \tag{1.10}$$

Global existence of weak solutions satisfying the above conditions was proved in [17].

2. Splitting Galerkin methods. In this section, we present a splitting FEM for solving the system of nonlinear equations (1.5)–(1.9). Due to the practical interest in a long time period, say, 8–24 hours, the backward Euler scheme is used in the time direction.

For any functions $u, v \in L^2(\Omega)$, we denote the $L^2(\Omega)$ inner product and norm by

$$(u,v) = \int_\Omega u(x)v(x)\,dx, \qquad \|u\|^2 = (u,u),$$

respectively. Let H^k, k being a positive integer, denote the usual Sobolev space with the norm $\|u\|_k := \|u\|_{H^k(\Omega)}$.

We rewrite the system (1.5)–(1.6) into

$$c_t - \nabla\cdot\nabla(c\theta) = -\Gamma(c,\theta), \quad x\in\Omega,\ t>0, \tag{2.1}$$

$$(c+\sigma)\theta_t - \nabla(c\theta)\cdot\nabla\theta - \nabla\cdot(\kappa(c)\nabla\theta) = (\lambda+\theta)\Gamma(c,\theta), \quad x\in\Omega,\ t>0, \tag{2.2}$$

where the second equation is obtained by adding (1.5) times $-\theta$ into (1.6). With the boundary conditions (1.7)–(1.8), the weak formulation of the above equations is to find $c, \theta \in L^2(0,\mathcal{T}; H^1(\Omega)) \cap L^\infty(0,\mathcal{T}; L^\infty(\Omega))$ and $c_t, \theta_t \in L^2(0,\mathcal{T}; H^{-1}(\Omega))$ such that for all $\xi_c, \xi_\theta \in H^1(\Omega)$

$$(c_t,\, \xi_c) + (\nabla(c\theta),\, \nabla\xi_c) + \langle \alpha c,\, \xi_c\rangle_{\partial\Omega} = -\,(\Gamma(c,\theta),\, \xi_c) + \langle \alpha\mu,\, \xi_c\rangle_{\partial\Omega}, \tag{2.3}$$

$$\begin{aligned}(\theta_t,\, (c+\sigma)\xi_\theta) - (\nabla(c\theta),\, \xi_\theta\nabla\theta) + (\kappa(c)\nabla\theta,\, \nabla\xi_\theta) + \langle\beta\theta,\, \xi_\theta\rangle_{\partial\Omega}\\ = ((\lambda+\theta)\Gamma(c,\theta),\, \xi_\theta) + \langle\beta\nu, \xi_\theta\rangle_{\partial\Omega}\end{aligned} \tag{2.4}$$

for almost all $t > 0$.

Let τ_h be a regular division of Ω with $\Omega = \cup_e \Omega_e$ and let $h = \max_{\Omega_e\in\tau_h}\{\operatorname{diam}\Omega_e\}$ denote the mesh size. For a given division τ_h, we introduce the C^0-finite element space

$$V_h^r = \{v_h \in C(\overline{\Omega}) \cap H^1(\Omega) : v_h|_{\Omega_e} \in P_r(\Omega_e)\},$$

where, for a given positive integer r, $P_r(\Omega_e)$ is a r-order polynomial space defined on Ω_e. Let I_h be the Lagrange finite element interpolant on V_h^r. For $0 \le l \le r+1$ and $2 \le p \le \infty$, we have

$$\|v - I_h v\|_{W_h^{l,p}(\Omega)} := \left(\sum_{\Omega_e\in\tau_h} \|v - I_h v\|^p_{W^{l,p}(\Omega_e)}\right)^{\frac{1}{p}} \le E_c h^{r+1-l}|v|_{W^{r+1,p}(\Omega)} \tag{2.5}$$

for $v \in W^{r+1,p}(\Omega)$, where E_c is a positive constant. Let (R_h^n, P_h^n), $n = 0,1,2,\dots$, be certain projection pairs from $H^1(\Omega)\times H^1(\Omega)$ onto $V_h^r \times V_h^r$, which will be defined in section 3. Moreover, let $\{t_n\}_{n=0}^N$ be a partition in the time direction with $N = [\frac{\mathcal{T}}{\tau}]$, where $\mathcal{T}$ is a fixed positive constant. We denote

$$D_t c^n := \frac{c^n - c^{n-1}}{\tau},$$

where $\tau > 0$ is a given time-step size.

The fully discrete FEM scheme is to find $c_h^n,\ \theta_h^n \in V_h^r$ such that for all $\xi_c,\ \xi_\theta \in V_h^r$

$$\begin{aligned}&(D_t c_h^n,\, \xi_c) + \left(c_h^{n-1}\nabla\theta_h^{n-1} + \theta_h^{n-1}\nabla c_h^n,\, \nabla\xi_c\right) + \langle \alpha c_h^n, \xi_c\rangle_{\partial\Omega}\\ &\quad = -\left(\Gamma(c_h^{n-1}, \theta_h^{n-1}),\, \xi_c\right) + \langle \alpha\mu, \xi_c\rangle_{\partial\Omega},\end{aligned} \tag{2.6}$$

$$\begin{aligned}&((c_h^{n-1} + \sigma)D_t\theta_h^n, \xi_\theta) - ((c_h^{n-1}\nabla\theta_h^{n-1} + \theta_h^{n-1}\nabla c_h^n)\cdot\nabla\theta_h^n, \xi_\theta)\\ &\quad + (\kappa(c_h^{n-1})\nabla\theta_h^n, \nabla\xi_\theta) + \langle \beta\theta_h^n, \xi_\theta\rangle_{\partial\Omega}\\ &\quad = ((\lambda + \theta_h^{n-1})\Gamma(c_h^{n-1}, \theta_h^{n-1}), \xi_\theta) + \langle \beta\nu, \xi_\theta\rangle_{\partial\Omega}\end{aligned} \tag{2.7}$$

with initial conditions $c_h^0 = I_h c^0$ and $\theta_h^0 = I_h\theta^0$, where $c^n = c(x, t_n)$, $\theta^n = \theta(x, t_n)$ are defined by the solution (c, θ) of the system (2.1)–(2.2).

In the above scheme, we have used the same discrete mass flux $F_h^n = c_h^{n-1}\nabla\theta_h^{n-1} + \theta_h^{n-1}\nabla c_h^n$ for both the mass and energy equations. This consistency approximately maintains the physical conservation due to mass convection and makes the analysis in section 3.2 possible.

We define error functions by

$$e_c^n = c_h^n - R_h^n c^n, \quad e_\theta^n = \theta_h^n - P_h^n\theta^n, \quad 0 \le n \le N,$$

and the truncation error functions by

$$\eta_c^n = c^n - R_h^n c^n, \quad \eta_\theta^n = \theta^n - P_h^n\theta^n, \quad 0 \le n \le N.$$

We present our results in the following theorem, and the proof will be presented in the next section.

THEOREM 2.1. *Let Ω be a either a smooth bounded domain or a convex polygonal domain in $\mathbb{R}^d$, $d = 1, 2, 3$. Suppose that the solution (c, θ) of the system* (2.3)–(2.4) *with the initial condition* (1.9) *satisfies* (1.10). *If the solution satisfies*

$$\begin{aligned}&c, \theta \in L^\infty(0, \mathcal{T}; W^{1,\infty}(\Omega)) \cap L^\infty(0, \mathcal{T}; H^{r+1}(\Omega)),\\ &c_t, \theta_t \in L^\infty(0, \mathcal{T}; H^{r+1}(\Omega)), \quad c_{tt}, \theta_{tt} \in L^\infty(0, \mathcal{T}; H^{-1}(\Omega))\end{aligned}$$

for $r \ge 1$, then there exist positive constants h_0, ρ_0, and E_0, independent of h and τ, such that when

$$h \le h_0, \quad \gamma_h^d \tau^{\frac12} \le \rho_0, \quad \textit{where}\ \ \gamma_h^d = \begin{cases} 1, & \textit{when}\ \ d = 1,\\ (1 + |\ln h|)^{\frac12}, & \textit{when}\ \ d = 2,\\ h^{-\frac12}, & \textit{when}\ \ d = 3,\end{cases}$$

the finite element system (2.6)–(2.7) *is uniquely solvable and*

$$\|e_c^n\|^2 + \|e_\theta^n\|^2 + \tau\sum_{m=0}^{n}[\|\nabla e_c^m\|^2 + \|\nabla e_\theta^m\|^2] \le E_0(\tau + h^{r+1})^2, \quad 0 \le n \le N. \tag{2.8}$$

In particular,

$$\max_{0\le n\le N}\|c_h^n - c^n\| + \max_{0\le n\le N}\|\theta_h^n - \theta^n\| \le E_0(\tau + h^{r+1}). \tag{2.9}$$

3. The proof of Theorem 2.1. In the rest of this paper, we denote by C a generic positive constant and by $C_1, C_2, \ldots$ some fixed positive constants, which depend solely upon the physical parameters $\kappa_1, \kappa_2, \sigma,\ \lambda$, the solution of the system

(2.1)–(2.2) and the parameters involved in initial and boundary conditions, independent of n, k, h, τ, and E_0.

3.1. Preliminaries. By (2.1)–(2.2), (c^n, θ^n) satisfies the following system:

$$\begin{aligned}&(D_t c^n,\, \xi_c) + \big(c^{n-1}\nabla\theta^{n-1} + \theta^{n-1}\nabla c^n,\, \nabla\xi_c\big) + \langle \alpha c^n,\, \xi_c\rangle_{\partial\Omega}\\ &\qquad = -\big(\Gamma(c^{n-1},\theta^{n-1}),\xi_c\big) + \langle \alpha\mu,\xi_c\rangle_{\partial\Omega} + (R_c^n,\, \xi_c) + (R^n,\, \nabla\xi_c)\,,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&((c^{n-1}+\sigma)D_t\theta^n,\xi_\theta) - \big((c^{n-1}\nabla\theta^{n-1} + \theta^{n-1}\nabla c^n)\cdot\nabla\theta^n,\, \xi_\theta\big)\\ &\qquad + \big(\kappa(c^{n-1})\nabla\theta^n,\, \nabla\xi_\theta\big) + \langle\beta\theta^n,\xi_\theta\rangle_{\partial\Omega}\\ &\qquad = \big((\lambda+\theta^{n-1})\Gamma(c^{n-1},\theta^{n-1}),\, \xi_\theta\big) + \langle\beta\nu,\xi_\theta\rangle_{\partial\Omega}\\ &\qquad\quad + (R_{\theta1}^n,\, \xi_\theta) + (R^n\cdot\nabla\theta^n,\, \xi_\theta) + (R_{\theta2}^n,\, \nabla\xi_\theta),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}R_c^n &= (D_t c^n - \partial_t c^n) + (\Gamma(c^n,\theta^n) - \Gamma(c^{n-1},\theta^{n-1}))\\ R^n &= c^{n-1}\nabla\theta^{n-1} + \theta^{n-1}\nabla c^n - \nabla(c^n\theta^n)\\ R_{\theta1}^n &= (D_t\theta^n - \partial_t\theta^n) + (\lambda+\theta^n)\Gamma(c^n,\theta^n) - (\lambda+\theta^{n-1})\Gamma(c^{n-1},\theta^{n-1})\\ R_{\theta2}^n &= (\kappa(c^n) - \kappa(c^{n-1}))\nabla\theta^n.\end{aligned}$$

With the regularity assumption in Theorem 2.1, we have

$$|(R_c^n,\xi_c)| \le C\tau\|\xi_c\|_1, \quad |(R_{\theta1}^n,\xi_\theta)| \le C\tau\|\xi_c\|_1, \quad \|R^n\|,\ \|R_{\theta2}^n\| \le C\tau. \tag{3.3}$$

