

UNIQUENESS IN INVERSE SCATTERING PROBLEMS WITH PHASELESS FAR-FIELD DATA AT A FIXED FREQUENCY. II*

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Abstract. This paper is concerned with uniqueness in inverse acoustic scattering with phaseless far-field data at a fixed frequency. In our previous work [*SIAM J. Appl. Math.*, 78 (2018), pp. 1737–1753], by utilizing spectral properties of the far-field operator we proved for the first time that the obstacle and the index of refraction of an inhomogeneous medium can be uniquely determined by the phaseless far-field patterns generated by infinitely many sets of superpositions of two plane waves with different directions at a fixed frequency under the a priori assumption that the obstacle is known to be a sound-soft or nonabsorbing impedance obstacle and the index of refraction n of the inhomogeneous medium is real-valued and satisfies that either $n - 1 \geq c_1$ or $n - 1 \leq -c_1$ in the support of $n - 1$ for some positive constant c_1 . In this paper, we remove the a priori assumption on the obstacle and the index of refraction of the inhomogeneous medium by adding a reference ball to the scattering system together with a simpler method of using Rellich's lemma and Green's representation formula for the scattering solutions. Further, our new method is also used to prove uniqueness in determining a locally rough surface from the phaseless far-field patterns corresponding to infinitely many sets of superpositions of two plane waves with different directions as the incident fields at a fixed frequency.

Key words. uniqueness, inverse scattering, phaseless far-field pattern, Dirichlet boundary conditions, impedance boundary conditions, inhomogeneous medium, locally rough surfaces

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1. Introduction. Inverse scattering theory has wide applications in such fields as radar, sonar, geophysics, medical imaging, and nondestructive testing (see, e.g., [11, 12, 17]). This paper is concerned with inverse scattering with phaseless far-field data generated by incident plane waves. We will consider three different scattering models: the scattering problems by a bounded obstacle, an inhomogeneous medium, and a locally perturbed infinite plane surface (called a locally rough surface in this paper).

Inverse scattering problems with full data have been widely studied both mathematically and numerically over the past decades (see, e.g., [11, 12, 17] and the references quoted there). However, in many applications, the phase of the near-field or far-field pattern cannot be measured accurately compared with its modulus or intensity. Therefore, it is often desirable to recover the obstacle and the medium

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from the modulus or intensity of the near-field or the far-field pattern (or the phaseless near-field data or the phaseless far-field data). Inverse scattering problems with phaseless near-field data, also called the (near-field) *phase retrieval problems* in optics and other physical and engineering sciences, have been widely studied numerically in recent decades (see, e.g., [2, 4, 8, 9, 10, 18, 20, 21]). Recently, several results have become available for the mathematical study (such as uniqueness and stability) of the near-field phase retrieval problem (see, e.g., [18, 19, 21, 30, 32, 33]).

In contrast to the case with phaseless near-field data, inverse scattering with phaseless far-field data is more challenging due to the *translation invariance property* of the phaseless far-field pattern (see [22, 27, 37, 38]). Thus, it is impossible to reconstruct the location of the scatterers from the phaseless far-field pattern with one plane wave as the incident field. Nevertheless, several reconstruction algorithms have been developed to reconstruct the shape of the scatterer from the phaseless far-field data with one plane wave as the incident field (see [1, 13, 14, 15, 22, 23, 24, 25, 35]). Further, by assuming a priori the scatterer to be a sound-soft ball centered at the origin, uniqueness was established in determining the radius of the ball from a single phaseless far-field datum in [28]. By studying the high-frequency asymptotics of the far-field pattern, it was proved in [29] that the shape of a general smooth convex sound-soft obstacle can be recovered from the modulus of the far-field pattern associated with one plane wave as the incident field. However, for plane wave incidence no uniqueness results are available for the inverse problem of recovering the scatterers from phaseless far-field data generated by one incident plane wave.

