Applied Mathematics Letters 57 (2016) 90–96

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Stokes–Darcy model☆

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ARTICLE INFO

Article history:
Received 29 December 2015
Received in revised form 13 January 2016
Accepted 13 January 2016
Available online 21 January 2016

Keywords:
Porous media flow
Stokes equation
Darcy's law
Two-grid method
Optimal error estimate

ABSTRACT

Although the numerical results suggest the optimal convergence order of the two-grid finite element decoupled scheme for mixed Stokes–Darcy model with Beavers–Joseph–Saffman interface condition in literatures, the numerical analysis only gets the optimal error order for porous media flow and a non-optimal error order that is half order lower than the optimal one in fluid flow. The purpose of this paper is to fill in the gap between the numerical results and the theoretical analysis.

1. Introduction

The mixed Stokes–Darcy model has a wide range of applications in science and engineering, especially in cases where a free flowing fluid moves over a porous medium. Since its important applications in real world, many sorts of numerical methods have been proposed and studied for this model in the past years, for examples, see [1–8]. To overcome the mathematical difficulties in simulation of coupled multiphysics models, especially for the steady state problems, some decoupled schemes based on two-grid or multi-grid finite element for coupled problems like mixed Stokes–Darcy problem have been proposed and investigated in [9], [10–12]. For transient coupled problems, the decoupling algorithm based on interface approximation via temporal extrapolation are studied in [13,14] with same time step length in both subproblems and [15] with different time step length in different subproblems.

To our knowledge, the first two-grid decoupled numerical scheme for the steady state mixed Stokes–Darcy model was successfully introduced by Mu and Xu in [10]. Later on, Cai and Mu studied the multi-grid decoupled scheme for such problem based on the same interface condition treatment strategy in [16].

☆ Subsidized by NSFC (Grant Nos. 11571274 & 11171269) and the Ph.D. Programs Foundation of Ministry of Education of China (Grant No. 20110201110027).

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http://dx.doi.org/10.1016/j.aml.2016.01.007

Although such decoupled schemes are easy to be implemented and numerical experiments show that the convergence orders of velocity and pressure in fluid flow region and piezometric head in porous media flow region are all optimal with proper configuration of the coarse and fine mesh sizes (for example, see [16]), only an optimal error estimate for piezometric head in the porous media flow region was obtained in the previous mentioned papers. For velocity and pressure in the fluid flow region, the estimates are half order lower than the optimal one. Subsequently, to improve the convergence order of the velocity and pressure in fluid flow region, some modified two-grid algorithms were proposed in [11] and [12] for mixed Stokes–Darcy model. The optimal convergence orders for the velocity and pressure in fluid flow region were obtained at the price of changing the parallel implementation in [10] of the decoupled scheme in fine grid with a serial implementation.

In this paper, we re-visit the decoupled scheme in [10] and try to give the optimal error estimates for fluid flow.

The rest of the paper is organized as follows. A mixed Stokes–Darcy model with Beavers–Joseph–Saffman interface condition is described in the next section. In Section 3, the two-grid decoupled scheme proposed in [10] is presented. The main result is presented in Section 4 which shows that the error orders of the velocity and pressure in fluid flow region are optimal.

2. Mixed Stokes–Darcy model

Let us consider a mixed model of the Stokes equations and the Darcy equation for coupling a fluid flow and porous media flow in $\Omega \subset R^d$, $d = 2, 3$. Here $\Omega = \Omega_f \cup \Gamma \cup \Omega_p$, where Ω_f and Ω_p are two disjoint domains occupied by fluid flow and porous media flow respectively and $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$ is the interface. Furthermore, we denote by $\Gamma_f = \partial\Omega_f \cap \partial\Omega$, $\Gamma_p = \partial\Omega_p \cap \partial\Omega$ and n_p, n_f the unit outward normal vectors on $\partial\Omega_p$ and $\partial\Omega_f$, respectively. And we decompose Γ_p into two disjoint segments Γ_{pd} and Γ_{pn}.