By (2.6)–(2.7) and (3.1)–(3.2), the error functions (e_c^n, e_θ^n) satisfy the system

$$\begin{aligned}&(D_t e_c^n,\xi_c) + (c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n,\nabla\xi_c) + \langle\alpha e_c^n,\, \xi_c\rangle_{\partial\Omega}\\ &\ = ((\theta^{n-1}\nabla\eta_c^n + c^{n-1}\nabla\eta_\theta^n),\nabla\xi_c) + (\eta_c^n\nabla(P_h^{n-1}\theta^{n-1}),\nabla\xi_c) + \eta_\theta^n\nabla(R_h^{n-1}c^{n-1}),\nabla\xi_c)\\ &\quad + \langle\alpha\eta_c^n,\xi_c\rangle_{\partial\Omega} + m(\eta_c^n,\xi_c) + J_{c1}^n(\xi_c) + J_{c2}^n(\xi_c),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&((c_h^{n-1}+\sigma)D_t e_\theta^n,\xi_\theta) - ((c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n)\cdot\nabla e_\theta^n,\xi_\theta)\\ &\qquad + (\kappa(c_h^{n-1})\nabla e_\theta^n,\nabla\xi_\theta) + \langle\beta e_\theta^n,\xi_\theta\rangle_{\partial\Omega}\\ &\quad = \langle\beta\eta_\theta^n,\xi_\theta\rangle_{\partial\Omega} + (\kappa(c^{n-1})\nabla\eta_\theta^n,\nabla\xi_\theta) + ((\theta^{n-1}\nabla\eta_c^n + c^{n-1}\nabla\eta_\theta^n)\cdot\nabla\theta^n,\xi_\theta)\\ &\qquad + ((\theta^{n-1}\nabla(R_h^{n-1}c^{n-1}) + c^{n-1}\nabla(P_h^{n-1}\theta^{n-1}))\cdot\nabla\eta_\theta^n,\xi_\theta)\\ &\qquad + m(\eta_\theta^n,\xi_\theta) + J_{\theta1}^n(\xi_\theta) + J_{\theta2}^n(\xi_\theta),\end{aligned} \tag{3.5}$$

where m is a positive constant to be determined later and

$$\begin{aligned}J_{c1}^n(\xi_c) &= -\big(e_c^{n-1}\nabla(P_h^{n-1}\theta^{n-1}) + e_\theta^{n-1}\nabla(R_h^n c^n),\nabla\xi_c\big)\\ &\quad - \big(\Gamma(c_h^{n-1},\theta_h^{n-1}) - \Gamma(R_h^{n-1}c^{n-1},P_h^{n-1}\theta^{n-1}),\xi_c\big)\\ &:= J_1^n(\xi_c) + J_2^n(\xi_c),\end{aligned} \tag{3.6}$$

$$\begin{aligned}J_{c2}^n(\xi_c) &= \big((\eta_c^{n-1}-\eta_c^n)\nabla(P_h^{n-1}\theta^{n-1}) + (\eta_\theta^{n-1}-\eta_\theta^n)\nabla(R_h^n c^n),\nabla\xi_c\big)\\ &\quad + \big(\eta_\theta^n\nabla(R_h^n c^n - R_h^{n-1}c^{n-1}),\nabla\xi_c\big) - \big(c^{n-1}\nabla(\eta_\theta^n - \eta_\theta^{n-1}),\nabla\xi_c\big)\\ &\quad - \big(D_t\eta_c^n,\xi_c\big) - m\big(\eta_c^n,\xi_c\big) + \big(\Gamma(c^{n-1},\theta^{n-1}) - \Gamma(R_h^{n-1}c^{n-1},P_h^{n-1}\theta^{n-1}),\xi_c\big)\\ &\quad + \big(R_c^n,\xi_c\big) + \big(R^n,\nabla\xi_c\big),\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
J_{\theta 1}^n(\xi_\theta) &= -(e_c^{n-1} D_t P_h^n \theta^n, \xi_\theta) + ((c_h^{n-1} \nabla (P_h^{n-1}\theta^{n-1}) + \theta_h^{n-1} \nabla (R_h^n c^n)) \cdot \nabla e_\theta^n, \xi_\theta) \\
&\quad + ((c_h^{n-1} \nabla e_\theta^{n-1} + \theta_h^{n-1} \nabla e_c^n) \cdot \nabla (P_h^n \theta^n), \xi_\theta) \\
&\quad + ((e_c^{n-1} \nabla (P_h^{n-1}\theta^{n-1}) + e_\theta^{n-1} \nabla (R_h^n c^n)) \cdot \nabla (P_h^n \theta^n), \xi_\theta) \\
&\quad - (\kappa_2 e_c^{n-1} \nabla (P_h^n \theta^n), \nabla \xi_\theta) + ((\lambda + \theta_h^{n-1})(\Gamma(c_h^{n-1}, \theta_h^{n-1}) \\
&\quad - \Gamma(R_h^{n-1} c^{n-1}, P_h^{n-1}\theta^{n-1})), \xi_\theta) + (e_\theta^{n-1} \Gamma(R_h^{n-1}c^{n-1}, P_h^{n-1}\theta^{n-1}), \xi_\theta) \\
&:= \sum_{i=3}^{9} J_i^n(\xi_\theta), \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
J_{\theta 2}^n(\xi_\theta) &= ((\eta_\theta^{n-1} \nabla (R_h^n c^n) + \eta_c^{n-1} \nabla (P_h^{n-1}\theta^{n-1})) \cdot \nabla (P_h^n \theta^n), \xi_\theta) \\
&\quad + (\theta^{n-1} \nabla (R_h^n c^n - R_h^{n-1} c^{n-1}) \cdot \nabla \eta_\theta^n, \xi_\theta) + (\theta^{n-1} \nabla (\eta_c^n - \eta_c^{n-1}) \cdot \nabla \theta^n, \xi_\theta) \\
&\quad - (c^{n-1} \nabla (\eta_\theta^n - \eta_\theta^{n-1}) \cdot \nabla \theta^n, \xi_\theta) \\
&\quad + ((\lambda + P_h^{n-1}\theta^{n-1})(\Gamma(R_h^{n-1}c^{n-1}, P_h^{n-1}\theta^{n-1}) - \Gamma(c^{n-1}, \theta^{n-1})), \xi_\theta) \\
&\quad - (\eta_\theta^{n-1} \Gamma(c^{n-1}, \theta^{n-1}), \xi_\theta) + (\kappa_2 \eta_c^{n-1} \nabla (P_h^n \theta^n), \nabla \xi_\theta) \\
&\quad ((c^{n-1} + \sigma) D_t \eta_\theta^n, \xi_\theta) + (\eta_c^{n-1} D_t (P_h^n \theta^n), \xi_\theta) \\
&\quad + (R_{\theta 1}^n, \xi_\theta) + (R^n \cdot \nabla \theta^n, \xi_\theta) + (R_{\theta 2}, \nabla \xi_\theta) - m(\eta_\theta^n, \xi_\theta). \qquad (3.9)
\end{aligned}
$$

Now we introduce the projection pairs (R_h^k, P_h^k). For given sequences θ^{n-1} and c^{n-1}, $n = 1, \ldots, N$, satisfying

$$
0 \le c^{n-1} \le c_{\max}, \quad 0 < \theta_{\min} \le \theta^{n-1} \le \theta_{\max}, \quad \kappa(c^{n-1}) \ge \kappa_1 > 0,
$$

and two positive constants $\zeta, m > 0$, we define a bilinear form

$$
\begin{aligned}
B^n[(u, v), (\xi_c, \xi_\theta)] &:= (\theta^{n-1} \nabla u, \nabla \xi_c) + (c^{n-1} \nabla v, \nabla \xi_c) + (u \nabla (P_h^{n-1}\theta^{n-1}), \nabla \xi_c) \\
&\quad + (v \nabla (R_h^{n-1} c^{n-1}), \nabla \xi_c) + \langle \alpha u, \xi_c \rangle_{\partial\Omega} + \langle \beta v, \xi_\theta \rangle_{\partial\Omega} \\
&\quad + \zeta \{ (\theta^{n-1} \nabla u + c^{n-1} \nabla v) \cdot \nabla \theta^n, \xi_\theta) + ((c^{n-1} \nabla (P_h^{n-1}\theta^{n-1}) \\
&\quad + \theta^{n-1} \nabla (R_h^{n-1} c^{n-1})) \cdot \nabla v, \xi_\theta) \} + \zeta (\kappa(c^{n-1}) \nabla v, \nabla \xi_\theta) \\
&\quad + m \{ (u, \xi_c) + (v, \xi_\theta) \}, \qquad (3.10)
\end{aligned}
$$

where m and ζ are positive constants to be determined. For any given $(u, v) \in H^1(\Omega) \times H^1(\Omega)$, we define $(R_h^n u, P_h^n v) \in V_h^r \times V_h^r$ such that

$$
B^n[(u - R_h^n u, v - P_h^n v), (\xi_c, \xi_\theta)] = 0 \quad \forall (\xi_c, \xi_\theta) \in V_h^r \times V_h^r, \quad n > 0\,. \qquad (3.11)
$$

Here, $(R_h^0 c^0, P_h^0 \theta^0)$ is the solution of the system

$$
B^1[(c^0 - R_h^0 c^0, \theta^0 - P_h^0 \theta^0), (\xi_c, \xi_\theta)] = 0 \quad \forall\ (\xi_c, \xi_\theta) \in V_h^r \times V_h^r\,. \qquad (3.12)
$$

Clearly, $(R_h^0 c^0, P_h^0 \theta^0)$ is a solution of a system of nonlinear elliptic equations. The existence and uniqueness of solution of the system (3.12) is given in the following lemma.

LEMMA 3.1. *There exist positive constants C_1 and $h_{m,\zeta}$ such that if the positive constants m and ζ satisfy*

$$
m \ge 2\zeta \left(1 + \frac{C_1(2C_1 + 1)}{2\kappa_1} \right) + C_1 + 1 \qquad (3.13)
$$

and $h < h_{m,\zeta}$*, then the nonlinear equation* (3.12) *has a solution* $(R_h^0 c^0, P_h^0\theta^0) \in V_h^r \times V_h^r$ *which satisfies the following estimates:*

$$\|c^0 - R_h^0 c^0\| + h\|\nabla(c^0 - R_h^0 c^0)\| \le \bar{C}_0(m,\zeta)h^{r+1},$$
$$\|\theta^0 - P_h^0\theta^0\| + h\|\nabla(\theta^0 - P_h^0\theta^0)\| \le \bar{C}_0(m,\zeta)h^{r+1}. \tag{3.14}$$

Proof. First, we show that there exist $R_h^0 c^0$ and $P_h^0\theta^0$ in V_h^r, which satisfy the truncated equation

(3.15)
$$\begin{aligned}
&(\theta^0\nabla(c^0 - R_h^0c^0), \nabla\xi_c) + (c^0\nabla(\theta^0 - P_h^0\theta^0), \nabla\xi_c)\\
&\quad + (F_1(R_h^0c^0 - I_hc^0)\nabla(P_h^0\theta^0 - \theta^0), \nabla\xi_c) + ((c^0 - R_h^0c^0)\nabla\theta^0, \nabla\xi_c)\\
&\quad + (F_2(P_h^0\theta^0 - I_h\theta^0)\nabla(R_h^0c^0 - c^0), \nabla\xi_c) + ((\theta^0 - P_h^0\theta^0)\nabla c^0, \nabla\xi_c)\\
&\quad + \langle\alpha(c^0 - R_h^0c^0), \xi_c\rangle_{\partial\Omega} + \langle\beta(\theta^0 - P_h^0\theta^0), \xi_\theta\rangle_{\partial\Omega} + m(c^0 - R_h^0c^0, \xi_c)\\
&\quad + m(\theta^0 - P_h^0\theta^0, \xi_\theta) + \zeta\{(\kappa(c^0)\nabla(\theta^0 - P_h^0\theta^0), \nabla\xi_\theta) + ((\theta^0\nabla(c^0 - R_h^0c^0)\cdot\nabla\theta^1, \xi_\theta)\\
&\quad + (c^0\nabla(\theta^0 - P_h^0\theta^0))\cdot\nabla\theta^1, \xi_\theta) + (F_3(R_h^0c^0 - I_hc^0, P_h^0\theta^0 - I_h\theta^0)\nabla(\theta^0 - P_h^0\theta^0), \xi_\theta)\\
&\quad + ((c^0\nabla\theta^0 + \theta^0\nabla c^0)\cdot\nabla(\theta^0 - P_h^0\theta^0), \xi_\theta)\} = 0
\end{aligned}$$

for all $\xi_c, \xi_\theta \in V_h^r$, where F_1, F_2, and F_3 are defined by

$$F_1(u) = \begin{cases} c^0 - I_hc^0 - u & \text{if } |c^0 - I_hc^0 - u| < \epsilon,\\ \frac{c^0 - I_hc^0 - u}{|c^0 - I_hc^0 - u|}\epsilon & \text{if } |c^0 - I_hc^0 - u| \ge \epsilon,\end{cases}$$
$$F_2(v) = \begin{cases} \theta^0 - I_h\theta^0 - v & \text{if } |\theta^0 - I_h\theta^0 - v| < \epsilon,\\ \frac{\theta^0 - I_h\theta^0 - v}{|\theta^0 - I_h\theta^0 - v|}\epsilon & \text{if } |\theta^0 - I_h\theta^0 - v| \ge \epsilon,\end{cases}$$

$$\begin{aligned}&F_3(u,v)\\ &= \begin{cases} -(c^0\nabla(\theta^0 - I_h\theta^0 - v) + \theta^0\nabla(c^0 - I_hc^0 - u)) & \text{if } |v| < \epsilon,\\ -\frac{\epsilon}{|v|}(c^0\nabla(\theta^0 - I_h\theta^0 - v) + \theta^0\nabla(c^0 - I_hc^0 - u)) & \text{if } |v| \ge \epsilon.\end{cases}\end{aligned}$$

Clearly, if $\|c^0 - R_h^0c^0\|_{L^\infty} < \epsilon$ and $\|I_h\theta^0 - P_h^0\theta^0\|_{L^\infty} < \epsilon$, then (3.15) reduces to (3.12).

