On the Solution of Coupled Stokes/Darcy Model with Beavers-Joseph Interface Condition *

Yanren Hou†and Yi Qin‡

Abstract

In this paper, we first get the uniqueness result of the possible solution to the steady-state coupled Stokes/Darcy model with Beavers-Joseph interface condition for any physical parameters, especially for any $\alpha > 0$. Then we show the existence of solutions for any $\alpha > 0$ by using Galerkin method. Furthermore, we analyze the error of the corresponding coupled finite element scheme and derive the optimal error estimates.

Keywords: coupled Stokes/Darcy model, Beavers-Joesph interface condition, well-posedness, error estimate

AMS Subject Classification: 76D05, 76S05, 76D03, 35D05

1 Introduction

Because of the important applications in real world, the mixed Stokes/Darcy and Navier-Stokes/Darcy model received much attention in both theoretical and numerical aspects in last decades. Many numerical methods have been studied for such mixed models, including coupled finite element methods [1, 2, 7, 10, 26, 33, 35], discontinuous Galerkin methods [11, 21, 25, 31, 32], domain decomposition methods [8, 13, 14, 15, 16, 17, 18, 19, 23], Lagrange multiplier methods [22, 27], interface relaxation methods [28, 29], and decoupled methods based on two-grid or multi-grid finite element [4, 5, 24, 30, 36, 37, 38]. Although there are so many literatures that made great contribution to the numerical simulation of the steady-state mixed Stokes/Darcy and Navier-Stokes/Darcy model with different interface conditions, some basic mathematical problems related to these coupled systems still remain unresolved. For examples, the existence of a weak solution to the steady-state mixed Navier-Stokes/Darcy model with Beavers-Joseph (BJ) or even more simpler Beavers-Joseph-Saffman (BJS) interface condition for general data and therefore the global uniqueness of the

*Subsidized by NSFC(Grant No. 11571274).

†School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China. (yrhou@mail.xjtu.edu.cn)

‡School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China. (qinyi1991@stu.xjtu.edu.cn)

weak solution, the well-posedness of the steady-state Stokes/Darcy model with BJ interface condition for any physical parameters keep unresolved. For the steady-state Stokes/Darcy model with BJ interface condition, the authors in [6] show that the coupled system is well-posed for small physical experiments determined parameter $\alpha > 0$ and then they established the error estimations of the coupled finite element approximation based on this small α assumption. In this paper, we focus our attention in investigating the well-posedness of this steady-state Stokes/Darcy model with BJ interface condition for any given physical parameters, especially any $\alpha > 0$ and try to establish the error estimates of the coupled finite element approximation for any $\alpha > 0$. The difficulty to do this is that, for large $\alpha > 0$, some interface bilinear form in the weak formulation of this model could not be absorbed in the other positive terms as is pointed out in [6]. The main idea in our investigation is to expanding the coupled model to a larger coupled system and try to make that bilinear form be absorbed in some newly appeared positive terms.

The rest of this paper is arranged as follows. In section 2, we give a brief introduction to the steady-state Stokes/Darcy model with BJ interface condition and its weak forms. In section 3, we give some technique lemmas for the later analysis. In section 4, we first get the uniqueness result of the possible solution to the steady-state coupled Stokes/Darcy model with Beavers-Joseph interface condition for any physical parameters, especially for any $\alpha > 0$. Finally, we show the existence of solutions for any $\alpha > 0$ by using Galerkin method. Furthermore, we analyze the error of the corresponding coupled finite element scheme and derive the optimal error estimates in section 5.

2 Mixed model with BJ interface condition

Let us consider the following mixed model for coupling a fluid flow and a porous media flow in a bounded smooth domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$. Here $\Omega = \Omega_f \cup \Gamma \cup \Omega_p$, where Ω_f and Ω_p are two disjoint, connected and bounded domains occupied by fluid flow and porous media flow and $\Gamma = \overline{\Omega}_f \cap \overline{\Omega}_p$ is the interface. For simplicity, we assume $\partial\Omega_p$ and $\partial\Omega_f$ are smooth enough in the rest of this paper. We denote $\Gamma_f = \partial\Omega_f \cap \partial\Omega$, $\Gamma_p = \partial\Omega_p \cap \partial\Omega$ and we also denote by $\mathbf{n}_p$ and $\mathbf{n}_f$ the unit outward normal vectors on $\partial\Omega_p$ and $\partial\Omega_f$, respectively. Furthermore, Γ_p consists of two disjoint parts Γ_{pd} and Γ_{pn}. We also assume $|\Gamma_f|$, $|\Gamma_{pd}| > 0$. See Figure 1 for a sketch.

In the rest of this paper, we always use boldface characters to denote vectors or vector valued spaces. For examples, for any given bounded domain D, we denote

$$\mathbf{L}^2(D) = L^2(D)^d, \quad \mathbf{H}^1(D) = H^1(D)^d, \quad \mathbf{H}^{\frac{1}{2}}(\partial D) = H^{\frac{1}{2}}(\partial D)^d.$$

The fluid motion in the fluid region Ω_f is governed by the Stokes equations

$$\begin{cases} -\nabla \cdot (\mathbb{T}_\nu(\mathbf{u}_f, p_f)) = \mathbf{g}_f, & \text{in } \Omega_f, \\ \nabla \cdot \mathbf{u}_f = 0, & \text{in } \Omega_f, \end{cases} \tag{2.1}$$

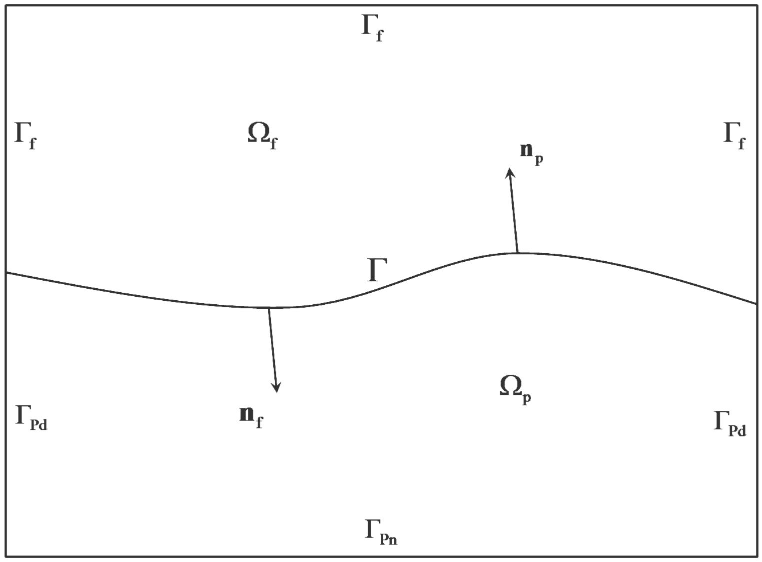

Figure 1: A global domain Ω consisting of a fluid flow region Ω_f and a porous media flow region Ω_p separated by an interface Γ.

where

$$\mathbb{T}_\nu(\mathbf{u}_f, p_f) = -p_f \mathbb{I} + 2\nu \mathbb{D}(\mathbf{u}_f), \quad \mathbb{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f),$$

are the stress tensor and the deformation rate tensor, $\nu > 0$ is the kinetic viscosity and $\mathbf{g}_f$ is the external force.

The fluid motion in the porous medium region Ω_p is governed by

$$\begin{cases} \nabla \cdot \mathbf{u}_d = g_p, & \text{in } \Omega_p, \\ \mathbf{u}_d = -\mathbb{K}\nabla\phi_p, & \text{in } \Omega_p, \end{cases} \tag{2.2}$$

where $\mathbb{K}$ denotes the hydraulic conductivity in Ω_p, which is a positive symetric tensor and is allowed to vary in space, and g_p is a source term. The first equation is the saturated flow model and the second equation is the Darcy's law. Here $\phi_p = z + \frac{p_p}{\rho g}$ is the piezometric (hydraulic) head, where p_p represents the dynamic pressure, z the height from a reference level, ρ the density and g the gravitational constant, and $\mathbf{u}_d$ is the flow velocity in the porous medium which is proportional to the gradient of ϕ_p, namely, the Darcy's law.

Combining the two equations in (2.2), we get the equation for the piezometric head, which we will refer to it simply as the Darcy equation:

$$-\nabla \cdot (\mathbb{K}\nabla\phi_p) = g_p, \quad \text{in } \Omega_p. \tag{2.3}$$

The above equations (2.1) and (2.3) are completed and coupled together by the following boundary conditions:

$$\mathbf{u}_f = 0 \quad \text{on } \Gamma_f, \quad \mathbb{K}\nabla\phi_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_{pn}, \quad \phi_p = 0 \quad \text{on } \Gamma_{pd}, \tag{2.4}$$

and the interface conditions on Γ:

$$\begin{cases} \mathbf{u}_f \cdot \mathbf{n}_f - \mathbb{K}\nabla\phi_p \cdot \mathbf{n}_p = 0, \\ -[\mathbb{T}_\nu(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f] \cdot \mathbf{n}_f = g(\phi_p - z), \\ -[\mathbb{T}_\nu(\mathbf{u}_f, p_f) \cdot \mathbf{n}_f] \cdot \boldsymbol{\tau}_i = \frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}(\mathbf{u}_f + \mathbb{K}\nabla\phi_p) \cdot \boldsymbol{\tau}_i, \end{cases} \tag{2.5}$$

where $\boldsymbol{\tau}_i$, $i = 1, \cdots, d-1$, are the orthonormal tangential unit vectors along Γ, α is an experimentally determined parameter and $\mathbf{\Pi}$ represents the permeability, which has the following relation with the hydraulic conductivity, $\mathbb{K} = \frac{\mathbf{\Pi} g}{\nu}$. The first condition is the mass conservation, the second one is the balance of normal force and the third interface condition means the tangential components of the normal stress force is proportional to the difference of the tangential components of the fluid flow and the porous media flow velocities, which is called the Beavers-Joseph (BJ) interface condition (see [3] and [34]). For more details of these equations, we refer readers to [21] and [30].

Furthermore, we assume

$$\mathbf{g}_f \in \mathbf{L}^2(\Omega_f), \quad g_p \in L^2(\Omega_p), \quad \mathbb{K} \in L^\infty(\Omega_p)^{d\times d}. \tag{2.6}$$

In the rest of the paper, we assume that there exist two constants $\lambda_{max} > 0$, $\lambda_{min} > 0$ such that

$$0 < \lambda_{min}|\mathbf{x}|^2 \le \mathbb{K}\mathbf{x} \cdot \mathbf{x} \le \lambda_{max}|\mathbf{x}|^2, \quad \forall \mathbf{x} \in \Omega_p. \tag{2.7}$$

From now on, we always use $(\cdot,\cdot)_D$ and $\|\cdot\|_D$ to denote the L^2 inner product and the corresponding norm on any given domain D.

Now let us introduce some Hilbert spaces

$$\begin{aligned} &\mathbf{X}_f = \{\mathbf{v}_f \in \mathbf{H}^1(\Omega_f) : \mathbf{v}_f|_{\Gamma_f} = 0\}, \\ &X_p = \{\psi_p \in H^1(\Omega_p) : \psi_p|_{\Gamma_{pd}} = 0\}, \\ &Q_f = L_0^2(\Omega_f) = \{q_f \in L^2(\Omega_f) : \int_{\Omega_f} q_f = 0\}, \end{aligned}$$

where $\mathbf{X}_f$ is equipped with the following inner product and the associated norm

$$(\mathbf{v}_f, \mathbf{w}_f)_{\mathbf{X}_f} := (\mathbb{D}(\mathbf{v}_f), \mathbb{D}(\mathbf{w}_f))_{\Omega_f}, \quad \|\mathbb{D}(\mathbf{v}_f)\|_{\Omega_f} = \sqrt{(\mathbb{D}(\mathbf{v}_f), \mathbb{D}(\mathbf{v}_f))_{\Omega_f}},$$

while X_p is equipped with the following inner product and its associated norm

$$(\psi_p, \xi_p)_{X_p} := (\mathbb{K}^{\frac{1}{2}}\nabla\psi_p, \mathbb{K}^{\frac{1}{2}}\nabla\xi_p)_{\Omega_p}, \quad \|\mathbb{K}^{\frac{1}{2}}\nabla\psi_p\|_{\Omega_p} := \sqrt{(\mathbb{K}^{\frac{1}{2}}\nabla\psi_p, \mathbb{K}^{\frac{1}{2}}\nabla\psi_p)_{\Omega_p}},$$

and Q_f is equipped with the usual L^2 inner product and its associated norm. Since $|\Gamma_f|, |\Gamma_{pd}| > 0$, thanks to the Korn's inequality and the Poincaré inequality, the above norms are equivalent to the usual Sobolev norms.

