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Numerical analysis of a second order algorithm for a non-stationary Navier–Stokes/Darcy model ☆

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ABSTRACT

In this paper, we construct a second order algorithm based on the semi-implicit spectral deferred correction method for the non-stationary coupled Navier–Stokes/Darcy equations with the Beavers–Joseph–Saffman's interface condition. We present a complete theoretical analysis to prove that this algorithm is stable and possesses second order accuracy in time. Numerical examples confirm the convergence rate and the effectiveness of our algorithm.

1. Introduction

The coupling of incompressible flow and porous media flow has a wide range of applications in science and engineering, such as the interaction between surface and subsurface flows, blood motion in vessels, industrial filtrations and so on. The fluid flow and the porous media flow are modeled by the Navier–Stokes equations and the Darcy equations, respectively, which are coupled through certain transmission conditions on the interface.

There is a rich literature on the mathematical analysis, numerical methods and applications for the coupled Stokes–Darcy model, see [1–10] and the references therein. In contrast, there are relatively few works on the coupling of Navier–Stokes and Darcy equations. In [11–16], there are the analysis of the steady-state coupled Navier–Stokes/Darcy equations. The non-stationary case has only been mathematically and numerically analyzed in [17–20]. In [17,18], the authors derived the well-posedness of the weak solution and convergence of the numerical algorithms for the non-stationary coupled Navier–Stokes/Darcy model. However, the interface conditions in these two papers contain the inertial forces. Although it makes the analysis easier in the case, physical justification of this model is not clear. In [19], the situations without the inertial term had been argued and the well-posed of the solutions were obtained under a mall data condition. We are interested in the non-stationary coupled Navier–Stokes and Darcy model without the inertial term. Unlike the algorithm in [18,20], we employ the semi-implicit spectral deferred correction (SISDC) [21] method in time and the finite element methods in space. And we not only get the second order convergence in time, but also the optimal error estimate in space.

The spectral deferred correction (SDC) method was proposed for stiff ordinary differential equations in [22] and further developed in [21,23–28] and the references therein. The SDC method allows one to increase the accuracy of a stable low order time stepping method through using spectral integration and constructing the corrections. This can avoid instabilities and conditioning problems associated with repeated differentiations, such as the backward differentiation formulas (BDF) based high order methods. In [21], the SISDC method is used to construct the high-order method. The

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SISDC method treats the non-stiff terms explicitly and the stiff terms implicitly. In this paper, we try to construct a second order algorithm based on the SISDC method for the coupling of Navier–Stokes and Darcy equations, in which we treat the trilinear term in the Navier–Stokes equations explicitly and other terms implicitly. Due to the comparison in [21] with existing methods such as multi-step methods, Runge–Kutta methods and operator splitting, the most notable advantage of the SDC methods is that one can use a simple numerical method, such as the backward-Euler method, to get a numerical solution with a high order of accuracy only by solving a sequence of deferred correction equations. That is to say, the high-order SDC methods are more easier to construct than other high-order methods. For the sake of simplicity, we analyze the second-order algorithm as an example in this paper. In addition, through numerical experiments, we can present some advantages of our scheme compared with other second order accurate methods.

This paper is organized as follows. In Section 2, we introduce the necessary notations and preliminary results. In Section 3, we present a complete theoretical analysis of stability and error estimate of the first order Euler method. The second order scheme based on the SISDC method with its stability and error estimate is given in Section 4. Numerical experiments are reported in Section 5, followed by conclusions in Section 6.

2. Model problem

In this section we introduce the model in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3), which consists of a fluid region Ω_f and a porous media region Ω_p. Here $\Omega_f \cap \Omega_p = \emptyset$, $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$. Two domains are separated by the interface $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Denote $\Gamma_f = \partial\Omega_f \backslash \Gamma$, $\Gamma_p = \partial\Omega_p \backslash \Gamma$. Let n_f and n_p denote the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$ respectively. Obviously, $n_p = -n_f$ on Γ.

Let $T > 0$ be a finite time. The Navier–Stokes equations for the fluid velocity u and the pressure p describe the fluid flow in Ω_f:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f_1 & \text{in } \Omega_f \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \Omega_f \times (0,T), \\ u(x,0) = u^0 = 0 & \text{in } \Omega_f, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinetic viscosity and f_1 is the external force.

The Darcy equations for the piezometric head φ govern the porous media flow in Ω_p (see [29,30]):

$$\begin{cases} S_0\varphi_t - \nabla\cdot(\mathbf{K}\nabla\varphi) = f_2 & \text{in } \Omega_p \times (0,T), \\ \varphi(x,0) = \varphi^0 = 0 & \text{in } \Omega_p, \end{cases} \tag{2}$$

where S_0 is the specific mass storativity coefficient, f_2 is the source term and $\mathbf{K} = \{K_{ij}\}_{d\times d}$, $K_{ij} \in L^\infty(\Omega_p)$ is a positive definite symmetric matrix corresponding to the permeability of Ω_p. In addition, we assume that there exist $k_{min}, k_{max} > 0$ such that

$$k_{min}|x|^2 \le \mathbf{K}x\cdot x \le k_{max}|x|^2 \quad a.e. \;\; x \in \Omega_p. \tag{3}$$

On the interface Γ, we impose the following coupling conditions:

$$\begin{cases} u\cdot n_f - \mathbf{K}\nabla\varphi\cdot n_p = 0 & \text{on } \Gamma\times(0,T), \\ p - \nu n_f\frac{\partial u}{\partial n_f} = g\varphi & \text{on } \Gamma\times(0,T), \\ -\nu\tau_i\frac{\partial u}{\partial n_f} = \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}(u\cdot\tau_i),\ 1\le i\le(d-1) & \text{on } \Gamma\times(0,T), \end{cases} \tag{4}$$

where g is the gravitational acceleration, α is a positive parameter and matters with the properties of the porous medium and $\{\tau_i\}_{i=1}^{d-1}$ are the unit tangential vectors on the interface Γ. The first condition in (4) ensures the mass conservation across the interface. The second one is the balance of normal forces on Γ. The third interface condition states that the tangential components of the normal stress force is proportional to the tangential components of the fluid velocity, which is called the Beavers–Joseph–Saffman's (BJS) interface condition [31,32].

For simplicity, we assume that the fluid velocity u and the piezometric head φ satisfy the homogeneous Dirichlet boundary conditions:

$$u = 0 \quad \text{on } \Gamma_f\times(0,T), \qquad \varphi = 0 \quad \text{on } \Gamma_p\times(0,T). \tag{5}$$

Let us introduce the following spaces

$$\begin{aligned} &W_f = \{v \in (H^1(\Omega_f))^d : v = 0 \text{ on } \Gamma_f\}, \\ &W_p = \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \Gamma_p\}, \\ &W = W_f \times W_p, \\ &Q = L^2(\Omega_f). \end{aligned}$$

The space W is equipped with the following norms: $\forall\ \underline{w} = (u, \varphi) \in W$,

$$\|\underline{w}\|_0 = \sqrt{(u, u)_{\Omega_f} + gS_0(\varphi, \varphi)_{\Omega_p}},$$
$$\|\underline{w}\|_W = \sqrt{\nu(\nabla u, \nabla u)_{\Omega_f} + g(\mathbf{K}\nabla\varphi, \nabla\varphi)_{\Omega_p}},$$

where $(\cdot,\cdot)_D$ refers to the scalar product $(\cdot,\cdot)$ in the corresponding domain D for $D = \Omega_f$ or Ω_p.

Assume that

$$f_1 \in L^\infty(0, T; L^2(\Omega_f)^d), \quad f_2 \in L^\infty(0, T; L^2(\Omega_p)), \quad \mathbf{K} \in L^\infty(\Omega_p)^{d\times d}. \tag{6}$$

Then the weak formulation of the non-stationary mixed Navier–Stokes/Darcy model with BJS interface condition reads as follows: find $\underline{w} = (u, \varphi) \in (L^2(0, T; W_f) \cap L^\infty(0, T; L^2(\Omega_f))^d) \times (L^2(0, T; W_p) \cap L^\infty(0, T; L^2(\Omega_p)))$ and $p \in L^2(0, T; Q)$, such that

$$\begin{cases} [\underline{w}_t, \underline{z}] + a(\underline{w}, \underline{z}) + N(u; u, v) + b(\underline{z}, p) = (f, \underline{z}) & \forall\ \underline{z} = (v, \psi) \in W, \\ b(\underline{w}, q) = 0 & \forall\ q \in Q, \\ \underline{w}(0) = \underline{w}^0 = 0, \end{cases} \tag{7}$$

where

$$[\underline{w}_t, \underline{z}] = (u_t, v) + gS_0(\varphi_t, \psi),$$
$$a(\underline{w}, \underline{z}) = a_f(u, v) + a_p(\varphi, \psi) + a_\Gamma(\underline{w}, \underline{z}),$$
$$a_f(u, v) = \nu(\nabla u, \nabla v)_{\Omega_f} + \sum_{i=1}^{d-1} \int_\Gamma \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}} (u\cdot\tau_i)(v\cdot\tau_i),$$
$$a_p(\varphi, \psi) = g(\mathbf{K}\nabla\varphi, \nabla\psi)_{\Omega_p},$$
$$a_\Gamma(\underline{w}, \underline{z}) = g\int_\Gamma (\varphi v\cdot n_f - \psi u\cdot n_f),$$
$$N(u; u, v) = (u\cdot\nabla u, v)_{\Omega_f},$$
$$b(\underline{z}, q) = -(q, \nabla\cdot v)_{\Omega_f},$$
$$(f, \underline{z}) = (f_1, v)_{\Omega_f} + g(f_2, \psi)_{\Omega_p}.$$

In [21], it has been proved the existence and uniqueness of the weak solution for the coupled Navier–Stokes and Darcy model (1)–(4).

And the following property about $a_\Gamma(\underline{w}, \underline{z})$ is useful in our later analysis:

$$a_\Gamma(\underline{w}, \underline{z}) = -a_\Gamma(\underline{z}, \underline{w}), \quad a_\Gamma(\underline{z}, \underline{z}) = 0 \quad \forall\ \underline{w}, \underline{z} \in W. \tag{8}$$

Moreover, we shall recall the Poincaré and trace inequalities. There exist constants C_p, C_t which only depend on the region Ω_f, and $\tilde{C}_p, \tilde{C}_t$ which only depend on the region Ω_p, such that for all $v \in W_f$ and $\psi \in W_p$,

$$\begin{aligned} &\|v\|_{L^2(\Omega_f)} \le C_p\|\nabla v\|_{L^2(\Omega_f)}, && \|\psi\|_{L^2(\Omega_p)} \le \tilde{C}_p\|\nabla\psi\|_{L^2(\Omega_p)}, \\ &\|v\|_{L^2(\Gamma)} \le C_t\|v\|^{\frac12}_{L^2(\Omega_f)}\|\nabla v\|^{\frac12}_{L^2(\Omega_f)}, && \|\psi\|_{L^2(\Gamma)} \le \tilde{C}_t\|\psi\|^{\frac12}_{L^2(\Omega_p)}\|\nabla\psi\|^{\frac12}_{L^2(\Omega_p)}. \end{aligned} \tag{9}$$

Moreover, owing to (3), for all $\psi \in H^1(\Omega_p)$,

$$\frac{1}{\sqrt{k_{max}}}\|\mathbf{K}^{\frac12}\nabla\psi\|_{L^2(\Omega_p)} \le \|\nabla\psi\|_{L^2(\Omega_p)} \le \frac{1}{\sqrt{k_{min}}}\|\mathbf{K}^{\frac12}\nabla\psi\|_{L^2(\Omega_p)}. \tag{10}$$

We consider a quasi-uniform triangulation τ_h of the domain $\overline{\Omega}$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$ or tetrahedra if $d = 3$. Let $W_h = W_{fh} \times W_{ph} \subset W$ and $Q_h \subset Q$ denote the finite element subspaces. Let $k_1 \ge 1$ and $k_2 \ge 1$ be two integers. The spaces W_{fh} and Q_h are chosen to be the continuous finite element spaces of piecewise polynomials of degree k_1 and $k_1 - 1$ respectively, while W_{ph} is chosen to be the continuous finite element space of piecewise polynomials of degree k_2. We assume that the triangulations induced on the subdomains Ω_f and Ω_p are compatible on the interface Γ, that is, they share the same edges therein. Furthermore, we assume (W_{fh}, Q_h) satisfy the well-known discrete inf–sup condition: for all $q_h \in Q_h$, there exists a positive constant β independent of h and $\underline{z}_h = (v_h, 0) \in W_h$, $v_h \ne 0$, such that

$$b(\underline{z}_h, q_h) \ge \beta\|\underline{z}_h\|_W\|q_h\|_{L^2(\Omega_f)}.$$

For all $v_h \in W_{fh}$ and $\psi_h \in W_{ph}$, we have the inverse inequality

$$\|\nabla v_h\|_{L^2(\Omega_f)} \le C_I h^{-1}\|v_h\|_{L^2(\Omega_f)}, \quad \|\nabla\psi_h\|_{L^2(\Omega_p)} \le \tilde{C}_I h^{-1}\|\psi_h\|_{L^2(\Omega_p)}. \tag{11}$$

Following [3], we define a projection

$$P_h : (\underline{w}(t), p(t)) \in W \times Q \mapsto (P_h^w \underline{w}(t), P_h^p p(t)) \in W_h \times Q_h,\ \ \forall\, t \in [0, T]$$

by requiring

$$\begin{cases} a(P_h^w \underline{w}(t), \underline{z}_h) + b(\underline{z}_h, P_h^p p(t)) = a(\underline{w}(t), \underline{z}_h) + b(\underline{z}_h, p(t)) & \forall\, \underline{z}_h \in W_h, \\ b(P_h^w \underline{w}(t), q_h) = 0 & \forall\, q_h \in Q_h. \end{cases} \tag{12}$$

Besides, for any $t > 0$, we assume that $(\underline{w}(t), p(t)) = (u(t), \varphi(t), p(t))$ is smooth enough and there is an approximation $(\tilde{\underline{w}}(t), \tilde{p}(t)) = (P_h^w \underline{w}(t), P_h^p p(t)) \in W_h \times Q_h$ of $(\underline{w}(t), p(t))$. Then the following approximation properties hold (see [3]):

$$\begin{aligned} &(\nabla \cdot (u(t) - \tilde{u}(t)), q)_{\Omega_f} = 0 \ \ \forall\, q \in Q_h, \\ &\|u(t) - \tilde{u}(t)\|_{L^2(\Omega_f)} \le Ch^{k_1+1} \|u(t)\|_{H^{k_1+1}(\Omega_f)}, \\ &\|\nabla(u(t) - \tilde{u}(t))\|_{L^2(\Omega_f)} \le Ch^{k_1} \|u(t)\|_{H^{k_1+1}(\Omega_f)}, \\ &\|p(t) - \tilde{p}(t)\|_{L^2(\Omega_f)} \le Ch^{k_1} \|p(t)\|_{H^{k_1}(\Omega_f)}, \\ &\|\varphi(t) - \tilde{\varphi}(t)\|_{L^2(\Omega_p)} \le Ch^{k_2+1} \|\varphi(t)\|_{H^{k_2+1}(\Omega_p)}, \\ &\|\nabla(\varphi(t) - \tilde{\varphi}(t))\|_{L^2(\Omega_p)} \le Ch^{k_2} \|\varphi(t)\|_{H^{k_2+1}(\Omega_p)}, \end{aligned} \tag{13}$$

where C is a positive constant which is different in different places but independent of mesh size and time step length.

Moreover, we assume that the true solution satisfies the following regularities:

$$\begin{aligned} &u \in L^\infty(0, T; H^{k_1+1}(\Omega_f)^d),\ \ p \in L^\infty(0, T; H^{k_1}(\Omega_f)),\ \ \varphi(t) \in L^\infty(0, T; H^{k_2+1}(\Omega_p)), \\ &u_t \in L^2(0, T; H^{k_1+1}(\Omega_f)^d),\ \ \varphi_t \in L^2(0, T; H^{k_2+1}(\Omega_p)), \\ &u_{tt} \in L^2(0, T; H^{k_1}(\Omega_f)^d),\ \ \varphi_{tt} \in L^2(0, T; H^{k_2}(\Omega_p)), \\ &u_{ttt} \in L^2(0, T; L^2(\Omega_f)^d),\ \ \varphi_{ttt} \in L^2(0, T; L^2(\Omega_p)). \end{aligned} \tag{14}$$

The following lemmas are basic and widely used in the study of Navier–Stokes and Darcy model, i.e., [8,11].

Lemma 2.1. *There exist constants $C_{N1} > 0$, $C_{N2} > 0$ and $C_{N2} > 0$, such that $\forall\, u, l, v \in W_f$,*

$$N(u, l, v) \le C_{N1} \|\nabla u\|_{L^2(\Omega_f)} \|\nabla l\|_{L^2(\Omega_f)} \|\nabla v\|_{L^2(\Omega_f)}. \tag{15}$$

In addition, if $u,\ \ l \in H^2(\Omega_f)$, we have

$$N(u, l, v) \le C_{N2} \|u\|_{L^2(\Omega_f)} \|l\|_{H^2(\Omega_f)} \|\nabla v\|_{L^2(\Omega_f)}, \tag{16}$$

$$N(u, l, v) \le C_{N3} \|\nabla u\|_{L^2(\Omega_f)} \|l\|_{H^2(\Omega_f)} \|v\|_{L^2(\Omega_f)}, \tag{17}$$

$$N(u, l, v) \le C_{N4} \|u\|_{H^2(\Omega_f)} \|\nabla l\|_{L^2(\Omega_f)} \|v\|_{L^2(\Omega_f)}. \tag{18}$$

Lemma 2.2. *For all $\underline{w} = (u, \varphi)$, $\underline{z} = (v, \psi) \in W$, there exists a constant $C_\Gamma = \frac{g C_p \tilde{C}_p C_t^2 \tilde{C}_t^2}{\nu k_{min}} > 0$ such that $\forall\, \lambda > 0$*

$$|a_\Gamma(\underline{w}, \underline{z})| \le \lambda \|\underline{w}\|_W^2 + \frac{C_\Gamma}{4\lambda} \|\underline{z}\|_W^2. \tag{19}$$

Lemma 2.3 (*The Discrete Gronwall's Lemma*)**.** *Suppose that n and N are non-negative integers, $n \le N$. The real numbers $a_n, b_n, c_n, \kappa_n, \Delta t, C$ are non-negative and satisfy that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \Delta t \sum_{n=0}^{N} \kappa_n a_n + \Delta t \sum_{n=0}^{N} c_n + C.$$

If $\Delta t \kappa_n < 1$ for each n, then

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le exp(\Delta t \sum_{n=0}^{N} \frac{\kappa_n}{1 - \Delta t \kappa_n})(\Delta t \sum_{n=0}^{N} c_n + C). \tag{20}$$

3. A first order algorithm for NS-Darcy model

In this section, we employ the first order Euler method for the temporal discretization. We first present the stability and then provide the error estimate. Besides, we estimate the time difference of the error, which is useful to the error analysis of the second order method in the next section.

The first order algorithm for (7) is: find $\underline{w}_{1,h}^{n+1} = (u_{1,h}^{n+1}, \varphi_{1,h}^{n+1}) \in W_h$ and $p_{1,h}^{n+1} \in Q_h$, such that for all $\underline{z}_h = (v_h, \psi_h) \in W_h$, $q_h \in Q_h$ with $n = 0, 1, \ldots, N-1$ ($N = \frac{T}{\Delta t}$),

$$\begin{cases} [\frac{\underline{w}_{1,h}^{n+1} - \underline{w}_{1,h}^{n}}{\Delta t}, \underline{z}_h] + a(\underline{w}_{1,h}^{n+1}, \underline{z}_h) + N(u_{1,h}^n; u_{1,h}^{n+1}, v_h) + b(\underline{z}_h, p_{1,h}^{n+1}) = (f^{n+1}, \underline{z}_h), \\ b(\underline{w}_{1,h}^{n+1}, q_h) = 0, \\ \underline{w}_{1,h}^0 = P_h \underline{w}^0 = 0. \end{cases} \tag{21}$$

Note that we linearize the trilinear term by time-lagging.