For any given $(u_0, v_0) \in V_h^r \times V_h^r$, we define (u, v) as the solution of the linear system

$$\begin{aligned}
&(\theta^0\nabla(c^0 - I_hc^0 - u), \nabla\xi_c) + s(c^0\nabla(\theta^0 - I_h\theta^0 - v), \nabla\xi_c)\\
&\quad + s(F_1(u_0)\nabla(v - (\theta^0 - I_h\theta^0)), \nabla\xi_c) + s((c^0 - I_hc^0 - u)\nabla\theta^0, \nabla\xi_c)\\
&\quad + s(F_2(v_0)\nabla(u - (c^0 - I_hc^0)), \nabla\xi_c) + s((\theta^0 - I_h\theta^0 - v)\nabla c^0, \nabla\xi_c)\\
&\quad + \langle\alpha(c^0 - I_hc^0 - u), \xi_c\rangle_{\partial\Omega} + \langle\beta(\theta^0 - I_h\theta^0 - v), \xi_\theta\rangle_{\partial\Omega}\\
&\quad + m\{(c^0 - I_hc^0 - u, \xi_c) + (\theta^0 - I_h\theta^0 - v, \xi_\theta)\} + \zeta\{(\kappa(c^0)\nabla(\theta^0 - I_h\theta^0 - v), \nabla\xi_\theta)\\
&\quad + s((\theta^0\nabla(c^0 - I_hc^0 - u)\cdot\nabla\theta^1, \xi_\theta) + s(c^0\nabla(\theta^0 - I_h\theta^0 - v))\cdot\nabla\theta^1, \xi_\theta)\\
&\quad + s(F_3(u_0, v_0)\nabla(\theta^0 - I_h\theta^0 - v), \xi_\theta) + s((c^0\nabla\theta^0 + \theta^0\nabla c^0)\cdot\nabla(\theta^0 - I_h\theta^0 - v), \xi_\theta)\} = 0
\end{aligned}$$

for any $(\xi_c, \xi_\theta) \in V_h^r \times V_h^r$. We can see that the map $M : V_h^r \times V_h^r \times [0,1] \to V_h^r \times V_h^r$ defined by $M(u_0, v_0, s) = (u, v)$ is bounded.

To prove the existence of a fixed point for the map, we use the Leray–Schauder fixed point theorem. If (u,v) is a fixed point, then $R_h^0 c^0 = I_h c^0 - u$ and $P_h^0\theta^0 = I_h\theta^0 - v$ is a solution to the nonlinear problem (3.15). By the Leray–Schauder fixed point theorem, it suffices to establish a priori estimates for any (u,v) such that $(u,v) = M(u,v,s)$ for some $s \in [0,1]$. In this case, (u,v) is the solution to the following equation:

(3.16)
$$\begin{aligned}
&(\theta^0\nabla(c^0 - I_h c^0 - u), \nabla\xi_c) + s(c^0\nabla(\theta^0 - I_h\theta^0 - v), \nabla\xi_c)\\
&+ s(F_1(u)\nabla(v - (\theta^0 - I_h\theta^0)), \nabla\xi_c) + s((c^0 - I_h c^0 - u)\nabla\theta^0, \nabla\xi_c)\\
&+ s(F_2(v)\nabla(u - (c^0 - I_h c^0)), \nabla\xi_c) + s((\theta^0 - I_h\theta^0 - v)\nabla c^0, \nabla\xi_c)\\
&+ \langle\alpha(c^0 - I_h c^0 - u), \xi_c\rangle_{\partial\Omega} + \langle\beta(\theta^0 - I_h\theta^0 - v), \xi_\theta\rangle_{\partial\Omega}\\
&+ m\{(c^0 - I_h c^0 - u, \xi_c) + (\theta^0 - I_h\theta^0 - v, \xi_\theta)\} + \zeta\{(\kappa(c^0)\nabla(\theta^0 - I_h\theta^0 - v), \nabla\xi_\theta)\\
&+ s((\theta^0\nabla(c^0 - I_h c^0 - u)\cdot\nabla\theta^1, \xi_\theta) + s(c^0\nabla(\theta^0 - I_h\theta^0 - v))\cdot\nabla\theta^1, \xi_\theta)\\
&+ s(F_3(u,v)\nabla(\theta^0 - I_h\theta^0 - v), \xi_\theta) + s((c^0\nabla\theta^0 + \theta^0\nabla c^0)\cdot\nabla(\theta^0 - I_h\theta^0 - v), \xi_\theta)\} = 0
\end{aligned}$$

for any $(\xi_c, \xi_\theta) \in V_h^r \times V_h^r$. Substituting $(\xi_c, \xi_\theta) = (-u, 0)$ and $(\xi_c, \xi_\theta) = (0, -v)$ into the above equation, respectively, and by noting that $\theta^0 \geq \theta_{\min}$, $|F_1(u)| \leq \epsilon$, $|F_2(v)| \leq \epsilon$, $|F_3(u,v)v| \leq \epsilon|\nabla(\theta^0 - I_h\theta^0 - v)| + \epsilon|\nabla(c^0 - I_h c^0 - u)|$, and the inequality $|(w_1, w_2)_{\partial\Omega}| \leq \|w_1\|_{L^2(\partial\Omega)}\|w_2\|_{L^2(\partial\Omega)} \leq \|w_1\|_1\|w_2\|_1$, we obtain

(3.17)
$$\begin{aligned}
&m\|u\|^2 + \|\nabla u\|^2\\
&\quad\leq C_1\|u\|^2 + C_1\|\nabla v\|^2 + (C_1 + C_1 m)\|c^0 - I_h c^0\|_1^2 + C_1\|\theta^0 - I_h\theta^0\|_1^2,
\end{aligned}$$

(3.18)
$$\begin{aligned}
&\frac{m}{2\zeta}\|v\|^2 + \kappa_1\|\nabla v\|^2\\
&\quad\leq \epsilon\|\nabla u\|^2 + \epsilon\|\nabla v\|^2 + \frac{C_1}{\epsilon}\|v\|^2 + \Big(C_1\epsilon + \frac{C_1 m}{\zeta}\Big)\|\theta^0 - I_h\theta^0\|_1^2 + C_1\epsilon\|c^0 - I_h c^0\|_1^2
\end{aligned}$$

for some positive constant C_1 (independent of m and ζ) and any small positive constant ϵ to be determined. The inequality (3.17) times 2ϵ plus (3.18) gives

$$\begin{aligned}
&2\epsilon(m - C_1)\|u\|^2 + \epsilon\|\nabla u\|^2 + \Big(\frac{m}{2\zeta} - \frac{C_1}{\epsilon}\Big)\|v\|^2 + \big(\kappa_1 - (2C_1 + 1)\epsilon\big)\|\nabla v\|^2\\
&\quad\leq CC_1\Big(\frac{m}{\zeta} + \epsilon + m\epsilon\Big)h^{2r}.
\end{aligned}$$

By choosing ϵ, m, and ζ in such a way that

$$\epsilon = \frac{\kappa_1/2}{2C_1 + 1} \quad \text{and} \quad m \geq 2\zeta\Big(1 + \frac{C_1}{\epsilon}\Big) + C_1 + 1, \tag{3.19}$$

we get the estimate $\|u\|_1^2 + \|v\|_1^2 \leq C_2 h^{2r}$ for some positive constant C_2, which together with the Leray–Schauder fixed point theorem implies the existence of a fixed point (u,v) for the map M. Thus $R_h^0 c^0 = u + I_h c^0$ and $P_h^0\theta^0 = v + I_h\theta^0$ form a solution of

(3.15), which satisfies the estimates

$$\|\theta^0 - P_h^0\theta^0\| + \|\nabla(\theta^0 - P_h^0\theta^0)\| \le C_3 h^r,$$
$$\|c^0 - R_h^0 c^0\| + \|\nabla(c^0 - R_h^0 c^0)\| \le C_3 h^r.$$

Second, we consider $F_1(u)$, $F_2(v)$, and $F_3(u,v)$ again. Let φ_c and φ_θ be the solution of the elliptic boundary value problems

$$\begin{cases} -\nabla\cdot(\theta^0\nabla\varphi_c) + \nabla\theta^0\cdot\nabla\varphi_c + m\varphi_c - \nabla\cdot(\zeta\theta^0\nabla\theta^1\varphi_\theta) = c^0 - R_h^0 c^0 & \text{in } \Omega, \\ -\theta^0\nabla\varphi_c\cdot\nu = \alpha\varphi_c + \zeta\theta^0\nabla\theta^1\cdot\nu\varphi_\theta & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\nabla\cdot(\zeta\kappa(c^0)\nabla\varphi_\theta) + m\varphi_\theta - c^0\Delta\varphi_c \\ \quad -\nabla\cdot[\zeta(\nabla(c^0\theta^0) + c^0\nabla\theta^1)\varphi_\theta] = \zeta(\theta^0 - P_h^0\theta^0) & \text{in } \Omega, \\ -\zeta\kappa(c^0)\nabla\varphi_\theta\cdot\nu = \beta\varphi_\theta + \big(c^0\nabla\theta^0\cdot\nu + \theta^0\nabla c^0\cdot\nu\big)\zeta\varphi_\theta - \frac{\alpha c^0}{\theta^0}\varphi_c & \text{on } \partial\Omega, \end{cases}$$

respectively. It is not difficult to prove the existence of a solution to the above boundary value problems, and the solutions φ_c and φ_θ satisfy the estimate

$$\|\varphi_c\|_2 + \|\varphi_\theta\|_2 \le C(m,\zeta)\|c^0 - R_h^0 c^0\| + C(m,\zeta)\|\theta^0 - P_h^0\theta^0\|.$$

Substituting $\xi_c = I_h\varphi_c$ and $\xi_\theta = I_h\varphi_\theta$ into (3.15), we can derive that

$$\|u\| + \|v\| \le C(m,\zeta)h(\|u\|_1 + \|v\|_1) \le \bar{C}_0(m,\zeta)h^{r+1},$$

where $\bar{C}_0(m,\zeta)$ is some positive constant which depends on m and ζ. With the above estimates, we have (by the inverse inequalities)

$$\begin{aligned} \|c^0 - R_h^0 c^0\|_{L^\infty} + \|\theta^0 - P_h^0\theta^0\|_{L^\infty} &\le \|c^0 - I_h c^0\|_{L^\infty} + \|\theta^0 - I_h\theta^0\|_{L^\infty} \\ &\quad + Ch^{-d/2}(\|I_h c^0 - R_h^0 c^0\| + \|I_h\theta^0 - P_h^0\theta^0\|) \\ &\le Ch + C\bar{C}_0(m,\zeta)h^{r+1-d/2}. \end{aligned}$$

Therefore, $F_1(u) = c^0 - R_h^0 c^0$ and $F_2(v) = \theta^0 - P_h^0\theta^0$ a.e. when h is small enough in the sense that $Ch + C\bar{C}_0(m,\zeta)h^{r+1-d/2} \le \epsilon$.