Let us denote by (u_f, p_f) the velocity field and pressure of the fluid flow in Ω_f and ϕ_p the piezometric head in Ω_p. Now we give the mixed Stokes–Darcy model

$$\begin{cases} -\nabla\cdot(\mathbb{T}(u_f,p_f)) = g_f, & \text{in } \Omega_f \text{ (conservation of momentum)},\\ \nabla\cdot u_f = 0, & \text{in } \Omega_f \text{ (conservation of mass)},\\ -\nabla\cdot\dfrac{\mathbb{K}}{n}\nabla\phi_p = g_p, & \text{in } \Omega_p \text{ (conservation of mass)}, \end{cases} \tag{2.1}$$

which is completed by the following homogeneous boundary conditions:

$$u_f = 0 \quad \text{on } \Gamma_f, \qquad \phi_p = 0 \quad \text{on } \Gamma_{pd}, \qquad \frac{\partial\phi_p}{\partial n_p} = 0 \quad \text{on } \Gamma_{pn}, \tag{2.2}$$

and the interface conditions on Γ:

$$\begin{cases} u_f\cdot n_f - \dfrac{\mathbb{K}}{n}\nabla\phi_p\cdot n_p = 0,\\ -[\mathbb{T}(u_f,p_f)\cdot n_f]\cdot n_f = \rho_f g\phi_p,\\ -[\mathbb{T}(u_f,p_f)\cdot n_f]\cdot\tau_i = \dfrac{\alpha}{\sqrt{\tau_i\cdot\mathbb{K}\cdot\tau_i}}u_f\cdot\tau_i, \quad i = 1,\ldots,d-1. \end{cases} \tag{2.3}$$

Here $\mathbb{K}$ is the hydraulic conductivity tensor which is assumed to be symmetric and positive, n is the volumetric porosity, ρ_f is the density, g is the gravity acceleration, α is an experimentally determined positive parameter depending on the properties of the porous medium, g_f and g_p are the source terms in fluid region and the porous medium region, respectively. And

$$\mathbb{T}(u_f,p_f) = -p_f\mathbb{I} + 2\nu\mathbb{D}(u_f), \qquad \mathbb{D}(u_f) = \frac{1}{2}(\nabla u_f + \nabla^T u_f),$$

are the stress and the deformation rate tensors, $\nu > 0$ is the kinetic viscosity. For the sake of simplicity, we regard ρ_f, g and n as positive constants (see [10]).

The first condition in (2.3) ensures the mass conservation across Γ. The second one is the balance of normal forces on Γ. And the third one is called Beavers–Joseph–Saffman law (BJS) which is a most accepted approximation of the Beavers–Joseph law (BJ) (see [17] and [18]), where $\{\tau_i\}_{i=1}^{d-1}$ are linearly independent unit tangential vectors on Γ.

Following the terminology in [10], let us denote

$$\begin{aligned}&H_f = \{v \in (H^1(\Omega_f))^d : v|_{\Gamma_f} = 0\}, \qquad H_p = \{\psi \in H^1(\Omega_p) : \psi|_{\Gamma_{pd}} = 0\},\\ &W = H_f \times H_p, \quad Q = L^2(\Omega_f).\end{aligned}$$

Then the weak formulation of the mixed Stokes–Darcy model reads as follows (see [10]): for $f \in W'$, find $u = (u_f, \phi_p) \in W$, $p_f \in Q$ such that

$$\begin{cases} a(u,v) + b(v,p_f) = f(v) & \forall v = (v_f, \psi_p) \in W,\\ b(u,q_f) = 0 & \forall q_f \in Q,\end{cases} \tag{2.4}$$

where

$$a(u,v) = a_\Omega(u,v) + a_\Gamma(u,v),$$

with

$$\begin{aligned}&a_\Omega(u,v) = a_{\Omega_f}(u_f,v_f) + a_{\Omega_p}(\phi_p,\psi_p),\\ &a_{\Omega_f}(u_f,v_f) = \int_{\Omega_f} 2n\nu(\mathbb{D}(u_f),\mathbb{D}(v_f)) + \sum_{i=1}^{d-1} \frac{\alpha n}{\sqrt{\tau_i \cdot \mathbb{K} \cdot \tau_i}} \int_\Gamma (u_f \cdot \tau_i)(v_f \cdot \tau_i),\\ &a_{\Omega_p}(\phi_p,\psi_p) = \int_{\Omega_p} \rho_f g \nabla\psi_p \cdot \mathbb{K}\nabla\phi_p, \quad a_\Gamma(u,v) = \int_\Gamma n\rho_f g[\phi_p v_f - \psi_p u_f]\cdot n_f\end{aligned}$$