Sometimes in the rest, we need the following subspaces of X_p

$$\begin{aligned} &X_{p,div} = \{\psi_p \in X_p : \nabla \cdot (\mathbb{K}\nabla\psi_p) \in L^2(\Omega_p)\}, \\ &X_{p,div}^0 = \{\psi_p \in X_p : \nabla \cdot (\mathbb{K}\nabla\psi_p) = 0\}. \end{aligned}$$

Let us denote

$$\mathbf{U} = \mathbf{X}_f \times X_p.$$

Henceforth, we use the notational convention that $\underline{\mathbf{u}} = (\mathbf{u}_f, \phi_p)$ and $\underline{\mathbf{v}} = (\mathbf{v}_f, \psi_p)$. They all belong to $\mathbf{U}$. Now the weak formulation of the mixed Stokes/Darcy model (2.1), (2.3), (2.4), (2.5) reads as follows (see [5], [21], [27] and [30] for details): for $\mathbf{g}_f \in \mathbf{L}^2(\Omega_f)$ and $g_p \in L^2(\Omega_p)$, find $(\underline{\mathbf{u}}, p_f) \in \mathbf{U} \times Q_f$ such that $\forall(\underline{\mathbf{v}}, q_f) \in \mathbf{U} \times Q_f$

$$(Q) \qquad a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) - (p_f, \nabla \cdot \mathbf{v}_f)_{\Omega_f} + (q_f, \nabla \cdot \mathbf{u}_f)_{\Omega_f} = < \mathbf{F}, \underline{\mathbf{v}} >_{\mathbf{U}'},$$

where

$$\begin{aligned} a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) &= 2\nu(\mathbb{D}(\mathbf{u}_f), \mathbb{D}(\mathbf{v}_f))_{\Omega_f} + g(\phi_p, \mathbf{v}_f \cdot \mathbf{n}_f)_\Gamma \\ &\quad + (\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbf{u}_f + \mathbb{K}\nabla\phi_p), \mathbf{v}_f)_\Gamma \\ &\quad + g(\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p} - g(\psi_p, \mathbf{u}_f \cdot \mathbf{n}_f)_\Gamma \\ < \mathbf{F}, \underline{\mathbf{v}} >_{\mathbf{U}'} &= (\mathbf{g}_f, \mathbf{v}_f)_{\Omega_f} + g(g_p, \psi_p)_{\Omega_p} + g(z, \mathbf{v}_f \cdot \mathbf{n}_f)_\Gamma, \end{aligned}$$

and $\mathbf{U}'$ is the dual space of $\mathbf{U}$, $P_{\boldsymbol{\tau}}(\cdot)$ is the projection onto the local tangential plane that can be explicitly expressed as $P_{\boldsymbol{\tau}}(\mathbf{v}_f) = \mathbf{v}_f - (\mathbf{v}_f \cdot \mathbf{n}_f)\mathbf{n}_f$.

Thanks to [21], we know that there exists a positive constant $\beta > 0$ such that the following Ladyzhenskaya-Babuška-Brezzi (LBB) condition holds:

$$\inf_{q_f \in Q_f} \sup_{\mathbf{v}_f \in \mathbf{X}_f} \frac{(q_f, \nabla \cdot \mathbf{v}_f)_{\Omega_f}}{\|q_f\|_{Q_f} \|\mathbf{v}_f\|_{\mathbf{X}_f}} \geq \beta. \tag{2.8}$$

If we define the following divergence-free space

$$\mathbf{V}_f = \{\mathbf{v}_f \in \mathbf{X}_f : \ \nabla \cdot \mathbf{v}_f = 0\},$$

and introduce

$$\mathbf{V} = \mathbf{V}_f \times X_p,$$

the restriction of the test function $\underline{\mathbf{v}}$ to $\mathbf{V}$ in (Q) leads to the following reduced weak form: find $\underline{\mathbf{u}} \in \mathbf{V}$ such that $\forall \underline{\mathbf{v}} \in \mathbf{V}$

$$(P) \qquad a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = < \mathbf{F}, \underline{\mathbf{v}} >_{\mathbf{V}'} .$$

By the same argument in [20], we know that the problem (Q) and (P) are equivalent.

For the purpose of later analysis, we recall some inequalities: $\forall v \in H^1(D)$

$$\begin{aligned} &\|v\|_{L^2(\partial D)} \leq c_0 \|v\|^{\frac{1}{2}}_{L^2(D)} \|v\|^{\frac{1}{2}}_{H^1(D)}, \\ &\|v\|_{L^2(\partial D)} \leq c_1 \|v\|_{H^1(D)}, \\ &\|\nabla v\|_{L^2(D)} \leq c_2 \|\mathbb{D}(v)\|_{L^2(D)}. \end{aligned}$$

Here and hereafter, we always use c_i and C_i to denote positive constants which are not dependent on the data of the problem.

3 Technique Lemmas

In this section, we first give two lemmas which are curial for our later analysis.

We use $\boldsymbol{\mathcal{D}}(R^d)$ to denote the space of infinitely differentiable vector valued functions with compact supports in R^d and $\boldsymbol{\mathcal{D}}(\overline{\Omega}_p) = \{\mathbf{v}|_{\Omega_p} : \mathbf{v} \in \boldsymbol{\mathcal{D}}(R^d)\}$ the space of bounded infinitely differentiable functions in domain Ω_p.

Let

$$H_E^1(\Omega_p) = \{\psi_p \in H^1(\Omega_p) : \psi_p|_{\Gamma_p} = 0\}.$$

We denote by $H^{\frac{1}{2}}(\partial\Omega_p) = H^1(\Omega_p)|_{\partial\Omega_p}$ and $H^{\frac{1}{2}}(\Gamma) = H^1(\Omega_p)|_\Gamma$ the trace spaces on $\partial\Omega_p$ and Γ equipped with the following norm

$$\|\psi_p\|_{H^{\frac{1}{2}}(\gamma)} = \Big(\int_\gamma\int_\gamma \frac{|\psi_p(\mathbf{x}) - \psi_p(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{x}d\mathbf{y}\Big)^{\frac{1}{2}},$$

where $\gamma = \partial\Omega_p$ or Γ. Their dual spaces are denoted by $(H^{\frac{1}{2}}(\partial\Omega_p))'$ and $(H^{\frac{1}{2}}(\Gamma))'$. We also consider the trace space $H_{00}^{\frac{1}{2}}(\Gamma) = H_E^1(\Omega_p)|_\Gamma$, whose norm is as follows:

$$\|\psi_p\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = (\|\psi_p\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|\frac{\psi_p}{r^{\frac{1}{2}}}\|^2_{L^2(\Gamma)})^{\frac{1}{2}},$$

where r is the distance function to the end-points of Γ (see [21]). Sometimes we will use vector valued trace spaces, their definitions are the same as the above scalar valued trace space.

In addition, we introduce a Hilbert space

$$\mathbf{H}(div, \Omega_p) = \{\boldsymbol{\psi}_p \in \mathbf{L}^2(\Omega_p) : \mathrm{div}\boldsymbol{\psi}_p \in L^2(\Omega_p)\},$$

equipped with the inner product and its associated norm

$$[\boldsymbol{\psi}_p, \boldsymbol{\xi}_p]_{\Omega_p} = (\boldsymbol{\psi}_p, \boldsymbol{\xi}_p)_{\Omega_p} + (\mathrm{div}\boldsymbol{\psi}_p, \mathrm{div}\boldsymbol{\xi}_p)_{\Omega_p}, \quad |||\boldsymbol{\psi}_p|||^2_{\Omega_p} = [\boldsymbol{\psi}_p, \boldsymbol{\psi}_p]_{\Omega_p}.$$

Suppose $h > 0$ is a positive small constant, let us denote by $\mathcal{T}_h^p = \{K\}$ a regular triangulation of Ω_p, $X_h^p \subset H^1(\Omega_p)$ and $X_h^{pE} \subset X_h^p \cap H_E^1(\Omega_p)$ are two finite element spaces over Ω_p. In addition, we introduce some finite dimensional trace spaces $H_h^{\frac{1}{2}}(\partial\Omega_p) = X_h^p|_{\partial\Omega_p}$ and $H_h^{\frac{1}{2}}(\Gamma) = X_h^p|_\Gamma$ equipped with the norm of $H^{\frac{1}{2}}(\partial\Omega_p)$ and $H^{\frac{1}{2}}(\Gamma)$, respectively. Let $(H_h^{\frac{1}{2}}(\partial\Omega_p))'$ and $(H_h^{\frac{1}{2}}(\Gamma))'$ be the dual spaces of $H_h^{\frac{1}{2}}(\partial\Omega_p)$ and $H_h^{\frac{1}{2}}(\Gamma)$. We also introduce $H_{h,00}^{\frac{1}{2}}(\Gamma) = X_h^{pE}|_\Gamma$, which is a closed subspace of $H_h^{\frac{1}{2}}(\Gamma)$ in the sense of $H^{\frac{1}{2}}(\Gamma)$ norm because of the finite dimensionality of $H_h^{\frac{1}{2}}(\Gamma)$.

Now, for any $\boldsymbol{\psi}_p \in \mathbf{L}^2(\Omega_p)$, we give a discrete analog of the divergence operator div_h from $\mathbf{L}^2(\Omega_p)$ to X_h^p:

$$(\mathrm{div}_h\boldsymbol{\psi}_p, \psi_{ph})_{\Omega_p} = (\boldsymbol{\psi}_p \cdot \mathbf{n}_p, \psi_{ph})_{\partial\Omega_p} - (\boldsymbol{\psi}_p, \nabla\psi_{ph})_{\Omega_p}, \quad \forall \psi_{ph} \in X_h^p. \quad (3.1)$$

It is clear that

$$\mathrm{div}_h\boldsymbol{\psi}_p \in L^2(\Omega_p) \Longleftrightarrow \boldsymbol{\psi}_p \cdot \mathbf{n}_p \in (H_h^{\frac{1}{2}}(\partial\Omega_p))'.$$

Now for such a triangulation $\mathcal{T}_h^p = \{K\}$, we define a Hilbert subspace of $\mathbf{L}^2(\Omega_p)$:

$$\mathbf{H}(div_h, \mathcal{T}_h^p) = \{\boldsymbol{\psi}_p \in \mathbf{L}^2(\Omega_p) : \|\mathrm{div}_h \boldsymbol{\psi}_p\|_{\Omega_p} < \infty\},$$

which is equipped with the following inner product and its associated norm

$$[\boldsymbol{\psi}_p, \boldsymbol{\xi}_p]_{h,\Omega_p} = (\boldsymbol{\psi}_p, \boldsymbol{\xi}_p)_{\Omega_p} + (\mathrm{div}_h \boldsymbol{\psi}_p, \mathrm{div}_h \boldsymbol{\xi}_p)_{\Omega_p}, \quad |||\boldsymbol{\psi}_p|||^2_{h,\Omega_p} = [\boldsymbol{\psi}_p, \boldsymbol{\psi}_p]_{h,\Omega_p}.$$

Furthermore, from the definition of div_h, for any given $\boldsymbol{\psi}_p \in \mathbf{H}(\mathrm{div}, \Omega_p)$, we can easily verify that

$$(\mathrm{div}_h \boldsymbol{\psi}_p, \psi_{ph})_{\Omega_p} = (\mathrm{div} \boldsymbol{\psi}_p, \psi_{ph})_{\Omega_p}, \quad \forall \psi_{ph} \in X_h^p. \tag{3.2}$$

Lemma 3.1. *The space $\boldsymbol{\mathcal{D}}(\overline{\Omega}_p)$ is dense in $\mathbf{H}(div_h, \mathcal{T}_h^p)$.*

Proof. Readers can find the proof of this lemma in the Appendix. □

Corollary 3.1. *For any $\boldsymbol{\psi}_p \in \mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p)$, there holds*

$$\|\boldsymbol{\psi}_p \cdot \mathbf{n}_p\|_{(H_h^{\frac{1}{2}}(\partial\Omega_p))'} \leq |||\boldsymbol{\psi}_p|||_{h,\Omega_p}.$$

Once we have Lemma 3.1 and (3.2), the above result can be obtained by completely the same argument for the similar result in $\mathbf{H}(\mathrm{div}, \Omega_p)$ in textbooks.