3.1. Stability analysis for the first order algorithm

Since the interface conditions of the Navier–Stokes/Darcy model cannot completely compensate the nonlinear convection of the Navier–Stokes equations in the energy balance, one has to impose some small data and/or large viscosity restriction to guarantee the well-posedness of the coupled system, see [19]. So does its numerical scheme and their corresponding numerical analysis, see [20]. Similarly, we need an extra condition for the stability of our scheme.

Lemma 3.1. *If $\underline{w}_{1,h}^m = (u_{1,h}^m, \varphi_{1,h}^m)$, $1 \le m \le N$ is the solution of (21) and satisfies*

$$\|\nabla u_{1,h}^n\|_{L^2(\Omega_f)} \le \frac{\nu}{4C_{N1}}, \quad 1 \le n \le m-1, \tag{22}$$

then we have the following stability result

$$\|\underline{w}_{1,h}^m\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|\underline{w}_{1,h}^{n+1}\|_W^2 \le \Upsilon^2. \tag{23}$$

Here $\Upsilon^2 = \frac{2C_p^2}{\nu} \Delta t \sum_{n=0}^{m-1} \|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{2g\tilde{C}_p^2}{k_{min}} \Delta t \sum_{n=0}^{m-1} \|f_2^{n+1}\|_{L^2(\Omega_p)}^2$.

Proof. Setting $z_h = 2\Delta t \underline{w}_{1,h}^{n+1}$, $q_h = p_{1,h}^{n+1}$ in (21) and using (8) and the equality

$$2(a-b, a) = |a|^2 - |b|^2 + |a-b|^2, \quad \forall\, a, b \in \mathbb{R}^d,$$

we have

$$\begin{aligned} &\|\underline{w}_{1,h}^{n+1}\|_0^2 - \|\underline{w}_{1,h}^{n}\|_0^2 + \|\underline{w}_{1,h}^{n+1} - \underline{w}_{1,h}^{n}\|_0^2 + 2\Delta t \|\underline{w}_{1,h}^{n+1}\|_W^2 + 2\Delta t \sum_{i=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}} \|u_{1,h}^{n+1} \cdot \tau_i\|_{L^2(\Gamma)}^2 \\ &= -2\Delta t N(u_{1,h}^n; u_{1,h}^{n+1}, u_{1,h}^{n+1}) + 2\Delta t (f^{n+1}, \underline{w}_{1,h}^{n+1}). \end{aligned} \tag{24}$$

Now we bound each term in the right hand side of (24). By using (15) and (22), we have

$$-2\Delta t N(u_{1,h}^n; u_{1,h}^{n+1}, u_{1,h}^{n+1}) \le 2\Delta t C_{N1} \|\nabla u_{1,h}^n\|_{L^2(\Omega_f)} \|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 \le \frac{\nu}{2} \Delta t \|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 \le \frac{1}{2} \Delta t \|\underline{w}_{1,h}^{n+1}\|_W^2.$$

By using (9) and (10), we can obtain

$$\begin{aligned} &2\Delta t (f^{n+1}, \underline{w}_{1,h}^{n+1}) = 2\Delta t (f_1^{n+1}, u_{1,h}^{n+1})_{\Omega_f} + 2g\Delta t (f_2^{n+1}, \varphi_{1,h}^{n+1})_{\Omega_p} \\ &\le 2\Delta t \|f_1^{n+1}\|_{L^2(\Omega_f)} \|u_{1,h}^{n+1}\|_{L^2(\Omega_f)} + 2g\Delta t \|f_2^{n+1}\|_{L^2(\Omega_p)} \|\varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)} \\ &\le 2C_p \Delta t \|f_1^{n+1}\|_{L^2(\Omega_f)} \|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)} + 2g\tilde{C}_p \Delta t \|f_2^{n+1}\|_{L^2(\Omega_p)} \|\nabla \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)} \\ &\le \varepsilon \Delta t \big(\nu \|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 + g k_{min} \|\nabla \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)}^2\big) + \frac{C_p^2 \Delta t}{\varepsilon \nu} \|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2 \Delta t}{\varepsilon k_{min}} \|f_2^{n+1}\|_{L^2(\Omega_p)}^2 \\ &\le \varepsilon \Delta t \|\underline{w}_{1,h}^{n+1}\|_W^2 + \frac{C_p^2 \Delta t}{\varepsilon \nu} \|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2 \Delta t}{\varepsilon k_{min}} \|f_2^{n+1}\|_{L^2(\Omega_p)}^2. \end{aligned} \tag{25}$$

Combining the above estimates with (24) and setting $\varepsilon = 1/2$, we get

$$\|\underline{w}_{1,h}^{n+1}\|_0^2 - \|\underline{w}_{1,h}^{n}\|_0^2 + \Delta t \|\underline{w}_{1,h}^{n+1}\|_W^2 \le \frac{2C_p^2 \Delta t}{\nu} \|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{2g\tilde{C}_p^2 \Delta t}{k_{min}} \|f_2^{n+1}\|_{L^2(\Omega_p)}^2. \tag{26}$$

Summing (26) with respect to n from 0 to $m-1$ and considering $\underline{w}_{1,h}^0 = 0$, we get

$$\|\underline{w}_{1,h}^m\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|\underline{w}_{1,h}^{n+1}\|_W^2 \le \frac{2C_p^2 \Delta t}{\nu} \sum_{n=0}^{m-1} \|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{2g\tilde{C}_p^2 \Delta t}{k_{min}} \sum_{n=0}^{m-1} \|f_2^{n+1}\|_{L^2(\Omega_p)}^2.$$

Thus, we complete the proof. □

Lemma 3.2. *Denote* $d_t \underline{w}_{1,h}^{n+1} = \frac{\underline{w}_{1,h}^{n+1} - \underline{w}_{1,h}^n}{\Delta t}$, $0 \le n \le N-1$. *Under the assumption* (22), *we have*

$$\|\underline{w}_{1,h}^{n+1}\|_W^2 \le \Big(\frac{4C_p^2}{\nu} + \frac{4S_0\tilde{C}_p^2}{k_{\min}}\Big)\|d_t \underline{w}_{1,h}^{n+1}\|_0^2 + \frac{4C_p^2}{\nu}\|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{4g\tilde{C}_p^2}{k_{min}}\|f_2^{n+1}\|_{L^2(\Omega_p)}^2. \tag{27}$$

Proof. Setting $\underline{z}_h = 2\underline{w}_{1,h}^{n+1}$ and $q_h = p_{1,h}^{n+1}$ in (21) and using (8), we have

$$2\|\underline{w}_{1,h}^{n+1}\|_W^2 \le -2[d_t \underline{w}_{1,h}^{n+1}, \underline{w}_{1,h}^{n+1}] - 2N(u_{1,h}^n; u_{1,h}^{n+1}, u_{1,h}^{n+1}) + 2(\underline{f}^{n+1}, \underline{w}_{1,h}^{n+1}). \tag{28}$$

Using Young's inequality, (9), (10) and Holder's inequality, we have

$$\begin{aligned}
&-2[d_t \underline{w}_{1,h}^{n+1}, \underline{w}_{1,h}^{n+1}] = -2[d_t u_{1,h}^{n+1}, u_{1,h}^{n+1}] - 2gS_0[d_t \varphi_{1,h}^{n+1}, \varphi_{1,h}^{n+1}] \\
&\le 2C_p\|d_t u_{1,h}^{n+1}\|_{L^2(\Omega_f)}\|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)} + 2gS_0\tilde{C}_p\|d_t \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)}\|\nabla \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)} \\
&\le \varepsilon\nu\|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_p^2}{\varepsilon\nu}\|d_t u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 + \varepsilon g k_{\min}\|\nabla \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)}^2 + \frac{gS_0^2\tilde{C}_p^2}{\varepsilon k_{\min}}\|d_t \varphi_{1,h}^{n+1}\|_{L^2(\Omega_p)}^2 \\
&\le \varepsilon\|\underline{w}_{1,h}^{n+1}\|_W^2 + \Big(\frac{C_p^2}{\varepsilon\nu} + \frac{S_0\tilde{C}_p^2}{\varepsilon k_{\min}}\Big)\|d_t \underline{w}_{1,h}^{n+1}\|_0^2.
\end{aligned} \tag{29}$$

By using (15) and (22), we have

$$-2N(u_{1,h}^n; u_{1,h}^{n+1}, u_{1,h}^{n+1}) \le 2C_{N1}\|\nabla u_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 \le \frac{\nu}{2}\|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)}^2 \le \frac{1}{2}\|\underline{w}_{1,h}^{n+1}\|_W^2.$$

Similarly to (25), we can obtain

$$2(\underline{f}^{n+1}, \underline{w}_{1,h}^{n+1}) \le \varepsilon\|\underline{w}_{1,h}^{n+1}\|_W^2 + \frac{C_p^2}{\varepsilon\nu}\|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2}{\varepsilon k_{min}}\|f_2^{n+1}\|_{L^2(\Omega_p)}^2.$$

Combining the above estimates with (28) and setting $\varepsilon = 1/4$, we complete the proof. □

Lemma 3.3. *Under the assumption* (22) *and following regularities*

$$(f_1)_t \in L^2(0,T; L^2(\Omega_f)^d), \quad (f_2)_t \in L^2(0,T; L^2(\Omega_p)), \tag{30}$$

we have, for all $1 \le m \le N$,

$$\|d_t \underline{w}_{1,h}^m\|_0^2 + \Delta t\|d_t \underline{w}_{1,h}^m\|_W^2 \le \mathcal{H}^2. \tag{31}$$

Here $\mathcal{H}^2 = \|f_1^1\|_{L^2(\Omega_f)}^2 + gS_0\|f_2^1\|_{L^2(\Omega_p)}^2 + \frac{16C_p^2}{7\nu}\|(f_1)_t\|_{L^2(0,T;L^2(\Omega_f))}^2 + \frac{16g\tilde{C}_p^2}{7k_{min}}\|(f_2)_t\|_{L^2(0,T;L^2(\Omega_p))}^2$.

Proof. Considering (21), we have the equation at time t^{n+1},

$$\begin{cases}
[d_t \underline{w}_{1,h}^{n+1}, \underline{z}_h] + a(\underline{w}_{1,h}^{n+1}, \underline{z}_h) + N(u_{1,h}^n; u_{1,h}^{n+1}, v_h) + b(\underline{z}_h, p_{1,h}^{n+1}) = (\underline{f}^{n+1}, \underline{z}_h), \\
b(\underline{w}_{1,h}^{n+1}, q_h) = 0.
\end{cases} \tag{32}$$

Then we take the difference of above equation at time t^{n+1} and t^n leading to

$$\begin{cases}
[d_t \underline{w}_{1,h}^{n+1} - d_t \underline{w}_{1,h}^n, \underline{z}_h] + \Delta t a(d_t \underline{w}_{1,h}^{n+1}, \underline{z}_h) + b(\underline{z}_h, p_{1,h}^{n+1} - p_{1,h}^n) = -N(u_{1,h}^n; u_{1,h}^{n+1}, v_h) \\
\quad + N(u_{1,h}^{n-1}; u_{1,h}^n, v_h) + (\underline{f}^{n+1} - \underline{f}^n, \underline{z}_h), \\
b(d_t \underline{w}_{1,h}^{n+1}, q_h) = 0.
\end{cases}$$

Taking $\underline{z}_h = 2d_t \underline{w}_{1,h}^{n+1}$ and $q_h = p_{1,h}^{n+1} - p_{1,h}^n$, we obtain

$$\|d_t \underline{w}_{1,h}^{n+1}\|_0^2 - \|d_t \underline{w}_{1,h}^n\|_0^2 + 2\Delta t\|d_t \underline{w}_{1,h}^{n+1}\|_W^2 \le -2N(u_{1,h}^n; u_{1,h}^{n+1}, d_t u_{1,h}^{n+1}) + 2N(u_{1,h}^{n-1}; u_{1,h}^n, d_t u_{1,h}^{n+1}) + 2(\underline{f}^{n+1} - \underline{f}^n, d_t \underline{w}_{1,h}^{n+1}). \tag{33}$$

Using (15), (22) and Holder's inequality, we have

$$\begin{aligned}
&-2N(u_{1,h}^n;u_{1,h}^{n+1},d_tu_{1,h}^{n+1})+2N(u_{1,h}^{n-1};u_{1,h}^n,d_tu_{1,h}^{n+1})\\
=&-2\Delta tN(u_{1,h}^n;d_tu_{1,h}^{n+1},d_tu_{1,h}^{n+1})-2\Delta tN(d_tu_{1,h}^n;u_{1,h}^n,d_tu_{1,h}^{n+1})\\
\le&\,2\Delta tC_{N1}\|\nabla u_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}+2\Delta tC_{N1}\|\nabla d_tu_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla u_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{1,h}^{n+1}\|_{L^2(\Omega_f)}\\
\le&\,\frac{\nu}{2}\Delta t\|\nabla d_tu_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}+\frac{\nu}{2}\Delta t\|\nabla d_tu_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{1,h}^{n+1}\|_{L^2(\Omega_f)}\\
\le&\,\frac{\nu}{2}\Delta t\|\nabla d_tu_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}+\big(\frac{\nu}{16}\Delta t\|\nabla d_tu_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}+\nu\Delta t\|\nabla d_tu_{1,h}^n\|^2_{L^2(\Omega_f)}\big).
\end{aligned}$$

Similarly to (25), we can obtain

$$\begin{aligned}
&2(f^{n+1}-f^n,d_t\underline{w}_{1,h}^{n+1})=2\Delta t(\frac{f_1^{n+1}-f_1^n}{\Delta t},u_{1,h}^{n+1})_{\Omega_f}+2g\Delta t(\frac{f_2^{n+1}-f_2^n}{\Delta t},\varphi_{1,h}^{n+1})_{\Omega_p}\\
\le&\,\varepsilon\Delta t\|\underline{w}_{1,h}^{n+1}\|_W^2+\frac{C_p^2}{\varepsilon\nu}\Delta t\|\frac{f_1^{n+1}-f_1^n}{\Delta t}\|^2_{L^2(\Omega_f)}+\frac{g\tilde{C}_p^2}{\varepsilon k_{min}}\Delta t\|\frac{f_2^{n+1}-f_2^n}{\Delta t}\|^2_{L^2(\Omega_p)}\\
\le&\,\varepsilon\Delta t\|\underline{w}_{1,h}^{n+1}\|_W^2+\frac{C_p^2}{\varepsilon\nu}\int_{t^n}^{t^{n+1}}\|(f_1)_t\|^2_{L^2(\Omega_f)}dt+\frac{g\tilde{C}_p^2}{\varepsilon k_{min}}\int_{t^n}^{t^{n+1}}\|(f_2)_t\|^2_{L^2(\Omega_p)}dt.
\end{aligned}$$

Combining these estimates with (33) and setting $\varepsilon=7/16$, we have

$$\|d_t\underline{w}_{1,h}^{n+1}\|_0^2-\|d_t\underline{w}_{1,h}^n\|_0^2+\Delta t\|d_t\underline{w}_{1,h}^{n+1}\|_W^2-\Delta t\|d_t\underline{w}_{1,h}^n\|_W^2\le\frac{16C_p^2}{7\nu}\int_{t^n}^{t^{n+1}}\|(f_1)_t\|^2_{L^2(\Omega_f)}dt+\frac{16g\tilde{C}_p^2}{7k_{min}}\int_{t^n}^{t^{n+1}}\|(f_2)_t\|^2_{L^2(\Omega_p)}dt.$$

Since we do not define $d_t\underline{w}_{1,h}^0$, we only sum the equation from $n=1$ to $n=m-1$. This leads to

$$\begin{aligned}
\|d_t\underline{w}_{1,h}^m\|_0^2+\Delta t\|d_t\underline{w}_{1,h}^m\|_W^2\le&\,\|d_t\underline{w}_{1,h}^1\|_0^2+\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2+\frac{16C_p^2}{7\nu}\sum_{n=1}^{m-1}\int_{t^n}^{t^{n+1}}\|(f_1)_t\|^2_{L^2(\Omega_f)}dt\\
&+\frac{16g\tilde{C}_p^2}{7k_{min}}\sum_{n=1}^{m-1}\int_{t^n}^{t^{n+1}}\|(f_2)_t\|^2_{L^2(\Omega_p)}dt\\
\le&\,\|d_t\underline{w}_{1,h}^1\|_0^2+\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2+\frac{16C_p^2}{7\nu}\|(f_1)_t\|^2_{L^2(0,T;L^2(\Omega_f))}+\frac{16g\tilde{C}_p^2}{7k_{min}}\|(f_2)_t\|^2_{L^2(0,T;L^2(\Omega_p))}.
\end{aligned}\tag{34}$$

Next, we need to estimate the results of $d_t\underline{w}_{1,h}^1$. Considering (32) at time t^1 and the fact $\underline{w}_{1,h}^0=0$, we have

$$\begin{cases}[d_t\underline{w}_{1,h}^1,\underline{z}_h]+\Delta ta(d_t\underline{w}_{1,h}^1,\underline{z}_h)+b(\underline{z}_h,p_{1,h}^1)=(f^1,\underline{z}_h),\\ b(d_t\underline{w}_{1,h}^1,q_h)=0.\end{cases}$$

Taking $\underline{z}_h=2d_t\underline{w}_{1,h}^1$ and $q_h=p_{1,h}^1$, we get

$$2\|d_t\underline{w}_{1,h}^1\|_0^2+2\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2\le2(f^1,d_t\underline{w}_{1,h}^1).$$

However,

$$2(f^1,d_t\underline{w}_{1,h}^1)\le2\|f^1\|_0\|d_t\underline{w}_{1,h}^1\|_0\le\|d_t\underline{w}_{1,h}^1\|_0^2+\|f^1\|_0^2=\|d_t\underline{w}_{1,h}^1\|_0^2+\|f_1^1\|^2_{L^2(\Omega_f)}+gS_0\|f_2^1\|^2_{L^2(\Omega_p)}.$$

So that

$$\|d_t\underline{w}_{1,h}^1\|_0^2+\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2\le\|d_t\underline{w}_{1,h}^1\|_0^2+2\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2\le\|f_1^1\|^2_{L^2(\Omega_f)}+gS_0\|f_2^1\|^2_{L^2(\Omega_p)}.\tag{35}$$

Then substituting (35) into (34), we get the result (31). □

Finally, we get the stability result for algorithm (21).