and

$$b(v,p_f) \equiv b(v_f,p_f) = -\int_{\Omega_f} np_f \nabla\cdot v_f, \quad f(v) = \int_{\Omega_f} ng_f v_f + \int_{\Omega_p} ng_p\psi_p.$$

Thanks to [10], we know that (i) $a(\cdot,\cdot)$ is continuous and coercive on W, (ii) $b(\cdot,\cdot)$ is continuous on $W \times Q$ and satisfies the Ladyzhenskaya–Babuška–Brezzi (LBB) condition: there exists a positive constant $\beta > 0$ such that

$$\inf_{q_f \in Q} \sup_{v_f \in H_f} \frac{b(v_f,q_f)}{\|q_f\|_Q \|v_f\|_{H_f}} \ge \beta, \tag{2.5}$$

and (iii) the model (2.4) is well-posed.

3. Two-grid decoupled scheme

Let $W_h = H_{fh} \times H_{ph} \subset W$ and $Q_h \subset Q$ be two finite element spaces with mesh size $h > 0$. In the rest, we assume that MINI element and piecewise linear continuous element are applied in the fluid and porous media regions, respectively. Here we assume that the triangulation of the entire domain Ω is regular, as well as compatible and quasi-uniform on Γ as described in [19] and [10], and (H_{fh}, Q_h) satisfies the following discrete LBB condition: there exists a positive constant $\beta^* > 0$, independent of h, such that

$$\inf_{q_{fh} \in Q_h} \sup_{v_{fh} \in H_{fh}} \frac{b(v_{fh},q_{fh})}{\|q_{fh}\|_Q \|v_{fh}\|_{H_f}} \ge \beta^*. \tag{3.1}$$

In addition, we assume the local regularity

$$u \in H^2(\Omega_f)^d \times H^2(\Omega_p), \quad p_f \in H^1(\Omega_f). \tag{3.2}$$

Then the finite element discretization of the coupled model (2.4) reads: find $u_h = (u_{fh}, \phi_{ph}) \in W_h$, $p_{fh} \in Q_h$ such that $\forall v_h = (v_{fh}, \psi_{ph}) \in W_h$, $q_{fh} \in Q_h$

$$a(u_h, v_h) + b(u_h, q_{fh}) + b(v_h, p_{fh}) = f(v_h). \tag{3.3}$$

The well-posedness and error estimates of this discrete model can be found in [19]. That is

$$\|u - u_h\|_W \lesssim h, \qquad \|p_f - p_{fh}\|_Q \lesssim h. \tag{3.4}$$

Here and after, $x \lesssim y$ means that there exists a mesh size independent positive constant $c > 0$ such that $x \le cy$. And for L^2 estimate, we know that (see [10])

$$\|u - u_h\|_{L^2(\Omega_f)^d \times L^2(\Omega_p)} \lesssim h^2. \tag{3.5}$$

Now we state the two-grid algorithm proposed in [10] as follows.

Two-grid algorithm

1. Solve (3.3) with a coarse mesh size $H > h$: find $u_H = (u_{fH}, \phi_{pH}) \in W_H$, $p_{fH} \in Q_H$ such that $\forall v_H = (v_{fH}, \psi_{pH}) \in W_H$, $q_{fH} \in Q_H$

$$a(u_H, v_H) + b(u_H, q_{fH}) + b(v_H, p_{fH}) = f(v_H). \tag{3.6}$$

2. Solve a modified fine grid problem: find $u^h = (u^{fh}, \phi^{ph}) \in W_h$, $p^{fh} \in Q_h$ such that $\forall v_h = (v_{fh}, \psi_{ph}) \in W_h$, $q_{fh} \in Q_h$

$$a_\Omega(u^h, v_h) + b(u^h, q_{fh}) + b(v_h, p^{fh}) = f(v_h) - a_\Gamma(u_H, v_h). \tag{3.7}$$