Lemma 3.2. *There exists a positive constant C_1 that may depend on $\mathbb{K}$ and Γ such that*

$$\|P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_p)\|_{(\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'} \leq C_1 |||\mathbb{K}\nabla\psi_p|||_{\Omega_p}, \tag{3.3}$$

for any $\psi_p \in X_p$ with $\mathbb{K}\nabla\psi_p \in \mathbf{H}(\mathrm{div}, \Omega_p)$.

Proof. The detail proof can be found in the Appendix. □

By the same procedure in the proof of Lemma 3.2, we can easily get the following corollary.

Corollary 3.2. *There exists a positive constant C_2 that may depend on $\mathbb{K}$ and Γ such that*

$$\|P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_p)\|_{(\mathbf{H}_{h,00}^{\frac{1}{2}}(\Gamma))'} \leq C_2 |||\mathbb{K}\nabla\psi_p|||_{h,\Omega_p}, \tag{3.4}$$

for any $\psi_p \in X_p$ with $\mathbb{K}\nabla\psi_p \in \mathbf{H}(div_h, \mathcal{T}_h^p)$.

4 Uniqueness Result of the Coupled Model

In literatures, many authors have already discussed the well-posedness of the weak solution to the steady-state Stokes/Darcy model with BJS interface condition, which regard $P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p) = 0$ since it is usually very small compared with $P_{\boldsymbol{\tau}}(\mathbf{u}_f)$. In this case, the left hand side of (P) is obvious $\mathbf{V}-$ elliptic and the well-posedness of (P) is a straight result of Lax-Milgram theorem. But for the coupled system with BJ interface condition, since one of the interface bilinear terms, that is $(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p), \mathbf{v}_f)_\Gamma$, is not non-negative definite and can not be absorbed in the other positive definite bilinear forms unless the physical parameter $\alpha > 0$ is small enough. For examples, we refer readers to [6] for details. This is also the crucial difficulty for showing the existence and uniqueness of the solution to the finite dimensional approximation to (P). On the other hand, if we expand the coupled system to a larger coupled system and produce some other positive terms, we have the possibility to make this term to be absorbed in some newly appeared positive terms. That is the main idea in the following sections.

In this section, if there exist weak solutions to the steady-state coupled Stokes/Darcy model with BJ interface condition, we want to show that the solution is unique for any $\alpha > 0$.

Since the Stokes/Darcy model with BJ interface condition, the problem (P), is a linear system, the uniqueness of the solution to such linear system is equivalent to the related homogeneous system

$$(P_0) \qquad a(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = 0 \quad \forall \underline{\mathbf{v}} \in \mathbf{W},$$

only has zero solution, where

$$\mathbf{W} = \mathbf{V}_f \times X^0_{p,div}.$$

That is, to show the uniqueness of the solution to (P), we only have to show that any solution $\underline{\mathbf{u}} \in \mathbf{W}$ to (P_0) is actual $\underline{\mathbf{u}} = 0$.

As we mentioned previously, the main difficulty here is the presence of the interface bilinear term $(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p), \mathbf{v}_f)_\Gamma$. In order to overcome this difficulty, our idea is to expand the above homogeneous equation to a larger coupled system such that this interface bilinear term could be absorbed in some newly appeared positive terms.

To construct the above mentioned larger coupled system, we first introduce a new Hilbert space defined on Ω_p as

$$\mathbf{E}_p = \{\mathbf{v}_p \in \mathbf{H}^1(\Omega_p) : \ \mathbf{v}_p|_{\Gamma_p} = 0\}.$$

For any given $\boldsymbol{\xi}_f \in \mathbf{X}_f$, we consider the following auxiliary elliptic problem defined in Ω_p:

$$\begin{cases} -\alpha_1 \Delta \mathbf{u}_p + \alpha_0 \mathbf{u}_p = 0, & \text{in } \Omega_p, \\ \mathbf{u}_p|_{\Gamma_p} = 0, \quad \mathbf{u}_p|_\Gamma = \boldsymbol{\xi}_f|_\Gamma, \end{cases} \tag{4.1}$$

where $\alpha_0, \alpha_1 > 0$ are two constants which will be determined later.

For any given small parameter $h > 0$, let us denote by $\mathcal{T}_h = \{K\}$ a regular triangulation of Ω such that the mesh aligns with Γ. We denote $\mathcal{T}_{fh} = \mathcal{T}_h|_{\Omega_f}$ and $\mathcal{T}_{ph} = \mathcal{T}_h|_{\Omega_p}$, which are regular triangulations of Ω_f and Ω_p, respectively. In addition, let $\mathbf{X}_{fh}$, Q_{fh}, X_{ph} and $\mathbf{E}_{ph}$ be finite element subspaces of $\mathbf{X}_f$, Q_f, X_p and $\mathbf{E}_p$, respectively.

In addition, we define the following orthogonal projection P_{fh} from $\mathbf{X}_f$ onto $\mathbf{X}_{fh}$ as: for any given $\mathbf{v}_f \in \mathbf{X}_f$, find $P_{fh}\mathbf{v}_f \in \mathbf{X}_{fh}$ such that

$$2\nu(\mathbb{D}(\mathbf{v}_f - P_{fh}\mathbf{v}_f), \mathbb{D}(\mathbf{v}_{fh}))_{\Omega_f} \\ +(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbf{v}_f - P_{fh}\mathbf{v}_f), \mathbf{v}_{fh})_\Gamma = 0 \quad \forall \mathbf{v}_{fh} \in \mathbf{X}_{fh}.$$

It is clear that

$$\lim_{h\to 0} \|\mathbb{D}(\mathbf{v}_f - P_{fh}\mathbf{v}_f)\|_{\Omega_f} = 0.$$

The finite element approximation of (4.1) reads: find $\mathbf{u}_{ph} \in \mathbf{E}_{ph}$ such that $\forall \mathbf{v}_{ph} \in \mathbf{E}_{ph}$

$$(AUX_h^1) \quad \begin{cases} \alpha_1(\nabla\mathbf{u}_{ph}, \nabla\mathbf{v}_{ph})_{\Omega_p} + \alpha_0(\mathbf{u}_{ph}, \mathbf{v}_{ph})_{\Omega_p} - \alpha_1(\dfrac{\partial \mathbf{u}_{ph}}{\partial \mathbf{n}_p}, \mathbf{v}_{ph})_\Gamma = 0, \\ \mathbf{u}_{ph}|_\Gamma = P_{fh}\mathbf{u}_f|_\Gamma. \end{cases}$$

where we choose $\boldsymbol{\xi}_f = P_{fh}\mathbf{u}_f$, the projection of the solution to (P_0).

Here we give the following inequality (see[9] and [12]) which will be used in the later analysis

$$\|\chi\|_{L^2(\partial K)} \le c_3 h^{-\frac{1}{2}} \|\chi\|_{L^2(K)} \quad \text{for any polynomial } \chi \text{ on } K \in \mathcal{T}_h. \tag{4.2}$$

Now the problems (P_0) and (AUX_h^1) form a new larger coupled system. The problem (AUX_h^1) is subjected to the problem (P_0), while the problem (P_0) is independent of the problem (AUX_h^1). The well-posedness of this larger coupled system implies the well-posedness of the problem (P_0), which ensures the uniqueness of the possible solutions to (P).

Taking $\underline{\mathbf{v}} = \underline{\mathbf{u}}$ in (P_0) and $\mathbf{v}_{ph} = \mathbf{u}_{ph}$ in (AUX_h^1),

$$a(\underline{\mathbf{u}}, \underline{\mathbf{u}}) + \alpha_1\|\nabla\mathbf{u}_{ph}\|^2_{\Omega_p} + \alpha_0\|\mathbf{u}_{ph}\|^2_{\Omega_p} - \alpha_1(\frac{\partial \mathbf{u}_{ph}}{\partial \mathbf{n}_p}, \mathbf{u}_{ph})_\Gamma \\ \ge 2\nu\|\mathbb{D}(\mathbf{u}_f)\|^2_{\Omega_f} + g\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} + \alpha_1\|\nabla\mathbf{u}_{ph}\|^2_{\Omega_p} + \alpha_0\|\mathbf{u}_{ph}\|^2_{\Omega_p} \\ -|(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p), \mathbf{u}_f)_\Gamma| - \alpha_1|(\frac{\partial \mathbf{u}_{ph}}{\partial \mathbf{n}_p}, \mathbf{u}_{ph})_\Gamma|.$$

Now we estimate the last two terms on the right hand side of the above inequality one by one. Firstly for the first term of them, noticing the boundary condition of the problem (AUX_h^1), Lemma 3.2, $\mathrm{div}(\mathbb{K}\nabla\phi_p) = 0$ and using the inverse inequality in finite element spaces, we have

$$
\begin{aligned}
&|(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p),\mathbf{u}_f)_\Gamma| \le |(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p),\mathbf{u}_f - P_{fh}\mathbf{u}_f)_\Gamma| \\
&\quad + |(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p),\mathbf{u}_{ph})_\Gamma| \\
&\le \frac{\alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda_{min}^{\frac{1}{2}}}\|P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p)\|_{(H_{00}^{\frac{1}{2}}(\Gamma))'}(\|\mathbf{u}_f - P_{fh}\mathbf{u}_f\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{u}_{ph}\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}) \\
&\le \frac{C_1\alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda_{min}^{\frac{1}{2}}}\|\mathbb{K}\nabla\phi_p\|_{\Omega_p}(c_1c_2\|\mathbb{D}(\mathbf{u}_f - P_{fh}\mathbf{u}_f)\|_{\Omega_f} + c_0h^{-\frac{1}{2}}\|\mathbf{u}_{ph}\|_{\Omega_p}^{\frac{1}{2}}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^{\frac{1}{2}}) \\
&\le \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 + \frac{C_1^2c_1^2c_2^2\alpha^2\nu\lambda_{max}}{\lambda_{min}}\|\mathbb{D}(\mathbf{u}_f - P_{fh}\mathbf{u}_f)\|_{\Omega_f}^2 \\
&\quad + \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 + \frac{C_1^2c_0^2\alpha^2\nu\lambda_{max}}{\lambda_{min}h}\|\mathbf{u}_{ph}\|_{\Omega_p}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p} \\
&\le \frac{g}{2}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 + \frac{\alpha_1}{4}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^2 + \frac{c_4\nu^2\alpha^4}{\alpha_1h^2}\|\mathbf{u}_{ph}\|_{\Omega_p}^2 \\
&\quad + c_5\|\mathbb{D}(\mathbf{u}_f - P_{fh}\mathbf{u}_f)\|_{\Omega_f}^2.
\end{aligned}
$$

We want to emphasis that the constant $c_5 > 0$ is independent of h, α_0 and α_1.

For the second term, we have

$$
\begin{aligned}
|\alpha_1(\frac{\partial\mathbf{u}_{ph}}{\partial\mathbf{n}_p},\mathbf{u}_{ph})_\Gamma| &\le \alpha_1\|\frac{\partial\mathbf{u}_{ph}}{\partial\mathbf{n}_p}\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\Gamma)} \\
&\le \alpha_1\|\nabla\mathbf{u}_{ph}\|_{\mathbf{L}^2(\partial\Omega_p)}\|\mathbf{u}_{ph}\|_{\mathbf{L}^2(\partial\Omega_p)} \\
&\le c_0\alpha_1(\sum_{K\in\mathcal{T}_{ph}}\|\nabla\mathbf{u}_{ph}\|_{\mathbf{L}^2(\partial K)}^2)^{\frac{1}{2}}\|\mathbf{u}_{ph}\|_{\Omega_p}^{\frac{1}{2}}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^{\frac{1}{2}} \\
&\le c_0c_3\alpha_1h^{-1}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}\|\mathbf{u}_{ph}\|_{\Omega_p} \\
&\le \frac{\alpha_1}{4}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^2 + c_6\alpha_1h^{-2}\|\mathbf{u}_{ph}\|_{\Omega_p}^2.
\end{aligned}
$$

Now for any given $h > 0$, choosing $\alpha_1 = \sqrt{\frac{c_4}{c_6}}\nu\alpha^2$, $\alpha_0 = 2\sqrt{c_4c_6}\nu\alpha^2h^{-2}$ admits

$$
\begin{aligned}
0 &= a(\underline{\mathbf{u}},\underline{\mathbf{u}}) + \alpha_1\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^2 + \alpha_0\|\mathbf{u}_{ph}\|_{\Omega_p}^2 - \alpha_1(\frac{\partial\mathbf{u}_{ph}}{\partial\mathbf{n}_p},\mathbf{u}_{ph})_\Gamma \\
&\ge 2\nu\|\mathbb{D}(\mathbf{u}_f)\|_{\Omega_f}^2 + \frac{g}{2}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 + \frac{\alpha_1}{2}\|\nabla\mathbf{u}_{ph}\|_{\Omega_p}^2 - c_5\|\mathbb{D}(\mathbf{u}_f - P_{fh}\mathbf{u}_f)\|_{\Omega_f}^2.
\end{aligned}
$$

We actually get

$$
2\nu\|\mathbb{D}(\mathbf{u}_f)\|_{\Omega_f}^2 + \frac{g}{2}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|_{\Omega_p}^2 \le c_5\|\mathbb{D}(\mathbf{u}_f - P_{fh}\mathbf{u}_f)\|_{\Omega_f}^2.
$$

Taking $h \to 0$ and notice the approximation property of the projection, we get

$$
\underline{\mathbf{u}} = 0.
$$

This means the homogeneous problem has only zero solution and proves the uniqueness of possible solutions to (P), the Stokes/Darcy model with BJ interface condition for any $\alpha > 0$. And the uniqueness of the pressure $p_f \in Q_f$ can be easily gotten due to (2.8). We summarize the above result in the following theorem.