Theorem 3.1. *Assume that*

$$\left((\frac{C_p^2}{\nu}+\frac{S_0\tilde{C}_p^2}{k_{\min}})\mathcal{H}^2+\frac{C_p^2}{\nu}\|f_1\|^2_{L^\infty(0,T;L^2(\Omega_f))}+\frac{g\tilde{C}_p^2}{k_{min}}\|f_2\|^2_{L^\infty(0,T;L^2(\Omega_p))}\right)^{\frac12}\le\frac{\nu^{3/2}}{8C_{N1}}.\tag{36}$$

Then under the assumptions (6) and (30)*, for all* $1\le m\le N$*, the solution* $\underline{w}_{1,h}^m$ *of algorithm* (21) *satisfies*

$$\|\underline{w}_{1,h}^m\|_0^2+\Delta t\sum_{n=0}^{m-1}\|\underline{w}_{1,h}^{n+1}\|_W^2\le\frac{2C_p^2\Delta t}{\nu}\sum_{n=0}^{m-1}\|f_1^{n+1}\|^2_{L^2(\Omega_f)}+\frac{2g\tilde{C}_p^2\Delta t}{k_{min}}\sum_{n=0}^{m-1}\|f_2^{n+1}\|^2_{L^2(\Omega_p)}.\tag{37}$$

Proof. Substituting (31) into (27), we get

$$\|\underline{w}_{1,h}^{n+1}\|_W^2 \leq (\frac{4C_p^2}{\nu} + \frac{4S_0\tilde{C}_p^2}{k_{\min}})\mathcal{H}^2 + \frac{4C_p^2}{\nu}\|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{4g\tilde{C}_p^2}{k_{min}}\|f_2^{n+1}\|_{L^2(\Omega_p)}^2.$$

Under the condition (36), we deduce

$$\begin{aligned}\nu^{\frac{1}{2}}\|\nabla u_{1,h}^{n+1}\|_{L^2(\Omega_f)} &\leq \|\underline{w}_{1,h}^{n+1}\|_W \leq 2\Big((\frac{C_p^2}{\nu} + \frac{S_0\tilde{C}_p^2}{k_{\min}})\mathcal{H}^2 + \frac{C_p^2}{\nu}\|f_1^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2}{k_{min}}\|f_2^{n+1}\|_{L^2(\Omega_p)}^2\Big)^{\frac{1}{2}} \\ &\leq 2\Big((\frac{C_p^2}{\nu} + \frac{S_0\tilde{C}_p^2}{k_{\min}})\mathcal{H}^2 + \frac{C_p^2}{\nu}\|f_1\|_{L^\infty(0,T;L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2}{k_{min}}\|f_2\|_{L^\infty(0,T;L^2(\Omega_p))}^2\Big)^{\frac{1}{2}} \leq \frac{\nu^{3/2}}{4C_{N1}}.\end{aligned} \tag{38}$$

Hence (22) is available for all n, $1 \leq n \leq N-1$. Then considering Lemma 3.1, we can get the final stability result. □

3.2. Convergence analysis for the first order algorithm

Next we show the convergence rate of algorithm (21). For any time $t > 0$, let $(\underline{\tilde{w}}(t), \tilde{p}(t)) = (P_h^w\underline{w}(t), P_h^p p(t))$ be the approximation of $(\underline{w}(t), p(t))$. For $n = 0, 1, \ldots, N-1$, we denote $(\underline{w}^{n+1}, p^{n+1}) = (\underline{w}(t^{n+1}), p(t^{n+1}))$ and $(\underline{\tilde{w}}^{n+1}, \tilde{p}^{n+1}) = (\underline{\tilde{w}}(t^{n+1}), \tilde{p}(t^{n+1}))$. Using these symbols, we decompose the errors as follows:

$$\begin{aligned}
\underline{e}_{1,w}^{n+1} &= \underline{w}_{1,h}^{n+1} - \underline{w}^{n+1} = \underline{w}_{1,h}^{n+1} - \underline{\tilde{w}}^{n+1} - (\underline{w}^{n+1} - \underline{\tilde{w}}^{n+1}) := \underline{e}_1^{n+1} - \underline{\xi}^{n+1},\\
e_{1,u}^{n+1} &= u_{1,h}^{n+1} - u^{n+1} = u_{1,h}^{n+1} - \tilde{u}^{n+1} - (u^{n+1} - \tilde{u}^{n+1}) := e_{1,f}^{n+1} - \xi_f^{n+1},\\
e_{1,\varphi}^{n+1} &= \varphi_{1,h}^{n+1} - \varphi^{n+1} = \varphi_{1,h}^{n+1} - \tilde{\varphi}^{n+1} - (\varphi^{n+1} - \tilde{\varphi}^{n+1}) := e_{1,p}^{n+1} - \xi_p^{n+1},\\
e_{1,\chi}^{n+1} &= p_{1,h}^{n+1} - p^{n+1} = p_{1,h}^{n+1} - \tilde{p}^{n+1} - (p^{n+1} - \tilde{p}^{n+1}) := \varepsilon_1^{n+1} - \eta^{n+1},
\end{aligned}$$

where $\underline{e}_{1,w}^{n+1} = (e_{1,u}^{n+1}, e_{1,\varphi}^{n+1})$, $\underline{e}_1^{n+1} = (e_{1,f}^{n+1}, e_{1,p}^{n+1})$ and $\underline{\xi}^{n+1} = (\xi_f^{n+1}, \xi_p^{n+1})$. From the approximation properties (13), it is easy to get

$$\|\underline{\xi}^{n+1}\|_0^2 \leq C(h^{2k_1+2} + h^{2k_2+2}), \quad \|\underline{\xi}^{n+1}\|_W^2 \leq C(h^{2k_1} + h^{2k_2}). \tag{39}$$

Then we have the following result about the convergence rate of (21).

Theorem 3.2. *Let the assumptions of Theorem 3.1 and (14) satisfy. In addition, we assume*

$$h^{-2}\Delta t + \frac{32C_{N1}^2C_I^2 + 32C_{N2}^2\|u\|_{L^\infty(0,T;H^2(\Omega_f))}^2}{\nu}\Delta t < 1. \tag{40}$$

Then, for all $1 \leq m \leq N$, we have the following estimate

$$\|\underline{e}_1^m\|_0^2 + \Delta t\sum_{n=0}^{m-1}\|\underline{e}_1^{n+1}\|_W^2 \leq C(\Delta t^2 + h^{2k_1+2} + h^{2k_2+2}). \tag{41}$$

Proof. At time t^{n+1}, the true solution $(\underline{w}, p) = (u, \varphi, p)$ satisfies: for all $\underline{z}_h \in W_h$, $q_h \in Q_h$,

$$\begin{cases}[\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t}, \underline{z}_h] + a(\underline{w}^{n+1}, \underline{z}_h) + N(u^{n+1}; u^{n+1}, v_h) + b(\underline{z}_h, p^{n+1}) = [\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} - \underline{w}_t^{n+1}, \underline{z}_h] + (f^{n+1}, \underline{z}_h),\\ b(\underline{w}^{n+1}, q_h) = 0.\end{cases} \tag{42}$$

Subtract (42) from (21) to obtain

$$\begin{cases}[\frac{\underline{e}_1^{n+1} - \underline{e}_1^n}{\Delta t}, \underline{z}_h] + a(\underline{e}_1^{n+1}, \underline{z}_h) + N(u_{1,h}^n; u_{1,h}^{n+1}, v_h) - N(u^{n+1}; u^{n+1}, v_h) + b(\underline{z}_h, \varepsilon_1^{n+1})\\ = [\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t}, \underline{z}_h] + [\frac{\underline{\xi}^{n+1} - \underline{\xi}^n}{\Delta t}, \underline{z}_h] + a(\underline{\xi}^{n+1}, \underline{z}_h) + b(\underline{z}_h, \eta^{n+1}),\\ b(\underline{e}_1^{n+1}, q_h) = 0.\end{cases} \tag{43}$$

Considering the definition of P_h (12), we have

$$a(\underline{\xi}^{n+1}, \underline{z}_h) + b(\underline{z}_h, \eta^{n+1}) = 0, \quad \forall\, \underline{z}_h \in W_h. \tag{44}$$

Taking $\underline{z}_h = 2\Delta t\underline{e}_1^{n+1},\ q_h = \varepsilon_1^{n+1}$ in (43) and using (44), we deduce that

$$\begin{aligned}
&\|\underline{e}_1^{n+1}\|_0^2 - \|\underline{e}_1^n\|_0^2 + \|\underline{e}_1^{n+1} - \underline{e}_1^n\|_0^2 + 2\Delta t\|\underline{e}_1^{n+1}\|_W^2 + 2\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|e_{1,f}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2 \\
=&2\Delta t[\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \frac{\underline{\xi}^{n+1} - \underline{\xi}^n}{\Delta t}, \underline{e}_1^{n+1}] - 2\Delta tN(u_{1,h}^n; u_{1,h}^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u^{n+1}; u^{n+1}, e_{1,f}^{n+1}).
\end{aligned} \tag{45}$$

Now we bound each term on the right hand side of (45). First, we consider that

$$\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \frac{\underline{\xi}^{n+1} - \underline{\xi}^n}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}(t - t^n)\underline{w}_{tt}(t)dt + \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\underline{\xi}_t(t)dt.$$

Then similarly to (29), we get, for all $\varepsilon > 0$,

$$\begin{aligned}
I &= 2\Delta t[\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \frac{\underline{\xi}^{n+1} - \underline{\xi}^n}{\Delta t}, \underline{e}_1^{n+1}] \\
&\le \varepsilon\Delta t\|\underline{e}_1^{n+1}\|_W^2 + (\frac{C_p^2}{\varepsilon\nu} + \frac{S_0\tilde{C}_p^2}{\varepsilon k_{\min}})\Delta t\|\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \frac{\underline{\xi}^{n+1} - \underline{\xi}^n}{\Delta t}\|_0^2. \\
&\le \varepsilon\Delta t\|\underline{e}_1^{n+1}\|_W^2 + \left(\frac{2C_p^2}{\varepsilon\nu} + \frac{2S_0\tilde{C}_p^2}{\varepsilon k_{\min}}\right)\left(\Delta t^2\int_{t^n}^{t^{n+1}}\|\underline{w}_{tt}\|_0^2dt + \int_{t^n}^{t^{n+1}}\|\underline{\xi}_t\|_0^2dt\right).
\end{aligned} \tag{46}$$

For the trilinear terms, it is easy to verify the following identity

$$\begin{aligned}
II &= -2\Delta tN(u_{1,h}^n; u_{1,h}^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u^{n+1}; u^{n+1}, e_{1,f}^{n+1}) \\
&= -2\Delta tN(u_{1,h}^n; u_{1,h}^{n+1} - u^{n+1}, e_{1,f}^{n+1}) - 2\Delta tN(u_{1,h}^n; u^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u^{n+1}; u^{n+1}, e_{1,f}^{n+1}) \\
&= -2\Delta tN(u_{1,h}^n; u_{1,h}^{n+1} - u^{n+1}, e_{1,f}^{n+1}) - 2\Delta tN(u_{1,h}^n - u^n; u^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u^{n+1} - u^n; u^{n+1}, e_{1,f}^{n+1}).
\end{aligned}$$

For the first term, we get

$$\begin{aligned}
i &= -2\Delta tN(u_{1,h}^n; u_{1,h}^{n+1} - u^{n+1}, e_{1,f}^{n+1}) = -2\Delta tN(u_{1,h}^n; e_{1,f}^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u_{1,h}^n; \xi_f^{n+1}, e_{1,f}^{n+1}) \\
&= -2\Delta tN(u_{1,h}^n; e_{1,f}^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(e_{1,f}^n; \xi_f^{n+1}, e_{1,f}^{n+1}) - 2\Delta tN(\xi_f^n; \xi_f^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(u^n; \xi_f^{n+1}, e_{1,f}^{n+1}).
\end{aligned} \tag{47}$$

For the first three terms, by using (11), (15) and (22), we have

$$\begin{aligned}
&-2\Delta tN(u_{1,h}^n; e_{1,f}^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(e_{1,f}^n; \xi_f^{n+1}, e_{1,f}^{n+1}) - 2\Delta tN(\xi_f^n; \xi_f^{n+1}, e_{1,f}^{n+1}) \\
\le& 2\Delta tC_{N1}\|\nabla u_{1,h}^n\|_{L^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + 2\Delta tC_{N1}C_Ih^{-1}\|e_{1,f}^n\|_{L^2(\Omega_f)}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\
&+ 2\Delta tC_{N1}\|\nabla\xi_f^n\|_{L^2(\Omega_f)}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\
\le& (\frac{\nu}{2} + 2\varepsilon\nu)\Delta t\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|e_{1,f}^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\
\le& (\frac{1}{2} + 2\varepsilon)\Delta t\|\underline{e}_1^{n+1}\|_W^2 + \frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2\|\underline{e}_1^n\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2.
\end{aligned}$$

Using (18) and Holder's inequality, we have

$$\begin{aligned}
2\Delta tN(u^n; \xi_f^{n+1}, e_{1,f}^{n+1}) &\le 2\Delta tC_{N4}\|u^n\|_{H^2(\Omega_f)}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}\|e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\
&\le h^{-2}\Delta t\|e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N4}^2}{h^{-2}}\Delta t\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\
&\le h^{-2}\Delta t\|\underline{e}_1^{n+1}\|_0^2 + C_{N4}^2h^2\Delta t\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
i \le& (\frac{1}{2} + 2\varepsilon)\Delta t\|\underline{e}_1^{n+1}\|_W^2 + h^{-2}\Delta t\|\underline{e}_1^{n+1}\|_0^2 + \frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2\|\underline{e}_1^n\|_0^2 \\
&+ \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + C_{N4}^2h^2\Delta t\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2.
\end{aligned}$$

Using (16), we have

$$ii = -2\Delta tN(u_{1,h}^n - u^n; u^{n+1}, e_{1,f}^{n+1}) = -2\Delta tN(e_{1,f}^n; u^{n+1}, e_{1,f}^{n+1}) + 2\Delta tN(\xi_f^n; u^{n+1}, e_{1,f}^{n+1})$$
$$\leq 2\Delta tC_{N2}\|e_{1,f}^n\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)} + 2\Delta tC_{N2}\|\xi_f^n\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\leq 2\varepsilon\Delta t\|\underline{e}_1^{n+1}\|_0^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|\underline{e}_1^n\|_0^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|\xi_f^n\|_{L^2(\Omega_f)}^2,$$
$$iii = 2\Delta tN(u^{n+1} - u^n; u^{n+1}, e_{1,f}^{n+1}) \leq 2\Delta C_{N2}\|u^{n+1} - u^n\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla e_{1,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\leq \varepsilon\Delta t\|\underline{e}_1^{n+1}\|_W^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u^{n+1}\|_{H^2(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_t\|_{L^2(\Omega_f)}^2 dt.$$

Considering the above estimates, we have

$$\begin{aligned} II \leq (\frac{1}{2} + 5\varepsilon)\Delta t\|\underline{e}_1^{n+1}\|_W^2 + h^{-2}\Delta t\|\underline{e}_1^{n+1}\|_0^2 + \big(\frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\big)\|\underline{e}_1^n\|_0^2 \\ + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + C_{N4}^2h^2\Delta t\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\ + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|\xi_f^n\|_{L^2(\Omega_f)}^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u^{n+1}\|_{H^2(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_t\|_{L^2(\Omega_f)}^2 dt. \end{aligned} \tag{48}$$

Finally, combining (46) and (48) with (45) and setting $\varepsilon = 1/12$, we have

$$\begin{aligned} \|\underline{e}_1^{n+1}\|_0^2 - \|\underline{e}_1^n\|_0^2 + \Delta t\|\underline{e}_1^{n+1}\|_W^2 \leq h^{-2}\Delta t\|\underline{e}_1^{n+1}\|_0^2 + \big(\frac{12C_{N1}^2C_I^2}{\nu}\Delta th^{-2}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{12C_{N2}^2}{\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\big)\|\underline{e}_1^n\|_0^2 \\ + \left(\frac{24C_p^2}{\nu} + \frac{24S_0\tilde{C}_p^2}{k_{\min}}\right)\left(\Delta t^2\int_{t^n}^{t^{n+1}}\|\underline{w}_{tt}(\tau)\|_0^2 d\tau + \int_{t^n}^{t^{n+1}}\|\underline{\xi}_t(\tau)\|_0^2 d\tau\right) + \frac{12C_{N1}^2}{\nu}\Delta t\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\ + C_{N4}^2h^2\Delta t\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{12C_{N2}^2}{\nu}\big(\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|\xi_f^n\|_{L^2(\Omega_f)}^2 + \Delta t^2\|u^{n+1}\|_{H^2(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_t\|_{L^2(\Omega_f)}^2 dt\big). \end{aligned} \tag{49}$$

Considering (13) and (40), we get

$$\begin{aligned} \kappa_n\Delta t &= h^{-2}\Delta t + \frac{12C_{N1}^2C_I^2}{\nu}\Delta th^{-2}\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{12C_{N2}^2}{\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2 \\ &\leq h^{-2}\Delta t + \frac{12C_{N1}^2C_I^2}{\nu}\Delta th^{2k_1-2} + \frac{12C_{N2}^2}{\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2 < 1. \end{aligned}$$

Then sum (49) from $n = 0$ to $n = m - 1$ and note again $\underline{e}_1^0 = 0$. Using (20) in Lemma 2.3, we get

$$\begin{aligned} \|\underline{e}_1^m\|_0^2 + \Delta t\sum_{n=0}^{m-1}\|\underline{e}_1^{n+1}\|_W^2 \leq \exp\big(\Delta t\sum_{n=0}^{N}\frac{\kappa_n}{1-\Delta t\kappa_n}\big)\Bigg(C\Bigg(\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{w}_{tt}\|_0^2 dt + \sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{\xi}_t\|_0^2 dt\Bigg) \\ + C\Delta t\sum_{n=0}^{m-1}\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + Ch^2\Delta t\sum_{n=0}^{m-1}\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^{n+1}\|_{L^2(\Omega_f)}^2 + C\Delta t\sum_{n=0}^{m-1}\|u^{n+1}\|_{H^2(\Omega_f)}^2\|\xi_f^n\|_{L^2(\Omega_f)}^2 \\ + C\Delta t^2\sum_{n=0}^{m-1}\|u^{n+1}\|_{H^2(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_t\|_{L^2(\Omega_f)}^2 dt\Bigg). \end{aligned}$$

Using (14) and (39), we can bound these terms as follows.

$$\Delta t^2\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{w}_{tt}\|_0^2 dt \leq \Delta t^2\int_0^T\|\underline{w}_{tt}\|_0^2 dt \leq \Delta t^2\big(\|u_{tt}\|_{L^2(0,T;L^2(\Omega_f))}^2 + \|\varphi_{tt}\|_{L^2(0,T;L^2(\Omega_p))}^2\big),$$
$$\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{\xi}_t\|_0^2 dt \leq C(h^{2k_1+2} + h^{2k_2+2})\big(\|u_t\|_{L^2(0,T;H^{k_1+1}(\Omega_f))}^2 + \|\varphi_t\|_{L^2(0,T;H^{k_2+1}(\Omega_p))}^2\big),$$

$$\Delta t\sum_{n=0}^{m-1}\|\nabla\xi_f^n\|^2_{L^2(\Omega_f)}\|\nabla\xi_f^{n+1}\|^2_{L^2(\Omega_f)}\le C(T)h^{4k_1}\|u\|^2_{L^\infty(0,T;H^{k_1+1}(\Omega_f))},$$

$$h^2\Delta t\sum_{n=0}^{m-1}\|u^n\|^2_{H^2(\Omega_f)}\|\nabla\xi_f^{n+1}\|^2_{L^2(\Omega_f)}\le C(T)h^2\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}h^{2k_1}\|u\|^2_{L^\infty(0,T;H^{k_1+1}(\Omega_f))}$$
$$\le C(T)\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}h^{2k_1+2}\|u\|^2_{L^\infty(0,T;H^{k_1+1}(\Omega_f))},$$

$$\Delta t\sum_{n=0}^{m-1}\|u^{n+1}\|^2_{H^2(\Omega_f)}\|\xi_f^n\|^2_{L^2(\Omega_f)}\le C(T)\|u\|_{L^\infty(0,T;H^2(\Omega_f))}h^{2k_1+2}\|u\|^2_{L^\infty(0,T;H^{k_1+1}(\Omega_f))},$$

$$\Delta t^2\sum_{n=0}^{m-1}\|u^{n+1}\|^2_{H^2(\Omega_f)}\int_{t^n}^{t^{n+1}}\|u_t\|^2_{L^2(\Omega_f)}dt\le C\Delta t^2\|u\|_{L^\infty(0,T;H^2(\Omega_f))}\|u_t\|^2_{L^2(0,T;L^2(\Omega_f))}.$$

Thus we complete the proof by considering (14). □

Corollary 3.1. *Under the assumptions of Theorem 3.2, it holds that, for all $1\le m\le N$,*

$$\begin{aligned}\|\underline{w}^m_{1,h}-\underline{w}^m\|_0^2&\le C(\Delta t^2+h^{2k_1+2}+h^{2k_2+2}),\\ \Delta t\sum_{n=1}^{m}\|\underline{w}^m_{1,h}-\underline{w}^m\|^2_W&\le C(\Delta t^2+h^{2k_1}+h^{2k_2}).\end{aligned}\tag{50}$$

Proof. Using the triangle inequalities and combining (13) with Theorem 3.2 will lead to the result. □

Next, we estimate the time difference $\frac{\underline{e}^{n+1}_{1,w}-\underline{e}^n_{1,w}}{\Delta t}$, which will be used in the error analysis of the second order scheme. Denote

$$\begin{aligned}d_t\underline{e}^{n+1}_{1,w}&:=\frac{\underline{e}^{n+1}_{1,w}-\underline{e}^n_{1,w}}{\Delta t}=\frac{\underline{e}^{n+1}_1-\underline{e}^n_1}{\Delta t}-\frac{\underline{\xi}^{n+1}-\underline{\xi}^n}{\Delta t}:=d_t\underline{e}^{n+1}_1-d_t\underline{\xi}^{n+1},\\ d_te^{n+1}_{1,u}&:=\frac{e^{n+1}_{1,u}-e^n_{1,u}}{\Delta t}=\frac{e^{n+1}_{1,f}-e^n_{1,f}}{\Delta t}-\frac{\xi_f^{n+1}-\xi_f^n}{\Delta t}:=d_te^{n+1}_{1,f}-d_t\xi_f^{n+1},\\ d_te^{n+1}_{1,\varphi}&:=\frac{e^{n+1}_{1,\varphi}-e^n_{1,\varphi}}{\Delta t}=\frac{e^{n+1}_{1,p}-e^n_{1,p}}{\Delta t}-\frac{\xi_p^{n+1}-\xi_p^n}{\Delta t}:=d_te^{n+1}_{1,p}-d_t\xi_p^{n+1},\end{aligned}$$

where $d_t\underline{e}^{n+1}_{1,w}=(d_te^{n+1}_{1,u},d_te^{n+1}_{1,\varphi})$, $d_t\underline{e}^{n+1}_1=(d_te^{n+1}_{1,f},d_te^{n+1}_{1,p})$ and $d_t\underline{\xi}^{n+1}=(d_t\xi_f^{n+1},d_t\xi_p^{n+1})$.