For simplicity, we always assume that the two triangulations are nested and

$$(W_H, Q_H) \subset (W_h, Q_h) \subset (W, Q).$$

In [10], the authors got the following error estimates

$$\|\phi_{ph} - \phi^{ph}\|_{H_p} \lesssim H^2, \quad \|u_{fh} - u^{fh}\|_{H_f} \lesssim H^{\frac{3}{2}} \quad \text{and} \quad \|p_{fh} - p^{fh}\|_Q \lesssim H^{\frac{3}{2}}. \tag{3.8}$$

And they also claim that the error estimates in the fluid region should be half order higher than the estimates obtained. And the numerical experiments in [9] do suggest that $\|u_{fh} - u^{fh}\|_{H_f}, \|p_{fh} - p^{fh}\|_Q = O(H^2)$.

4. Optimal error estimates

In this section, we will give some more rigorous estimates on $\|u_{fh} - u^{fh}\|_{H_f}$ and $\|p_{fh} - p^{fh}\|_Q$ than that of the estimates in [10] and show that they do reach the optimum convergence order.

Theorem 4.1. *Let u_h, p_h and u^h, p^h be defined by the two discrete models (3.3) and (3.7). We assume that both Ω_f and Ω_p are smooth domains and the source terms are all square integrable in corresponding domain* *such that the mixed Stokes–Darcy problem satisfies the local regularity assumption (3.2). Then there hold the following optimal error estimates:*

$$\|\phi_{ph} - \phi^{ph}\|_{H_p} \lesssim H^2, \tag{4.1}$$

$$\|u_{fh} - u^{fh}\|_{H_f} \lesssim H^2, \tag{4.2}$$

$$\|p_{fh} - p^{fh}\|_{Q} \lesssim H^2. \tag{4.3}$$

Proof. The estimate (4.1) was obtained in [10] already. Thus we refer readers to [10] for its details.

For the estimates of the approximation to the velocity and pressure in the fluid flow region, we introduce the following auxiliary problem in the porous media region: find $\Phi \in H^1(\Omega_p)$ satisfies

$$(\textbf{Auxiliary Problem}) \begin{cases} -\nabla \cdot (\mathbb{K}\nabla\Phi) = 0 & \text{in } \Omega_p, \\ \mathbb{K}\nabla\Phi \cdot n_p = \xi \cdot n_p & \text{on } \Gamma, \\ \mathbb{K}\nabla\Phi \cdot n_p = 0 & \text{on } \Gamma_p. \end{cases}$$

Here for any given $v_{fh} \in H_{fh}$, we take $\xi = n\rho_f g v_{fh}$. And we can easily show that

$$\|\mathbb{K}^{\frac{1}{2}}\nabla\Phi\|_{L^2(\Omega_p)^d} \lesssim \|v_{fh}\|_{H_f}. \tag{4.4}$$

Since Γ is smooth, we have for $v_{fh} \in H_f$,

$$\|\Phi\|_{H^2(\Omega_p)} \lesssim \|v_{fh}\|_{H_f}. \tag{4.5}$$

Now by taking $v_h = (v_{fh}, 0)$ in (3.7), it yields

$$a_{\Omega_f}(u_{fh} - u^{fh}, v_{fh}) + b(v_{fh}, p_{fh} - p^{fh}) = \int_\Gamma n\rho_f g(\phi_{ph} - \phi_{pH})v_{fh} \cdot n_f.$$