In conclusion, the solution of (3.15) satisfies (3.12) and the estimates (3.14). ◻

Since $(R_h^1 c^0, P_h^1\theta^0)$ is the unique solution of the linear problem

$$B^1((c^0 - R_h^1 c^0, \theta^0 - P_h^1\theta^0),\ (\xi_c,\xi_\theta)) = 0 \in V_h^r \times V_h^r,$$

it follows that $R_h^1 c^0 = R_h^0 c^0$ and $P_h^1\theta^0 = P_h^0\theta^0$. Therefore, we can simply define $(R_h^0 u, P_h^0 v) = (R_h^1 u, P_h^1 v)$ for any $(u,v) \in H^1(\Omega)\times H^1(\Omega)$ without contradiction.

LEMMA 3.2. *If ζ and m are chosen as certain fixed positive constants, then there exists a positive constant $h_1 > 0$ such that when $h < h_1$ the bilinear form $B^n(\cdot,\cdot)$ is continuous and coercive for $n \ge 1$.*

Proof. We prove the lemma by mathematical induction. With Lemma 3.1, we can assume that the projection pair $(R_h^n, P_h^n) : H^1(\Omega)\times H^1(\Omega) \to V_h^r \times V_h^r$ is well defined for $1 \le n \le k$ $(k \ge 1)$ and satisfies

$$\|c^{n-1} - R_h^{n-1}c^{n-1}\| + h\|\nabla(c^{n-1} - R_h^{n-1}c^{n-1})\| \le \bar{C}_{n-1}(m,\zeta)h^{r+1},$$
$$\|\theta^{n-1} - P_h^{n-1}\theta^n\| + h\|\nabla(\theta^{n-1} - P_h^{n-1}\theta^n)\| \le \bar{C}_{n-1}(m,\zeta)h^{r+1}.$$

By the inverse inequalities, we have

$$\|\nabla(P_h^{n-1}\theta^{n-1} - I_h\theta^{n-1})\|_{L^3} \leq Ch^{-d/6}\|\nabla(P_h^{n-1}\theta^{n-1} - I_h\theta^{n-1})\| \leq C\bar{C}_{n-1}(m,\zeta)h^{1-d/6}, \tag{3.20}$$

$$\|\nabla(R_h^{n-1}c^{n-1} - I_hc^{n-1})\|_{L^3} \leq Ch^{-d/6}\|\nabla(R_h^{n-1}c^{n-1} - I_hc^{n-1})\| \leq C\bar{C}_{n-1}(m,\zeta)h^{1-d/6}, \tag{3.21}$$

we have

$$\begin{aligned}
&B^n[(u,v),(u,v)] \\
&\geq \theta_{\min}\|\nabla u\|^2 + \zeta\kappa_1\|\nabla v\|^2 + m(\|u\|^2 + \|v\|^2) \\
&\quad + (c^{n-1}\nabla v, \nabla u) + \zeta\{(\theta^{n-1}\nabla u + c^{n-1}\nabla v)\cdot\nabla\theta^n, v) \\
&\quad + ((c^{n-1}\nabla(P_h^{n-1}\theta^{n-1}) + \theta^{n-1}\nabla(R_h^{n-1}c^{n-1}))\cdot\nabla v, v)\} \\
&\quad + (u\nabla(P_h^{n-1}\theta^{n-1}), \nabla\xi_c) + (v\nabla(R_h^{n-1}c^{n-1}), \nabla u) \\
&\geq \theta_{\min}\|\nabla u\|^2 + \zeta\kappa_1\|\nabla v\|^2 + m(\|u\|^2 + \|v\|^2) \\
&\quad - c_{\max}\|\nabla u\|\,\|\nabla v\| - \zeta(\theta_{\max}\|\nabla u\| + c_{\max}\|\nabla v\|)\|\nabla\theta^n\|_{L^\infty}\|v\| \\
&\quad - \zeta c_{\max}(C\|\nabla\theta^{n-1}\|_{L^\infty}\|v\| + C\bar{C}_{n-1}(m,\zeta)\|\theta^{n-1}\|_2 h^{1-d/6}\|v\|_{L^6})\|\nabla v\| \\
&\quad - \zeta\theta_{\max}(C\|\nabla c^{n-1}\|_{L^\infty}\|v\| + C\bar{C}_{n-1}(m,\zeta)\|c^{n-1}\|_2 h^{1-d/6}\|v\|_{L^6})\|\nabla v\| \\
&\quad - (C\|\nabla\theta^{n-1}\|_{L^\infty}\|u\| + C\bar{C}_{n-1}(m,\zeta)\|c^{n-1}\|_2 h^{1-d/6}\|u\|_{L^6})\|\nabla u\| \\
&\quad - (C\|\nabla c^{n-1}\|_{L^\infty}\|v\| + C\bar{C}_{n-1}(m,\zeta)\|c^{n-1}\|_2 h^{1-d/6}\|v\|_{L^6})\|\nabla u\| \\
&\geq \Big(\frac{\theta_{\min}}{2} - C_4\bar{C}_{n-1}(m,\zeta)h^{1-d/6}\Big)\|\nabla u\|^2 \\
&\quad + \Big(\zeta(\kappa_1 - C_4\bar{C}_{n-1}(m,\zeta)h^{1-d/6}) - C_4\Big)\|\nabla v\|^2 \\
&\quad + \Big(m - (C_4 + C_4\bar{C}_{n-1}(m,\zeta)^2h^{2-d/3})(1+\zeta^2)\Big)(\|u\|^2 + \|v\|^2),
\end{aligned}$$

where C_4 is some positive constant independent of m, ζ. If we choose

$$\zeta = \frac{2(C_4+1)}{\kappa_1}, \tag{3.22}$$

$$m \geq \left[2\zeta\left(1 + \frac{C_1(2C_1+1)}{2\kappa_1}\right) + C_1 + 1\right] + 2C_4(1+\zeta^2) + 1 \tag{3.23}$$

(clearly, m and ζ satisfy the condition (3.13)) and, with m and ζ being fixed, choose h to be small enough such that

$$h < h_{m,\zeta} \quad \text{and} \quad \bar{C}_{n-1}(m,\zeta)h^{1-d/6} < \min\Big(\theta_{\min}/(4C_4), \kappa_1/(2C_4), 1\Big), \tag{3.24}$$

where $h_{m,\zeta}$ is the positive constant in Lemma 3.1, then we get

$$B^n[(u,v),(u,v)] \geq C_5(\|u\|_1^2 + \|v\|_1^2), \tag{3.25}$$

where $C_5 = \min(\theta_{\min}/4, 1)$.

With m and ζ fixed, the dependence of $\bar{C}_{n-1}(m,\zeta)$ on m and ζ can be omitted. Moreover, with the help of (3.20)–(3.24), we find that

$$B^n[(u,v),(\xi_c,\xi_\theta)] \le C_7(\|u\|_1 + \|v\|_1)(\|\xi_c\|_1 + \|\xi_\theta\|_1), \tag{3.26}$$

where $C_7 = 2\theta_{\min} + 5(C_4+1) + 2m + C_6^2(\alpha+\beta)$, where m is given by (3.22)–(3.23) and C_6 is the constant in the Sobolev inequality $\|u\|_{L^2(\partial\Omega)} \le C_6\|u\|_1$.

Thanks to the Lax–Milgram theorem, for any given $(u,v) \in H^1(\Omega)\times H^1(\Omega)$, there exists a unique $(R_h^n u, P_h^n v) \in V_h^r \times V_h^r$ which satisfies (3.11) and

$$\|R_h^n u\|_1 + \|P_h^n v\|_1 \le C_8(\|u\|_1 + \|v\|_1) \tag{3.27}$$

for some positive constant C_8 which depends only C_5 and C_7. Furthermore, by the classical finite element theory for linear elliptic equations (using the approach of Lemma 3.1 for the L^2 error estimates), we can derive that

$$\|u - R_h^n u\| + h\|\nabla(u - R_h^n u)\| \le C_9(\|u\|_{r+1} + \|v\|_{r+1})h^{r+1},$$
$$\|v - P_h^n v\| + h\|\nabla(v - P_h^n v)\| \le C_9(\|u\|_{r+1} + \|v\|_{r+1})h^{r+1}$$

for some positive constant C_9 which depends only on C_5, C_7, and certain norms of the exact solution (c,θ), independent of $\bar{C}_k$ for $0 \le k \le n-1$. We see that $\bar{C}_n \le C_9(\|c^n\|_{r+1} + \|\theta^n\|_{r+1})$. If we choose

$$C_{10} = \bar{C}_0 + \max_{1\le k\le N} C_9(\|c^k\|_{r+1} + \|\theta^k\|_{r+1}),$$

then $\bar{C}_k \le C_{10}$ for $1 \le k \le n-1$ implies that $\bar{C}_n \le C_{10}$ with the condition (3.24) which reduces to

$$h < h_{m,\zeta} \quad \text{and} \quad C_{10}h^{1-d/6} < \min\Big(\theta_{\min}/(4C_4), \kappa_1/(2C_4), 1\Big). \tag{3.28}$$

The proof of the lemma is complete. ∎

From the above proof, we also get the following result.

LEMMA 3.3. *There exists a positive constant h_1 such that when $h < h_1$ we have*

$$\|c^n - R_h^n c^n\| + h\|\nabla(c^n - R_h^n c^n)\| \le C_{10}h^{r+1}, \tag{3.29}$$

$$\|\theta^n - P_h^n \theta^n\| + h\|\nabla(\theta^n - P_h^n \theta^n)\| \le C_{10}h^{r+1} \tag{3.30}$$

for $n = 0,1,2,\ldots,N$.

In addition to the above result, we also need the following lemma.

LEMMA 3.4.

$$\|\eta_c^n - \eta_c^{n-1}\| + \|\eta_\theta^n - \eta_\theta^{n-1}\| + h\|\nabla(\eta_c^n - \eta_c^{n-1})\| + h\|\nabla(\eta_\theta^n - \eta_\theta^{n-1})\| \le C_{11}\tau h^{r+1}. \tag{3.31}$$

Proof. Note that

$$B^n((c^n - R_h^n c^n, \theta^n - P_h^n \theta^n),(\xi_c,\xi_\theta)) = 0,$$
$$B^{n-1}((c^n - R_h^{n-1} c^n, \theta^n - P_h^{n-1} \theta^n),(\xi_c,\xi_\theta)) = 0$$

for any $(\xi_c,\xi_\theta) \in V_h^r \times V_h^r$. The difference of the above two equations with $\xi_\theta = 0$

gives

$$
\begin{aligned}
&\int_\Omega \big[\theta^{n-2}\nabla(R_h^n c^n - R_h^{n-1}c^n)\cdot\nabla\xi_c + c^{n-2}\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\cdot\nabla\xi_c\big]\,dx \\
&\qquad + \int_\Omega m(R_h^n c^n - R_h^{n-1}c^n)\xi_c dx + \langle \alpha(R_h^n c^n - R_h^{n-1}c^n), \xi_c\rangle_{\partial\Omega} \\
&\quad = \int_\Omega \Big((\theta^{n-1}-\theta^{n-2})\nabla\eta_c^n + (c^{n-1}-c^{n-2})\nabla\eta_\theta^n\Big)\cdot\nabla\xi_c\ dx \\
&\qquad + \int_\Omega \Big[\Big(\eta_c^n\nabla(P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1}) + \eta_c^n\nabla(P_h^{n-2}(\theta^{n-1}-\theta^{n-2}))\Big)\cdot\nabla\xi_c \\
&\qquad\qquad + \Big(\eta_\theta^n\nabla(R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1}) + \eta_\theta^n\nabla(R_h^{n-2}(c^{n-1}-c^{n-2}))\Big)\cdot\nabla\xi_c\Big]dx \\
&\qquad - \int_\Omega \Big[(R_h^n c^n - R_h^{n-1}c^n)\nabla(P_h^{n-2}\theta^{n-2})\cdot\nabla\xi_c \\
&\qquad\qquad + (P_h^n\theta^n - P_h^{n-1}\theta^n)\nabla(R_h^{n-2}c^{n-2})\cdot\nabla\xi_c\Big]dx \\
&\quad := \widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3
\end{aligned}
$$

and the difference of the two equations with $\xi_c = 0$ gives

$$
\begin{aligned}
&\int_\Omega \Big[\kappa(c^{n-2})\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\cdot\nabla\xi_\theta + \theta^{n-2}\nabla\theta^{n-1}\cdot\nabla(R_h^n c^n - R_h^{n-1}c^n)\cdot\xi_\theta \\
&\qquad + \zeta^{-1}m(P_h^n\theta^n - P_h^{n-1}\theta^n)\xi_\theta + \zeta^{-1}\langle\beta(P_h^n\theta^n - P_h^{n-1}\theta^n), \xi_\theta\rangle_{\partial\Omega} \\
&\qquad = \int_\Omega \Big[(\theta^{n-1}\nabla\theta^n - \theta^{n-2}\nabla\theta^{n-1})\cdot\nabla\eta_c^n\,\xi_\theta + (c^{n-1}\nabla\theta^n - c^{n-2}\nabla\theta^{n-1})\cdot\nabla\eta_\theta^n\,\xi_\theta \\
&\qquad\qquad + (\kappa(c^{n-1}) - \kappa(c^{n-2}))\nabla\eta_\theta^n\cdot\nabla\xi_\theta - c^{n-2}\nabla\theta^{n-1}\cdot\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\cdot\xi_\theta \\
&\qquad\qquad + \Big(c^{n-1}\nabla(P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1}) + (c^{n-1}-c^{n-2})\nabla(P_h^{n-2}\theta^{n-1}) \\
&\qquad\qquad + c^{n-2}\nabla P_h^{n-2}(\theta^{n-1}-\theta^{n-2}) + \theta^{n-1}\nabla(R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1}) \\
&\qquad\qquad + (\theta^{n-1}-\theta^{n-2})\nabla(R_h^{n-2}c^{n-1}) + \theta^{n-2}\nabla R_h^{n-2}(c^{n-1}-c^{n-2})\Big)\cdot\nabla\eta_\theta^n\,\xi_\theta\Big]dx \\
&\qquad + (c^{n-2}\nabla(P_h^{n-2}\theta^{n-2}) + \theta^{n-2}\nabla(R_h^{n-2}c^{n-2}))\cdot\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\cdot\xi_\theta\Big]dx \\
&\ := \sum_{j=4}^{9}\widetilde{J}_j.
\end{aligned}
$$