Theorem 4.1. *For any physical parameter $\alpha > 0$, if there exist weak solutions to the mixed Stokes/Darcy model with BJ interface condition, the possible solution is unique in $\mathbf{U} \times Q_f$.*

5 Finite Element Approximation

In this section, we investigate the solvability of the steady-state mixed Stokes /Darcy model with BJ interface condition for any $\alpha > 0$ by the traditional Galerkin method. Moreover, although the error estimates for the finite element approximation for coupled Stokes/Darcy model with BJ interface condition have already been discussed in formal literatures, for example see [6], the error estimates were established heavily depending on the small α restriction and could not be adopted in the case we consider here. Therefore we re-visit the error estimate for the coupled finite element Galerkin scheme in this paper.

For any given small parameter $h > 0$, as what has been done in the previous section, we construct the regular triangulations $\mathcal{T}_h$, $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ of Ω, Ω_f and Ω_p. For simplicity, we assume that Ω_f and Ω_p are smooth domains, for example, polygons or polyhedrons. Let $\mathbf{X}_{fh} \subset \mathbf{X}_f$, $X_{ph} \subset X_p$ and $Q_{fh} \subset Q_f$ are finite element spaces such that the space pair $(\mathbf{X}_{fh}, Q_{fh})$ satisfies the discrete LBB condition:

$$\inf_{q_{fh} \in Q_{fh}} \sup_{\mathbf{v}_{fh} \in \mathbf{X}_{fh}} \frac{(q_{fh}, \nabla \cdot \mathbf{v}_{fh})_{\Omega_f}}{\|q_{fh}\|_{Q_f} \|\mathbf{v}_{fh}\|_{\mathbf{X}_f}} \geq \beta. \tag{5.1}$$

For examples, in the rest of this paper, we always choose MINI finite element pair for $(\mathbf{X}_{fh}, Q_{fh})$ and $P1$ finite element for X_{ph}.

If we define

$$\mathbf{U}_h = \mathbf{X}_{fh} \times X_{ph},$$

the coupled finite element Galerkin approximation of (Q) reads:
Coupled Finite Element Scheme: find $\underline{\mathbf{u}}_h = (\mathbf{u}_{fh}, \phi_{ph}) \in \mathbf{U}_h$, $p_{fh} \in Q_{fh}$ such that for any $\underline{\mathbf{v}}_h = (\mathbf{v}_{fh}, \phi_{ph}) \in \mathbf{U}_h$ and $q_{fh} \in Q_{fh}$

$$(Q_h) \qquad a(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) - (p_{fh}, \nabla \cdot \mathbf{v}_{fh})_{\Omega_f} + (q_{fh}, \nabla \cdot \mathbf{u}_{fh})_{\Omega_f} =< \mathbf{F}, \underline{\mathbf{v}}_h >_{\mathbf{U}'},$$

Furthermore, we introduce

$$\mathbf{V}_{fh} = \{\mathbf{v}_{fh} \in \mathbf{X}_{fh} : \ (\nabla \cdot \mathbf{v}_{fh}, q_{fh})_{\Omega_f} = 0, \ \forall q_{fh} \in Q_{fh}\},$$

and denote

$$\mathbf{V}_h = \mathbf{V}_{fh} \times X_{ph}.$$

Now the coupled finite element scheme for the problem (P) reads: find $\underline{\mathbf{u}}_h \in \mathbf{V}_h$ such that $\forall \underline{\mathbf{v}}_h \in \mathbf{V}_h$

$$(P_h) \qquad a(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) =< \mathbf{F}, \underline{\mathbf{v}}_h >_{\mathbf{V}'} .$$

Theorem 5.1. *For any given small positive parameter $h > 0$, there exists a unique solution $\underline{\mathbf{u}}_h = (\mathbf{u}_{fh}, \phi_{ph}) \in \mathbf{V}_h$ of the coupled system (P_h). And there holds*

$$\nu\|\mathbb{D}(\mathbf{u}_{fh})\|^2_{\Omega_f} + \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_{ph}\|^2_{\Omega_p} \le C(\frac{1}{2\nu}\|\mathbf{g}_f\|^2_{\Omega_f} + \frac{g}{\lambda_{min}}\|g_p\|^2_{\Omega_p} + \frac{g^2}{2}\|z\|^2_{L^2(\Gamma)}),$$

where $C > 0$ is a constant independent of h. Furthermore, there exists a unique $p_{fh} \in Q_{fh}$ such that $(\underline{\mathbf{u}}_h, p_{fh})$ is the unique solution to the problem (Q_h) with

$$\|p_{fh}\|_{\Omega_f} \le \tilde{C}\beta^{-1}(\nu\|\mathbb{D}(\mathbf{u}_{fh})\|_{\Omega_f} + \frac{g + \alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}_{min}}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_{ph}\|_{\Omega_p} + \|\mathbf{g}_f\|_{\Omega_f},)$$

$\tilde{C}$ is a constant independent of h.

Proof. To show the unique solvability of (P_h), we will use the same method in the previous section. Let us recall the auxiliary system (AUX^1_h) with $\boldsymbol{\xi}_f = \mathbf{u}_{fh}$, the possible solution of (P_h). We construct a new auxiliary problem: find $\bar{\mathbf{u}}_{ph} \in \mathbf{E}_{ph}$ such that $\forall \mathbf{v}_{ph} \in \mathbf{E}_{ph}$

$$(AUX^2_h) \qquad \begin{cases} \alpha_1(\nabla\bar{\mathbf{u}}_{ph}, \nabla\mathbf{v}_{ph})_{\Omega_p} + \alpha_0(\bar{\mathbf{u}}_{ph}, \mathbf{v}_{ph})_{\Omega_p} - \alpha_1(\frac{\partial\bar{\mathbf{u}}_{ph}}{\partial\mathbf{n}_p}, \mathbf{v}_{ph})_\Gamma = 0, \\ \bar{\mathbf{u}}_{ph}|_\Gamma = \mathbf{u}_{fh}|_\Gamma. \end{cases}$$

It is obvious that (P_h) and (AUX^2_h) form a new larger coupled finite dimensional system. The unique solvability of this new coupled system implies the unique solvability of (P_h) since (AUX^2_h) is subjected to (P_h) while (P_h) is independent of (AUX^2_h).

For any $\underline{\mathbf{v}}_h = (\mathbf{v}_{fh}, \psi_{ph}) \in \mathbf{V}_h$ and $\mathbf{v}_{ph} \in \mathbf{E}_{ph}$ with $\mathbf{v}_{ph}|_\Gamma = \mathbf{v}_{fh}|_\Gamma$, we have

$$\begin{aligned} &a(\underline{\mathbf{v}}_h, \underline{\mathbf{v}}_h) + \alpha_1\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p} + \alpha_0\|\mathbf{v}_{ph}\|^2_{\Omega_p} - \alpha_1(\frac{\partial\mathbf{v}_{ph}}{\partial\mathbf{n}_p}, \mathbf{v}_{ph})_\Gamma \\ &\quad \ge 2\nu\|\mathbb{D}(\mathbf{v}_{fh})\|^2_{\Omega_f} + g\|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{ph}\|^2_{\Omega_p} + \alpha_1\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p} + \alpha_0\|\mathbf{v}_{ph}\|^2_{\Omega_p} \\ &\quad -|(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_{ph}), \mathbf{v}_{fh})_\Gamma| - \alpha_1|(\frac{\partial\mathbf{v}_{ph}}{\partial\mathbf{n}_p}, \mathbf{v}_{ph})_\Gamma|. \end{aligned}$$

Now let us estimate the last two terms on the right hand side of the above inequality. Firstly, by using (4.2), we have

$$
\begin{aligned}
&|(\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_{ph}),\mathbf{v}_{fh})_\Gamma| \le \frac{\alpha g^{\frac12}\nu^{\frac12}\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbb{K})}}\|\mathbb{K}\nabla\psi_{ph}\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{v}_{fh}\|_{\mathbf{L}^2(\Gamma)}\\
&\quad\le \frac{\alpha g^{\frac12}\nu^{\frac12}\lambda_{max}}{\lambda_{min}^{\frac12}}\|\nabla\psi_{ph}\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{v}_{ph}\|_{\mathbf{L}^2(\Gamma)}\\
&\quad\le \frac{\alpha g^{\frac12}\nu^{\frac12}\lambda_{max}}{\lambda_{min}^{\frac12}}\|\nabla\psi_{ph}\|_{\mathbf{L}^2(\partial\Omega_p)}\|\mathbf{v}_{ph}\|_{\mathbf{L}^2(\partial\Omega_p)}\\
&\quad\le \frac{c_0\alpha g^{\frac12}\nu^{\frac12}\lambda_{max}}{\lambda_{min}^{\frac12}}(\sum_{K\in\mathcal{T}_{ph}}\|\nabla\psi_{ph}\|^2_{\mathbf{L}^2(\partial K)})^{\frac12}\|\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\|\nabla\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\\
&\quad\le \frac{c_0c_3\alpha g^{\frac12}\nu^{\frac12}\lambda_{max}}{h^{\frac12}\lambda_{min}^{\frac12}}\|\nabla\psi_{ph}\|_{\Omega_p}\|\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\|\nabla\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\\
&\quad\le \frac{c_0c_3\alpha g^{\frac12}\nu^{\frac12}\lambda_{max}}{h^{\frac12}\lambda_{min}}\|\mathbb{K}^{\frac12}\nabla\psi_{ph}\|_{\Omega_p}\|\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\|\nabla\mathbf{v}_{ph}\|^{\frac12}_{\Omega_p}\\
&\quad\le \frac{g}{2}\|\mathbb{K}^{\frac12}\nabla\psi_{ph}\|^2_{\Omega_p}+\frac{c_0^2c_3^2\alpha^2\nu\lambda_{max}^2}{2\lambda_{min}^2h}\|\mathbf{v}_{ph}\|_{\Omega_p}\|\nabla\mathbf{v}_{ph}\|_{\Omega_p}\\
&\quad\le \frac{g}{2}\|\mathbb{K}^{\frac12}\nabla\psi_{ph}\|^2_{\Omega_p}+\frac{\alpha_1}{4}\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p}+\frac{c_7\nu^2\alpha^4}{\alpha_1h^2}\|\mathbf{v}_{ph}\|^2_{\Omega_p}.
\end{aligned}
$$

For the last term, the estimation is completely the same as the estimation of the same term in the previous section, we just copy the estimation here:

$$|\alpha_1(\frac{\partial\mathbf{v}_{ph}}{\partial\mathbf{n}_p},\mathbf{v}_{ph})_\Gamma| \le \frac{\alpha_1}{4}\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p}+c_8\alpha_1h^{-2}\|\mathbf{v}_{ph}\|^2_{\Omega_p}.$$

Now, for fixed $h>0$, if we take

$$\alpha_1=\sqrt{\frac{c_7}{c_8}}\nu\alpha^2,\quad \alpha_0=2\sqrt{c_7c_8}\nu\alpha^2h^{-2}, \tag{5.2}$$

we have

$$
\begin{aligned}
&a(\underline{\mathbf{v}}_h,\underline{\mathbf{v}}_h)+\alpha_1\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p}+\alpha_0\|\mathbf{v}_{ph}\|^2_{\Omega_p}-\alpha_1(\frac{\partial\mathbf{v}_{ph}}{\partial\mathbf{n}_p},\mathbf{v}_{ph})_\Gamma \\
&\qquad\ge 2\nu\|\mathbb{D}(\mathbf{v}_{fh})\|^2_{\Omega_f}+\frac{g}{2}\|\mathbb{K}^{\frac12}\nabla\psi_{ph}\|^2_{\Omega_p}+\frac{\alpha_1}{2}\|\nabla\mathbf{v}_{ph}\|^2_{\Omega_p}.
\end{aligned} \tag{5.3}
$$

If we denote $\bar{\mathbf{X}}_h$ a finite element space on the whole domain Ω associated with the regular triangulation $\mathcal{T}_h$ and

$$\mathbf{X}_h=\{\mathbf{v}_h\in\bar{\mathbf{X}}_h:\ (\mathbf{v}_h,q_{fh})_{\Omega_f}=0,\forall q_{fh}\in Q_{fh},\ \mathbf{v}_h|_{\Gamma_f\cup\Gamma_{pd}}=0\},$$

the above estimates actually mean that the bilinear form associated to the coupled system (P_h) and (AUX_h^2) is $\mathbf{X}_h\times X_{ph}-$elliptic, which guarantees the well-posedness of this coupled system in $\mathbf{X}_h \times X_{ph}$ by Lax-Milgram theory. This implies the well-posedness of the problem (P_h).