Theorem 3.3. *Under the assumptions of Theorem 3.2, we have, for all $1\le m\le N$,*

$$\|d_t\underline{e}^m_1\|_0^2+\Delta t\sum_{n=0}^{m-1}\|d_t\underline{e}^{n+1}_1\|^2_W\le C(\Delta t^2+h^{2k_1}+h^{2k_2}).\tag{51}$$

Proof. From (43) and (44), we get the error equation at time t^{n+1}: for all $\underline{z}_h\in W_h,\ q_h\in Q_h$,

$$\begin{cases}[d_t\underline{e}^{n+1}_1,\underline{z}_h]+a(\underline{e}^{n+1}_1,\underline{z}_h)+N(u^n_{1,h};u^{n+1}_{1,h},v_h)-N(u^{n+1};u^{n+1},v_h)+b(\underline{z}_h,\varepsilon^{n+1}_1)\\ =[\underline{w}^{n+1}_t-\dfrac{\underline{w}^{n+1}-\underline{w}^n}{\Delta t},\underline{z}_h]+[\dfrac{\underline{\xi}^{n+1}-\underline{\xi}^n}{\Delta t},\underline{z}_h],\\ b(\underline{e}^{n+1}_{1,w},q_h)=0.\end{cases}\tag{52}$$

Consider (52) at the previous time t^n to get new equations and subtract it from (52). Then setting $\underline{z}_h=2d_t\underline{e}^{n+1}_1$ and $q_h=\varepsilon^{n+1}_1-\varepsilon^n_1$, we obtain

$$\begin{aligned}&\|d_t\underline{e}^{n+1}_1\|_0^2-\|d_t\underline{e}^n_1\|_0^2+\|d_t\underline{e}^{n+1}_1-d_t\underline{e}^n_1\|_0^2+2\Delta t\|d_t\underline{e}^{n+1}_1\|^2_W+2\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|d_te^{n+1}_{1,f}\cdot\tau_i\|^2_{L^2(\Gamma)}\\ &=2[(\underline{w}^{n+1}_t-\frac{\underline{w}^{n+1}-\underline{w}^n}{\Delta t})-(\underline{w}^n_t-\frac{\underline{w}^n-\underline{w}^{n-1}}{\Delta t}),d_t\underline{e}^{n+1}_1]+2[d_t\underline{\xi}^{n+1}-d_t\underline{\xi}^n,d_t\underline{e}^{n+1}_1]\\ &\quad-2N(u^n_{1,h};u^{n+1}_{1,h},d_te^{n+1}_{1,f})+2N(u^{n+1};u^{n+1},d_te^{n+1}_{1,f})+2N(u^{n-1}_{1,h};u^n_{1,h},d_te^{n+1}_{1,f})-2N(u^n;u^n,d_te^{n+1}_{1,f}).\end{aligned}\tag{53}$$

First, we bound the first two terms on the right hand side of (53). Similarly to (29), we get

$$\begin{aligned} I &= 2[(\underline{w}_t^{n+1} - \frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t}) - (\underline{w}_t^n - \frac{\underline{w}^n - \underline{w}^{n-1}}{\Delta t}), d_t \underline{e}_1^{n+1}] + 2[d_t \underline{\xi}^{n+1} - d_t \underline{\xi}^n, d_t \underline{e}_1^{n+1}] \\ &\le 2\varepsilon \Delta t \|d_t \underline{e}_1^{n+1}\|_W^2 + \left(\frac{C_p^2}{\varepsilon\nu} + \frac{S_0 \tilde{C}_p^2}{\varepsilon k_{\min}}\right)\left(\Delta t^2 \int_{t^n}^{t^{n+1}} \|\underline{w}_{ttt}\|_0^2 dt + \int_{t^n}^{t^{n+1}} \|\underline{\xi}_{tt}\|_0^2 dt\right). \end{aligned} \tag{54}$$

To bound the trilinear terms, we have

$$\begin{aligned} II &= -2N(u_{1,h}^n; u_{1,h}^{n+1}, d_t e_{1,f}^{n+1}) + 2N(u^{n+1}; u^{n+1}, d_t e_{1,f}^{n+1}) + 2N(u_{1,h}^{n-1}; u_{1,h}^n, d_t e_{1,f}^{n+1}) - 2N(u^n; u^n, d_t e_{1,f}^{n+1}) \\ &= -2N(u_{1,h}^n; u_{1,h}^{n+1} - u^{n+1}, d_t e_{1,f}^{n+1}) - 2N(u_{1,h}^n - u^n; u^{n+1}, d_t e_{1,f}^{n+1}) + 2N(u^{n+1} - u^n; u^{n+1}, d_t e_{1,f}^{n+1}) \\ &\quad + 2N(u_{1,h}^{n-1}; u_{1,h}^n - u^n, d_t e_{1,f}^{n+1}) + 2N(u_{1,h}^{n-1} - u^{n-1}; u^n, d_t e_{1,f}^{n+1}) - 2N(u^n - u^{n-1}; u^n, d_t e_{1,f}^{n+1}). \end{aligned} \tag{55}$$

Next, we bound the first and fourth terms on the right hand side of (55). It is easy to verify the following identity:

$$\begin{aligned} i &= -2N(u_{1,h}^n; u_{1,h}^{n+1} - u^{n+1}, d_t e_{1,f}^{n+1}) + 2N(u_{1,h}^{n-1}; u_{1,h}^n - u^n, d_t e_{1,f}^{n+1}) \\ &= -2N(u_{1,h}^n - u_{1,h}^{n-1}; e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}) - 2N(u_{1,h}^{n-1}; e_{1,u}^{n+1} - e_{1,u}^n, d_t e_{1,f}^{n+1}) \\ &= -2\Delta t N(d_t u_{1,h}^n; e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}) - 2\Delta t N(u_{1,h}^{n-1}; d_t e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}). \end{aligned}$$

Using (11), (15), (22) and the facts $d_t e_{1,u}^{n+1} = d_t e_{1,f}^{n+1} - d_t \xi_f^{n+1}$,

$$d_t e_{1,u}^n = \frac{e_{1,u}^n - e_{1,u}^{n-1}}{\Delta t} = \frac{(u_{1,h}^n - u^n) - (u_{1,h}^{n-1} - u^{n-1})}{\Delta t} = \frac{u_{1,h}^n - u_{1,h}^{n-1}}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t} := d_t u_{1,h}^n - d_t u^n,$$

we have

$$\begin{aligned} &- 2\Delta t N(d_t u_{1,h}^n; e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}) = -2\Delta t N(d_t e_{1,f}^n - d_t \xi_f^n + d_t u^n; e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}) \\ &\le 2\Delta t C_{N1}\big(\|\nabla d_t e_{1,f}^n\|_{L^2(\Omega_f)} + \|\nabla d_t \xi_f^n\|_{L^2(\Omega_f)} + \|\nabla d_t u^n\|_{L^2(\Omega_f)}\big)\|\nabla e_{1,u}^{n+1}\|_{L^2(\Omega_f)}\|\nabla d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\ &\le 3\varepsilon\nu\Delta t\|\nabla d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2 C_I^2}{\varepsilon\nu}\Delta t h^{-2}\|\nabla e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2\|d_t e_{1,f}^n\|_{L^2(\Omega_f)}^2 \\ &\quad + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2\big(\|\nabla d_t \xi_f^n\|_{L^2(\Omega_f)}^2 + \|\nabla d_t u^n\|_{L^2(\Omega_f)}^2\big) \\ &\le 3\varepsilon\Delta t\|d_t \underline{e}_1^{n+1}\|_W^2 + \frac{C_{N1}^2 C_I^2}{\varepsilon\nu^2}h^{-2}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\|d_t \underline{e}_1^n\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\big(\|d_t \underline{\xi}^n\|_W^2 + \|d_t u^n\|_{H^1(\Omega_f)}^2\big). \end{aligned}$$

Similar to the estimate of (47), we get

$$\begin{aligned} &- 2\Delta t N(u_{1,h}^{n-1}; d_t e_{1,u}^{n+1}, d_t e_{1,f}^{n+1}) = -2\Delta t N(u_{1,h}^{n-1}; d_t e_{1,f}^{n+1}, d_t e_{1,f}^{n+1}) + 2\Delta t N(u_{1,h}^{n-1}; d_t \xi_f^{n+1}, d_t e_{1,f}^{n+1}) \\ &= - 2\Delta t N(u_{1,h}^{n-1}; d_t e_{1,f}^{n+1}, d_t e_{1,f}^{n+1}) + 2\Delta t N(e_{1,u}^{n-1}; d_t \xi_f^{n+1}, d_t e_{1,f}^{n+1}) + 2\Delta t N(u^{n-1}; d_t \xi_f^{n+1}, d_t e_{1,f}^{n+1}) \\ &\le 2\Delta t C_{N1}\|\nabla u_{1,h}^{n-1}\|_{L^2(\Omega_f)}\|\nabla d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + 2\Delta t C_{N1}\|\nabla e_{1,u}^{n-1}\|_{L^2(\Omega_f)}\|\nabla d_t \xi_f^{n+1}\|_{L^2(\Omega_f)}\|\nabla d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\ &\quad + 2\Delta t C_{N4}\|u^{n-1}\|_{H^2(\Omega_f)}\|\nabla d_t \xi_f^{n+1}\|_{L^2(\Omega_f)}\|d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)} \\ &\le \frac{\nu}{2}\Delta t\|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2 + \varepsilon\Delta t\|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla e_{1,u}^{n-1}\|_{L^2(\Omega_f)}^2\|\nabla d_t \xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\ &\quad + h^{-2}\Delta t\|d_t e_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \Delta t h^2 C_{N4}^2\|u^{n-1}\|_{H^2(\Omega_f)}^2\|\nabla d_t \xi_f^{n+1}\|_{L^2(\Omega_f)}^2 \\ &\le (\frac{1}{2} + \varepsilon)\Delta t\|d_t \underline{e}_1^{n+1}\|_W^2 + h^{-2}\Delta t\|d_t \underline{e}_1^{n+1}\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n-1}\|_W^2\|d_t \underline{\xi}^{n+1}\|_W^2 + \Delta t h^2 C_{N4}^2\|u^{n-1}\|_{H^2(\Omega_f)}^2\|d_t \underline{\xi}^{n+1}\|_W^2. \end{aligned}$$

So

$$\begin{aligned} i &\le (\frac{1}{2} + 4\varepsilon)\Delta t\|d_t \underline{e}_1^{n+1}\|_W^2 + h^{-2}\Delta t\|d_t \underline{e}_1^{n+1}\|_0^2 + \frac{C_{N1}^2 C_I^2}{\varepsilon\nu^2}h^{-2}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\|d_t \underline{e}_1^n\|_0^2 \\ &\quad + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\big(\|d_t \underline{\xi}^n\|_W^2 + \|d_t u^n\|_{H^1(\Omega_f)}^2\big) \\ &\quad + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n-1}\|_W^2\|d_t \underline{\xi}^{n+1}\|_W^2 + \Delta t h^2 C_{N4}^2\|u^{n-1}\|_{H^2(\Omega_f)}^2\|d_t \underline{\xi}^{n+1}\|_W^2. \end{aligned} \tag{56}$$

Then we consider the second and fifth terms on the right hand side of (55),

$$
\begin{aligned}
ii &= -2N(u_{1,h}^{n}-u^{n};u^{n+1},d_te_{1,f}^{n+1})+2N(u_{1,h}^{n-1}-u^{n-1};u^{n},d_te_{1,f}^{n+1})\\
&= -2N(e_{1,u}^{n}-e_{1,u}^{n-1};u^{n+1},d_te_{1,f}^{n+1})-2N(e_{1,u}^{n-1};u^{n+1}-u^{n},d_te_{1,f}^{n+1})\\
&= -2\Delta tN(d_te_{1,u}^{n};u^{n+1},d_te_{1,f}^{n+1})-2\Delta tN(e_{1,u}^{n-1};\frac{u^{n+1}-u^{n}}{\Delta t},d_te_{1,f}^{n+1}).
\end{aligned}
$$

Using (14)–(16), we have

$$
\begin{aligned}
ii &\le 2\Delta tC_{N2}\big(\|d_te_{1,f}^{n}\|_{L^2(\Omega_f)}+\|d_t\xi_f^{n}\|_{L^2(\Omega_f)}\big)\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}\\
&\quad +2\Delta tC_{N1}\|\nabla e_{1,u}^{n-1}\|_{L^2(\Omega_f)}\|d_tu^{n+1}\|_{H^1(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}\\
&\le 3\varepsilon\Delta t\|d_t\underline{e}_1^{n+1}\|_W^2+\frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|d_t\underline{e}_1^{n}\|_0^2+\frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|_{H^2(\Omega_f)}^2\|d_t\underline{\xi}^{n}\|_0^2+\frac{C_{N1}^2}{\varepsilon\nu^2}\Delta t\|d_tu^{n+1}\|_{H^1(\Omega_f)}^2\|\underline{e}_{1,w}^{n-1}\|_W^2.
\end{aligned}
\tag{57}
$$

Then, we consider the third and sixth terms on the right hand side of (56),

$$
\begin{aligned}
iii &= 2N(u^{n+1}-u^{n};u^{n+1},d_te_{1,f}^{n+1})-2N(u^{n}-u^{n-1};u^{n},d_te_{1,f}^{n+1})\\
&= 2\Delta tN(\frac{u^{n+1}-u^{n}}{\Delta t}-u_t^{n+1};u^{n+1},d_te_{1,f}^{n+1})-2\Delta tN(\frac{u^{n}-u^{n-1}}{\Delta t}-u_t^{n};u^{n},d_te_{1,f}^{n+1})\\
&\quad +2\Delta tN(u_t^{n+1};u^{n+1},d_te_{1,f}^{n+1})-2\Delta tN(u_t^{n};u^{n},d_te_{1,f}^{n+1}).
\end{aligned}
$$

Using (14)–(16), we have

$$
\begin{aligned}
&2\Delta tN(\frac{u^{n+1}-u^{n}}{\Delta t}-u_t^{n+1};u^{n+1},d_te_{1,f}^{n+1})-2\Delta tN(\frac{u^{n}-u^{n-1}}{\Delta t}-u_t^{n};u^{n},d_te_{1,f}^{n+1})\\
&\le 2\Delta tC_{N2}\|\frac{u^{n+1}-u^{n}}{\Delta t}-u_t^{n+1}\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}\\
&\quad +2\Delta tC_{N2}\|\frac{u^{n}-u^{n-1}}{\Delta t}-u_t^{n}\|_{L^2(\Omega_f)}\|u^{n}\|_{H^2(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}\\
&\le 2\varepsilon\nu\Delta t\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2+\frac{C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u\|_{L^\infty(0,T;H^2(\Omega_f))}^2\Big(\int_{t^n}^{t^{n+1}}\|u_{tt}\|_{L^2(\Omega_f)}^2dt+\int_{t^{n-1}}^{t^n}\|u_{tt}\|_{L^2(\Omega_f)}^2dt\Big),\\
&2\Delta tN(u_t^{n+1};u^{n+1},d_te_{1,f}^{n+1})-2\Delta tN(u_t^{n};u^{n},d_te_{1,f}^{n+1})\\
&=2\Delta tN(u_t^{n+1}-u_t^{n};u^{n+1},d_te_{1,f}^{n+1})+2\Delta tN(u_t^{n};u^{n+1}-u^{n},d_te_{1,f}^{n+1})\\
&\le 2\Delta tC_{N2}\|u_t^{n+1}-u_t^{n}\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}+2\Delta tC_{N1}\|\nabla u_t^{n}\|_{L^2(\Omega_f)}\|\nabla(u^{n+1}-u^{n})\|_{L^2(\Omega_f)}\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}\\
&\le 4\varepsilon\nu\Delta t\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2+\frac{C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u^{n+1}\|_{H^2(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_{tt}\|_{L^2(\Omega_f)}^2dt+\frac{C_{N1}^2}{\varepsilon\nu}\Delta t^2\|u_t^{n}\|_{H^1(\Omega_f)}^2\int_{t^n}^{t^{n+1}}\|u_t\|_{H^1(\Omega_f)}^2dt\\
&\le 4\varepsilon\nu\Delta t\|\nabla d_te_{1,f}^{n+1}\|_{L^2(\Omega_f)}^2+\frac{C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u\|_{L^\infty(0,T;H^2(\Omega_f))}^2\int_{t^n}^{t^{n+1}}\|u_{tt}\|_{L^2(\Omega_f)}^2dt+\frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\|u_t^{n}\|_{L^\infty(0,T;H^1(\Omega_f))}^4.
\end{aligned}
$$

So that

$$
\begin{aligned}
iii &\le 4\varepsilon\Delta t\|d_t\underline{e}_1^{n+1}\|_W^2+\frac{2C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u\|_{L^\infty(0,T;H^2(\Omega_f))}^2\Big(\int_{t^n}^{t^{n+1}}\|u_{tt}\|_{L^2(\Omega_f)}^2dt+\int_{t^{n-1}}^{t^n}\|u_{tt}\|_{L^2(\Omega_f)}^2dt\Big)\\
&\quad +\frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\|u_t\|_{L^\infty(0,T;H^1(\Omega_f))}^4.
\end{aligned}
\tag{58}
$$

Hence, we combine (56)–(58) with (55) leading to

$$
\begin{aligned}
II \le & (\frac{1}{2}+11\varepsilon)\Delta t\|d_t\underline{e}_1^{n+1}\|_W^2 + h^{-2}\Delta t\|d_t\underline{e}_1^{n+1}\|_0^2 + \big(\frac{C_{N1}^2C_I^2}{\varepsilon\nu^2}h^{-2}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|^2_{H^2(\Omega_f)}\big)\|d_t\underline{e}_1^n\|_0^2 \\
& + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\big(\|d_t\underline{\xi}^n\|_W^2 + \|d_tu^n\|^2_{H^1(\Omega_f)}\big) + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n-1}\|_W^2\|d_t\underline{\xi}^{n+1}\|_W^2 + \Delta t h^2C_{N4}^2\|u^{n-1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^{n+1}\|_W^2. \\
& + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^n\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu^2}\Delta t\|d_tu^{n+1}\|^2_{H^1(\Omega_f)}\|\underline{e}_{1,w}^{n-1}\|_W^2 \\
& + \frac{2C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}\big(\int_{t^n}^{t^{n+1}}\|u_{tt}\|^2_{L^2(\Omega_f)}dt + \int_{t^{n-1}}^{t^n}\|u_{tt}\|^2_{L^2(\Omega_f)}dt\big) + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\|u_t\|^4_{L^\infty(0,T;H^1(\Omega_f))}.
\end{aligned} \tag{59}
$$