Set $\xi_c = R_h^n c^n - R_h^{n-1}c^n$ and $\xi_\theta = P_h^n\theta^n - P_h^{n-1}\theta^n$ in the above equations. By using (3.29)–(3.30), we obtain the following estimates:

$$
\begin{aligned}
\widetilde{J}_1 &\le \int_\Omega \big[C\varepsilon^{-1}\tau^2(\|\nabla\eta_c^n\|^2 + \|\nabla\eta_\theta^n\|^2) + \varepsilon\|\nabla(R_h^n c^n - R_h^{n-1}c^n)\|^2\big]dx, \\
\widetilde{J}_2 &\le \epsilon\|\eta_c^n\|_{L^\infty}^2\|\nabla(P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1})\|^2 + \epsilon^{-1}\|\eta_c^n\|_{L^6}^2\|\nabla P_h^{n-2}(\theta^{n-1}-\theta^{n-2})\|_{L^3}^2 \\
&\quad + \epsilon\|\eta_\theta^n\|_{L^\infty}^2\|\nabla(R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1})\|^2 + \epsilon^{-1}\|\eta_\theta^n\|_{L^6}^2\|\nabla R_h^{n-2}(c^{n-1}-c^{n-2})\|_{L^3}^2 \\
&\quad + C\epsilon^{-1}\ \|\nabla(R_h^n c^n - R_h^{n-1}c^n)\|^2 \\
&\le \epsilon\|\nabla(P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1})\|^2 + \epsilon\|\nabla(R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1})\|^2 \\
&\quad + C\epsilon^{-1}\tau^2(\|\eta_c^n\|_1^2 + \|\eta_\theta^n\|_1^2) + \epsilon\|R_h^n c^n - R_h^{n-1}c^n\|_1^2,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{J}_3 &\le (C\|R_h^n c^n - R_h^{n-1}c^n\| + \|R_h^n c^n - R_h^{n-1}c^n\|_{L^6} Ch^{1-d/6})\|\nabla(R_h^n c^n - R_h^{n-1}c^n)\| \\
&\quad + (C\|P_h^n\theta^n - P_h^{n-1}\theta^n\| + \|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} Ch^{1-d/6})\|\nabla(R_h^n c^n - R_h^{n-1}c^n)\| \\
&\le C^*\epsilon^{-1}\|R_h^n c^n - R_h^{n-1}c^n\|^2 + (C\epsilon + Ch^{1-d/6})\|R_h^n c^n - R_h^{n-1}c^n\|_1^2, \\
\widetilde{J}_4 &\le C\epsilon^{-1}\tau^2\|\nabla\eta_c^n\|^2 + C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2 + \epsilon\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6}^2 \\
&\le C\epsilon^{-1}\tau^2\|\nabla\eta_c^n\|^2 + C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2 + \epsilon\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2, \\
\widetilde{J}_5 &\le C^*\epsilon^{-1}\|P_h^n\theta^n - P_h^{n-1}\theta^n\|^2 + \epsilon\|\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\|^2 + C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2, \\
\widetilde{J}_6 &\le \|\nabla\eta_\theta^n\|_{L^3}\|\nabla(P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1})\|\|\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\|_{L^6} \\
&\quad + C\|c^{n-1} - c^{n-2}\|_{L^\infty}\|\nabla P_h^{n-2}\theta^{n-1}\|_{L^3}\|\nabla\eta_\theta^n\|\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} \\
&\le Ch^{r-d/6}\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2 + Ch^{r-d/6}\|P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1}\|_1^2 \\
&\quad + C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2 + \epsilon\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2, \\
\widetilde{J}_7 &\le \|\nabla P_h^{n-2}(\theta^n - \theta^{n-1})\|_{L^3}\|\nabla\eta_\theta^n\|\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} \\
&\quad + \|\nabla(R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1})\|\|\nabla\eta_\theta^n\|_{L^3}\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} \\
&\le C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2 + (\epsilon + Ch^{r-d/6})\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2 \\
&\quad + Ch^{r-d/6}\|R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1}\|_1^2, \\
\widetilde{J}_8 &\le C\|\theta^{n-1} - \theta^{n-2}\|_{L^\infty}\|\nabla R_h^{n-2}c^{n-1}\|_{L^3}\|\nabla\eta_\theta^n\|\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} \\
&\quad + \|\nabla R_h^{n-2}(c^n - c^{n-1})\|_{L^3}\|\nabla\eta_\theta^n\|\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^6} \\
&\le C\epsilon^{-1}\tau^2\|\nabla\eta_\theta^n\|^2 + \epsilon\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2, \\
\widetilde{J}_9 &\le C(\|\nabla P_h^{n-2}\theta^{n-2}\|_{L^6} + \|\nabla R_h^{n-2}c^{n-2}\|_{L^6}) \\
&\quad \|\nabla(P_h^n\theta^n - P_h^{n-1}\theta^n)\|\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_{L^3} \\
&\le \epsilon\|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2 + C^*\epsilon^{-1}\|P_h^n\theta^n - P_h^{n-1}\theta^n\|^2.
\end{aligned}
$$

Since C^* does not depend on m, if we choose m to be larger than $3C^*$ and choose ε and h small enough, then we derive that

$$
\begin{aligned}
&\|R_h^n c^n - R_h^{n-1}c^n\|_1^2 + \|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2 \\
&\quad \le \frac{1}{2}\big(\|R_h^{n-1}c^{n-1} - R_h^{n-2}c^{n-1}\|_1^2 + \|P_h^{n-1}\theta^{n-1} - P_h^{n-2}\theta^{n-1}\|_1^2\big) + C\tau^2 h^{2r}.
\end{aligned}
$$

By our definition of (R_h^0, P_h^0), we note that $R_h^1 = R_h^0$ and $P_h^1 = P_h^0$. Hence, the above inequality implies that

$$
\|R_h^n c^n - R_h^{n-1}c^n\|_1 + \|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1^2 \le C\tau^2 h^{2r}. \tag{3.32}
$$

Then

$$
\begin{aligned}
&\|\eta_c^n - \eta_c^{n-1}\|_1 + \|\eta_\theta^n - \eta_\theta^{n-1}\|_1 \\
&\quad \le \|c^n - R_h^{n-1}c^n - (c^{n-1} - R_h^{n-1}c^{n-1})\|_1 + \|R_h^n c^n - R_h^{n-1}c^n\|_1 \\
&\qquad + \|\theta^n - P_h^{n-1}\theta^n - (\theta^{n-1} - P_h^{n-1}\theta^{n-1})\|_1 + \|P_h^n\theta^n - P_h^{n-1}\theta^n\|_1 \\
&\quad \le C(\|c^n - c^{n-1}\|_{r+1} + \|\theta^n - \theta^{n-1}\|_{r+1})h^r + C\tau h^r \\
&\quad \le C\tau h^r.
\end{aligned}
$$

The L^2 estimates in (3.31) can be performed in a routine way, as in Lemma 3.1. ∎

The following lemmas can be proved by the inverse inequality and classical interpolation formulas.

LEMMA 3.5. *Suppose that τ_h is a regular division of Ω. Then for any $u \in V_h^r$,*

$$\|u\|_{L^\infty} \le C_{12}\gamma_h^d \|u\|_1,$$
$$\|u\|_{L^\infty} \le C_{12} h^{-d/2}\|u\|,$$
$$\|\nabla u\| \le C_{12} h^{-1}\|u\|,$$

and moreover,

$$\|u\|_{L^\infty} \le C_{12}\min(\gamma_h^d \tau^{-\frac{1}{2}}, h^{-d/2})(\|u\|^2 + \tau\|\nabla u\|^2)^{1/2},$$
$$\|\nabla u\| \le C_{12}\min(\tau^{-\frac{1}{2}}, h^{-1})(\|u\|^2 + \tau\|\nabla u\|^2)^{1/2}.$$

Hereafter we assume that the mesh size τ and h satisfy the following condition:

$$(E_0+1)(\gamma_h^d \tau^{\frac{1}{2}} + h^{1/4}) < \epsilon_1, \tag{3.33}$$

where ϵ_1 is a small positive constant to be determined.

By Lemmas 3.2 and 3.3 and the assumption of induction for (2.8), we get

$$\|e_c^{n-1}\|_{L^\infty},\, \|e_\theta^{n-1}\|_{L^\infty},\, \|\eta_c^{n-1}\|_{L^\infty},\, \|\eta_\theta^{n-1}\|_{L^\infty},\, \|e_c^{n-1}\|_1,\, \|e_\theta^{n-1}\|_1 \le C_{12}\epsilon_1\,. \tag{3.34}$$

LEMMA 3.6. *For $u \in V_h^r$,*

$$\|I_h(u^2) - u^2\|_{L^1} \le Ch^2\|\nabla u\|^2,$$
$$\|I_h(u^2) - u^2\|_{W^{1,1}} \le Ch\|\nabla u\|^2.$$

3.2. A priori estimates. We prove Theorem 2.1 by mathematical induction. By the initial conditions for (2.6)–(2.7) and Lemma 3.3, (2.8) holds for $n = 0$. We assume that for some $E_0 > 0$, (2.8) holds for $0 \le n \le k-1$. We need to show that there exists such a E_0 so that (2.8) also holds for $n = k$ with the same E_0.