Thanks to the following estimates

$$
\begin{aligned}
&(\mathbf{g}_f, \mathbf{v}_{fh})_{\Omega_f} \leq c_2 \|\mathbf{g}_f\|_{\Omega_f} \|\mathbb{D}(\mathbf{v}_{fh})\|_{\Omega_f} \leq \frac{\nu}{2}\|\mathbb{D}(\mathbf{v}_{fh})\|^2_{\Omega_f} + \frac{c_2^2}{2\nu}\|\mathbf{g}_f\|^2_{\Omega_f},\\
&g(g_p, \psi_{ph})_{\Omega_p} \leq \frac{c_2 g}{\lambda_{min}^{\frac{1}{2}}}\|g_p\|_{\Omega_p}\|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{ph}\|_{\Omega_p} \leq \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\psi_{ph}\|^2_{\Omega_p} + \frac{c_2^2 g}{\lambda_{min}}\|g_p\|^2_{\Omega_p},\\
&g(z, \mathbf{v}_{fh}\cdot\mathbf{n}_f)_\Gamma \leq g\|z\|_{L^2(\Gamma)}\|\mathbf{v}_{fh}\|_{\mathbf{L}^2(\Gamma)} \leq c_1 c_2 g\|z\|_{L^2(\Gamma)}\|\mathbb{D}(\mathbf{v}_{fh})\|_{\Omega_f}\\
&\qquad \leq \frac{\nu}{2}\|\mathbb{D}(\mathbf{v}_{fh})\|^2_{\Omega_f} + \frac{c_1^2 c_2^2 g^2}{2}\|z\|^2_{L^2(\Gamma)},
\end{aligned}
$$

we can easily get

$$
\nu\|\mathbb{D}(\mathbf{u}_{fh})\|^2_{\Omega_f} + \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_{ph}\|^2_{\Omega_p} \leq \frac{c_2^2}{2\nu}\|\mathbf{g}_f\|^2_{\Omega_f} + \frac{c_2^2 g}{\lambda_{min}}\|g_p\|^2_{\Omega_p} + \frac{c_1^2c_2^2g^2}{2}\|z\|^2_{L^2(\Gamma)}.
$$

Let us choose

$$
C = \max\{c_2^2, c_1^2 c_2^2\},
$$

and this ends the proof of the first part of this theorem.

Moreover, we can deduce that

$$
\begin{aligned}
(p_{fh}, \nabla\cdot\mathbf{u}_{fh})_{\Omega_f} &= 2\nu(\mathbb{D}(\mathbf{u}_{fh}), \mathbb{D}(\mathbf{u}_{fh})_{\Omega_f} + g(\phi_{ph}, \mathbf{u}_{fh}\cdot\mathbf{n}_f)_\Gamma\\
&\quad + (\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbf{u}_{fh} + \mathbb{K}\nabla\phi_{ph}), \mathbf{u}_{fh})_\Gamma - (\mathbf{g}_f, \mathbf{u}_{fh})_{\Omega_f},
\end{aligned}
$$

and thanks to the discrete LBB condition (5.1)

$$
\|p_{fh}\|_{\Omega_f} \leq \tilde{C}\beta^{-1}(\nu\|\mathbb{D}(\mathbf{u}_{fh})\|_{\Omega_f} + \frac{g + \alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda_{min}^{\frac{1}{2}}}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_{ph}\|_{\Omega_p} + \|\mathbf{g}_f\|_{\Omega_f})
$$

□

For any mesh size $h > 0$, Theorem 5.1 ensures the existence of a bounded sequence $\{\underline{\mathbf{u}}_h\}_{h>0}$ in $\mathbf{X}_f \times X_p$. Then we can extract a subsequence, which is still denoted by h, such that the subsequence $\{\underline{\mathbf{u}}_h\}_{h>0}$ weakly converges to a function $\underline{\mathbf{u}} = (\mathbf{u}_f, \phi_p) \in \mathbf{V}$. Taking $h \to 0$ in (P_h), we can show that $\underline{\mathbf{u}} \in \mathbf{V}$ is a solution of (P) and shares the same bound of the sequence.

Because of the weakly convergence of $\{\underline{\mathbf{u}}_h\}_{h>0}$ to $\underline{\mathbf{u}} \in \mathbf{V}$, the following limits hold:

$$
\begin{aligned}
&\lim_{h\to 0}[2\nu(\mathbb{D}(\mathbf{u}_{fh}), \mathbb{D}(\mathbf{v}_f))_{\Omega_f} + g(\phi_{ph}, \mathbf{v}_f\cdot\mathbf{n}_f)_\Gamma + (\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}\mathbf{u}_{fh}, \mathbf{v}_f)_\Gamma\\
&\qquad + g(\mathbb{K}\nabla\phi_{ph}, \nabla\psi_p)_{\Omega_p} - g(\psi_p, \mathbf{u}_{fh}\cdot\mathbf{n}_f)_\Gamma]\\
&\quad = 2\nu(\mathbb{D}(\mathbf{u}_f), \mathbb{D}(\mathbf{v}_f))_{\Omega_f} + g(\phi_p, \mathbf{v}_f\cdot\mathbf{n}_f)_\Gamma + (\frac{\alpha\nu\sqrt{d}}{\sqrt{\mathrm{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}\mathbf{u}_f, \mathbf{v}_f)_\Gamma\\
&\qquad + g(\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p} - g(\psi_p, \mathbf{u}_f\cdot\mathbf{n}_f)_\Gamma], \quad \forall\underline{\mathbf{v}} = (\mathbf{v}_f, \psi_p) \in \mathbf{V}.
\end{aligned}
$$

Then we get $\phi_p \in X_p$ satisfies the following variational form:

$$(\mathbb{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p} = (\psi_p, \mathbf{u}_f \cdot \mathbf{n}_f)_\Gamma + (g_p, \psi_p), \quad \forall \psi_p \in X_p.$$

Being aware of that ϕ_{ph} satisfies the following equation

$$(\mathbb{K}\nabla\phi_{ph}, \nabla\psi_{ph})_{\Omega_p} = (\psi_{ph}, \mathbf{u}_{fh} \cdot \mathbf{n}_f)_\Gamma + (g_p, \psi_{ph}), \quad \forall \psi_{ph} \in X_{ph},$$

and $\|\mathbf{u}_f - \mathbf{u}_{fh}\|_{\Omega_f} \to 0$ as $h \to 0$, simple calculation shows that

$$\|\mathbb{K}\nabla(\phi_{ph} - \phi_p)\|_{\Omega_p} \to 0 \quad \text{as} \quad h \to 0.$$

That is ϕ_{ph} also strongly converges to $\phi_p \in X_p$. Furthermore, from the equation that ϕ_p satisfies, we can show that

$$\operatorname{div}_h(\mathbb{K}\nabla\phi_p) = \operatorname{div}_h(\mathbb{K}\nabla\phi_{ph}). \tag{5.4}$$

To show that $\underline{\mathbf{u}}$ is a solution of (P), the only thing left is to show

$$\lim_{h\to 0}\left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla(\phi_{ph} - \phi_p)), \mathbf{v}_f\right)_\Gamma = 0, \quad \forall \mathbf{v}_f \in \mathbf{V}_f.$$

Let us denote by $\mathbf{v}_{fh} = P_{fh}\mathbf{v}_f$, the H^1-orthogonal projection of $\mathbf{v}_f \in \mathbf{V}_f$ onto $\mathbf{V}_{fh}$. And there holds

$$\lim_{h\to 0} \|\mathbb{D}(\mathbf{v}_f - \mathbf{v}_{fh})\|_{\Omega_f} = 0.$$

We have

$$\begin{aligned}\left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla(\phi_{ph} - \phi_p)), \mathbf{v}_f\right)_\Gamma &= \left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla(\phi_{ph} - \phi_p)), \mathbf{v}_{fh}\right)_\Gamma \\ &+ \left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla(\phi_{ph} - \phi_p)), \mathbf{v}_f - \mathbf{v}_{fh}\right)_\Gamma \triangleq I_1 + I_2.\end{aligned}$$

Being aware of the result of Corollary 3.2 and $\operatorname{div}_h(\mathbb{K}\nabla(\phi_{ph} - \phi_p)) = 0$, we have

$$|I_1| = \left|\left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla(\phi_{ph} - \phi_p)), \mathbf{v}_{fh}\right)_\Gamma\right| \le c\|\mathbb{K}\nabla(\phi_{ph} - \phi_p)\|_{\Omega_p} \|D(\mathbf{v}_{fh})\|_{\Omega_f}.$$

Then I_1 tends to zero as $h \to 0$ since $\phi_{ph} - \phi_p$ strongly converges to zero in X_p.

The second term I_2 tends to zero as $h \to 0$ since the two terms,

$$\left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_{ph}), \mathbf{v}_f - \mathbf{v}_{fh}\right)_\Gamma \quad \text{and} \quad \left(\frac{\alpha\nu\sqrt{d}}{\sqrt{\operatorname{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\phi_p), \mathbf{v}_f - \mathbf{v}_{fh}\right)_\Gamma,$$

go to zero, respectively.

We conclude the above investigation by giving the following theorem.

Theorem 5.2. *For $\mathbf{g}_f \in L^2(\Omega_f)$, $g_p \in L^2(\Omega_p)$ and any given physical parameter $\alpha > 0$, the problem (P) is unique solvable and its unique solution $\underline{\mathbf{u}} = (\mathbf{u}_f, \phi_p) \in \mathbf{V}$ satisfies*

$$\nu\|\mathbb{D}(\mathbf{u}_f)\|^2_{\Omega_f} + \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla\phi_p\|^2_{\Omega_p} \leq C(\frac{1}{2\nu}\|\mathbf{g}_f\|^2_{\Omega_f} + \frac{g}{\lambda_{min}}\|g_p\|^2_{\Omega_p} + \frac{g^2}{2}\|z\|^2_{L^2(\Gamma)}),$$

where $C > 0$ is the same constant appeared in Theorem 5.1.

The existence a solution to the problem (Q) is obvious because the space pair $(\mathbf{X}_f, Q_f)$ satisfies the LBB condition.

Remark From the results of Theorem 4.1 and 5.2, it is clear that the bound for fluid velocity $\mathbf{u}_f$ and the piezometric head ϕ_p are independent of the parameter $\alpha > 0$.