Recently, it was suggested in [38] to use superpositions of two plane waves rather than one plane wave as the incident fields with an interval of frequencies to break the translation invariance property of the phaseless far-field pattern. A recursive Newton iteration algorithm in frequencies was further developed in [38] to recover both the location and the shape of the obstacle simultaneously from multifrequency phaseless far-field data. The idea was also extended to inverse scattering by locally rough surfaces with phaseless far-field data in [39]. On the other hand, a fast imaging algorithm was developed in [40] to numerically recover the scattering obstacles from phaseless far-field data at a fixed frequency. Furthermore, it was rigorously proved in [37] that the obstacle and the index of refraction of an inhomogeneous medium can be uniquely determined by the phaseless far-field patterns generated by infinitely many sets of superpositions of two plane waves at a fixed frequency under *the a priori assumption* that the obstacle is a sound-soft obstacle or an impedance obstacle with a real-valued impedance function and the refractive index n of the inhomogeneous medium is real-valued and satisfies the condition that either $n - 1 \geq c_1$ or $n - 1 \leq -c_1$ in the support of $n - 1$ for some positive constant c_1 . The purpose of the present paper is to remove the a priori assumption on the obstacle and the refractive index of the inhomogeneous medium by adding a known reference ball to the scattering system in conjunction with a simple technique based on Rellich's lemma and Green's representation formula for the scattering solutions.

It should be remarked that the idea of adding a reference ball to the scattering system was first used in [26] to improve the linear sampling method numerically. It was recently used in [41] to prove uniqueness results for inverse scattering with phaseless far-field data. Precisely, it was proved in [41] that an obstacle D and its boundary condition can be uniquely determined by the phaseless far-field data

$$\begin{aligned} & \{|u_{D \cup B}^\infty(\hat{x}, d_0)| : \hat{x} \in \mathbb{S}^2\}, \quad \{|v_{D \cup B}^\infty(\hat{x}, z)| : \hat{x} \in \mathbb{S}^2, z \in \Pi\}, \\ & \{|u_{D \cup B}^\infty(\hat{x}, d_0) + v_{D \cup B}^\infty(\hat{x}, z)| : \hat{x} \in \mathbb{S}^2, z \in \Pi\} \end{aligned}$$

for a fixed wave number $k > 0$ and a fixed $d_0 \in \mathbb{S}^2$, where $\mathbb{S}^2 := \{d \in \mathbb{R}^3 : |d| = 1\}$ denotes the unit sphere in \mathbb{R}^3 , B is a reference sound-soft ball, $u_{D \cup B}^\infty(\hat{x}, d_0)$ and $u_{D \cup B}^\infty(\hat{x}, z)$ denote the far-field pattern of the scattering solution of the Helmholtz equation $\Delta u + k^2 u = 0$ associated with the obstacle $D \cup B$ corresponding to the incident plane wave $e^{ikx \cdot d_0}$ and the incident point source $e^{ik|x-z|}/(4\pi|x-z|)$ ($x \neq z$), respectively, and Π is the boundary of a simply connected convex polyhedron P such that $P \subset \mathbb{R}^3 \setminus \overline{D \cup B}$ and k^2 is not a Dirichlet eigenvalue of $-\Delta$ in P . A similar uniqueness result was also proved in [41] for the inhomogeneous medium case by adding a reference homogeneous medium ball.

The remaining part of the paper is organized as follows. In section 2, we introduce the direct and inverse scattering problems considered in this paper. Sections 3 and 4 are devoted to the inverse obstacle and medium scattering problems, respectively, while section 5 is devoted the inverse rough surface scattering problem. Conclusions are given in section 6.

2. The direct scattering problems. To give a precise description of the scattering problems, we assume that D is an open and bounded domain in \mathbb{R}^3 with C^2 -boundary $\Gamma := \partial D$ satisfying that the exterior $\mathbb{R}^3 \setminus \overline{D}$ of \overline{D} is connected. Note that D may not be connected and thus may consist of several (finitely many) connected components. Suppose the time-harmonic ($e^{-i\omega t}$ time dependence) plane wave

$$u^i = u^i(x, d) := e^{ikx \cdot d}$$

is incident on the obstacle D from the unbounded part $\mathbb{R}^3 \setminus \overline{D}$. Here, $d \in \mathbb{S}^2$ is the incident direction, $k = \omega/c > 0$ is the wave number, and ω and c are the wave frequency and speed in the homogeneous medium in $\mathbb{R}^3 \setminus \overline{D}$. Let u^s be the scattered field. Then the total field $u := u^i + u^s$ outside an impenetrable obstacle D satisfies the exterior boundary value problem:

$$(2.1a) \quad \Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D},$$

$$(2.1b) \quad \mathcal{B}u = 0 \quad \text{on } \partial D,$$

$$(2.1c) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|,$$

where (2.1a) is the Helmholtz equation and (2.1c) is the Sommerfeld radiation condition. The boundary condition \mathcal{B} in (2.1b) depends on the physical property of the obstacle D :

$$\begin{cases} \mathcal{B}u = u & \text{if } D \text{ is a sound-soft obstacle,} \\ \mathcal{B}u = \frac{\partial u}{\partial \nu} + \eta u & \text{if } D \text{ is an impedance obstacle,} \\ \mathcal{B}u = u \text{ on } \Gamma_D, \mathcal{B}u = \frac{\partial u}{\partial \nu} + \eta u \text{ on } \Gamma_I & \text{if } D \text{ is a partially coated obstacle,} \end{cases}$$

where ν is the unit outward normal to the boundary ∂D or Γ_I and η is the impedance function satisfying that $\text{Im}[\eta(x)] \geq 0$ for all $x \in \partial D$ or Γ_I . In this paper, we assume that $\eta \in C(\partial D)$ or $C(\Gamma_I)$, that is, η is continuous on ∂D or Γ_I . When $\eta = 0$, the impedance boundary condition becomes the Neumann boundary condition (a sound-hard obstacle). For a partially coated obstacle D , we assume that the boundary $\Gamma := \partial D$ has a Lipschitz dissection $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of Γ , having Π as their common boundary in Γ (see, e.g., [31]). Furthermore, boundary conditions of Dirichlet and impedance type are specified on Γ_D and Γ_I , respectively.

The problem of scattering of a plane wave by an inhomogeneous medium is modeled by the medium scattering problem:

$$(2.2a) \quad \Delta u + k^2 n(x)u = 0 \quad \text{in } \mathbb{R}^3,$$

$$(2.2b) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|,$$

where $u = u^i + u^s$ is the total field, u^s is the scattered field, and n in the reduced wave equation (2.2a) is the index of refraction characterizing the inhomogeneous medium. In this paper, we assume that $m := n - 1$ has the compact support \bar{D} and $n \in L^\infty(D)$ with $\text{Re}[n(x)] > 0$ and $\text{Im}[n(x)] \geq 0$ for almost all $x \in D$.

Well-posedness of the problems (2.1a)–(2.1c) and (2.2a)–(2.2b) has been established in [11, 16, 17] by using either the integral equation method or the variational method (see Theorem 3.11 in [11] and Theorem 1.1 in [17] for the exterior Dirichlet problem, Theorem 2.2 in [17] for the exterior impedance problem, Theorem 8.5 in [7] for the partially coated problem, and Theorem 6.9 in [16] for the medium scattering problem (2.2a)–(2.2b)). In particular, it is well-known that the scattered field u^s has the asymptotic behavior,

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|} \left[u^\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right], \quad |x| \rightarrow \infty,$$

uniformly for all observation directions $\hat{x} = x/|x| \in \mathbb{S}^2$, where $u^\infty(\hat{x}, d)$ is the far-field pattern of u^s which is an analytic function of $\hat{x} \in \mathbb{S}^2$ for each $d \in \mathbb{S}^2$ and of $d \in \mathbb{S}^2$ for each $\hat{x} \in \mathbb{S}^2$ (see, e.g., [11]).

The last model we consider is the scattering of acoustic plane waves by a locally perturbed plane surface. Precisely, let $h \in C^2(\mathbb{R}^2)$ with a compact support in \mathbb{R}^2 . Denote by $\Gamma := \{x = (x_1, x_2, x_3) : x_3 = h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ the locally rough surface and by $D_+ := \{x = (x_1, x_2, x_3) : x_3 > h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ the half-plane above the locally rough surface Γ which is filled with a homogeneous medium. Let $k = \omega/c > 0$ be the wave number in D_+ with ω and c being the wave frequency and speed, respectively. Assume that the time-harmonic ($e^{-i\omega t}$ time dependence) plane wave

$$u^i = u^i(x, d) = e^{ikx \cdot d}$$

is incident on the locally rough surface Γ from D_+ , where $d \in \mathbb{S}_-^2$ is the incident direction and $\mathbb{S}_-^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| = 1, x_3 < 0\}$ is the lower part of the unit sphere \mathbb{S}^2 . Then the total field u satisfies the Helmholtz equation