Here we assume that the system (2.6)–(2.7) has a solution (c_h^n, θ_h^n). Since the projection pair $(R_h^n u, P_h^n v)$ satisfies (3.11), (3.4) can be written by

$$\begin{aligned}(D_t e_c^n, \xi_c) &+ (\theta_h^{n-1}\nabla e_c^n, \nabla\xi_c) + \langle \alpha e_c^n, \xi_c\rangle_{\partial\Omega} \\ &= -(c_h^{n-1}\nabla e_\theta^{n-1}, \nabla \xi_c) + J_{c1}^n(\xi_c) + J_{c2}^n(\xi_c).\end{aligned} \tag{3.35}$$

By (3.3) and Lemmas 3.3 and 3.4, for $1 \le n \le k$ we have

$$\begin{aligned}J_{c1}^n(\xi_c) &= -(e_c^{n-1}\nabla(P_h^{n-1}\theta^{n-1}) + e_\theta^{n-1}\nabla(R_h^n c^n), \nabla\xi_c) \\ &\quad - (\Gamma(c_h^{n-1}, \theta_h^{n-1}) - \Gamma(R_h^{n-1}c^{n-1}, P_h^{n-1}\theta^{n-1}), \xi_c) \\ &\le (\|e_c^{n-1}\|_{L^6}\|\nabla(P_h^{n-1}\theta^{n-1} - \theta^{n-1})\|_{L^3} + \|e_c^{n-1}\|\|\nabla\theta^{n-1}\|_{L^\infty})\|\nabla\xi_c\| \\ &\quad + (\|e_\theta^{n-1}\|_{L^6}\|\nabla(R_h^n c^n - c^n)\|_{L^3} + \|e_\theta^{n-1}\|\|\nabla c^n\|_{L^\infty})\|\nabla\xi_c\| \\ &\quad + C(\|e_c^{n-1}\| + \|e_\theta^{n-1}\|)\|\xi_c\| \\ &\le C_{13}\big[h^{1-d/6}(\|e_c^{n-1}\|_1 + \|e_\theta^{n-1}\|_1) + (\|e_c^{n-1}\| + \|e_\theta^{n-1}\|)\big]\|\xi_c\|_1,\end{aligned} \tag{3.36}$$

(3.37)

$$
\begin{aligned}
J_{c2}^n(\xi_c) &= ((\eta_c^{n-1} - \eta_c^n)\nabla(P_h^{n-1}\theta^{n-1}), \nabla\xi_c) \\
&\quad + ((\eta_\theta^{n-1} - \eta_\theta^n)\nabla(R_h^n c^n), \nabla\xi_c) + (\eta_\theta^n \nabla(R_h^{n-1}c^{n-1} - R_h^n c^n), \nabla\xi_c) \\
&\quad + (\Gamma(c^{n-1}, \theta^{n-1}) - \Gamma(R_h^{n-1}c^{n-1}, P_h^{n-1}\theta^{n-1}), \xi_c) \\
&\quad - (D_t\eta_c^n, \xi_c) + (R_c^n, \xi_c) + (R^n, \nabla\xi_c) - m(\eta_c^n, \xi_c) - (c^{n-1}\nabla(\eta_\theta^n - \eta_\theta^{n-1}), \nabla\xi_c) \\
&\le C_{14}(\tau + h^{r+1})\|\xi_c\|_1.
\end{aligned}
$$

Setting $\xi_c = e_c^n$ in the above estimates and choosing ϵ_2, h to be small enough such that $C_{13}^2\epsilon_2^{-1}h^{2-d/3} < \epsilon_2 < \theta_{\min}/8$, we get

$$
\begin{aligned}
&J_{c1}^n(e_c^n) + J_{c2}^n(e_c^n) \\
&\quad \le C_{15}\epsilon_2^{-1}\left[(\tau + h^{r+1})^2 + \|e_c^{n-1}\|^2 + \|e_\theta^{n-1}\|^2\right] + \epsilon_2(\|\nabla e_c^{n-1}\|^2 + \|\nabla e_\theta^{n-1}\|^2 + \|e_c^n\|_1^2),
\end{aligned}
$$

where $C_{15} = C_{13}^2 + C_{14}^2$. The equation (3.35) with $\xi_c = e_c^n$ reduces to

$$
\begin{aligned}
\frac{1}{2}D_t\|e_c^n\|^2 + \frac{\tau}{2}\|D_t e_c^n\|^2 + \frac{\theta_{\min}}{4}\|\nabla e_c^n\|^2 \qquad\qquad (3.38)\\
\le C_{15}\epsilon_2^{-1}(\tau + h^{r+1})^2 + \epsilon_2(\|\nabla e_c^{n-1}\|^2 + \|\nabla e_\theta^{n-1}\|^2) \\
+ C_{15}\epsilon_2^{-1}(\|e_c^n\|^2 + \|e_\theta^{n-1}\|^2 + \|e_c^{n-1}\|^2).
\end{aligned}
$$

The above inequality holds for any $\epsilon_2 > 0$. If we choose $\epsilon_2 = \theta_{\min}/16$, then by the induction assumption the above inequality reduces to

$$
\begin{aligned}
\left(\frac{1}{2} - C_{16}\tau\right)\|e_c^n\|^2 + \frac{\tau^2}{2}\|D_t e_c^n\|^2 + \frac{\tau\theta_{\min}}{4}\|\nabla e_c^n\|^2 \qquad\qquad (3.39)\\
\le (C_{17}E_0 + C_{17}E_0\tau + C_{17}\tau)(\tau + h^{r+1})^2,
\end{aligned}
$$

where $C_{16} = 16C_{15}/\theta_{\min}$ and $C_{17} = 3C_{16} + \theta_{\min}/16 + 1/2$. Therefore, if τ is small enough such that

$$
C_{16}\tau < \frac{1}{4},
$$

the inequality (3.40) together with Lemma 3.5 implies that

$$
\|e_c^n\|_{L^\infty} \le C_{18}(E_0 + 1)^{\frac{1}{2}}(\gamma_h^d\tau^{\frac{1}{2}} + h^{r+1-d/2}), \tag{3.40}
$$

$$
\|\nabla e_c^n\| \le C_{18}(E_0 + 1)(\tau^{\frac{1}{2}} + h^r), \tag{3.41}
$$

where $C_{18} = \sqrt{C_{12}C_{17}(4 + 4/\theta_{\min})}$. If we choose ϵ_1 and h so small that $(C_{12} + C_{18})\epsilon_1 < \sigma/4$ and $C_{10}h^{r+1} < \epsilon_1 < 1$, then Lemmas 3.3, 3.4, and 3.5, the estimates (3.33) and (3.40)–(3.41) imply that

$$
\|e_c^{n-1}\|_{L^\infty} + \|e_c^n\|_{L^\infty} \le (C_{12} + C_{18})\epsilon_1, \quad \|\eta_c^{n-1}\|_{L^\infty} + \|\eta_c^n\|_{L^\infty} \le 2\epsilon_1,
$$

and

$$
\|c_h^n\|_{L^\infty} + \|c_h^{n-1}\|_{L^\infty} \le C_{19}, \quad \frac{\sigma}{2} + c_h^n \ge 0, \quad \frac{\sigma}{2} + c_h^{n-1} \ge 0, \tag{3.42}
$$

where $C_{19} = 2\|c\|_{L^\infty(\Omega\times(0,T))} + \sigma/4 + \sigma/(2C_{12} + 2C_{18})$.

By taking $\xi_\theta = e_\theta^n$ in (3.5) and noting that

$$((\sigma + c_h^{n-1})D_t e_\theta^n, e_\theta^n) = \frac{1}{2}D_t((\sigma + R_h^n c^n), (e_\theta^n)^2) + (e_c^{n-1}D_t e_\theta^n, e_\theta^n) - \frac{1}{2}(D_t R_h^n c^n, (e_\theta^n)^2) + \frac{\tau}{2}((\sigma + R_h^{n-1}c_h^{n-1}), (D_t e_\theta^n)^2)$$

and (3.11), we get

$$\begin{aligned}
&\frac{1}{2}D_t\Big(\big(\frac{\sigma}{2} + R_h^n c^n\big), (e_\theta^n)^2\Big) + \frac{\sigma}{4}D_t(\|e_\theta^n\|^2) + \frac{\tau}{2}((\sigma + R_h^{n-1}c_h^{n-1}), (D_t e_\theta^n)^2)\\
&\quad + (e_c^{n-1}D_t e_\theta^n, e_\theta^n) + (\kappa(c_h^{n-1})\nabla e_\theta^n, \nabla e_\theta^n) + \langle \beta e_\theta^n, e_\theta^n\rangle_{\partial\Omega}\\
&\quad - ((c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n)\cdot\nabla e_\theta^n, e_\theta^n)\\
&= \frac{1}{2}(D_t(R_h^n c^n), (e_\theta^n)^2) + J_{\theta 1}^n(e_\theta^n) + J_{\theta 2}^n(e_\theta^n).
\end{aligned}$$

By taking $\xi_c = \frac{1}{2}I_h(e_\theta^n)^2$ in (3.35),

$$\begin{aligned}
&\frac{1}{2}(D_t e_c^n, (e_\theta^n)^2) + \frac{1}{2}(D_t e_c^n, I_h(e_\theta^n)^2 - (e_\theta^n)^2) + \frac{1}{2}(c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n, \nabla I_h(e_\theta^n)^2)\\
&\quad + \frac{1}{2}\langle \alpha e_c^n, I_h(e_\theta^n)^2\rangle_{\partial\Omega}\\
&= J_{c1}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big) + J_{c2}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big).
\end{aligned}$$

Summing up the last two equations and noting that

$$\frac{1}{2}(D_t e_c^n, (e_\theta^n)^2) + (e_c^{n-1}D_t e_\theta^n, e_\theta^n) = \frac{1}{2}D_t(e_c^n, (e_\theta^n)^2) + \frac{\tau}{2}(e_c^{n-1}, (D_t e_\theta^n)^2),$$

we get (using the notation defined in (3.6)–(3.9))

$$\begin{aligned}
&\frac{1}{2}D_t\Big(\frac{\sigma}{2} + c_h^n, (e_\theta^n)^2\Big) + \frac{\sigma}{4}D_t(\|e_\theta^n\|^2) + \frac{\tau}{2}(\sigma + c_h^{n-1}, (D_t e_\theta^n)^2)\\
&\qquad + (\kappa(c_h^{n-1})\nabla e_\theta^n, \nabla e_\theta^n) + \langle \beta e_\theta^n, e_\theta^n\rangle_{\partial\Omega}\\
&\quad = J_{c1}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big) + J_{c2}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big) + J_{\theta 1}^n(e_\theta^n) + J_{\theta 2}^n(e_\theta^n) - \frac{1}{2}\langle \alpha e_c^n, I_h(e_\theta^n)^2\rangle\\
&\qquad + \frac{1}{2}(D_t e_c^n, (e_\theta^n)^2 - I_h(e_\theta^n)^2) + \frac{1}{2}(D_t c^n, (e_\theta^n)^2) - \frac{1}{2}(D_t \eta_c^n, (e_\theta^n)^2)\\
&\qquad + \frac{1}{2}\big(c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n, \nabla[(e_\theta^n)^2 - I_h(e_\theta^n)^2]\big)\\
&\quad = J_{c1}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big) + J_{c2}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big) + \sum_{i=3}^{9} J_i^n(e_\theta^n) + J_{\theta 2}^n(e_\theta^n) + \sum_{i=10}^{14}\widehat{J}_i. \qquad (3.43)
\end{aligned}$$

Since the discrete energy flux due to the mass convection is just the mass flux multiplied by temperature, the major nonlinear term $((c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n)\cdot\nabla e_\theta^n, e_\theta^n)$ from the convection vanishes by combining these two discrete systems. The remainder $\widehat{J}_{14}$ can be dealt with using Lemma 3.6.

We need to estimate each term on the right-hand side of the above equation. By using (3.31), (3.34), (3.36), (3.37), (3.40)–(3.41), the condition (3.33), and Lemmas