For later analysis, we introduce the following orthogonal projections ρ_{fh} and P_{ph} from Q_f and X_p onto Q_{fh} and X_{ph}: for any $q_f \in Q_f$, $\psi_p \in X_p$, find $\rho_{fh}q_f \in Q_{fh}$, $P_{ph}\psi_p \in X_{ph}$ such that

$$\begin{aligned}&(q_f - \rho_{fh}q_f, q_{fh})_{\Omega_f} = 0 \quad \forall q_{fh} \in Q_{fh},\\ &g(\mathbb{K}\nabla(\psi_p - P_{ph}\psi_p), \nabla\psi_{ph})_{\Omega_p} = 0 \quad \forall \psi_{ph} \in X_{ph}.\end{aligned}$$

For these two projections ρ_{fh}, P_{ph} and the projection P_{fh} from $\mathbf{X}_f$ onto $\mathbf{X}_{fh}$ in the previous section, we make the following assumption: for any given $\mathbf{v}_f \in \mathbf{H}^2(\Omega_f) \cap \mathbf{X}_f$, $\psi_p \in H^2(\Omega_p) \cap X_p$ and $q_f \in H^1(\Omega_f)$, there holds

$$\|\mathbb{D}(\mathbf{v}_f - P_{fh}\mathbf{v}_f)\| + \|\mathbb{K}^{\frac{1}{2}}\nabla(\psi_p - P_{ph}\psi_p)\| + \|q_f - \rho_{fh}q_f\| \leq ch. \tag{5.5}$$

Suppose the weak solution $(\underline{\mathbf{u}}, p_f)$ of the mixed Stokes/Darcy problem is local H^2-regular, that is

$$\mathbf{u}_f \in \mathbf{H}^2(\Omega_f) \cap \mathbf{X}_f, \quad p_f \in H^1(\Omega_f) \quad \text{and} \quad \phi_p \in H^2(\Omega_p) \cap X_p.$$

Now we discuss the error estimate of the finite element approximation $(\underline{\mathbf{u}}_h, p_{fh})$ to the above coupled finite element scheme.

Theorem 5.3. *For any given small positive parameter $h > 0$, there holds*

$$\nu\|\mathbb{D}(\mathbf{u}_f - \mathbf{u}_{fh})\|^2_{\Omega_f} + \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p - \phi_{ph})\|^2_{\Omega_p} \leq C(\frac{\lambda_{min}g\nu + \lambda_{min}g^2 + \alpha^2 g\nu}{\lambda^2_{min}\nu})h^2.$$

where $C > 0$ is a constant independent of h.

Proof. Since the bilinear form $a(\cdot,\cdot)$ is non-coercive in $\mathbf{V}_h$ unless $\alpha > 0$ is small enough, we first try to give an H^1 estimate of $\mathbf{u}_f - \mathbf{u}_{fh}$ and $\phi_p - \phi_{ph}$ by considering the difference of the corresponding expanded coupled system $(P) \sim (AUX^1_h)$ and $(P_h) \sim (AUX^2_h)$.

If we denote

$$\mathbf{u}_f - \mathbf{u}_{fh} = \mathbf{e}_{fh} + \hat{\mathbf{u}}_f, \quad \phi_p - \phi_{ph} = e_{ph} + \hat{\phi}_p, \quad \mathbf{u}_{ph} - \bar{\mathbf{u}}_{ph} = \mathbf{e}_{ph},$$

where

$$\mathbf{e}_{fh} = P_{fh}\mathbf{u}_f - \mathbf{u}_{fh}, \quad \hat{\mathbf{u}}_f = (I - P_{fh})\mathbf{u}_f,$$
$$e_{ph} = P_{ph}\phi_p - \phi_{ph}, \quad \hat{\phi}_p = (I - P_{ph})\phi_p,$$

and notice the definitions of the orthogonal projections, we have

$$\begin{aligned}
&2\nu(\mathbb{D}(\mathbf{e}_{fh}), \mathbb{D}(\mathbf{v}_{fh})_{\Omega_f} + g(e_{ph}, \mathbf{v}_{fh} \cdot \mathbf{n}_f)_\Gamma \\
&\quad +(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}(\mathbf{e}_{fh} + \mathbb{K}\nabla e_{ph}), \mathbf{v}_{fh})_\Gamma + g(\mathbb{K}\nabla e_{ph}, \nabla\psi_{ph})_{\Omega_p} \\
&\quad -g(\psi_{ph}, \mathbf{e}_{fh} \cdot \mathbf{n}_f)_\Gamma = g(\psi_{ph}, \hat{\mathbf{u}}_f \cdot \mathbf{n}_f)_\Gamma - g(\hat{\phi}_p, \mathbf{v}_{fh} \cdot \mathbf{n}_f)_\Gamma \\
&\qquad -(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}\mathbb{K}\nabla\hat{\phi}_p, \mathbf{v}_{fh})_\Gamma, \\
&\alpha_1(\nabla\mathbf{e}_{ph}, \nabla\mathbf{v}_{ph})_{\Omega_p} + \alpha_0(\mathbf{e}_{ph}, \mathbf{v}_{ph})_{\Omega_p} - \alpha_1(\frac{\partial\mathbf{e}_{ph}}{\partial\mathbf{n}_p}, \mathbf{v}_{ph})_\Gamma = 0, \\
&\mathbf{e}_{ph}|_\Gamma = \mathbf{e}_{fh}|_\Gamma.
\end{aligned}$$

For any given $h > 0$, choosing α_0, α_1 satisfying (5.2) and taking $\mathbf{v}_{fh} = \mathbf{e}_{fh}$, $\psi_{ph} = e_{ph}$, $\mathbf{v}_{ph} = \mathbf{e}_{ph}$ and noticing that $\mathbf{e}_{fh}|_\Gamma = \mathbf{e}_{ph}|_\Gamma$, by completely the same procedure for getting (5.3), we can get

$$\begin{aligned}
&2\nu\|\mathbb{D}(\mathbf{e}_{fh})\|^2_{\Omega_f} + \frac{g}{2}\|\mathbb{K}^{\frac{1}{2}}\nabla e_{ph}\|^2_{\Omega_p} \qquad (5.6)\\
&\le g|(e_{ph}, \hat{\mathbf{u}}_f \cdot \mathbf{n}_f)_\Gamma| + g|(\hat{\phi}_p, \mathbf{e}_{fh} \cdot \mathbf{n}_f)_\Gamma| + |(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}} P_{\boldsymbol{\tau}}\mathbb{K}\nabla\hat{\phi}_p, \mathbf{e}_{fh})_\Gamma|.
\end{aligned}$$

Let us estimate the three terms on the right hand side of the above inequality. For the first two terms, we have

$$\begin{aligned}
g|(e_{ph}, \hat{\mathbf{u}}_f \cdot \mathbf{n}_f)_\Gamma| &\le g\|e_{ph}\|_{L^2(\Gamma)}\|\hat{\mathbf{u}}_f\|_{\mathbf{L}^2(\Gamma)} \le \frac{c_1^2 c_2 g}{\lambda_{min}^{\frac{1}{2}}}\|\mathbb{K}^{\frac{1}{2}}\nabla e_{ph}\|_{\Omega_p}\|\mathbb{D}(\hat{\mathbf{u}}_f)\|_{\Omega_f} \\
&\le \frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla e_{ph}\|^2_{\Omega_p} + \frac{c_1^4 c_2^2 g}{\lambda_{min}}\|\mathbb{D}(\hat{\mathbf{u}}_f)\|^2_{\Omega_f}, \\
g|(\hat{\phi}_p, \mathbf{e}_{fh} \cdot \mathbf{n}_f)_\Gamma| &\le g\|\hat{\phi}_p\|_{L^2(\Gamma)}\|\mathbf{e}_{fh}\|_{\mathbf{L}^2(\Gamma)} \le \frac{c_1^2 c_2 g}{\lambda_{min}^{\frac{1}{2}}}\|\mathbb{K}^{\frac{1}{2}}\nabla\hat{\phi}_p\|_{\Omega_p}\|\mathbb{D}(\mathbf{e}_{fh})\|_{\Omega_f} \\
&\le \frac{c_1^4 c_2^2 g^2}{2\lambda_{min}\nu}\|\mathbb{K}^{\frac{1}{2}}\nabla\hat{\phi}_p\|^2_{\Omega_p} + \frac{\nu}{2}\|\mathbb{D}(\mathbf{e}_{fh})\|^2_{\Omega_f}.
\end{aligned}$$

For the third term, being aware of that $\text{div}_h(\mathbb{K}\nabla\hat{\phi}_p) = 0$ and Corollary 3.2, we obtain

$$\begin{aligned}|(\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\mathbf{\Pi})}}P_{\boldsymbol{\tau}}\mathbb{K}\nabla\hat{\phi}_p,\mathbf{e}_{fh})_\Gamma| &\le \frac{\alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda_{min}^{\frac{1}{2}}}|||P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\hat{\phi}_p)|||_{(H^{\frac{1}{2}}_{h,00}\Gamma)'}\|\mathbf{e}_{fh}\|_{\mathbf{H}^{\frac{1}{2}}_{h,00}(\Gamma)}\\ &\le \frac{C_2c_1c_2\alpha g^{\frac{1}{2}}\nu^{\frac{1}{2}}}{\lambda_{min}}\|\mathbb{K}^{\frac{1}{2}}\nabla\hat{\phi}_p\|_{\Omega_p}\|\mathbb{D}(\mathbf{e}_{fh})\|_{\Omega_f}\\ &\le \frac{C_2^2c_1^2c_2^2\alpha^2 g}{2\lambda_{min}^2}\|\mathbb{K}^{\frac{1}{2}}\nabla\hat{\phi}_p\|^2_{\Omega_p}+\frac{\nu}{2}\|\mathbb{D}(\mathbf{e}_{fh})\|^2_{\Omega_f}.\end{aligned}$$

Finally, by using (5.5), we can get

$$\nu\|\mathbb{D}(\mathbf{e}_{fh})\|^2_{\Omega_f}+\frac{g}{4}\|\mathbb{K}^{\frac{1}{2}}\nabla e_{ph}\|^2_{\Omega_p}\le C(\frac{\lambda_{min}g\nu+\lambda_{min}g^2+\alpha^2 g\nu}{\lambda_{min}^2\nu})h^2.$$

$C>0$ is a constant independent of any physical parameters and h. Then we end the proof of this theorem with triangle inequalities. □

Remark Let us introduce the formal adjoint problem of (P): find $\underline{\mathbf{w}} = (\mathbf{w}_f,\zeta_p)\in\mathbf{V}$ such that

$$a(\underline{\mathbf{v}},\underline{\mathbf{w}}) = (\mathbf{u}_f-\mathbf{u}_{fh},\mathbf{v}_f)_{\Omega_f}+g(\phi_p-\phi_{ph},\psi_p)_{\Omega_p}\quad\forall\underline{\mathbf{v}}=(\mathbf{v}_f,\psi_p)\in\mathbf{V}.\quad(5.7)$$

By the same method, we can show that this formal adjoint problem is well-posed. If we further assume the unique solution of (5.7) is sufficiently regular, by using the classical duality argument we can get the L^2 estimate of $\underline{\mathbf{u}}-\underline{\mathbf{u}}_h$ given by

$$\|\mathbf{u}_f-\mathbf{u}_{fh}\|_{\Omega_f}+g\|\phi_p-\phi_{ph}\|_{\Omega_p}\le ch^2.$$

In fact, let $\underline{\mathbf{v}}=\underline{\mathbf{u}}-\underline{\mathbf{u}}_h$ in (5.7),

$$a(\underline{\mathbf{u}}-\underline{\mathbf{u}}_h,\underline{\mathbf{w}}) = (\mathbf{u}_f-\mathbf{u}_{fh},\mathbf{u}_f-\mathbf{u}_{fh})_{\Omega_f}+g(\phi_p-\phi_{ph},\phi_p-\phi_{ph})_{\Omega_p}.$$

Let us consider the problem (P) and (P_h), we have

$$a(\underline{\mathbf{u}}-\underline{\mathbf{u}}_h,\underline{\mathbf{w}}_h)=0\quad\forall\underline{\mathbf{w}}_h=(\mathbf{w}_{fh},\zeta_{ph})\in\mathbf{V},$$

then

$$a(\underline{\mathbf{u}}-\underline{\mathbf{u}}_h,\underline{\mathbf{w}}-\underline{\mathbf{w}}_h) = (\mathbf{u}_f-\mathbf{u}_{fh},\mathbf{u}_f-\mathbf{u}_{fh})_{\Omega_f}+g(\phi_p-\phi_{ph},\phi_p-\phi_{ph})_{\Omega_p}.$$

Now, we have

$$\begin{aligned}\|\mathbf{u}_f-\mathbf{u}_{fh}\|^2_{\Omega_f}+g\|\phi_p-\phi_{ph}\|^2_{\Omega_p}&\le C(\|\mathbb{D}(\mathbf{u}_f-\mathbf{u}_{fh})\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p-\phi_{ph})\|_{\Omega_p})\\ &(\inf_{\mathbf{w}_{fh}\in\mathbf{V}_{fh}}\|\mathbb{D}(\mathbf{w}_f-\mathbf{w}_{fh})\|_{\Omega_f}+\inf_{\zeta_{ph}\in X_{ph}}\|\mathbb{K}^{\frac{1}{2}}\nabla(\zeta_p-\zeta_{ph})\|_{\Omega_p}).\end{aligned}$$

The approximation properties of the finite element spaces and the regularity of the unique solution of (5.7) give us

$$\|\mathbf{u}_f-\mathbf{u}_{fh}\|_{\Omega_f}+g\|\phi_p-\phi_{ph}\|_{\Omega_p}\le Ch(\|\mathbb{D}(\mathbf{u}_f-\mathbf{u}_{fh})\|_{\Omega_f}+\|\mathbb{K}^{\frac{1}{2}}\nabla(\phi_p-\phi_{ph})\|_{\Omega_p}).$$

By using the Theorem 5.3, we have the result.