Finally, combining (54) and (59) with (53) and setting $\varepsilon = 1/26$, we obtain

$$
\begin{aligned}
& \|d_t\underline{e}_1^{n+1}\|_0^2 - \|d_t\underline{e}_1^n\|_0^2 + \|d_t\underline{e}_1^{n+1} - d_t\underline{e}_1^n\|_0^2 + \Delta t\|d_t\underline{e}_1^{n+1}\|_W^2 \\
& \le h^{-2}\Delta t\|d_t\underline{e}_1^{n+1}\|_0^2 + \big(\frac{26C_{N1}^2C_I^2}{\nu^2}\Delta t h^{-2}\|\underline{e}_{1,w}^{n+1}\|_W^2 + \frac{26C_{N2}^2}{\nu}\Delta t\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}\big)\|d_t\underline{e}_1^n\|_0^2 + \left(\frac{26C_p^2}{\nu} + \frac{26S_0\tilde{C}_p^2}{k_{\min}}\right) \\
& \left(\Delta t^2\int_{t^n}^{t^{n+1}}\|\underline{w}_{ttt}(\tau)\|_0^2d\tau + \int_{t^n}^{t^{n+1}}\|\underline{\xi}_{tt}(\tau)\|_0^2d\tau\right) + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n+1}\|_W^2\big(\|d_t\underline{\xi}^n\|_W^2 + \|d_tu^n\|^2_{H^1(\Omega_f)}\big) \\
& + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\|\underline{e}_{1,w}^{n-1}\|_W^2\|d_t\underline{\xi}^{n+1}\|_W^2 \\
& + \Delta t h^2C_{N4}^2\|u^{n-1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^{n+1}\|_W^2 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^{n+1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^n\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu^2}\Delta t\|d_tu^{n+1}\|^2_{H^1(\Omega_f)}\|\underline{e}_{1,w}^{n-1}\|_W^2 \\
& + \frac{2C_{N2}^2}{\varepsilon\nu}\Delta t^2\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}\big(\int_{t^n}^{t^{n+1}}\|u_{tt}\|^2_{L^2(\Omega_f)}dt + \int_{t^{n-1}}^{t^n}\|u_{tt}\|^2_{L^2(\Omega_f)}dt\big) + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\|u_t\|^4_{L^\infty(0,T;H^1(\Omega_f))}.
\end{aligned}
$$

Furthermore, since $d_t\underline{e}_1^n = \frac{\underline{e}_1^n - \underline{e}_1^{n-1}}{\Delta t}$ is not defined for $n = 0$, we can only sum the above equation from $n = 1$ to $n = m-1$. Considering (40) and (50), we have

$$
\kappa_n\Delta t := h^{-2}\Delta t + \frac{26C_{N1}^2C_I^2}{\nu^2}\Delta t h^{-2}\|\underline{e}_{1,w}^{n+1}\|_W^2 + \frac{26C_{N2}^2}{\nu}\Delta t\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))} < 1.
$$

Using (20), we have

$$
\begin{aligned}
& \|d_t\underline{e}_1^m\|_0^2 + \Delta t\sum_{n=1}^{m-1}\|d_t\underline{e}_1^{n+1}\|_W^2 \le \exp\big(\Delta t\sum_{n=1}^{m-1}\frac{\kappa_n}{1-\kappa_n\Delta t}\big)\Bigg(\|d_t\underline{e}_1^1\|_0^2 \\
& + C\Delta t^2\sum_{n=1}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{w}_{ttt}(\tau)\|_0^2d\tau + C\sum_{n=1}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{\xi}_{tt}(\tau)\|_0^2d\tau \\
& + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\sum_{n=1}^{m-1}\|\underline{e}_{1,w}^{n+1}\|_W^2\big(\|d_t\underline{\xi}^n\|_W^2 + \|d_tu^n\|^2_{H^1(\Omega_f)}\big) + \frac{C_{N1}^2}{\varepsilon\nu^3}\Delta t\sum_{n=1}^{m-1}\|\underline{e}_{1,w}^{n-1}\|_W^2\|d_t\underline{\xi}^{n+1}\|_W^2 \\
& + \Delta t\sum_{n=1}^{m-1}h^2C_{N41}^2\|u^{n-1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^{n+1}\|_W^2 \\
& + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\sum_{n=1}^{m-1}\|u^{n+1}\|^2_{H^2(\Omega_f)}\|d_t\underline{\xi}^n\|_0^2 + \frac{C_{N1}^2}{\varepsilon\nu^2}\Delta t\sum_{n=1}^{m-1}\|d_tu^{n+1}\|^2_{H^1(\Omega_f)}\|\underline{e}_{1,w}^{n-1}\|_W^2 \\
& + \frac{2C_{N2}^2}{\varepsilon\nu}\Delta t^2\sum_{n=1}^{m-1}\|u\|^2_{L^\infty(0,T;H^2(\Omega_f))}\big(\int_{t^n}^{t^{n+1}}\|u_{tt}\|^2_{L^2(\Omega_f)}dt + \int_{t^{n-1}}^{t^n}\|u_{tt}\|^2_{L^2(\Omega_f)}dt\big) + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\sum_{n=1}^{m-1}\|u_t\|^4_{L^\infty(0,T;H^1(\Omega_f))}\Bigg).
\end{aligned}
$$

Then using (20) and considering (13), (14) and (50), we can obtain

$$
\|d_t\underline{e}_1^m\|_0^2 + \Delta t\sum_{n=1}^{m-1}\|d_t\underline{e}_1^{n+1}\|_W^2 \le C\big(\|d_t\underline{e}_1^1\|_0^2 + \Delta t^2 + h^{2k_1} + h^{2k_2}\big). \tag{60}
$$

Next we need to bound the terms $\|d_t\underline{e}_1^1\|_0^2$ and $\Delta t\|d_t\underline{e}_1^1\|_W^2$. Note that $\underline{w}_{1,h}^0 = \tilde{\underline{w}}^0 = \underline{w}^0 = 0$, then we can get $\underline{e}_1^0 = e_{1,f}^0 = e_{1,p}^0 = 0$. At time t^1, setting $\underline{z}_h = 2d_t\underline{e}_1^1$, $q_h = 2\varepsilon_1^1$ in (52) and considering (44), we have

$$2\|d_t\underline{e}_1^1\|_0^2 + 2\Delta t\|d_t\underline{e}_1^1\|_W^2 \le 2[\underline{w}_t^1 - \frac{\underline{w}^1 - \underline{w}^0}{\Delta t}, d_t\underline{e}_1^1] + 2[d_t\underline{\xi}^1, d_t\underline{e}_1^1] + 2N(u^1; u^1, d_te_{1,f}^1). \tag{61}$$

Using (13), (14) and (17), for all $\mu > 0$, we get

$$\begin{aligned}
&2[\underline{w}_t^1 - \frac{\underline{w}^1 - \underline{w}^0}{\Delta t}, d_t\underline{e}_1^1] + 2[d_t\underline{\xi}^1, d_t\underline{e}_1^1] \le 2\mu\|d_t\underline{e}_1^1\|_0^2 + \frac{1}{\mu}\Delta t\int_{t^0}^{t^1}\|\underline{w}_{tt}\|_0^2dt + \frac{1}{\mu}\Delta t\int_{t^0}^{t^1}\|(\underline{\xi})_t\|_0^2dt \\
&\le 2\mu\|d_t\underline{e}_1^1\|_0^2 + C(h^{2k_1+2} + h^{2k_2+2} + \Delta t^2). \\
&2N(u^1; u^1, d_te_{1,f}^1) = 2\Delta tN(\frac{u^1 - u^0}{\Delta t}; u^1, d_te_{1,f}^1) \le 2\Delta tC_{N3}\|\nabla\frac{u^1 - u^0}{\Delta t}\|_{L^2(\Omega_f)}\|u^1\|_{H^2(\Omega_f)}\|d_te_{1,f}^1\|_{L^2(\Omega_f)} \\
&\le \mu\|d_t\underline{e}_1^1\|_0^2 + \frac{C_{N3}^2}{\mu}\Delta t\|u^1\|_{H^2(\Omega_f)}^2\int_{t^0}^{t^1}\|u_t(\tau)\|_{H^1(\Omega_f)}^2d\tau \le \mu\|d_t\underline{e}_1^1\|_0^2 + C\Delta t^2.
\end{aligned}$$

Combining the above estimates with (61) and setting $\mu = 1/3$, we have

$$\|d_t\underline{e}_1^1\|_0^2 + \Delta t\|d_t\underline{e}_1^1\|_W^2 \le C(\Delta t^2 + h^{2k_1+2} + h^{2k_2+2}). \tag{62}$$

Hence combining (62) with (60), we get

$$\|d_t\underline{e}_1^m\|_0^2 + \Delta t\sum_{n=1}^{m}\|d_t\underline{e}_1^n\|_W^2 \le C\big(\Delta t^2 + h^{2k_1} + h^{2k_2}\big). \tag{63}$$

Then the proof is completed. □

Corollary 3.2. *Under the assumptions of Theorem 3.3, it holds that, for $m = 1, \ldots, N$,*

$$\|d_t\underline{e}_{1,w}^m\|_0^2 + \Delta t\sum_{n=1}^{m}\|d_t\underline{e}_{1,w}^n\|_W^2 \le C(\Delta t^2 + h^{2k_1} + h^{2k_2}). \tag{64}$$

Proof. Using triangle inequalities and combining the approximation properties (13) with (51) will lead to (64). □

4. A second order algorithm for the NS-Darcy model

In this section, we present a second order scheme based on the SISDC method. We first prove the stability of the scheme. Then we establish the error estimation of our scheme, which shows that it is second order accurate.

Based on the scheme (21), we now present a second order scheme by using SISDC discretization method in time. Given $(\underline{w}_{1,h}^n, p_{1,h}^n) = (u_{1,h}^n, \varphi_{1,h}^n, p_{1,h}^n)$ and $(\underline{w}_{1,h}^{n+1}, p_{1,h}^{n+1}) = (u_{1,h}^{n+1}, \varphi_{1,h}^{n+1}, p_{1,h}^{n+1})$ of the algorithm (21), find $(\underline{w}_{2,h}^{n+1}, p_{2,h}^{n+1}) = (u_{2,h}^{n+1}, \varphi_{2,h}^{n+1}, p_{2,h}^{n+1}) \in W_h \times Q_h$ such that, for all $\underline{z}_h \in W_h$, $q_h \in Q_h$,

$$\begin{cases}
[\dfrac{\underline{w}_{2,h}^{n+1} - \underline{w}_{2,h}^n}{\Delta t}, \underline{z}_h] + a(\underline{w}_{2,h}^{n+1}, \underline{z}_h) + N(u_{2,h}^n; u_{2,h}^n, v_h) + b(\underline{z}_h, p_{2,h}^{n+1}) \\
= \dfrac{1}{2}a(\underline{w}_{1,h}^{n+1}, \underline{z}_h) - \dfrac{1}{2}a(\underline{w}_{1,h}^n, \underline{z}_h) - \dfrac{1}{2}N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, v_h) \\
\quad + \dfrac{1}{2}N(u_{1,h}^n; u_{1,h}^n, v_h) + \dfrac{1}{2}b(\underline{z}_h, p_{1,h}^{n+1} - p_{1,h}^n) + (\dfrac{f^{n+1} + f^n}{2}, \underline{z}_h), \\
b(\underline{w}_{2,h}^{n+1}, q_h) = 0, \\
\underline{w}_{2,h}^0 = P_h\underline{w}^0 = 0.
\end{cases} \tag{65}$$

4.1. Stability analysis for the second order algorithm

Similar to the stability analysis in Section 3.1. We also need an extra condition to overcome the difficulty caused by the trilinear term in Navier–Stokes equations.

Lemma 4.1. *Let $\underline{w}_{2,h}^m = (u_{2,h}^m, \varphi_{2,h}^m)$, $0 \le m \le N$, be the solution of (65). Then under the assumptions of Theorem 3.3 and*

$$\|\nabla u_{2,h}^n\|_{L^2(\Omega_f)} \le \frac{\nu}{4C_{N1}}, \quad 0 \le n \le m - 1, \tag{66}$$

we have

$$\|\underline{w}_{2,h}^m\|_0^2 + \Delta t\sum_{n=0}^{m-1}\|\underline{w}_{2,h}^{n+1}\|_W^2 \le C\Big(\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + \mathcal{H}^2 + \|u_t\|^2_{L^2(0,T;H^1(\Omega_f))}\Upsilon^2 + \Upsilon^2\Big). \tag{67}$$

Proof. Setting $\underline{z}_h = 2\Delta t\underline{w}_{2,h}^{n+1}$ and $q_h = p_{2,h}^{n+1}$ in (65), we have

$$\begin{aligned}
&\big(\|\underline{w}_{2,h}^{n+1}\|_0^2 - \|\underline{w}_{2,h}^{n}\|_0^2 + \|\underline{w}_{2,h}^{n+1} - \underline{w}_{2,h}^{n}\|_0^2\big) + 2\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2 + 2\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|^2_{L^2(\Gamma)}\\
&= \Delta t a(\underline{w}_{1,h}^{n+1}, \underline{w}_{2,h}^{n+1}) - \Delta t a(\underline{w}_{1,h}^{n}, \underline{w}_{2,h}^{n+1}) - 2\Delta t N(u_{2,h}^n; u_{2,h}^n, u_{2,h}^{n+1})\\
&- \Delta t N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, u_{2,h}^{n+1}) + \Delta t N(u_{1,h}^{n}; u_{1,h}^{n}, u_{2,h}^{n+1}) + 2\Delta t(\frac{f^{n+1}+f^n}{2}, \underline{w}_{2,h}^{n+1}).
\end{aligned} \tag{68}$$

First, using (9), (19), Young's and Holder's inequalities, we get

$$\begin{aligned}
I &= \Delta t a(\underline{w}_{1,h}^{n+1}, \underline{w}_{2,h}^{n+1}) - \Delta t a(\underline{w}_{1,h}^{n}, \underline{w}_{2,h}^{n+1}) = \Delta t^2 a(d_t\underline{w}_{1,h}^{n+1}, \underline{w}_{2,h}^{n+1})\\
&\le \Delta t^2\|d_t w_{1,h}^{n+1}\|_W\|w_{2,h}^{n+1}\|_W + \Delta t^2 C_p^{\frac{1}{2}}C_t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|\nabla d_t u_{1,h}^{n+1}\|_{L^2(\Omega_f)}\|u_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}\\
&\quad + \varepsilon\Delta t\|w_{2,h}^{n+1}\|_W^2 + \frac{C_\Gamma}{4\varepsilon}\Delta t^3\|d_t w_{1,h}^{n+1}\|_W^2\\
&\le 2\varepsilon\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2 + \Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|^2_{L^2(\Gamma)} + \frac{1+C_\Gamma}{4\varepsilon}\Delta t^3\|d_t\underline{w}_{1,h}^{n+1}\|_W^2\\
&\quad + \frac{C_pC_t^2}{4}\Delta t^3\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|\nabla d_t u_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}\\
&\le 2\varepsilon\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2 + \Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|^2_{L^2(\Gamma)} + C\Delta t^3\|d_t\underline{w}_{1,h}^{n+1}\|_W^2.
\end{aligned} \tag{69}$$

Using (15) and (66), we get

$$\begin{aligned}
II &= -2\Delta t N(u_{2,h}^n; u_{2,h}^n, u_{2,h}^{n+1}) = 2\Delta t N(u_{2,h}^n; u_{2,h}^{n+1} - u_{2,h}^n, u_{2,h}^{n+1}) - 2\Delta t N(u_{2,h}^n; u_{2,h}^{n+1}, u_{2,h}^{n+1})\\
&= 2\Delta t^2 N(u_{2,h}^n; d_t u_{2,h}^{n+1}, u_{2,h}^{n+1}) - 2\Delta t N(u_{2,h}^n; u_{2,h}^{n+1}, u_{2,h}^{n+1})\\
&\le 2C_{N1}\Delta t^2\|\nabla u_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla d_t u_{2,h}^{n+1}\|_{L^2(\Omega_f)}\|\nabla u_{2,h}^{n+1}\|_{L^2(\Omega_f)} + 2C_{N1}\Delta t\|\nabla u_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla u_{2,h}^{n+1}\|^2_{L^2(\Omega_f)}\\
&\le \frac{\nu}{2}\Delta t^2\|\nabla d_t u_{2,h}^{n+1}\|_{L^2(\Omega_f)}\|\nabla u_{2,h}^{n+1}\|_{L^2(\Omega_f)} + \frac{\nu}{2}\Delta t\|\nabla u_{2,h}^{n+1}\|^2_{L^2(\Omega_f)}\\
&\le (\frac{\nu}{2} + \varepsilon\nu)\Delta t\|\nabla u_{2,h}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{\nu}{16\varepsilon}\Delta t^3\|\nabla d_t u_{2,h}^{n+1}\|^2_{L^2(\Omega_f)} \le (\frac{1}{2}+\varepsilon)\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2 + \frac{1}{16\varepsilon}\Delta t^3\|d_t\underline{w}_{2,h}^{n+1}\|_W^2.
\end{aligned}$$

For the remaining trilinear terms, we have

$$\begin{aligned}
III &= -\Delta t N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, u_{2,h}^{n+1}) + \Delta t N(u_{1,h}^{n}; u_{1,h}^{n}, u_{2,h}^{n+1})\\
&= -\Delta t N(u_{1,h}^{n+1} - u_{1,h}^n; u_{1,h}^{n+1}, u_{2,h}^{n+1}) - \Delta t N(u_{1,h}^{n}; u_{1,h}^{n+1} - u_{1,h}^n, u_{2,h}^{n+1}).
\end{aligned} \tag{70}$$

By using (13), (15) and the fact that $d_t e_{1,u}^{n+1} = \frac{e_{1,u}^{n+1} - e_{1,u}^n}{\Delta t} = \frac{u_{1,h}^{n+1} - u_{1,h}^n}{\Delta t} - \frac{u^{n+1} - u^n}{\Delta t}$, we obtain

$$\begin{aligned}
&- \Delta t N(u_{1,h}^{n+1} - u_{1,h}^n; u_{1,h}^{n+1}, u_{2,h}^{n+1}) = -\Delta t^2 N(\frac{u^{n+1} - u^n}{\Delta t} + d_t e_{1,f}^{n+1} - d_t\xi_f^{n+1}; u_{1,h}^{n+1}, u_{2,h}^{n+1})\\
&\le 3\varepsilon\nu\Delta t\|\nabla u_{2,h}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t^2\|\nabla u_{1,h}^{n+1}\|^2_{L^2(\Omega_f)}\Big(\int_{t^n}^{t^{n+1}}\|u_t\|^2_{H^1(\Omega_f)}d\tau + \Delta t\|\nabla d_t\underline{e}_{1,f}^{n+1}\|^2_{L^2(\Omega_f)} + \int_{t^n}^{t^{n+1}}\|(\xi_f)_t\|^2_{H^1(\Omega_f)}dt\Big)\\
&\le 3\varepsilon\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2 + \frac{C_{N1}^2}{\varepsilon\nu^2}\Delta t\|\underline{w}_{1,h}^{n+1}\|_W^2\Big(\int_{t^n}^{t^{n+1}}\|u_t\|^2_{H^1(\Omega_f)}d\tau + \Delta t\|\nabla d_t\underline{e}_{1,f}^{n+1}\|^2_{L^2(\Omega_f)} + \int_{t^n}^{t^{n+1}}\|(\xi_f)_t\|^2_{H^1(\Omega_f)}dt\Big).
\end{aligned}$$

Similar to bound $\Delta t N(u^n_{1,h}; u^{n+1}_{1,h} - u^n_{1,h}, u^{n+1}_{2,h})$. Then using (13), (14) and (51), we obtain

$$
\begin{aligned}
III \le\ & 6\varepsilon\Delta t\|\underline{w}^{n+1}_{2,h}\|^2_W + \frac{C^2_{N1}}{\varepsilon\nu^2}\Delta t\big(\|\underline{w}^{n+1}_{1,h}\|^2_W + \|\underline{w}^n_{1,h}\|^2_W\big)\Bigg(\int_{t^n}^{t^{n+1}}\|u_t\|^2_{H^1(\Omega_f)}d\tau \\
& + \Delta t\|\nabla d_t\underline{e}^{n+1}_{1,f}\|^2_{L^2(\Omega_f)} + \int_{t^n}^{t^{n+1}}\|(\xi_f)_t\|^2_{H^1(\Omega_f)}d\tau\Bigg) \\
\le\ & 6\varepsilon\Delta t\|\underline{w}^{n+1}_{2,h}\|^2_W + \frac{C^2_{N1}}{\varepsilon\nu^2}\Delta t\big(\|\underline{w}^{n+1}_{1,h}\|^2_W + \|\underline{w}^n_{1,h}\|^2_W\big)\Bigg(\|u_t\|^2_{L^2(0,T;H^1(\Omega_f))} \\
& + C(\Delta t^2 + h^{2k_1} + h^{2k_2}) + h^{2k_1}\|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))}\Bigg) \\
\le\ & 6\varepsilon\Delta t\|\underline{w}^{n+1}_{2,h}\|^2_W + \frac{C}{\varepsilon\nu^2}\big(1 + \|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))}\big)\Delta t\big(\|\underline{w}^{n+1}_{1,h}\|^2_W + \|\underline{w}^n_{1,h}\|^2_W\big).
\end{aligned}
$$