3.2–3.4, we have

$$
\begin{aligned}
\left|J_{c1}^n\Big(\frac{1}{2}I_h(e_\theta^n)^2\Big)\right| &\le C_{20}\big[h^{1-d/6}(\|e_c^{n-1}\|_1+\|e_\theta^{n-1}\|_1)+(\|e_c^{n-1}\|+\|e_\theta^{n-1}\|)\big]\|I_h(e_\theta^n)^2\|_1\\
&\le C_{20}\big(C_{12}\epsilon_1 h^{1-d/6}+E_0(\tau+h^{r+1})\big)\|e_\theta^n\|_{L^\infty}\|e_\theta^n\|_1\\
&\le 2C_{22}\epsilon_1\|e_\theta^n\|_1^2,\\
|J_{c2}^n(I_h^n(e_\theta^n)^2)| &\le 2C_{14}(\tau+h^{r+1})\|e_\theta^n\|_{L^\infty}\|e_\theta^n\|_1\le C_{23}\epsilon_1\|e_\theta^n\|_1^2,\\
|J_3^n(e_\theta^n)| &\le (\|e_c^{n-1}\|_{L^\infty}\|D_t\eta_\theta^n\|+\|D_t\theta^n\|_{L^\infty}\|e_c^{n-1}\|)\|e_\theta^n\|\\
&\le C_{24}(\tau h^{r+1-d/2}\|e_c^{n-1}\|+\|e_c^{n-1}\|)\|e_\theta^n\|\le C_{25}\epsilon_1^{-1}\|e_c^{n-1}\|^2+\epsilon_1\|e_\theta^n\|^2,\\
|J_4^n(e_\theta^n)| &\le \|c_h^{n-1}\|_{L^\infty}\|\nabla\eta_\theta^{n-1}\|_{L^3}\|\nabla e_\theta^n\|\|e_\theta^n\|_{L^6}\\
&\quad+\|c_h^{n-1}\|_{L^\infty}\|\nabla\theta^{n-1}\|_{L^\infty}\|\nabla e_\theta^n\|\|e_\theta^n\|\\
&\quad+\|\theta_h^{n-1}\|_{L^\infty}\|\nabla\eta_c^{n-1}\|_{L^3}\|\nabla e_\theta^n\|\|e_\theta^n\|_{L^6}\\
&\quad+\|\theta_h^{n-1}\|_{L^\infty}\|\nabla c^{n-1}\|_{L^\infty}\|\nabla e_\theta^n\|\|e_\theta^n\|\\
&\le C_{26}\epsilon_1^{-1}\|e_\theta^n\|^2+\epsilon_1\|\nabla e_\theta^n\|^2,\\
|J_5^n(e_\theta^n)| &\le C_{27}\epsilon_2^{-1}\|e_\theta^{n-1}\|^2+C_{27}\epsilon_1^{-1}\|e_c^n\|^2+\epsilon_1\|\nabla e_\theta^n\|^2,\\
|J_6^n(e_\theta^n)| &\le ((e_c^{n-1}(\nabla\theta^{n-1}-\nabla\eta_\theta^{n-1})+e_\theta^{n-1}(\nabla c^n-\nabla\eta_c^n))\cdot(\nabla\theta^n-\nabla\eta_\theta^n),e_\theta^n)\\
&\le \|e_c^{n-1}\|(\nabla\theta^{n-1}\nabla\theta^n-\nabla\theta^{n-1}\nabla\eta_\theta^n\\
&\qquad\qquad-\nabla\eta_\theta^{n-1}\nabla\theta^n+\nabla\eta_\theta^{n-1}\nabla\eta_\theta^n)\|_{L^3}\|e_\theta^n\|_{L^6}\\
&\quad+\|e_\theta^{n-1}\|(\nabla c^n\nabla\theta^n-\nabla c^n\nabla\eta_\theta^n-\nabla\eta_c^n\nabla\theta^n+\nabla\eta_c^n\nabla\eta_\theta^n)\|_{L^3}\|e_\theta^n\|_{L^6}\\
&\le C_{28}\epsilon_1^{-1}\|e_c^{n-1}\|^2+C_{28}\epsilon_1^{-1}\|e_\theta^{n-1}\|^2+\epsilon_1\|e_\theta^n\|_1^2,\\
|J_7^n(e_\theta^n)| &\le C_{29}(\|e_c^{n-1}\|_{L^6}\|\nabla\eta_\theta^n\|_{L^3}+\|\nabla\theta^n\|_{L^\infty}\|e_c^{n-1}\|)\|\nabla e_\theta^n\|\\
&\le C_{30}\epsilon_1^{-1}\|e_c^{n-1}\|^2+C_{30}\epsilon_1\|\nabla e_c^{n-1}\|^2+\epsilon_1\|\nabla e_\theta^n\|^2,\\
|J_8^n(e_\theta^n)| &\le C_{31}(\|e_\theta^{n-1}\|^2+\|e_c^{n-1}\|^2+\|e_\theta^n\|^2),\\
|J_9^n(e_\theta^n)| &\le C_{32}(\|e_\theta^{n-1}\|^2+\|e_\theta^n\|^2),\\
|\widehat{J}_{10}^n| &\le C_{33}\|e_c^n\|_{L^\infty}(\|I_h(e_\theta^n)^2-(e_\theta^n)^2\|_{L^1(\partial\Omega)}+\|(e_\theta^n)^2\|_{L^1(\partial\Omega)})\\
&\le C_{34}(C_{12}+C_{18})\epsilon_1(\|I_h(e_\theta^n)^2-(e_\theta^n)^2\|_{W^{1,1}}+\|(e_\theta^n)^2\|_{W^{1,1}})\\
&\le C_{35}\epsilon_1\|e_\theta^n\|^2+C_{35}\epsilon_1\|\nabla e_\theta^n\|^2,\\
|\widehat{J}_{12}^n| &\le C\|e_\theta^n\|_{L^4}^2\le C_{\epsilon_1}\|e_\theta^n\|^2+\epsilon_1\|\nabla e_\theta^n\|^2,\\
|\widehat{J}_{13}^n| &\le C\|e_\theta^n\|_{L^4}^2\le C_{\epsilon_1}\|e_\theta^n\|^2+\epsilon_1\|\nabla e_\theta^n\|^2,\\
|\widehat{J}_{14}^n| &\le C(\|\nabla e_\theta^{n-1}\|_{L^\infty}+\|\nabla e_c^n\|_{L^\infty})\|\nabla(I_h(e_\theta^n)^2-(e_\theta^n)^2)\|_{L^1}\\
&\le C_{37}(E_0+1)(\tau^{\frac{1}{2}}+h^r)h^{1-d/2}\|\nabla e_\theta^n\|^2\\
&\le C_{37}\epsilon_1\|\nabla e_\theta^n\|^2,\\
|J_{\theta2}^n(e_\theta^n)| &\le \epsilon_1\|e_\theta^n\|^2+\epsilon_1\|\nabla e_\theta^n\|^2+C_{39}\epsilon_1^{-1}(\tau+h^{r+1})^2.
\end{aligned}
$$

It remains to estimate $\widehat{J}_{11} = \frac{1}{2}(D_t e_c^n, (e_\theta^n)^2 - I_h(e_\theta^n)^2)$. Let $\xi_c = D_t e_c^n$ and use (3.4). We have

$$\begin{aligned}\|D_t e_c^n\|^2 &= -(c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n, \nabla D_t e_c^n) - \langle \alpha e_c^n,\, D_t e_c^n\rangle_{\partial\Omega} \\ &\quad + ((\theta^{n-1}\nabla\eta_c^n + c^{n-1}\nabla\eta_\theta^n), \nabla D_t e_c^n) + (\eta_c^n \nabla(P_h^{n-1}\theta^{n-1}), \nabla D_t e_c^n) \\ &\quad + \langle \alpha\eta_c^n, D_t e_c^n\rangle_{\partial\Omega} + J_{c1}^n(D_t e_c^n) + J_{c2}^n(D_t e_c^n)\,. \end{aligned} \tag{3.44}$$

With (3.34), it is easy to see that

$$\begin{aligned} &|(c_h^{n-1}\nabla e_\theta^{n-1} + \theta_h^{n-1}\nabla e_c^n, \nabla D_t e_c^n)| \\ &\quad \le C_{40}\epsilon_3^{-1}(E_0+1)(\tau + h^{r+1})^2 h^{-4} + \epsilon_3\|D_t e_c^n\|^2, \\ &|\langle \alpha e_c^n,\, D_t e_c^n\rangle_{\partial\Omega}| \\ &\quad \le C\|\alpha e_c^n D_t e_c^n\|_{W^{1,1}} \\ &\quad \le C(|e_c^n|, |D_t e_c^n|) + C(|\nabla e_c^n|, |D_t e_c^n|) + C(|e_c^n|, |\nabla D_t e_c^n|) \\ &\quad \le C_{41}\epsilon_3^{-1}(E_0+1)(\tau + h^{r+1})^2 h^{-2} + \epsilon_3\|D_t e_c^n\|^2, \\ &((\theta^{n-1}\nabla\eta_c^n + c^{n-1}\nabla\eta_\theta^n), \nabla D_t e_c^n) \\ &\quad \le C_{42}\epsilon_3^{-1}(\tau + h^{r+1})^2 h^{-4} + \epsilon_3\|D_t e_c^n\|^2, \\ &|(\eta_c^n \nabla(P_h^{n-1}\theta^{n-1}), \nabla D_t e_c^n)| \\ &\quad \le \|\eta_c^n\|_{L^6}\|\nabla(P_h^{n-1}\theta^{n-1} - I_h\theta^{n-1})\|_{L^3} h^{-1}\|D_t e_c^n\| \\ &\qquad + \|\eta_c^n\|\|\nabla I_h\theta^{n-1}\|_{L^\infty} h^{-1}\|D_t e_c^n\| \\ &\quad \le C_{43}\epsilon_3^{-1}(\tau + h^{r+1})^2 h^{-4} + \epsilon_3\|D_t e_c^n\|^2, \\ &|\langle \alpha\eta_c^n,\, D_t e_c^n\rangle_{\partial\Omega}| \\ &\quad \le C(|\eta_c^n|, |D_t e_c^n|) + C(|\nabla\eta_c^n|, |D_t e_c^n|) + C(|\eta_c^n|, |\nabla D_t e_c^n|) \\ &\quad \le C_{44}\epsilon_3^{-1}(\tau + h^{r+1})^2 h^{-2} + \epsilon_3\|D_t e_c^n\|^2, \end{aligned}$$

and by (3.36)–(3.37),

$$\begin{aligned} J_{c1}^n(D_t e_c^n) &\le C_{45}\epsilon_3^{-1}(E_0+1)(\tau + h^{r+1})^2 h^{-3} + \epsilon_3\|D_t e_c^n\|^2, \\ J_{c2}^n(D_t e_c^n) &\le C_{45}\epsilon_3^{-1}(\tau + h^{r+1})^2 h^{-2} + \epsilon_3\|D_t e_c^n\|^2. \end{aligned}$$

By choosing a small ϵ_3, (3.44) reduces to

$$\|D_t e_c^n\|^2 \le C_{46}(E_0+1)(\tau + h^{r+1})^2 h^{-4}.$$

On the other hand, from (3.40) we see that

$$\|D_t e_c^n\|^2 \le C_{47}(E_0+1)(\tau + h^{r+1})^2 \tau^{-2}.$$

Therefore,

$$\|D_t e_c^n\|^2 \le (E_0+1)\min(C_{47}\tau^{-2}, C_{46}h^{-4})(\tau + h^{r+1})^2 \le C_{48}(E_0+1).$$

Using Lemma 3.6, we get

$$\begin{aligned}|2\widehat{J}_{11}| &= |(D_t e_c^n, I_h^n(e_\theta^n)^2 - (e_\theta^n)^2)| \\ &\le \|D_t e_c^n\|_{L^\infty} \|I_h^n(e_\theta^n)^2 - (e_\theta^n)^2\|_{L^1} \\ &\le C_{49}(E_0+1)h^{2-d/2}\|\nabla e_\theta^n\|^2 \\ &\le C_{49}\epsilon_1 \|\nabla e_\theta^n\|^2,\end{aligned}$$

where we have used the assumption (3.33).

Substituting the above inequalities into (3.43) leads to

$$\begin{aligned}&\frac{\sigma}{4}\|e_\theta^n\|^2 + \frac{1}{2}\Big\|\sqrt{\frac{\sigma}{2}+c_h^n}e_\theta^n\Big\|^2 + \kappa_1\tau\|\nabla e_\theta^n\|^2 \\ &\le \frac{\sigma}{4}\|e_\theta^{n-1}\|^2 + \frac{1}{2}\Big\|\sqrt{\frac{\sigma}{2}+c_h^{n-1}}e_\theta^{n-1}\Big\|^2 + C_{\epsilon_1}\tau(\tau+h^{r+1})^2 \\ &\quad + C_{50}\epsilon_1\tau(\|\nabla e_c^{n-1}\|^2 + \|\nabla e_c^n\|^2 + \|\nabla e_\theta^{n-1}\|^2 + \|\nabla e_\theta^n\|^2) \\ &\quad + C_{\epsilon_1}\tau(\|e_c^{n-1}\|^2 + \|e_c^n\|^2 + \|e_\theta^{n-1}\|^2 + \|e_\theta^n\|^2),\end{aligned}$$

where C_{50} depends on the constants C_{20} up to C_{49}, and C_{ϵ_1} is a positive constant which depends on ϵ_1.