Appendix A

In this appendix, we give the proofs of the two lemmas in Section 3.

A.1 Proof of Lemma 3.1

Proof. Let $\mathbb{F} \in (\mathbf{H}(div_h, \mathcal{T}_h^p))'$, the dual space of the Hilbert space $\mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p)$, we can associate with the functional $\mathbb{F}$ a function $\mathbf{f}_p = (f_{p1}, \cdots, f_{pd}) \in \mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p)$ such that $\forall \boldsymbol{\psi}_p = (\psi_{p1}, \cdots, \psi_{pd}) \in \mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p)$

$$< \mathbb{F}, \boldsymbol{\psi}_p >= (\mathbf{f}_p, \boldsymbol{\psi}_p)_{\Omega_p} + (f_{p(d+1)}, \mathrm{div}_h \boldsymbol{\psi}_p)_{\Omega_p},$$

where

$$f_{p(d+1)} = \mathrm{div}_h \mathbf{f}_p \in X_h^p.$$

Now, assume that $\mathbb{F}$ vanishes on $\boldsymbol{\mathcal{D}}(\overline{\Omega}_p)$, that is

$$< \mathbb{F}, \mathbf{v}_p >= (\mathbf{f}_p, \mathbf{v}_p)_{\Omega_p} + (f_{p(d+1)}, \mathrm{div}_h \mathbf{v}_p)_{\Omega_p} = 0, \quad \forall \mathbf{v}_p \in \boldsymbol{\mathcal{D}}(\overline{\Omega}_p).$$

Because $f_{p(d+1)} \in X_h^p$ and (3.2), we know from the above assumption that

$$\begin{aligned} < \mathbb{F}, \mathbf{v}_p >&= (\mathbf{f}_p, \mathbf{v}_p)_{\Omega_p} + (f_{p(d+1)}, \mathrm{div}_h \mathbf{v}_p)_{\Omega_p} \\ &= (\mathbf{f}_p, \mathbf{v}_p)_{\Omega_p} + (f_{p(d+1)}, \mathrm{div} \mathbf{v}_p)_{\Omega_p} = 0, \quad \forall \mathbf{v}_p \in \boldsymbol{\mathcal{D}}(\overline{\Omega}_p). \end{aligned}$$

If we denote $\tilde{\mathbf{f}}_p$ and $\tilde{f}_{p(d+1)}$ the extensions of $\mathbf{f}_p$ and $f_{p(d+1)}$ by zero outside Ω_p, the above formula can be rewritten as follows:

$$\int_{R^d} \{\tilde{\mathbf{f}}_p \cdot \mathbf{v} + \tilde{f}_{p(d+1)} \mathrm{div} \mathbf{v}\} dx = 0, \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{D}}(R^d).$$

This means

$$\tilde{\mathbf{f}}_p = \nabla \tilde{f}_{p(d+1)},$$

in the sense of distribution in R^d.

Since $\tilde{\mathbf{f}}_p \in \mathbf{L}^2(R^d)$, we have $\tilde{f}_{p(d+1)} \in H^1(R^d)$, which means $f_{p(d+1)} \in H_0^1(\Omega_p)$. That is $f_{p(d+1)} \in X_h^p \cap H_0^1(\Omega_p)$. Then by the definition of div_h in (3.1) and the property (3.2),

$$< \mathbb{F}, \boldsymbol{\psi}_p >= (\nabla f_{p(d+1)}, \boldsymbol{\psi}_p)_{\Omega_p} + (f_{p(d+1)}, \mathrm{div}_h \boldsymbol{\psi}_p)_{\Omega_p} = 0, \quad \forall \boldsymbol{\psi}_p \in \mathbf{H}(div, \mathcal{T}_h^p).$$

In summary, we just showed that for any functional $\mathbb{F} \in (\mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p))'$, if $\mathbb{F}$ vanishes on $\boldsymbol{\mathcal{D}}(\overline{\Omega}_p)$, it vanishes on $\mathbf{H}(\mathrm{div}_h, \mathcal{T}_h^p)$. This completes the proof of this lemma. □

Proof of Lemma 3.2

For the purpose of proving Lemma 3.2, we need some Green's formula in curl form. First of all, we define the curl operator in 2D case for scalar and vector function by

$$\mathbf{curl}\phi = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}), \quad \forall \phi \in H^1(D),$$
$$\mathrm{curl}\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \forall \mathbf{v} = (v_1, v_2) \in \mathbf{H}^1(D),$$

and the curl operator in 3D case

$$\mathbf{curl}\mathbf{v} = \nabla \times \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}^1(D).$$

Then for the mapping $\gamma_{\boldsymbol{\tau}}$

$$\gamma_{\boldsymbol{\tau}}\mathbf{v} = \mathbf{v} \cdot \boldsymbol{\tau}|_{\partial D} \text{ for } \ d = 2 \quad \text{and} \quad \gamma_{\boldsymbol{\tau}}\mathbf{v} = \mathbf{v} \times \mathbf{n}|_{\partial D} \text{ for } \ d = 3,$$

we have the following Green's formula in 3D case (see Chapter I, Section 2.3 in [20])

$$\int_D \mathbf{curl}\mathbf{v} \cdot \boldsymbol{\phi} - \int_D \mathbf{v} \cdot \mathbf{curl}\boldsymbol{\phi} = \int_{\partial D} \boldsymbol{\phi} \cdot \gamma_{\boldsymbol{\tau}}\mathbf{v}, \tag{A.1}$$

for any $\mathbf{v}, \boldsymbol{\phi} \in \mathbf{H}^1(D)$ and the Green's formula in 2D case

$$\int_D \mathrm{curl}\mathbf{v} \cdot \phi - \int_D \mathbf{v} \cdot \mathbf{curl}\phi = \int_{\partial D} \phi \gamma_{\boldsymbol{\tau}}\mathbf{v}, \tag{A.2}$$

for any $\mathbf{v} \in \mathbf{H}^1(D)$ and $\phi \in H^1(D)$.

Now it is ready for us to begin the proof.

Proof. Note that

$$\begin{aligned} P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_p) &= \mathbb{K}\nabla\psi_p - [(\mathbb{K}\nabla\psi_p) \cdot \mathbf{n}_p]\mathbf{n}_p \\ &= \mathbb{K}(\nabla\psi_p \cdot \mathbf{n}_p)\mathbf{n}_p + \mathbb{K}\nabla_{\boldsymbol{\tau}}\psi_p - [(\mathbb{K}\nabla\psi_p) \cdot \mathbf{n}_p]\mathbf{n}_p, \end{aligned} \tag{A.3}$$

where

$$\nabla_{\boldsymbol{\tau}}\psi_p = P_{\boldsymbol{\tau}}(\nabla\psi_p) = \frac{\partial \psi_p}{\partial \boldsymbol{\tau}_1}\boldsymbol{\tau}_1 + \cdots + \frac{\partial \psi_p}{\partial \boldsymbol{\tau}_{d-1}}\boldsymbol{\tau}_{d-1}.$$

To show $P_{\boldsymbol{\tau}}(\mathbb{K}\nabla\psi_p) \in (\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'$, we have to show the three terms on the right hand side of the last identity in (A.3) belong to $(\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'$, respectively.

It is obvious that $(H^{\frac{1}{2}}(\partial\Omega_p))'|_\Gamma \subset (H_{00}^{\frac{1}{2}}(\Gamma))'$ (see [6]) and since $\mathbb{K}\nabla\psi_p \in \mathbf{L}^2(\Omega_p)$, we know

$$\|(\mathbb{K}\nabla\psi_p) \cdot \mathbf{n}_p\|_{(H_{00}^{\frac{1}{2}}(\Gamma))'} \leq \||\mathbb{K}\nabla\psi_p|\|_{\Omega_p}. \tag{A.4}$$

For smooth interface Γ, i.e., $\mathbf{n}_p(\mathbf{x})$ is continuous on Γ, we can easily get from (A.4)

$$\|[(\mathbb{K}\nabla\psi_p)\cdot\mathbf{n}_p]\mathbf{n}_p\|_{(\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'} \leq C_\Gamma \||\mathbb{K}\nabla\psi_p|\|_{\Omega_p}, \tag{A.5}$$

where $C_\Gamma > 0$ is a constant depending on the shape of Γ.

It is clear that $P_{\boldsymbol{\tau}}(\nabla\psi_p)$ is exactly the gradient on Γ, which ensures $\mathbb{K}P_{\boldsymbol{\tau}}(\nabla\psi_p)|_\Gamma \in (\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'$. In fact, thanks to the definition of the dual norm, we know

$$\|\mathbb{K}P_{\boldsymbol{\tau}}(\nabla\psi_p)\|_{(\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma))'} = \sup_{\mathbf{v}_p\in\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)} \frac{\int_\Gamma P_{\boldsymbol{\tau}}(\nabla\psi_p)\cdot\mathbb{K}\mathbf{v}_p}{\|\mathbf{v}_p\|_{\mathbf{H}_{00}^{\frac{1}{2}}\cdot(\Gamma)}}.$$

In the case of $d = 3$, for any given $\mathbf{v}_p \in \mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)$, there must be some $\tilde{\mathbf{v}}_p \in \mathbf{H}^1(\Omega_p)$ such that $\tilde{\mathbf{v}}_p|_\Gamma = \mathbb{K}\mathbf{v}_p$, $\tilde{\mathbf{v}}_p|_{\Gamma_p} = 0$ satisfying $\|\tilde{\mathbf{v}}_p\|_{\mathbf{H}^1(\Omega_p)} \leq C_\mathbb{K}\|\mathbf{v}_p\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}$, where $C_\mathbb{K} > 0$ is a constant that depends on $\mathbb{K}$. And we construct a function $\bar{\mathbf{v}}_p \in \mathbf{H}^1(\Omega_p)$ such that

$$\bar{\mathbf{v}}_p|_{\partial\Omega_p} = [(\tilde{\mathbf{v}}_p\cdot\boldsymbol{\tau}_2)\boldsymbol{\tau}_1 + (\tilde{\mathbf{v}}_p\cdot\boldsymbol{\tau}_1)\boldsymbol{\tau}_2 + (\tilde{\mathbf{v}}_p\cdot\mathbf{n}_p)\mathbf{n}_p]|_{\partial\Omega_p},$$

since Γ is smooth,

$$\|\bar{\mathbf{v}}_p\|_{\mathbf{H}^1(\Omega_p)} \leq C_{\mathbb{K}\Gamma}\|\tilde{\mathbf{v}}_p\|_{\mathbf{H}^1(\Omega_p)}.$$

Here $C_{\mathbb{K}\Gamma} > 0$ is a constant depending on $\mathbb{K}$ and Γ.