Similarly to (25), we deduce

$$
IV = 2\Delta t\Big(\frac{f^{n+1}+f^n}{2}, \underline{w}^{n+1}_{2,h}\Big) \le \varepsilon\Delta t\|\underline{w}^{n+1}_{2,h}\|^2_W + \frac{C^2_p\Delta t}{\varepsilon\nu}\Big\|\frac{f^{n+1}_1+f^n_1}{2}\Big\|^2_{L^2(\Omega_f)} + \frac{g\tilde{C}^2_p\Delta t}{\varepsilon k_{min}}\Big\|\frac{f^{n+1}_2+f^n_2}{2}\Big\|^2_{L^2(\Omega_p)}.
$$

Combine the above estimates with (68) and set $\varepsilon = 1/20$.

$$
\begin{aligned}
&\|\underline{w}^{n+1}_{2,h}\|^2_0 - \|\underline{w}^n_{2,h}\|^2_0 + \Delta t\|\underline{w}^{n+1}_{2,h}\|^2_W + \Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u^{n+1}_{2,h}\cdot\tau_i\|^2_{L^2(\Gamma)} \le \frac{5}{4}\Delta t^3\|d_t\underline{w}^{n+1}_{2,h}\|^2_W + C\Delta t^3\|d_t\underline{w}^{n+1}_{1,h}\|^2_W \\
&+ \frac{C}{\varepsilon\nu^2}\big(1 + \|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))}\big)\Delta t\big(\|\underline{w}^{n+1}_{1,h}\|^2_W + \|\underline{w}^n_{1,h}\|^2_W\big) + \frac{20C^2_p\Delta t}{\nu}\Big\|\frac{f^{n+1}_1+f^n_1}{2}\Big\|^2_{L^2(\Omega_f)} + \frac{20g\tilde{C}^2_p\Delta t}{k_{min}}\Big\|\frac{f^{n+1}_2+f^n_2}{2}\Big\|^2_{L^2(\Omega_p)}.
\end{aligned}
$$

Note that $\underline{w}^0_{2,h} = 0$. Then summing the above equation from $n = 0$ to $m-1$, we obtain

$$
\begin{aligned}
&\|\underline{w}^m_{2,h}\|^2_0 + \Delta t\sum_{n=0}^{m-1}\|\underline{w}^{n+1}_{2,h}\|^2_W \le \frac{5}{4}\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{w}^{n+1}_{2,h}\|^2_W + C\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{w}^{n+1}_{1,h}\|^2_W \\
&+ C\big(1 + \|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))}\big)\Delta t\sum_{n=0}^{m}\|\underline{w}^n_{1,h}\|^2_W + \frac{10C^2_p\Delta t}{\nu}\sum_{n=0}^{m}\|f^n_1\|^2_{L^2(\Omega_f)} + \frac{10g\tilde{C}^2_p\Delta t}{k_{min}}\sum_{n=0}^{m}\|f^n_2\|^2_{L^2(\Omega_p)}.
\end{aligned}
$$

Considering (13), (14), (23) and (31), we have

$$
\begin{aligned}
\|\underline{w}^m_{2,h}\|^2_0 + \Delta t\sum_{n=0}^{m-1}\|\underline{w}^{n+1}_{2,h}\|^2_W &\le \frac{5}{4}\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{w}^{n+1}_{2,h}\|^2_W + C\Delta t\mathcal{H}^2 + C(1 + \|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))})\Upsilon^2 + 5\Upsilon^2 \\
&\le C\Bigg(\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{w}^{n+1}_{2,h}\|^2_W + \mathcal{H}^2 + \|u_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f))}\Upsilon^2 + \Upsilon^2\Bigg).
\end{aligned}
\tag{71}
$$

So we complete the proof. □

Lemma 4.2. *Under the assumptions of Lemma 4.1, we have, for all* $1 \le m \le N$,

$$
\begin{aligned}
\|\underline{w}^{n+1}_{2,h}\|^2_W \le\ & \Big(\frac{4C^2_p}{\nu} + \frac{4gS_0\tilde{C}^2_p}{k_{\min}}\Big)\|d_t\underline{w}^{n+1}_{2,h}\|^2_0 + \frac{5}{2}\Delta t^2\|d_t\underline{w}^{n+1}_{2,h}\|^2_W + C\big(\mathcal{H}^2 + \Upsilon^2\big) \\
& + \frac{20C^2_p}{\nu}\big(\|f^{n+1}_1\|^2_{L^2(\Omega_f)} + \|f^n_1\|^2_{L^2(\Omega_f)}\big) + \frac{20g\tilde{C}^2_p}{k_{min}}\big(\|f^{n+1}_2\|^2_{L^2(\Omega_p)} + \|f^n_2\|^2_{L^2(\Omega_o)}\big).
\end{aligned}
\tag{72}
$$

Proof. Setting $\underline{z}_h = 2\underline{w}^{n+1}_{2,h}$ and $q_h = p^{n+1}_{2,h}$ in (65), we get

$$
\begin{aligned}
&2\|\underline{w}^{n+1}_{2,h}\|^2_W + 2\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u^{n+1}_{2,h}\cdot\tau_i\|^2_{L^2(\Gamma)} \le -2[d_t\underline{w}^{n+1}_{2,h}, \underline{w}^{n+1}_{2,h}] + a(\underline{w}^{n+1}_{1,h} - \underline{w}^n_{1,h}, \underline{w}^{n+1}_{2,h}) \\
&- 2N(u^n_{2,h}; u^n_{2,h}, u^{n+1}_{2,h}) - N(u^{n+1}_{1,h}; u^{n+1}_{1,h}, u^{n+1}_{2,h}) + N(u^n_{1,h}; u^n_{1,h}, u^{n+1}_{2,h}) + 2\Big(\frac{f^{n+1}+f^n}{2}, \underline{w}^{n+1}_{2,h}\Big).
\end{aligned}
\tag{73}
$$

Similarly to (29), we have

$$-2[d_t\underline{w}_{2,h}^{n+1}, \underline{w}_{2,h}^{n+1}] \le \frac{1}{2}\|\underline{w}_{2,h}^{n+1}\|_W^2 + (\frac{2C_p^2}{\nu} + \frac{2gS_0\tilde{C}_p^2}{k_{\min}})\|d_t\underline{w}_{2,h}^{n+1}\|_0^2.$$

The remaining terms on the right hand side of (73) have been bounded in Lemma 4.1. So we deduce that

$$\begin{aligned}&-2N(u_{2,h}^n; u_{2,h}^n, u_{2,h}^{n+1}) + a(\underline{w}_{1,h}^{n+1} - \underline{w}_{1,h}^n, \underline{w}_{2,h}^{n+1}) - N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, u_{2,h}^{n+1}) + N(u_{1,h}^n; u_{1,h}^n, u_{2,h}^{n+1}) + 2(\frac{f^{n+1}+f^n}{2}, \underline{w}_{2,h}^{n+1})\\
&\le (\frac{1}{2} + 10\varepsilon)\|\underline{w}_{2,h}^{n+1}\|_W^2 + \sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2 + \frac{1}{16\varepsilon}\Delta t^2\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + C\Delta t^2\|d_t\underline{w}_{1,h}^{n+1}\|_W^2\\
&+ \frac{C}{\varepsilon\nu^2}\big(1 + \|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\big(\|\underline{w}_{1,h}^{n+1}\|_W^2 + \|\underline{w}_{1,h}^n\|_W^2\big) + \frac{C_p^2}{\varepsilon\nu}\|\frac{f_1^{n+1}+f_1^n}{2}\|_{L^2(\Omega_f)}^2 + \frac{g\tilde{C}_p^2}{\varepsilon k_{min}}\|\frac{f_2^{n+1}+f_2^n}{2}\|_{L^2(\Omega_p)}^2.\end{aligned}$$

Combining these two estimates with (73) and taking $\varepsilon = 1/20$, we have

$$\begin{aligned}&\frac{1}{2}\|\underline{w}_{2,h}^{n+1}\|_W^2 \le (\frac{2C_p^2}{\nu} + \frac{2gS_0\tilde{C}_p^2}{k_{\min}})\|d_t\underline{w}_{2,h}^{n+1}\|_0^2 + \frac{5}{4}\Delta t^2\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + C\Delta t^2\|d_t\underline{w}_{1,h}^{n+1}\|_W^2\\
&+ C\Delta t\frac{C}{\varepsilon\nu^2}\big(1 + \|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\big(\|\underline{w}_{1,h}^{n+1}\|_W^2 + \|\underline{w}_{1,h}^n\|_W^2\big) + \frac{20C_p^2}{\nu}\|\frac{f_1^{n+1}+f_1^n}{2}\|_{L^2(\Omega_f)}^2 + \frac{20g\tilde{C}_p^2}{k_{min}}\|\frac{f_2^{n+1}+f_2^n}{2}\|_{L^2(\Omega_p)}^2.\end{aligned}$$

Then considering (13), (23), (31) and (51), we complete the proof. □

Next, we need to get the estimates of $d_t\underline{w}_{2,h}^{n+1}$.

Lemma 4.3. *Under the assumptions of Lemma 4.1, we have, for all* $1 \le m \le N$,

$$\|d_t\underline{w}_{2,h}^m\|_0^2 + \Delta t\|d_t\underline{w}_{2,h}^m\|_W^2 \le \mathcal{F}^2. \tag{74}$$

Here

$$\begin{aligned}\mathcal{F}^2 &= \big(\|f_1^1\|_0^2 + \|f_1^0\|_0^2 + gS_0\|f_2^1\|_0^2 + gS_0\|f_2^0\|_0^2\big) + C\Bigg((1 + \nu\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2 + gk_{\max}\|\varphi_t\|_{L^2(0,T;H^1(\Omega_p))}^2)(1+\Upsilon^2)\\
&+ \frac{C_p^2}{\nu}\|(f_1)_t\|_{L^2(0,T;L^2(\Omega_f))}^2 + \frac{g\tilde{C}_p^2}{k_{min}}\|(f_2)_t\|_{L^2(0,T;L^2(\Omega_p))}^2\Bigg).\end{aligned}$$

Proof. From (65), we can take the difference of it at time t^{n+1} and t^n as follows:

$$\begin{cases}[d_t\underline{w}_{2,h}^{n+1} - d_t\underline{w}_{2,h}^n, \underline{z}_h] + \Delta t a(d_t\underline{w}_{2,h}^{n+1}, \underline{z}_h) + N(u_{2,h}^n; u_{2,h}^n, v_h) - N(u_{2,h}^{n-1}; u_{2,h}^{n-1}, v_h) + b(\underline{z}_h, p_{2,h}^{n+1} - p_{2,h}^n)\\
= \frac{1}{2}\Delta t a(d_t\underline{w}_{1,h}^{n+1}, \underline{z}_h) - \frac{1}{2}\Delta t a(d_t\underline{w}_{1,h}^n, \underline{z}_h) - \frac{1}{2}N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, v_h) + \frac{1}{2}N(u_{1,h}^n; u_{1,h}^n, v_h) + \frac{1}{2}N(u_{1,h}^n; u_{1,h}^n, v_h)\\
-\frac{1}{2}N(u_{1,h}^{n-1}; u_{1,h}^{n-1}, v_h) + \frac{1}{2}b(\underline{z}_h, p_{1,h}^{n+1} - p_{1,h}^n) - \frac{1}{2}b(\underline{z}_h, p_{1,h}^n - p_{1,h}^{n-1}) + 2\Delta t(\frac{f^{n+1} - f^{n-1}}{2\Delta t}, \underline{z}_h),\\
b(d_t\underline{w}_{2,h}^{n+1}, q_h) = 0.\end{cases}$$

Taking $\underline{z}_h = 2d_t\underline{w}_{2,h}^{n+1}$ and considering (8), we obtain

$$\begin{aligned}&\|d_t\underline{w}_{2,h}^{n+1}\|_0^2 - \|d_t\underline{w}_{2,h}^n\|_0^2 + 2\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + 2\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2\\
&\le \Delta t a(d_t\underline{w}_{1,h}^{n+1}, d_t\underline{w}_{2,h}^{n+1}) - \Delta t a(d_t\underline{w}_{1,h}^n, d_t\underline{w}_{2,h}^{n+1}) - 2N(u_{2,h}^n; u_{2,h}^n, d_tu_{2,h}^{n+1})\\
&+2N(u_{2,h}^{n-1}; u_{2,h}^{n-1}, d_tu_{2,h}^{n+1}) - N(u_{1,h}^{n+1}; u_{1,h}^{n+1}, d_tu_{2,h}^{n+1})\\
&+N(u_{1,h}^n; u_{1,h}^n, d_tu_{2,h}^{n+1}) + N(u_{1,h}^n; u_{1,h}^n, d_tu_{2,h}^{n+1}) - N(u_{1,h}^{n-1}; u_{1,h}^{n-1}, d_tu_{2,h}^{n+1}) + 2\Delta t(\frac{f^{n+1}-f^{n-1}}{2\Delta t}, d_t\underline{w}_{2,h}^{n+1}).\end{aligned} \tag{75}$$

Similarly to (69), we bound the first two terms on the right hand side of (75) as follows.

$$\Delta t a(d_t\underline{w}_{1,h}^{n+1}, d_t\underline{w}_{2,h}^{n+1}) \le 2\varepsilon\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + \frac{1}{2}\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|d_tu_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2 + C\Delta t\|d_t\underline{w}_{1,h}^{n+1}\|_W^2,$$

$$-\Delta t a(d_t\underline{w}_{1,h}^n, d_t\underline{w}_{2,h}^{n+1}) \le 2\varepsilon\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2 + \frac{1}{2}\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|d_tu_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2 + C\Delta t\|d_t\underline{w}_{1,h}^n\|_W^2.$$

Using (15), (66) and Holder's inequality, we have

$$\begin{aligned}
&-2N(u_{2,h}^n;u_{2,h}^n,d_tu_{2,h}^{n+1})+2N(u_{2,h}^{n-1};u_{2,h}^{n-1},d_tu_{2,h}^{n+1})\\
=&-2\Delta tN(u_{2,h}^n;d_tu_{2,h}^n,d_tu_{2,h}^{n+1})-2\Delta tN(d_tu_{2,h}^n;u_{2,h}^{n-1},d_tu_{2,h}^{n+1})\\
\leq&\,2\Delta tC_{N1}\|\nabla u_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^{n+1}\|_{L^2(\Omega_f)}+2\Delta tC_{N1}\|\nabla d_tu_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla u_{2,h}^{n-1}\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^{n+1}\|_{L^2(\Omega_f)}\\
\leq&\,\Delta t\nu\|\nabla d_tu_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^{n+1}\|_{L^2(\Omega_f)}^2\\
\leq&\,\frac{\nu}{4}\Delta t\|\nabla d_tu_{2,h}^{n+1}\|_{L^2(\Omega_f)}^2+\Delta t\nu\|\nabla d_tu_{2,h}^n\|_{L^2(\Omega_f)}^2\leq\frac14\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2+\Delta t\|d_t\underline{w}_{2,h}^n\|_W^2.
\end{aligned}$$

Similar to the estimate of (70) and (25), we get

$$\begin{aligned}
&-N(u_{1,h}^{n+1};u_{1,h}^{n+1},d_tu_{2,h}^{n+1})+N(u_{1,h}^n;u_{1,h}^n,d_tu_{2,h}^{n+1})\leq6\varepsilon\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2\\
&\quad+\frac{C}{\varepsilon\nu^2}\big(1+\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\big(\|\underline{w}_{1,h}^{n+1}\|_W^2+\|\underline{w}_{1,h}^n\|_W^2\big),\\
&N(u_{1,h}^n;u_{1,h}^n,d_tu_{2,h}^{n+1})-N(u_{1,h}^{n-1};u_{1,h}^{n-1},d_tu_{2,h}^{n+1})\leq6\varepsilon\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2\\
&\quad+\frac{C}{\varepsilon\nu^2}\big(1+\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\big(\|\underline{w}_{1,h}^n\|_W^2+\|\underline{w}_{1,h}^{n-1}\|_W^2\big),\\
&2\Delta t(\frac{f^{n+1}-f^{n-1}}{2\Delta t},d_t\underline{w}_{2,h}^{n+1})\leq\varepsilon\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2+\frac{C_p^2\Delta t}{\varepsilon\nu}\|\frac{f_1^{n+1}-f_1^{n-1}}{2\Delta t}\|_{L^2(\Omega_f)}^2+\frac{g\tilde{C_p^2}\Delta t}{\varepsilon k_{min}}\|\frac{f_1^{n+1}-f_1^{n-1}}{2\Delta t}\|_{L^2(\Omega_p)}^2\\
&\quad\leq\varepsilon\Delta t\|\underline{w}_{2,h}^{n+1}\|_W^2+\frac{C_p^2}{2\varepsilon\nu}\int_{t^{n-1}}^{t^{n+1}}\|(f_1)_t\|_{L^2(\Omega_f)}^2dt+\frac{g\tilde{C_p^2}}{2\varepsilon k_{min}}\int_{t^{n-1}}^{t^{n+1}}\|(f_2)_t\|_{L^2(\Omega_p)}^2dt.
\end{aligned}$$

Considering these estimates with (75) and setting $\varepsilon=3/68$, we obtain

$$\begin{aligned}
&\|d_t\underline{w}_{2,h}^{n+1}\|_0^2-\|d_t\underline{w}_{2,h}^n\|_0^2+\Delta t\|d_t\underline{w}_{2,h}^{n+1}\|_W^2-\Delta t\|d_t\underline{w}_{2,h}^n\|_W^2+\Delta t\sum_{n=1}^{m-1}\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|u_{2,h}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2\\
\leq&\,C\Delta t(\|d_t\underline{w}_{1,h}^{n+1}\|_W^2+\|d_t\underline{w}_{1,h}^n\|_W^2)+C\big(1+\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\big(\|\underline{w}_{1,h}^{n+1}\|_W^2+\|\underline{w}_{1,h}^n\|_W^2+\|\underline{w}_{1,h}^{n-1}\|_W^2\big)\\
&+\frac{34C_p^2}{3\nu}\int_{t^{n-1}}^{t^{n+1}}\|(f_1)_t\|_{L^2(\Omega_f)}^2dt+\frac{34g\tilde{C_p^2}}{3k_{min}}\int_{t^{n-1}}^{t^{n+1}}\|(f_2)_t\|_{L^2(\Omega_p)}^2dt.
\end{aligned}$$

Then summing the equation from $n=1$ to $n=m-1$, we deduce

$$\begin{aligned}
&\|d_t\underline{w}_{2,h}^m\|_0^2+\Delta t\|d_t\underline{w}_{2,h}^m\|_W^2\leq\|d_t\underline{w}_{2,h}^1\|_0^2+\Delta t\|d_t\underline{w}_{2,h}^1\|_W^2+C\Delta t\sum_{n=1}^{m-1}\|d_t\underline{w}_{1,h}^n\|_W^2\\
&+C\big(1+\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\big)\Delta t\sum_{n=1}^{m}\|\underline{w}_{1,h}^n\|_W^2\\
&+\frac{34C_p^2}{3\nu}\sum_{n=0}^{m}\int_{t^{n-1}}^{t^{n+1}}\|(f_1)_t\|_{L^2(\Omega_f)}^2dt+\frac{34g\tilde{C_p^2}}{3k_{min}}\sum_{n=0}^{m}\int_{t^{n-1}}^{t^{n+1}}\|(f_2)_t\|_{L^2(\Omega_p)}^2dt.
\end{aligned}\tag{76}$$

In addition, $d_t\underline{w}_{1,h}^{n+1}=d_t\underline{e}_{1,w}^{n+1}+d_t\underline{w}^{n+1}$. Using (14) and (64), we get

$$\begin{aligned}
\Delta t\sum_{n=1}^{m-1}\|d_t\underline{w}_{1,h}^{n+1}\|_W^2&\leq\Delta t\sum_{n=1}^{m-1}\|d_t\underline{e}_{1,w}^{n+1}\|_W^2+\sum_{n=1}^{m-1}\int_{t^n}^{t^{n+1}}\|\underline{w}_t\|_W^2dt\leq C(\Delta t^2+h^{2k_1}+h^{2k_2})+\|\underline{w}_t\|_{L^2(0,T;H^1(\Omega))}^2\\
&\leq C(\Delta t^2+h^{2k_1}+h^{2k_2})+\nu\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2+gk_{\max}\|\varphi_t\|_{L^2(0,T;H^1(\Omega_p))}^2.
\end{aligned}$$