Finally, we recall that (3.38) holds for any ϵ_2. Add (3.38) times $16C_{50}\epsilon_1/\theta_{\min}$ to the above equation with $\epsilon_2 = \theta_{\min}/16$. Then we obtain that for $1 \le n \le k$,

$$\begin{aligned}&\frac{8C_{50}\epsilon_1}{\theta_{\min}}\|e_c^n\|^2 + \frac{\sigma}{4}\|e_\theta^n\|^2 + \frac{1}{2}\Big\|\sqrt{\frac{\sigma}{2}+c_h^n}e_\theta^n\Big\|^2 \\ &\quad + 3C_{50}\epsilon_1\tau\|\nabla e_c^n\|^2 + (\kappa_1 - C_{50}\epsilon_1)\tau\|\nabla e_\theta^n\|^2 \\ &\le \frac{8C_{50}\epsilon_1}{\theta_{\min}}\|e_c^{n-1}\|^2 + \frac{\sigma}{4}\|e_\theta^{n-1}\|^2 + \frac{1}{2}\Big\|\sqrt{\frac{\sigma}{2}+c_h^{n-1}}e_\theta^{n-1}\Big\|^2 \\ &\quad + 2C_{50}\epsilon_1\tau\|\nabla e_c^{n-1}\|^2 + C_{50}\epsilon_1\tau\|\nabla e_\theta^{n-1}\|^2 \\ &\quad + C_{\epsilon_1}\tau(\|e_c^n\|^2 + \|e_c^{n-1}\|^2 + \|e_\theta^n\|^2 + \|e_\theta^{n-1}\|^2) + C_{\epsilon_1}\tau(\tau+h^{r+1})^2.\end{aligned} \tag{3.45}$$

By the discrete Gronwall inequality, we obtain

$$\|e_c^n\|^2 + \|e_\theta^n\|^2 + \tau\sum_{m=1}^{n}(\|\nabla e_c^m\|^2 + \|\nabla e_\theta^m\|^2) \le C_{51}(\tau+h^{r+1})^2$$

for $0 \le n \le k$, where C_{51} depends only upon C_{50} and independent of n, h, τ, and E_0.

From the above equation, we can see that the constant E_0 in (2.8) can be chosen as C_{51}. Then we complete the mathematical induction. The condition (3.33) thus reduces to

$$\tau < \tau_0 \quad \text{and} \quad (C_{51}+1)(\gamma_h^d\tau^{\frac{1}{2}} + h^{1/4}) < \epsilon_1$$

for some positive constants τ_0 and ϵ_1, which hold if τ and h satisfy the condition of Theorem 2.1 with proper constants h_0 and ρ_0. Finally, the inequality (2.9) is a combination of (2.8) and Lemmas 3.3 and 3.4.

3.3. Existence and uniqueness. Clearly, the numerical solution (c_h^n, θ_h^n) can be obtained by solving the system (2.6)–(2.7). However, this system is not linear for the pair (c_h^n, θ_h^n). Thus the unique solvability of the system does not follow immediately from the boundedness of the numerical solution. Since (2.6) is a linear system for c_h^n, it is obvious that (2.6) is uniquely solvable once c_h^{n-1} and θ_h^{n-1} are known. If c_h^n, c_h^{n-1}, and θ_h^{n-1} are known, the estimate (2.8) still holds if we set the right-hand side of (2.7) to be zero. Hence, the homogeneous linear system (2.7) has only the trivial solution (otherwise there exist solutions which are unbounded). The proof of Theorem 2.1 is complete. □

4. Numerical examples. In this section, we present two numerical examples. The computations are performed with the software FreeFEM++.

Example 4.1. First we consider an artificial example to confirm our theoretical analysis. The system is defined by

$$\partial_t c - \nabla \cdot \nabla(c\theta) = -\Gamma(c,\theta) + f, \tag{4.1}$$

$$\partial_t(c\theta + \sigma\theta) - \nabla \cdot (\theta\nabla(c\theta)) - \nabla \cdot (\kappa(c)\nabla\theta) = (\lambda + \theta)\Gamma(c,\theta) + g, \tag{4.2}$$

where $\kappa(c) = 1 + c$, $\Gamma(c,\theta) = c\sqrt{\theta} - p_s(\theta)$, and $p_s(\theta) = \theta^2$. The boundary conditions are

$$\nabla(c\theta) \cdot \vec{n} = \mu - c, \quad \kappa(c)\nabla\theta \cdot \vec{n} = 2(\nu - \theta) \quad \text{on } \partial\Omega, \tag{4.3}$$

where $\Omega = (0,1) \times (0, \frac{1}{2})$ and $0 \le t \le 1$. The functions f, g, μ, and ν are chosen correspondingly to the exact solution

$$c(x,y,t) = 1 + e^{x+y-0.2t}, \qquad \theta(x,y,t) = 2 + e^{x-y-0.2t}.$$

The initial conditions are given by the above exact solution.

A uniform triangular partition with M nodes in the horizontal direction and $M/2$ nodes in the vertical direction is used in our computation. We solve the system by the proposed splitting Galerkin method with a linear element method and a quadratic element method, respectively. To confirm our error estimates, we choose $\tau = 1/M^2$ for the linear element method and $\tau = 1/M^3$ for the quadratic element method. Thus, our estimates in Theorem 4.1 become

$$\|c_h^n - c^n\| + \|\theta_h^n - \theta^n\| \le Ch^{r+1}\,. \tag{4.4}$$

We present in Table 1 the error of the linear Galerkin method and in Table 2 the error of the quadratic Galerkin method, respectively, in L^2 norm with $M = 20, 40, 80$

TABLE 1
L^2 error with piecewise linear FEM with $\tau = 1/M^2$.

	$\|e_c^n\|$			Order	$\|e_\theta^n\|$			Order
t	$M=20$	$M=40$	$M=80$	(h^γ)	$M=20$	$M=40$	$M=80$	(h^γ)
0.25	7.188E-4	1.787E-4	4.503E-5	1.99	2.415E-4	6.070E-5	1.534E-5	1.98
0.50	8.076E-4	1.484E-4	3.745E-5	2.21	2.686E-4	5.078E-5	1.284E-5	2.19
0.75	7.993E-4	1.364E-4	4.685E-5	2.04	2.696E-4	4.812E-5	1.612E-5	2.03
1.00	5.800E-4	1.292E-4	3.557E-5	2.01	1.999E-4	4.645E-5	1.243E-5	2.00

TABLE 2
L^2 error with piecewise quadratic FEM with $\tau = 1/M^3$.

	$\|e_c^n\|$			Order	$\|e_\theta^n\|$			Order
t	$M=10$	$M=20$	$M=40$	(h^γ)	$M=10$	$M=20$	$M=40$	(h^γ)
0.25	2.585E-4	3.235E-5	4.043E-6	2.99	9.808E-5	1.227E-5	1.534E-6	2.99
0.50	3.057E-4	2.761E-5	3.451E-6	3.23	1.110E-4	1.055E-5	1.318E-6	3.19
0.75	3.047E-4	2.546E-5	4.436E-6	3.05	1.103E-4	9.882E-6	1.631E-6	3.03
1.00	2.928E-4	3.352E-5	3.303E-6	3.23	1.068E-4	1.245E-5	1.243E-6	3.21

TABLE 3
L^2 error at each time level with $h = 1/80$ and $\tau = k/M$.

	$\|e_c^n\|$			$\|e_\theta^n\|$		
t	$k=5$	$k=10$	$k=20$	$k=5$	$k=10$	$k=20$
0.25	1.514E-2	2.856E-2	5.035E-2	5.998E-3	1.222E-2	2.414E-2
0.50	1.865E-2	3.640E-2	6.993E-2	6.800E-3	1.332E-2	2.636E-2
0.75	1.894E-2	3.759E-2	7.410E-2	6.846E-3	1.357E-2	2.638E-2
1.00	1.834E-2	3.670E-2	7.314E-2	6.665E-3	1.330E-2	2.646E-2

at different times t. We can see clearly that the numerical errors for both the components, temperature and vapor concentration, are proportional to h^{r+1}, $r = 1, 2$, which confirms our theoretical analysis in previous sections. We also test the linear Galerkin method with $M = 80$ and large time steps $\tau = 5/M, 10/M, 20/M$. The results are presented in Table 3. Numerical results show that the scheme is stable for the large time steps, although the numerical results with $\tau = 20/M$ seem not very accurate.

Example 4.2. The second example is a textile assembly with a porous batting sandwiched by two covering layers, which was investigated in [10, 27] for the single-component model with finite difference methods and finite volume methods, respectively. A polyester batting and two nylon covers are tested here. The values of all these physical parameters can be found in [10, 14]. We test the problem in the rectangular domain $\Omega = (0, 0.0492) \times (0, 0.5)$ up to 24 hours. We assume that the inner boundary ($x = 0$) is connected to a human body and the outer boundary ($x = 0.0492$) is exposed to a cold environment. The temperature at the inner and outer backgrounds is fixed at $\nu = 248\,\mathrm{K}$ and $\nu = 308\,\mathrm{K}$ with relative humidity of $R_H = 100\%$ and $R_H = 90\%$, respectively. Therefore the inner and outer background vapor concentrations are given by

$$\mu = R_H \frac{P_{\mathrm{sat}}(\nu)}{R\nu} \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 0.0492.$$

We apply the commonly used flux type boundary conditions [10, 14] in our simulation, where $\alpha_1 = 0.0070$ and $\beta_1 = 0.9542$ for $x = 0$ and $\alpha_2 = 0.0017$ and $\beta_2 = 0.1083$ for $x = 0.0492$, as given in [10, 14]. We assume that no flux passes through the boundary at $y = 0$ and $y = 0.5$ (i.e., $\alpha = \beta = 0$). To compare with the experimental data in [9], (1.1)–(1.2) are coupled with a a water equation [14]

$$\partial_t \left(\rho_W (1 - \epsilon) W\right) = \Gamma, \tag{4.5}$$

where ρ_W is the density of water and W is the water content relative to the fiber weight.

The initial conditions are given by

$$C(x, 0) \equiv 0.65 \frac{P_{\mathrm{sat}}(T(x, 0))}{RT(x, 0)}, \quad T(x, 0) \equiv 298\,\mathrm{K}, \quad W(x, 0) \equiv 0.$$

We apply the splitting linear Galerkin method for solving the vapor-temperature system defined in (1.1)–(1.2) and the Euler scheme for the water equation (4.5) with a uniform triangular partition of 200 linear triangular elements and $\tau = 10s$. The Galerkin method with a smaller time step and spatial step is also tested to confirm our numerical results. We present in Figure 2 numerical results of vapor concentration, temperature, and water content on the line $y = 0.25$ at 8 hours and 24 hours, respectively. Comparisons with experimental measurement of water content done in [9] are given in last two subfigures.

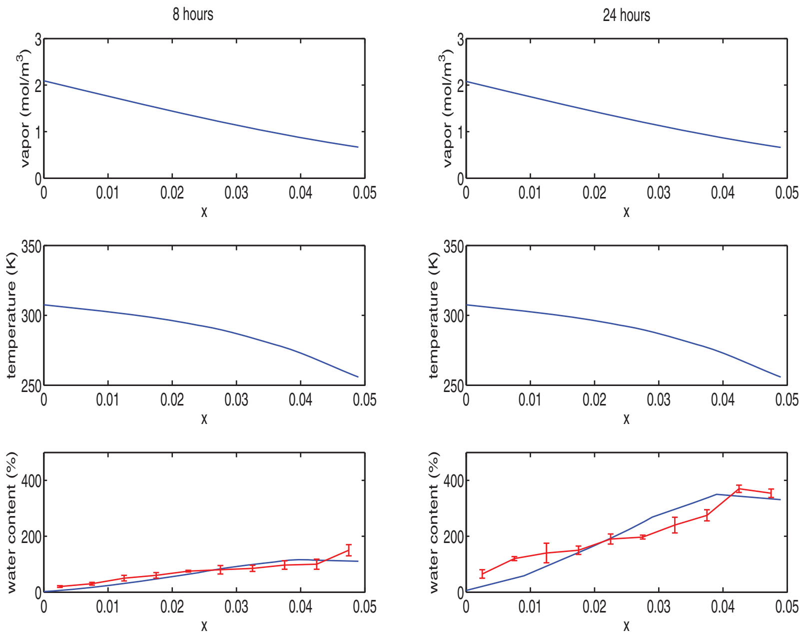

Fig. 2. *Numerical results of Example* 4.2.

5. Conclusions. We have presented a splitting Galerkin method with error analysis for a system of parabolic equations from heat and sweat transport in porous textile media. Similar models can be found in many other areas [2, 12, 13, 21, 22]. Numerical simulations have been done extensively for different applications. However, error analysis of existing numerical methods for such a nonlinear system has not been explored. The problem is especially challenging due to the strong nonlinearity, degeneracy, and coupling. Since the scheme is decoupled for the system, the method is efficient for problems in high-dimensional space. It is also noted that theoretical analysis for the system of nonlinear parabolic equations is very limited. Existence of strong solutions has not been proved yet, while we believe that the physical system has a unique classical solution.

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