Applying the Green's formula (A.1) leads to

$$\begin{aligned}
\int_\Gamma \nabla_{\boldsymbol{\tau}}\psi_p\cdot\mathbb{K}\mathbf{v}_p &= \int_\Gamma \nabla\psi_p\cdot P_{\boldsymbol{\tau}}(\tilde{\mathbf{v}}_p) = \int_\Gamma \nabla\psi_p\cdot\gamma_{\boldsymbol{\tau}}(\bar{\mathbf{v}}_p)\\
&= (\nabla\times\bar{\mathbf{v}}_p, \nabla\psi_p)_{\Omega_p} - (\bar{\mathbf{v}}_p, \nabla\times\nabla\psi_p)_{\Omega_p}\\
&= (\nabla\times\bar{\mathbf{v}}_p, \nabla\psi_p)_{\Omega_p}\\
&\leq \|\bar{\mathbf{v}}_p\|_{\mathbf{H}^1(\Omega_p)}\|\nabla\psi_p\|_{\Omega_p}\\
&\leq C_{\mathbb{K}\Gamma}\|\nabla\psi_p\|_{\Omega_p}\|\mathbf{v}_p\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}.
\end{aligned}$$

In the case of $d = 2$, for smooth Γ, we know $\mathbb{K}\mathbf{v}_p\cdot\boldsymbol{\tau} \in H_{00}^{\frac{1}{2}}(\Gamma)$ and

$$\|\mathbb{K}\mathbf{v}_p\cdot\boldsymbol{\tau}\|_{H_{00}^{\frac{1}{2}}(\Gamma)} \leq C_{\mathbb{K}\Gamma}\|\mathbf{v}_p\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)},$$

Then we know there must be some $v_{p\boldsymbol{\tau}} \in H^1(\Omega_p)$ such that $v_{p\boldsymbol{\tau}}|_\Gamma = \mathbb{K}\mathbf{v}_p\cdot\boldsymbol{\tau}$, $v_{p\boldsymbol{\tau}}|_{\Gamma_p} = 0$ and

$$\|v_{p\boldsymbol{\tau}}\|_{H^1(\Omega_p)} \leq \|\mathbb{K}\mathbf{v}_p\cdot\boldsymbol{\tau}\|_{H_{00}^{\frac{1}{2}}(\Gamma)} \leq C_{\mathbb{K}\Gamma}\|\mathbf{v}_p\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}.$$

By using the Green's formula (A.2), we have

$$\begin{aligned}\int_\Gamma \nabla_{\boldsymbol\tau}\psi_p\cdot \mathbb{K}\mathbf{v}_p &= \int_\Gamma \gamma_{\boldsymbol\tau}(\nabla\psi_p)(\mathbb{K}\mathbf{v}_p\cdot\boldsymbol\tau) = \int_\Gamma \gamma_{\boldsymbol\tau}(\nabla\psi_p)v_{p\boldsymbol\tau}\\ &= (\mathbf{curl}\ v_{p\boldsymbol\tau},\nabla\psi_p)_{\Omega_p} - (v_{p\boldsymbol\tau},\mathrm{curl}\ \nabla\psi_p)_{\Omega_p}\\ &= (\mathbf{curl}\ v_{p\boldsymbol\tau},\nabla\psi_p)_{\Omega_p}\\ &\le \|v_{p\boldsymbol\tau}\|_{H^1(\Omega_p)}\|\nabla\psi_p\|_{\Omega_p}\\ &\le C_{\mathbb{K}\Gamma}\|\mathbf{v}_p\|_{\mathbf{H}^{\frac12}_{00}(\Gamma)}\|\nabla\psi_p\|_{\Omega_p}.\end{aligned}$$

Combination of the estimations in 2D and 3D cases, we get

$$\|\mathbb{K}P_{\boldsymbol\tau}(\nabla\psi_p)\|_{(\mathbf{H}^{\frac12}_{00}(\Gamma))'} \le C_{\mathbb{K}\Gamma}\|\nabla\psi_p\|_{\Omega_p} \le \frac{C_{\mathbb{K}\Gamma}}{\lambda_{min}}\|\mathbb{K}\nabla\psi_p\|_{\Omega_p}. \tag{A.6}$$

Note that

$$(\mathbf{n}_p^T\mathbb{K}\mathbf{n}_p)\frac{\partial\psi_p}{\partial\mathbf{n}_p} = (\mathbb{K}\nabla\psi_p)\cdot\mathbf{n}_p - (\mathbb{K}\nabla_{\boldsymbol\tau}\psi_p)\cdot\mathbf{n}_p,$$

and (A.4) and (A.6), we have

$$\|(\mathbf{n}_p^T\mathbb{K}\mathbf{n}_p)\frac{\partial\psi_p}{\partial\mathbf{n}_p}\|_{(H^{\frac12}_{00}(\Gamma))'} \le (1+\frac{C_{\mathbb{K}\Gamma}}{\lambda_{min}})\||\mathbb{K}\nabla\psi_p|\|_{\Omega_p}. \tag{A.7}$$

Then, for smooth Γ, we can finally get

$$\|\mathbb{K}\frac{\partial\psi_p}{\partial\mathbf{n}_p}\mathbf{n}_p\|_{(\mathbf{H}^{\frac12}_{00}(\Gamma))'} \le C_\Gamma(1+\frac{C_{\mathbb{K}\Gamma}}{\lambda_{min}})\||\mathbb{K}\nabla\psi_p|\|_{\Omega_p}. \tag{A.8}$$

Combination of the above estimates (A.5), (A.6) and (A.8) with (A.3) leads to

$$\|P_{\boldsymbol\tau}(\mathbb{K}\nabla\psi_p)\|_{(\mathbf{H}^{\frac12}_{00}(\Gamma))'} \le C_1\||\mathbb{K}\nabla\psi_p|\|_{\Omega_p}, \tag{A.9}$$

where

$$C_1 = 2C_\Gamma + \frac{C_\Gamma C_{\mathbb{K}\Gamma}+C_{\mathbb{K}\Gamma}}{\lambda_{min}}. \tag{A.10}$$

This concludes the proof of this lemma. □

References

[1] T. Arbogast and D.S. Brunson. A computational method for approximating a Darcy-Stokes system poverning a vuggy porous medium. *Comput. Geosci.*, 11:207–218, 2007.

[2] L. Badea, M. Discacciati, and A. Quarteroni. Numerical analysis of the Navier-Stokes/Darcy coupling. *Numer. Math.*, 115:195–227, 2010.

[3] G. Beavers and D. Joseph. Boundary conditions at a naturally permeable wall. *J. Fluid Mech.*, 30:197–207, 1967.

[4] M.C. Cai and Mo Mu. A multilevel decoupled method for a mixed Stokes/Darcy model. *Journal of Computational and Applied Mathematics*, 236:2452–2465, 2012.

[5] M.C. Cai, Mo Mu, and Jinchao Xu. Numerical solution to a mixed navier-stokes/darcy model by the two-grid approach. *SIAM J. Numer. Anal.*, 47:3325–3338, 2009.

[6] Y. Cao, M. Gunzburger, F. Hua, and X. Wang. Coupled Stokes-Darcy model with beavers-joseph interface boundary condition. *Commun. Math. Sci.*, 8:1–25, 2010.

[7] Yanzhao Cao, M. Gunzburger, Xiaoming He, Xiaoming Wang, and W. Zhao. Finite element approximations for Stokes-Darcy flow with Beavers-Joseph interface conditions. *SIAM J. Numer. Anal.*, 47:4239–4256, 2010.

[8] Y.Z. Cao, M. Gunzburger, X.M. He, and X.M. Wang. Robin-Robin domain decomposition methods for the steady-state Stokes-Darcy system with the Beavers-Joseph interface condition. 117:601–629, 2011.

[9] P Castillo, B Cockburn, I Perugia, and D Schötzau. An a priori error analysis of the local discontinuous Galerkin method for elliptic problems. *SIAM J. Numer. Anal.*, 38(5):1676–1706, 2000.

[10] Wenbin Chen, P. Chen, M. Gunzburger, and Ningning Yan. Superconvergence analysis of fems for the Stokes-Darcy system. *Math. Methods Appl. Sci.*, 33:1605–1617, 2010.

[11] P. Chidyagwai and B. Rivière. On the solution of the coupled Navier-Stokes and Darcy equations. *Comput. Methods Appl. Mech. Engrg.*, 198:3806–3820, 2009.

[12] P Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[13] M. Discacciati. *Domain decomposition methods for the coupling of surface and groundwater flows.* PhD thesis, École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland, 2004.

[14] M Discacciati, E Miglio, and A Quarteroni. Mathematical and numerical models for coupling surface and groundwater flows. *Appl. Numer. Math.*, 43:57–74, 2002.

[15] M Discacciati and A Quarteroni. Analysis of a domain decomposition method for the coupling Stokes and Darcy equations. In F Brezzi et al, editor, *Numerical Analysis and Advanced Applications*, Enumath 2001, pages 3–20, Milan, 2003. Springer.

[16] M Discacciati and A Quarteroni. Convergence analysis of a subdomain iterative method for the finite element approximation of the coupling of Stokes and Darcy equations. *Comput. Visual Sci.*, 6:1001–1026, 2005.

[17] M Discacciati, A Quarteroni, and A Valli. Robin-Robin domain decomposition methods for the Stokes-Darcy coupling. *SIAM J. Numer. Anal.*, 45:1246–1268, 2007.

[18] W Q Feng, Xiaoming He, Z Wang, and X Zhang. Non-iterative domain decomposition methods for a non-stationary Stokes-Darcy model with the Beavers-Joseph interface condition. *Appl. Math. Comput.*, 219:453–463, 2012.

[19] J Galvis and M Sarkis. Balancing domain decomposition methods for mortar coupling Stokes-Darcy systems. In *Proceedings of the 16th International Conference on Domain Decomposition Methods*, 2005.

[20] V Girault and P Raviart. *Finite element methods for Navier-Stokes equations: Theory and algorithms*. Springer-Verlag, Berlin Heidelberg New York Tokyo, 1986.

[21] V Girault and B Rivière. DG approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman interface condition. *SIAM J. Numer. Anal.*, 47:2052–2089, 2009.

[22] Roland Glowinski, T Pan, and J Periaux. A Lagrange multiplier/fictitious domain method for the numerical simulation of incompressible viscous flow around moving grid bodies: I. Case where the rigid body motions are known a priori. *C. R. Acad. Sci. Paris Sér. I Math.*, 324:361–369, 1997.

[23] Xiaoming He, Jian Li, Yanping Lin, and J Ming. A domain decomposition method for the steady-state Navier-Stokes-Darcy model with Beavers-Joseph interface condition. *SIAM J. Sci. Comput.*, 37:264–290, 2015.

[24] Yanren Hou. Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Stokes-Darcy model. *Applied Mathematics Letters*, 57:90–96, 2016.

[25] G Kanschat and B Rivière. A strongly conservative finite element method for the coupling of Stokes and Darcy flow. *J. Comput. Phys.*, 229:5933–5943, 2010.

[26] T Karper, K A Marcal, and R Winter. Unified finite element discretizations of coupled Darcy-Stokes flow. *Numer. Methods Partial Differ. Eqns.*, 25:311–326, 2009.

[27] W J Layton, F Schieweck, and I Yotov. Coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 40:2195–2218, 2003.

[28] S Markus, E Houstis, A Catlin, J Rice, P Tsompanopoulou, E Vavalis, D Gottfried, K Su, and G Balakrishnan. An agent-based netcentric framework for multidisciplinary problem solving environments (MPSE). *Internat. J. Comput. Engrg. Sci.*, 1:33–60, 2000.

[29] Mo Mu. Solving composite problems with interface relaxation. *SIAM J. Sci. Comput.*, 20:1394–1416, 1999.

[30] Mo Mu and Jinchao Xu. A two-grid method of a mixed Stokes-Darcy model for coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 45:1801–1813, 2007.

[31] B Rivière. Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems. *J. Sci. Comput.*, 22:479–500, 2005.

[32] B Rivière and I Yotov. Locally conservative coupling of Stokes and Darcy flows. *SIAM J. Numer. Anal.*, 42:1959–1977, 2005.

[33] Hongxing Rui and R Zhang. A unified stabilized mixed finite element method for coupling Stokes and Darcy flows. *Comput. Methods Appl. Mech. Engrg.*, 198:2692–2699, 2009.

[34] P Saffman. On the boundary condition at the surface of a porous medium. *Stud. Appl. Math.*, 50:93–101, 1971.

[35] J M Urquiza, D N'Dri, A Garon, and M C Delfour. Coupling Stokes and Darcy equations. *Appl. Numer. Math.*, 58:525–538, 2008.

[36] Liyun Zuo and Yanren Hou. A decoupling two-grid algorithm for the mixed Stokes-Darcy model with the Beavers-Joseph interface condition. *Numerical Methods for Partial Differential Equations*, 30(3):1066–1082, 2014.

[37] Liyun Zuo and Yanren Hou. Numerical analysis for the mixed Navier-Stokes and Darcy problem with the Beavers-Joseph interface condition. *Numerical Methods for Partial Differential Equations*, 31(4):1009–1030, 2015.

[38] Liyun Zuo and Yanren Hou. A two-grid decoupling method for the mixed Stokes-Darcy model. *Journal of Computational and Applied Mathematics*, 275:139–147, 2015.