For the right hand side of the above equation, we have

$$\begin{aligned} \int_\Gamma n\rho_f g(\phi_{ph} - \phi_{pH})v_{fh} \cdot n_f &= -\int_\Gamma (\phi_{ph} - \phi_{pH})\mathbb{K}\nabla\Phi \cdot n_p \\ &= -\int_{\partial\Omega_p} (\phi_{ph} - \phi_{pH})\mathbb{K}\nabla\Phi \cdot n_p = -\int_{\Omega_p} \operatorname{div}((\phi_{ph} - \phi_{pH})\mathbb{K}\nabla\Phi) \\ &= -\int_{\Omega_p} (\phi_{ph} - \phi_{pH})\operatorname{div}(\mathbb{K}\nabla\Phi) - \int_{\Omega_p} \mathbb{K}\nabla(\phi_{ph} - \phi_{pH}) \cdot \nabla\Phi \\ &= -\int_{\Omega_p} \mathbb{K}\nabla(\phi_{ph} - \phi_{pH}) \cdot \nabla\Phi \\ &= -a_{\Omega_p}((\phi_{ph} - \phi^{ph}), \Phi) - a_{\Omega_p}(\phi^{ph} - \phi_{pH}, \Phi). \end{aligned}$$

For the first term on the right hand side of the last equality, it is easy to get from (4.1) and (4.4) that

$$|a_{\Omega_p}((\phi_{ph} - \phi^{ph}), \Phi)| \lesssim \|\phi_{ph} - \phi^{ph}\|_{H_p}\|\mathbb{K}^{\frac{1}{2}}\nabla\Phi\|_{L^2(\Omega_p)^d} \lesssim H^2\|v_{fh}\|_{H_f}. \tag{4.6}$$

For the second term, as we can easily verify that

$$a_{\Omega_p}(\phi^{ph} - \phi_{pH}, \psi_{pH}) = 0 \quad \forall\psi_{pH} \in H_{pH},$$

we have

$$a_{\Omega_p}(\phi^{ph} - \phi_{pH}, \Phi) = a_{\Omega_p}(\phi^{ph} - \phi_{pH}, \Phi - \psi_{pH}) \quad \forall\psi_{pH} \in H_{pH}.$$

Therefore by using (3.4), (4.1) and (4.5), we obtain

$$\begin{aligned}|a_{\Omega_p}(\phi^{ph}-\phi_{pH},\Phi)| &= \inf_{\psi\in H_{pH}} |a_{\Omega_p}(\phi^{ph}-\phi_{pH},\Phi-\psi_{pH})| \\ &\lesssim \|\phi^{ph}-\phi_{pH}\|_{H_p} \inf_{\psi_{pH}\in H_{pH}} \|\Phi-\psi_{pH}\|_{H_p} \\ &\lesssim H(\|\phi^{ph}-\phi_{ph}\|_{H_p}+\|\phi_{ph}-\phi_{pH}\|_{H_p})\|\Phi\|_{H^2(\Omega_p)} \\ &\lesssim H(H^2+H)\|v_{fh}\|_{H_f} \lesssim H^2\|v_{fh}\|_{H_f}. \end{aligned} \tag{4.7}$$

Taking into account (4.6) and (4.7), we have

$$\left|\int_\Gamma n\rho_f g(\phi_{ph}-\phi_{pH})v_{fh}\cdot n_f\right| \lesssim H^2\|v_{fh}\|_{H_f}.$$

If we take $v_{fh}=u_{fh}-u^{fh}$ and note that $b(u_{fh}-u^{fh},p_{fh}-p^{fh})=0$ and the H_p—coercive property of $a_{\Omega_p}(\cdot,\cdot)$, we have

$$\|u_{fh}-u^{fh}\|_{H_f} \lesssim H^2.$$

Now let us give the error estimate of the pressure. By the discrete LBB condition we can get

$$\begin{aligned}\|p_{fh}-p^{fh}\|_Q &\lesssim \sup_{v_{fh}\in H_{fh}} \frac{|a_{\Omega_f}(u_{fh}-u^{fh},v_{fh})|+|\int_\Gamma n\rho_f g(\phi_{ph}-\phi_{pH})v_{fh}\cdot n_f|}{\|v_{fh}\|_{H_f}} \\ &\lesssim H^2. \quad \square\end{aligned}$$

Remark. The numerical justifications of the optimality of the two-grid algorithm can be found in M.C. Cai's Ph.D. thesis [20] and also the paper [16] of M.C. Cai and M. Mu in 2012.

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