Obviously, we can believe that $\Delta t^2+h^{2k_1}+h^{2k_2}<1$. Using (23) and (31), we get

$$\begin{aligned}
\|d_t\underline{w}_{2,h}^m\|_0^2+\Delta t\|d_t\underline{w}_{2,h}^m\|_W^2\leq\|d_t\underline{w}_{2,h}^1\|_0^2+\Delta t\|d_t\underline{w}_{2,h}^1\|_W^2+C\Bigg(&(1+\nu\|u_t\|_{L^2(0,T;H^1(\Omega_f))}^2\\
&+gk_{\max}\|\varphi_t\|_{L^2(0,T;H^1(\Omega_p))}^2)(1+\Upsilon^2)\\
&+\frac{C_p^2}{\nu}\|(f_1)_t\|_{L^2(0,T;L^2(\Omega_f))}^2+\frac{g\tilde{C_p^2}}{k_{min}}\|(f_2)_t\|_{L^2(0,T;L^2(\Omega_p))}^2\Bigg).
\end{aligned}\tag{77}$$

Next, we consider (65) at time t^1 and $\underline{w}_{2,h}^0 = \underline{w}_{,h}^0 = 0$. Setting $\underline{z}_h = d_t\underline{w}_{2,h}^1$, we get

$$\|d_t\underline{w}_{2,h}^1\|_0^2 + \Delta t\|d_t\underline{w}_{2,h}^1\|_W^2 = \frac{1}{2}a(\underline{w}_{1,h}^1, d_t\underline{w}_{2,h}^1) - \frac{1}{2}N(u_{1,h}^1; u_{1,h}^1, d_tu_{2,h}^1) + (\frac{f^1+f^0}{2}, d_t\underline{w}_{2,h}^1).$$

For these terms on the right hand side of above equation, we have

$$\begin{aligned}
&\frac{1}{2}a(\underline{w}_{1,h}^1, d_t\underline{w}_{2,h}^1) = \frac{1}{2}\Delta t a(d_t\underline{w}_{1,h}^1, d_t\underline{w}_{2,h}^1) \le \frac{1}{2}\Delta t\|d_t\underline{w}_{1,h}^1\|_W\|d_t\underline{w}_{2,h}^1\|_W \le \frac{1}{4}\Delta t\|d_t\underline{w}_{2,h}^1\|_W^2 + \frac{1}{4}\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2,\\
&-\frac{1}{2}N(u_{1,h}^1; u_{1,h}^1, d_tu_{2,h}^1) = -\frac{\Delta t}{2}N(u_{1,h}^1; d_tu_{1,h}^1, d_tu_{2,h}^1) \le \frac{\Delta t}{2}C_{N1}\|\nabla u_{1,h}^1\|_{L^2(\Omega_f)}\|\nabla d_tu_{1,h}^1\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^1\|_{L^2(\Omega_f)}\\
&\quad \le \frac{\nu\Delta t}{8}\|\nabla d_tu_{1,h}^1\|_{L^2(\Omega_f)}\|\nabla d_tu_{2,h}^1\|_{L^2(\Omega_f)} \le \frac{1}{4}\Delta t\|d_t\underline{w}_{2,h}^1\|_W^2 + \frac{1}{64}\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2,\\
&(\frac{f^1+f^0}{2}, d_t\underline{w}_{2,h}^1) \le \|\frac{f^1+f^0}{2}\|_0\|d_t\underline{w}_{2,h}^1\|_0 \le \frac{1}{2}\|d_t\underline{w}_{2,h}^1\|_0^2 + \frac{1}{8}\|f_1^1+f_1^0\|_0^2 + \frac{1}{8}gS_0\|f_2^1+f_2^0\|_0^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2}\|d_t\underline{w}_{2,h}^1\|_0^2 + \frac{1}{2}\Delta t\|d_t\underline{w}_{2,h}^1\|_W^2 &\le \frac{17}{64}\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2 + \frac{1}{8}\|f_1^1+f_1^0\|_0^2 + \frac{1}{8}gS_0\|f_2^1+f_2^0\|_0^2\\
&\le \frac{17}{64}\Delta t\|d_t\underline{w}_{1,h}^1\|_W^2 + \frac{1}{4}(\|f_1^1\|_0^2 + \|f_1^0\|_0^2) + \frac{1}{4}gS_0(\|f_2^1\|_0^2 + \|f_2^0\|_0^2).
\end{aligned}$$

Then considering (35), we have

$$\|d_t\underline{w}_{2,h}^1\|_0^2 + \Delta t\|d_t\underline{w}_{2,h}^1\|_W^2 \le \|f_1^1\|_0^2 + \|f_1^0\|_0^2 + gS_0\|f_2^1\|_0^2 + gS_0\|f_2^0\|_0^2. \tag{78}$$

By substituting (78) into (77), we yield (74). □

Theorem 4.1. *Under the assumptions of Theorem 3.2 and*

$$\left((\frac{4C_p^2}{\nu} + \frac{4gS_0\tilde{C}_p^2}{k_{\min}} + \frac{5}{2})\mathcal{F}^2 + C\big(\mathcal{H}^2 + \Upsilon^2 + \frac{C_p^2}{\nu}\|f_1\|^2_{L^\infty(0,T;L^2(\Omega_f))} + \frac{g\tilde{C}_p^2}{k_{min}}\|f_2\|^2_{L^\infty(0,T;L^2(\Omega_p))}\big) \right)^{\frac{1}{2}} \le \frac{\nu^{3/2}}{4C_{N1}}, \tag{79}$$

the solution $\underline{w}_{2,h}^m$, $1 \le m \le N$, of algorithm (65) satisfies

$$\|\underline{w}_{2,h}^m\|_0^2 + \Delta t\sum_{n=0}^{m-1}\|\underline{w}_{2,h}^{n+1}\|_W^2 \le C\left(\mathcal{F}^2 + \mathcal{H}^2 + (1 + \|u_t\|^2_{L^2(0,T;H^1(\Omega_f))})\Upsilon^2\right). \tag{80}$$

Proof. The result (80) can be calculated by combining (79) with the results of Lemmas 4.1–4.3 . □

4.2. Convergence analysis for the second order algorithm

In this part, we analyze the convergence rate of algorithm (65) and show that it is second order accurate. Moreover, we define the function

$$\begin{aligned}
F &:= \underline{w}_t = u_t + gS_0\varphi_t\\
&= \nu\Delta u - (u\cdot\nabla)u - \nabla p + g\nabla\cdot(\mathbf{K}\nabla\varphi) + f_1 + gf_2.
\end{aligned}$$

Then the first continuous momentum equation can be written as

$$\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F d\tau.$$

Using the trapezoid rule that

$$\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F d\tau = \frac{F^{n+1} + F^n}{2} + C\Delta t^2 F_{tt}(\alpha^{n+1}),$$

where $\alpha^{n+1} \in (t^n, t^{n+1})$, we obtain

$$\begin{aligned}
\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} &= \frac{1}{2}\nu\Delta u^{n+1} + \frac{1}{2}\nu\Delta u^n + \frac{1}{2}g\nabla\cdot(\mathbf{K}\nabla\varphi^{n+1}) + \frac{1}{2}g\nabla\cdot(\mathbf{K}\nabla\varphi^n) - \frac{1}{2}(u^{n+1}\cdot\nabla)u^{n+1} - \frac{1}{2}(u^n\cdot\nabla)u^n\\
&\quad - \frac{1}{2}\nabla p^{n+1} - \frac{1}{2}\nabla p^n + \frac{f_1^{n+1}+f_1^n}{2} + g\frac{f_2^{n+1}+f_2^n}{2} + C\Delta t^2 F_{tt}(\alpha^{n+1}).
\end{aligned}$$

Add $-\nu\Delta u^{n+1} - g\nabla\cdot(\mathbf{K}\nabla\varphi^{n+1}) + (u^n\cdot\nabla)u^n + \nabla p^{n+1}$ to both sides of the relation and we can get

$$\begin{aligned}&\frac{\underline{w}^{n+1}-\underline{w}^n}{\Delta t} - \nu\Delta u^{n+1} - g\nabla\cdot(\mathbf{K}\nabla\varphi^{n+1}) + (u^n\cdot\nabla)u^n + \nabla p^{n+1}\\ =& -\frac{1}{2}\nu\Delta u^{n+1} + \frac{1}{2}\nu\Delta u^n - \frac{1}{2}g\nabla\cdot(\mathbf{K}\nabla\varphi^{n+1}) + \frac{1}{2}g\nabla\cdot(\mathbf{K}\nabla\varphi^n) - \frac{1}{2}(u^{n+1}\cdot\nabla)u^{n+1} + \frac{1}{2}(u^n\cdot\nabla)u^n\\ &+\frac{1}{2}\nabla p^{n+1} - \frac{1}{2}\nabla p^n + \frac{f_1^{n+1}+f_1^n}{2} + g\frac{f_2^{n+1}+f_2^n}{2} + C\Delta t^2 F_{tt}(\alpha^{n+1}).\end{aligned}$$

Then it is easy to get the following weak formulation: for all $\underline{z}_h \in W_h$,

$$\begin{aligned}&[\frac{\underline{w}^{n+1}-\underline{w}^n}{\Delta t}, \underline{z}_h] + a(\underline{w}^{n+1}, \underline{z}_h) + N(u^n; u^n, v_h) + b(\underline{z}_h, p^{n+1})\\ =& \frac{1}{2}a(\underline{w}^{n+1}, \underline{z}_h) - \frac{1}{2}a(\underline{w}^n, \underline{z}_h) - \frac{1}{2}N(u^{n+1}; u^{n+1}, v_h) + \frac{1}{2}N(u^n; u^n, v_h)\\ &+\frac{1}{2}b(\underline{z}_h, p^{n+1}-p^n) + (\frac{f_1^{n+1}+f_1^n}{2}, v_h) + g(\frac{f_2^{n+1}+f_2^n}{2}, \psi_h) + C\Delta t^2(F_{tt}(\alpha^{n+1}), \underline{z}_h).\end{aligned} \tag{81}$$

Before we give the convergence result of algorithm (65), we decompose the errors into numerical errors and approximation errors as follows.

$$\begin{aligned}\underline{e}_{2,w}^{n+1} &= \underline{w}_{2,h}^{n+1} - \underline{w}^{n+1} = \underline{w}_{2,h}^{n+1} - \tilde{\underline{w}}^{n+1} - (\underline{w}^{n+1} - \tilde{\underline{w}}^{n+1}) := \underline{e}_2^{n+1} - \underline{\xi}^{n+1},\\ e_{2,u}^{n+1} &= u_{2,h}^{n+1} - u^{n+1} = u_{2,h}^{n+1} - \tilde{u}^{n+1} - (u^{n+1} - \tilde{u}^{n+1}) := e_{2,f}^{n+1} - \xi_f^{n+1},\\ e_{2,\varphi}^{n+1} &= \varphi_{2,h}^{n+1} - \varphi^{n+1} = \varphi_{2,h}^{n+1} - \tilde{\varphi}^{n+1} - (\varphi^{n+1} - \tilde{\varphi}^{n+1}) := e_{2,p}^{n+1} - \xi_p^{n+1},\\ e_{2,\chi}^{n+1} &= p_{2,h}^{n+1} - p^{n+1} = p_{2,h}^{n+1} - \tilde{p}^{n+1} - (p^{n+1} - \tilde{p}^{n+1}) := \varepsilon_2^{n+1} - \eta^{n+1},\end{aligned}$$

where $\underline{e}_{2,w}^{n+1} = (e_{2,u}^{n+1}, e_{2,\varphi}^{n+1})$, $\underline{e}_2^{n+1} = (e_{2,f}^{n+1}, e_{2,p}^{n+1})$, $\underline{\xi}^{n+1} = (\xi_f^{n+1}, \xi_p^{n+1})$.

Theorem 4.2. *Under the assumptions of Theorem 4.1, we have for any $1 \le m \le N$,*

$$\|\underline{e}_2^m\|_0^2 \le C(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}). \tag{82}$$

Proof. At time t^{n+1}, let $(\underline{w}_{1,h}^{n+1}, p_{1,h}^{n+1})$ satisfy (21) and the true solution satisfy (81). Subtracting (81) from (65) and setting $\underline{z}_h = 2\Delta t\underline{e}_2^{n+1}$, $q_h = \varepsilon_2^{n+1}$, we obtain

$$\begin{aligned}&(\|\underline{e}_2^{n+1}\|_0^2 - \|\underline{e}_2^n\|_0^2 + \|\underline{e}_2^{n+1} - \underline{e}_2^n\|_0^2) + 2\Delta t\|\underline{e}_2^{n+1}\|_W^2 + 2\Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|e_{2,f}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2\\ =&2\Delta t[\frac{\underline{\xi}^{n+1}-\underline{\xi}^n}{\Delta t}, \underline{e}_2^{n+1}] + 2\Delta ta(\underline{\xi}^{n+1}, \underline{e}_2^{n+1}) + 2\Delta tb(e_2^{n+1}, \eta^{n+1}) + \Delta ta(\underline{e}_{1,w}^{n+1} - \underline{e}_{1,w}^n, \underline{e}_2^{n+1}) - \Delta tb(e_2^{n+1}, \eta^{n+1} - \eta^n)\\ &-2\Delta tN(u_{2,h}^n; u_{2,h}^n, e_{2,f}^{n+1}) + 2\Delta tN(u^n; u^n, e_{2,f}^{n+1}) - \Delta tN(u_{1,h}^{n+1}; u_{1,h}^{n+1}, e_{2,f}^{n+1}) + \Delta tN(u^{n+1}; u^{n+1}, e_{2,f}^{n+1})\\ &+\Delta tN(u_{1,h}^n; u_{1,h}^n, e_{2,f}^{n+1}) - \Delta tN(u^n; u^n, e_{2,f}^{n+1}) - C\Delta t^2(F_{tt}(\alpha^{n+1}), \underline{e}_2^{n+1}).\end{aligned} \tag{83}$$

For the first term on the right hand side of (83), similarly to (29), we get for all $\varepsilon > 0$,

$$I = 2\Delta t[\frac{\underline{\xi}^{n+1}-\underline{\xi}^n}{\Delta t}, \underline{e}_2^{n+1}] \le \varepsilon\Delta t\|\underline{e}_2^{n+1}\|_W^2 + (\frac{C_p^2}{\varepsilon\nu} + \frac{S_0\tilde{C}_p^2}{\varepsilon k_{\min}})\int_{t^n}^{t^{n+1}}\|\underline{\xi}_t(\tau)\|_0^2 d\tau. \tag{84}$$

From (12) and (19), we have

$$\begin{aligned}II &= 2\Delta ta(\underline{\xi}^{n+1}, \underline{e}_2^{n+1}) + 2\Delta tb(e_{2,f}^{n+1}, \eta^{n+1}) + \Delta ta(\underline{e}_{1,w}^{n+1} - \underline{e}_{1,w}^n, \underline{e}_2^{n+1}) - \Delta tb(e_{2,f}^{n+1}, \eta^{n+1} - \eta^n)\\ &= \Delta ta(\underline{e}_1^{n+1} - \underline{e}_1^n, \underline{e}_2^{n+1}) = \Delta t^2 a(d_t\underline{e}_1^{n+1}, \underline{e}_2^{n+1})\\ &\le \varepsilon\Delta t\|\underline{e}_2^{n+1}\|_W^2 + \Delta t\sum_{i=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\|e_{2,f}^{n+1}\cdot\tau_i\|_{L^2(\Gamma)}^2 + C\Delta t^3\|d_t\underline{e}_1^{n+1}\|_W^2.\end{aligned} \tag{85}$$

Next, by using the fact $e_{2,u}^{n+1} = u_{2,h}^{n+1} - u^{n+1} = e_{2,f}^{n+1} - \xi_f^{n+1}$, the trilinear terms are bounded as follows.

$$\begin{aligned}III &= -2\Delta tN(u_{2,h}^n; u_{2,h}^n, e_{2,f}^{n+1}) + 2\Delta tN(u^n; u^n, e_{2,f}^{n+1})\\ &= -2\Delta tN(u_{2,h}^n; u_{2,h}^n - u^n, e_{2,f}^{n+1}) - 2\Delta tN(u_{2,h}^n - u^n; u^n, e_{2,f}^{n+1}).\end{aligned}$$

First,

$$i = -2\Delta tN(u_{2,h}^n; u_{2,h}^n - u^n, e_{2,f}^{n+1}) = -2\Delta tN(u_{2,h}^n; e_{2,f}^n, e_{2,f}^{n+1}) + 2\Delta tN(u_{2,h}^n; \xi_f^n, e_{2,f}^{n+1})$$
$$= 2\Delta tN(u_{2,h}^n; e_{2,f}^n, e_{2,f}^{n+1}) + 2\Delta tN(e_{2,f}^n; \xi_f^n, e_{2,f}^{n+1}) - 2\Delta tN(\xi_f^n; \xi_f^n, e_{2,f}^{n+1}) + 2\Delta tN(u^n; \xi_f^n, e_{2,f}^{n+1}). \quad (86)$$

Using (11), (15) and (79), we have

$$-2\Delta tN(u_{2,h}^n; e_{2,f}^n, e_{2,f}^{n+1}) \le 2\Delta tC_{N1}\|\nabla u_{2,h}^n\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^n\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le \frac{\nu}{2}\Delta t\|\nabla e_{2,f}^n\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le \frac{\nu}{8}\Delta t\|\nabla e_{2,f}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{\nu}{2}\Delta t\|\nabla e_{2,f}^n\|^2_{L^2(\Omega_f)} \le \frac{1}{8}\Delta t\|\underline{e}_2^{n+1}\|^2_W + \frac{1}{2}\Delta t\|\underline{e}_2^n\|^2_W.$$

$$2\Delta tN(e_{2,f}^n; \xi_f^n, e_{2,f}^{n+1}) - 2\Delta tN(\xi_f^n; \xi_f^n, e_{2,f}^{n+1})$$
$$\le 2\Delta tC_{N1}\|\nabla e_{2,f}^n\|_{L^2(\Omega_f)}\|\nabla \xi_f^n\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} + 2\Delta tC_{N1}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le 2\Delta th^{-1}C_{N1}C_I\|e_{2,f}^n\|_{L^2(\Omega_f)}\|\nabla \xi_f^n\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} + 2\Delta tC_{N1}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le 2\varepsilon\nu\Delta t\|\nabla e_{2,f}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}\|e_{2,f}^n\|^2_{L^2(\Omega_f)} + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla \xi_f^n\|^4_{L^2(\Omega_f)}.$$

Using (18), we have

$$2\Delta tN(u^n; \xi_f^n, e_{2,f}^{n+1}) \le 2\Delta tC_{N4}\|u^n\|_{H^2(\Omega_f)}\|\nabla \xi_f^n\|_{L^2(\Omega_f)}\|e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le h^{-2}\Delta t\|e_{2,f}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N4}^2}{h^{-2}}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}.$$

So

$$i \le (\frac{1}{8} + 2\varepsilon)\Delta t\|\underline{e}_2^{n+1}\|^2_W + \frac{1}{2}\Delta t\|\underline{e}_2^n\|^2_W + h^{-2}\Delta t\|\underline{e}_2^{n+1}\|^2_0 + \frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}\|\underline{e}_2^n\|^2_0$$
$$+ \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla \xi_f^n\|^4_{L^2(\Omega_f)} + C_{N4}^2h^2\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)}.$$

Using (16), we have

$$ii - 2\Delta tN(u_{2,h}^n - u^n; u^n, e_{2,f}^{n+1}) = -2\Delta tN(e_{2,f}^n; u^n, e_{2,f}^{n+1}) + 2\Delta tN(\xi_f^n; u^n, e_{2,f}^{n+1})$$
$$\le 2\Delta tC_{N2}(\|e_{2,f}^n\|_{L^2(\Omega_f)} + \|\xi_f^n\|_{L^2(\Omega_f)})\|u^n\|_{H^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le 2\varepsilon\nu\Delta t\|\nabla e_{2,f}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|e_{2,f}^n\|^2_{L^2(\Omega_f)} + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\xi_f^n\|^2_{L^2(\Omega_f)}$$
$$\le 2\varepsilon\Delta t\|\underline{e}_2^{n+1}\|^2_W + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\underline{e}_2^n\|^2_0 + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\underline{\xi}^n\|^2_0.$$

So we get

$$III \le (\frac{1}{8} + 4\varepsilon)\Delta t\|\underline{e}_2^{n+1}\|^2_W + \frac{1}{2}\Delta t\|\underline{e}_2^n\|^2_W + h^{-2}\Delta t\|\underline{e}_2^{n+1}\|^2_0 + \big(\frac{C_{N1}^2C_I^2}{\varepsilon\nu}\Delta th^{-2}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)} + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\big)\|\underline{e}_2^n\|^2_0$$
$$+ \frac{C_{N1}^2}{\varepsilon\nu}\Delta t\|\nabla \xi_f^n\|^4_{L^2(\Omega_f)} + C_{N4}^2h^2\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\nabla \xi_f^n\|^2_{L^2(\Omega_f)} + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t\|u^n\|^2_{H^2(\Omega_f)}\|\underline{\xi}^n\|^2_0. \quad (87)$$

Next we manipulate the remaining nonlinear terms by adding and subtracting $\Delta tN(u^{n+1}; u_{1,h}^{n+1}, e_{2,f}^{n+1})$, $\Delta tN(u^n; u_{1,h}^n, e_{2,f}^{n+1})$, $\Delta tN(e_{1,u}^n; u_{1,h}^{n+1}, e_{2,f}^{n+1})$, $\Delta tN(u^n; e_{1,u}^{n+1}, e_{2,f}^{n+1})$, and using the fact $e_{1,u}^{n+1} = u_{1,h}^{n+1} - u^{n+1}$. This lead to the following expression

$$IV = -\Delta tN(u_{1,h}^{n+1}; u_{1,h}^{n+1}, e_{2,f}^{n+1}) + \Delta tN(u^{n+1}; u^{n+1}, e_{2,f}^{n+1}) + \Delta tN(u_{1,h}^n; u_{1,h}^n, e_{2,f}^{n+1}) - \Delta tN(u^n; u^n, e_{2,f}^{n+1})$$
$$= -\Delta tN(e_{1,u}^{n+1} - e_{1,u}^n; u_{1,h}^{n+1}, e_{2,f}^{n+1}) - \Delta tN(e_{1,u}^n; u_{1,h}^{n+1} - u_{1,h}^n, e_{2,f}^{n+1})$$
$$- \Delta tN(u^{n+1}; e_{1,u}^{n+1} - e_{1,u}^n, e_{2,f}^{n+1}) - \Delta tN(u^{n+1} - u^n; e_{1,u}^n, e_{2,f}^{n+1})..$$

Using (15), (16) and the fact $e_{1,u}^{n+1} = u_{1,h}^{n+1} - u^{n+1}$, we have

$$iii = -\Delta tN(e_{1,u}^{n+1} - e_{1,u}^n; u_{1,h}^{n+1}, e_{2,f}^{n+1}) = -\Delta t^2N(d_te_{1,u}^{n+1}; e_{1,u}^{n+1}, e_{2,f}^{n+1}) - \Delta t^2N(d_te_{1,u}^{n+1}; u^{n+1}, e_{2,f}^{n+1})$$
$$\le \Delta t^2C_{N1}\|\nabla d_te_{1,u}^{n+1}\|_{L^2(\Omega_f)}\|\nabla e_{1,u}^{n+1}\|_{L^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} + \Delta t^2C_{N2}\|d_te_{1,u}^{n+1}\|_{L^2(\Omega_f)}\|u^{n+1}\|_{H^2(\Omega_f)}\|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}$$
$$\le 2\varepsilon\nu\Delta t\|\nabla e_{2,f}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N1}^2}{\varepsilon\nu}\Delta t^3\|\nabla d_te_{1,u}^{n+1}\|^2_{L^2(\Omega_f)}\|\nabla e_{1,u}^{n+1}\|^2_{L^2(\Omega_f)} + \frac{C_{N2}^2}{\varepsilon\nu}\Delta t^3\|d_te_{1,u}^{n+1}\|^2_{L^2(\Omega_f)}\|u^{n+1}\|^2_{H^2(\Omega_f)}.$$

Using (15), (17) and Holder's inequality, we have

$$\begin{aligned}
iv &= -\Delta t N(e_{1,u}^n; u_{1,h}^{n+1} - u_{1,h}^n, e_{2,f}^{n+1}) = -\Delta t^2 N(e_{1,u}^n; \frac{u_{1,h}^{n+1} - u_{1,h}^n}{\Delta t}, e_{2,f}^{n+1}) = -\Delta t^2 N(e_{1,u}^n; d_t e_{1,u}^{n+1} + \frac{u^{n+1} - u^n}{\Delta t}, e_{2,f}^{n+1}) \\
&\leq \Delta t^2 C_{N1} \|\nabla e_{1,u}^n\|_{L^2(\Omega_f)} \left(\|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)} + \|\nabla \frac{u^{n+1} - u^n}{\Delta t}\|_{L^2(\Omega_f)} \right) \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} \\
&\leq 2\varepsilon\nu \Delta t \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2}{\varepsilon\nu} \Delta t^2 \|\nabla e_{1,u}^n\|_{L^2(\Omega_f)}^2 \big(\Delta t \|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2 + \int_{t^n}^{t^{n+1}} \|u_t(\tau)\|_{H^1(\Omega_f)}^2 d\tau \big), \\
v &= -\Delta t N(u^{n+1}; e_{1,u}^{n+1} - e_{1,u}^n, e_{2,f}^{n+1}) = -\Delta t^2 N(u^{n+1}; d_t e_{1,u}^{n+1}, e_{2,f}^{n+1}) \\
&\leq \Delta t^2 C_{N3} \|u^{n+1}\|_{H^2(\Omega_f)} \|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)} \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} \\
&\leq \varepsilon\nu \Delta t \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N3}^2}{\varepsilon\nu} \Delta t^3 \|u^{n+1}\|_{H^2(\Omega_f)}^2 \|\nabla d_t e_{1,u}^{n+1}\|_{L^2(\Omega_f)}^2, \\
vi &= -\Delta t N(u^{n+1} - u^n; e_{1,u}^n, e_{2,f}^{n+1}) \leq \Delta t C_{N1} \|\nabla (u^{n+1} - u^n)\|_{L^2(\Omega_f)} \|\nabla e_{1,u}^{n+1}\|_{L^2(\Omega_f)} \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)} \\
&\leq \varepsilon\nu \Delta t \|\nabla e_{2,f}^{n+1}\|_{L^2(\Omega_f)}^2 + \frac{C_{N1}^2}{\varepsilon\nu} \Delta t^2 \|\nabla e_{1,u}^n\|_{L^2(\Omega_f)}^2 \int_{t^n}^{t^{n+1}} \|u_t(\tau)\|_{H^1(\Omega_f)}^2 d\tau.
\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}
IV &\leq 6\varepsilon \Delta t \|\underline{e}_2^{n+1}\|_W^2 + \frac{C_{N1}^2}{\varepsilon\nu^3} \Delta t^3 \|d_t \underline{e}_{1,w}^{n+1}\|_W^2 \|\underline{e}_{1,w}^{n+1}\|_W^2 + \frac{C_{N2}^2}{\varepsilon\nu} \Delta t^3 \|d_t \underline{e}_{1,w}^{n+1}\|_0^2 \|u^{n+1}\|_{H^2(\Omega_f)}^2 \\
&\quad + \frac{C_{N3}^2}{\varepsilon\nu^2} \Delta t^3 \|u^{n+1}\|_{H^2(\Omega_f)}^2 \|d_t \underline{e}_{1,w}^{n+1}\|_W^2 + \frac{C_{N1}^2}{\varepsilon\nu^2} \Delta t^2 \|\underline{e}_{1,w}^n\|_W^2 \int_{t^n}^{t^{n+1}} \|u_t(\tau)\|_{H^1(\Omega_f)}^2 d\tau.
\end{aligned} \tag{88}$$

For the last term on the right hand side of (83), we have

$$V = -C\Delta t^3 (F_{tt}(\alpha^{n+1}), \underline{e}_2^{n+1}) \leq \Delta t \|\underline{e}_2^{n+1}\|_W^2 + C\Delta t^5 \|F_{tt}(\alpha^{n+1})\|_0^2. \tag{89}$$

Combining (84)–(89) with (83) and setting $\varepsilon = 1/32$, we obtain

$$\begin{aligned}
&\|\underline{e}_2^{n+1}\|_0^2 - \|\underline{e}_2^n\|_0^2 + \frac{1}{2}\Delta t \|\underline{e}_2^{n+1}\|_W^2 - \frac{1}{2}\Delta t \|\underline{e}_2^n\|_W^2 + \Delta t \sum_{i=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}} \|e_{2,f}^{n+1} \cdot \tau_i\|_{L^2(\Gamma)}^2 \\
&\leq h^{-2} \Delta t \|\underline{e}_2^{n+1}\|_0^2 + \left(\frac{32 C_{N1}^2 C_I^2}{\nu} \Delta t h^{-2} \|\nabla \xi_f^n\|_{L^2(\Omega_f)}^2 + \frac{32 C_{N2}^2}{\nu} \Delta t \|u^n\|_{H^2(\Omega_f)}^2 \right) \|\underline{e}_2^n\|_0^2 \\
&+ C \int_{t^n}^{t^{n+1}} \|\underline{\xi}_t(\tau)\|_0^2 d\tau + C\Delta t^3 \|d_t \underline{e}_1^{n+1}\|_W^2 \\
&+ \frac{32 C_{N1}^2}{\nu} \Delta t \|\nabla \xi_f^n\|_{L^2(\Omega_f)}^4 + C_{N4}^2 h^2 \Delta t \|u^n\|_{H^2(\Omega_f)}^2 \|\nabla \xi_f^n\|_{L^2(\Omega_f)}^2 + \frac{32 C_{N2}^2}{\nu} \Delta t \|u^n\|_{H^2(\Omega_f)}^2 \|\underline{\xi}^n\|_0^2 \\
&+ \frac{32 C_{N1}^2}{\nu^3} \Delta t^3 \|d_t \underline{e}_{1,w}^{n+1}\|_W^2 \|\underline{e}_{1,w}^{n+1}\|_W^2 + \frac{32 C_{N2}^3}{\nu} \Delta t^3 \|d_t \underline{e}_{1,w}^{n+1}\|_0^2 \|u^{n+1}\|_{H^2(\Omega_f)}^2 + \frac{32 C_{N3}^2}{\nu^2} \Delta t^3 \|u^{n+1}\|_{H^2(\Omega_f)}^2 \|d_t \underline{e}_{1,w}^{n+1}\|_W^2 \\
&+ \frac{32 C_{N1}^2}{\nu^2} \Delta t^2 \|\underline{e}_{1,w}^n\|_W^2 \int_{t^n}^{t^{n+1}} \|u_t(\tau)\|_{H^1(\Omega_f)}^2 d\tau + C\Delta t^5 \|F_{tt}(\alpha^{n+1})\|_0^2.
\end{aligned}$$

Considering (13) and (40), we get

$$\kappa_n \Delta t = h^{-2} \Delta t + \frac{32 C_{N1}^2 C_I^2}{\nu} \Delta t h^{-2} \|\nabla \xi_f^n\|_{L^2(\Omega_f)}^2 + \frac{32 C_{N2}^2}{\nu} \Delta t \|u^n\|_{H^2(\Omega_f)}^2 < 1.$$

Summing the above equation from $n = 0$ to $n = m - 1$ and using (20), we obtain

$$\|\underline{e}_2^m\|_0^2 + \Delta t\|\underline{e}_2^m\|_W^2 \le \exp\left(\Delta t \sum_{n=0}^{N} \frac{\kappa_n}{1-\Delta t\kappa_n}\right)\Bigg(\|\underline{e}_2^0\|_0^2 + \Delta t\|\underline{e}_2^0\|_W^2$$
$$+ C\sum_{n=0}^{m-1}\int_{t^n}^{t^{n+1}} \|\underline{\xi}_t(\tau)\|_0^2 d\tau + C\Delta t^3 \sum_{n=0}^{m-1} \|d_t\underline{e}_1^{n+1}\|_W^2$$
$$+ C\Delta t\sum_{n=0}^{m-1}\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^4 + Ch^2\Delta t\sum_{n=0}^{m-1}\|u^n\|_{H^2(\Omega_f)}^2\|\nabla\xi_f^n\|_{L^2(\Omega_f)}^2 + C\Delta t\sum_{n=0}^{m-1}\|\underline{\xi}^n\|_0^2 + C\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{e}_{1,w}^{n+1}\|_W^2\|\underline{e}_{1,w}^{n+1}\|_W^2$$
$$+ C\Delta t^3\sum_{n=0}^{m-1}\|d_t\underline{e}_{1,w}^{n+1}\|_0^2\|u^{n+1}\|_{H^2(\Omega_f)}^2 + C\Delta t^2\sum_{n=0}^{m-1}\|\underline{e}_{1,w}^n\|_W^2\int_{t^n}^{t^{n+1}}\|u_t(\tau)\|_{H^1(\Omega_f)}^2 d\tau + C\Delta t^5\sum_{n=0}^{m-1}\|F_{tt}(\alpha^{n+1})\|_0^2\Bigg).$$

By using (13), (14) and (51), we complete the proof. □

Corollary 4.1. *Under the assumptions of Theorem 4.2, it holds that*

$$\|\underline{w}_{2,h}^m - \underline{w}^m\|_0^2 \le C(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}). \tag{90}$$

Proof. Using the triangle inequality and considering (13), we can obtain the results. □

5. Numerical experiment

In this section, we provide some numerical experiments to testify the convergence and effectiveness of the second order method based on the SISDC method presented in Section 4. A Galerkin finite element method is employed, using the MINI element for the Navier–Stokes equations and the linear Lagrangian elements (P1) for the Darcy equation. The results presented were obtained by using the software FreeFEM++.

Let the computational domain $\Omega = (0, 1) \times (0, 2)$ with $\Omega_f = (0, 1) \times (1, 2)$, $\Omega_p = (0, 1) \times (0, 1)$ and the interface $\Gamma = (0, 1) \times \{1\}$. For simplicity, we set all the physical parameters ν, S_0, g, α equal to 1. The exact solution satisfying the interface condition (4) is given by

$$u(x, t) = \Big((x^2(y-1)^2 + y)\sin(t), [-\frac{2}{3}x(y-1)^3 + 2 - \pi\sin(\pi x)]\sin(t)\Big),$$
$$p(x, t) = [2 - \pi\sin(\pi x)]\sin(\frac{\pi}{2}y)\cos(t),$$
$$\varphi(x, t) = [2 - \pi\sin(\pi x)][1 - y - \cos(\pi y)]\sin(t) + \frac{1}{2}[1 + (2 - \pi\sin(\pi x))^2][(y-1)^2 + 1]\sin^2(t).$$

Next, we examine the orders of convergence with respect to the mesh size h and the time step Δt. If we assume that

$$v_h^{\Delta t}(x, t_m) \approx v(x, t_m) + C_1(x, t_m)\Delta t^\gamma + C_2(x, t_m)h^\mu.$$

Thus,

$$\rho_{v,h,0} = \frac{\|v_h^{\Delta t}(x, t_m) - v_{\frac{h}{2}}^{\Delta t}(x, t_m)\|_0}{\|v_{\frac{h}{2}}^{\Delta t}(x, t_m) - v_{\frac{h}{4}}^{\Delta t}(x, t_m)\|_0} \approx \frac{4^\mu - 2^\mu}{2^\mu - 1},$$
$$\rho_{v,\Delta t,0} = \frac{\|v_h^{\Delta t}(x, t_m) - v_h^{\frac{\Delta t}{2}}(x, t_m)\|_0}{\|v_h^{\frac{\Delta t}{2}}(x, t_m) - v_h^{\frac{\Delta t}{4}}(x, t_m)\|_0} \approx \frac{4^\gamma - 2^\gamma}{2^\gamma - 1}.$$

Here v can be u, p, φ. While $\rho_{v,h,0}, \rho_{v,\Delta t,0}$ approach 4.0, the convergence order will be 2.0 .

In Table 1, we study the convergence orders with a fixed time step $\Delta t = 1/100$ and varying spacing . Observe that $\rho_{u,h,0}, \rho_{\varphi,h,0}$ are near 4.0, which suggest that the error estimates $O(h^2)$ for the L^2-norm of u, φ when $k_1 = 1$ and $k_2 = 1$.

In Table 2, we study the convergence orders with a fixed spacing $h = 1/8$ and varying time step Δt. The numerical experiments strongly suggest that the error of convergence in time for all should be $O(\Delta t^2)$ as long as $k_1 = 1$ and $k_2 = 1$, which strongly agree with the theoretical analysis.

In order to express the advantage of our scheme over other numerical method, we compute the error between the exact solution and numerical solutions as well as the computed time. For simplicity, we denote u_h and φ_h as the numerical solutions, obtained by the SISDC method, Crank–Nicolson (CN) method and BDF2 method. We take the fixed time step $\Delta t = 1/500$ and mesh step $h = 1/10$. The data in Table 3 show that although the accuracy compared to CN method is almost the same, our SISDC scheme costs less computed time than it. In addition, BDF2 method is less accurate and costs more computed time than SISDC scheme.

Table 1
Convergence orders of $O(h^r)$ of SISDC at time $t_m = 1.0$, with the same time step $\Delta t = 1/100$..

h	$\|u^m_{2,h} - u^m_{2,\frac{h}{2}}\|_{L^2(\Omega_f)}$	$\rho_{u,h,0}$	$\|\varphi^m_{2,h} - \varphi^m_{2,\frac{h}{2}}\|_{L^2(\Omega_p)}$	$\rho_{\varphi,h,0}$
1/2	3.24518e−1	3.93392	8.61537e−1	3.32173
1/4	8.24923e−2	3.97985	2.59364e−1	4.17544
1/8	2.07275e−2	4.01935	6.21166e−2	3.74357
1/16	5.15693e−3	3.75066	1.65929e−2	4.41961
1/32	1.37494e−3		3.75438e−3	

Table 2
Convergence orders of $O(\Delta t^r)$ of SISDC at time $t_m = 1.0$, with varying time step Δt but fixed mesh $h = 1/8$.

Δt	$\|u^m_{2,\Delta t} - u^m_{2,\frac{\Delta t}{2}}\|_{L^2(\Omega_f)}$	$\rho_{u,\Delta t,0}$	$\|\varphi^m_{2,\Delta t} - \varphi^m_{2,\frac{\Delta t}{2}}\|_{L^2(\Omega_p)}$	$\rho_{\varphi,\Delta t,0}$
1/50	5.92270e−6	3.75343	2.38792e−5	3.83750
1/100	1.57794e−6	3.88048	5.99322e−6	3.91982
1/200	4.06636e−7	3.94115	1.50067e−6	3.96020
1/400	1.03177e−7	3.97080	3.75426e−7	3.98017
1/800	2.59839e−8		9.38867e−8	

Table 3
Comparisons of the error between the exact solutions and numerical solutions at the time T and the computed time, with the fixed mesh size $h = 1/10$ and time step $\Delta t = 1/500$.

	T=1			T=10		
	$\|(\boldsymbol{u}_h - \boldsymbol{u})(T)\|_{L^2(\Omega_f)}$	$\|(\varphi_h - \varphi)(T)\|_{L^2(\Omega_p)}$	CPU	$\|(\boldsymbol{u}_h - \boldsymbol{u})(T)\|_{L^2(\Omega_f)}$	$\|(\varphi_h - \varphi)(T)\|_{L^2(\Omega_p)}$	CPU
SISDC	1.76301e−2	6.03257e−2	75.428	1.32808e−2	2.48604e−2	78.123
CN	1.76301e−2	6.03258e−2	137.39	1.32816e−2	2.48600e−2	160.40
BDF2	4.76521e−2	6.68059e−1	100.74	2.34523e−2	1.35661e−1	112.01

6. Conclusions

In this paper, we introduce a second order scheme based on the SISDC method to solve the non-stationary mixed Navier–Stokes/Darcy model with the Beavers–Joseph–Saffman's interface condition. We give a complete theoretical analysis about the stability and error estimate of the algorithm. We prove that the algorithm is stable and second order accurate. The numerical experiments testify the rightness of our theoretical analysis and the advantages of our scheme compared with other second-order methods.

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