$$(2.3) \quad \Delta u + k^2 u = 0 \quad \text{in } D_+.$$

Here, the total field $u = u^i + u^r + u^s$ satisfies the Dirichlet boundary condition

$$(2.4) \quad u = u^i + u^r + u^s = 0 \quad \text{on } \Gamma$$

for a sound-soft surface or the Neumann boundary condition

$$(2.5) \quad \frac{\partial u}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \frac{\partial u^r}{\partial \nu} + \frac{\partial u^s}{\partial \nu} = 0 \quad \text{on } \Gamma$$

for a sound-hard surface, where ν is the unit outward normal vector on Γ directed into D_+ , u^r is the reflected wave by the infinite plane $x_3 = 0$,

$$u^r(x, d) := \begin{cases} -e^{ikx \cdot d'} & \text{for the Dirichlet boundary condition on } \Gamma, \\ e^{ikx \cdot d'} & \text{for the Neumann boundary condition on } \Gamma \end{cases}$$

with $d' = (d_1, d_2, -d_3) \in \mathbb{S}^2$, and u^s is the unknown scattered field to be determined. The scattered field u^s is required to satisfy the Sommerfeld radiation condition

$$(2.6) \quad \lim_{r \rightarrow \infty} r \left(\frac{\partial u^s}{\partial \nu} - iku^s \right) = 0, \quad r = |x|, \quad x \in D_+.$$

The problem (2.3)–(2.4) and (2.6) and the problem (2.3) and (2.5)–(2.6) model the problem of scattering of acoustic plane waves by a two-dimensional sound-soft surface and a two-dimensional sound-hard surface, respectively. In the case when the local perturbation is invariant in the x_2 -direction, these two problems can be reduced to the two-dimensional case, which also models the problem of electromagnetic plane waves by a locally perturbed, perfectly reflecting, infinite plane in the transverse-electric and transverse-magnetic polarization cases, respectively.

The existence of a unique solution to the problem (2.3)–(2.4) and (2.6) has been proved in [36] by the integral equation method, while the existence of a unique solution to the problem (2.3)–(2.4) and (2.6) can also be established similarly by using the integral equation method (see [36, Remark 4.7 (i)]). We remark that, in the two-dimensional case, the well-posedness of the problem (2.3)–(2.4) and (2.6) was proved in [42] by a new integral equation formulation and in [3] by the variational method, while the existence and uniqueness of solutions to the problem (2.3) and (2.5)–(2.6) was established in [34] by the integral equation method. From the results in [36] it is known that u^s has the asymptotic behavior at infinity

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|} \left[u^\infty(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right], \quad |x| \rightarrow \infty,$$

uniformly for all observation directions $\hat{x} \in \mathbb{S}_+^2$ with $\mathbb{S}_+^2 := \{x = (x_1, x_2, x_3) : |x| = 1, x_3 > 0\}$ the upper part of the unit sphere \mathbb{S}^2 , where $u^\infty(\hat{x}, d)$ is the far-field pattern of the scattered field u^s which is an analytic function of $\hat{x} \in \mathbb{S}_+^2$ for each $d \in \mathbb{S}_-^2$ and of $d \in \mathbb{S}_-^2$ for each $\hat{x} \in \mathbb{S}_+^2$.

Throughout this paper, we assume that the wave number k is arbitrarily fixed. Following [37, 38, 40], we make use of the following superposition of two plane waves as the incident field:

$$u^i = u^i(x; d_1, d_2) = u^i(x, d_1) + u^i(x, d_2) = e^{ikx \cdot d_1} + e^{ikx \cdot d_2},$$

where the incident directions $d_1, d_2 \in \mathbb{S}^2$ for the obstacle and medium scattering problems and $d_1, d_2 \in \mathbb{S}_-^2$ for the locally rough surface scattering problems. Then the scattered field u^s has the asymptotic behavior

$$u^s(x; d_1, d_2) = \frac{e^{ik|x|}}{|x|} \left\{ u^\infty(\hat{x}; d_1, d_2) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty,$$

uniformly for all observation directions $\hat{x} \in \mathbb{S}^2$ for the obstacle and medium scattering problems or the asymptotic behavior

$$u^s(x; d_1, d_2) = \frac{e^{ik|x|}}{|x|} \left\{ u^\infty(\hat{x}; d_1, d_2) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \rightarrow \infty,$$

uniformly for all observation directions $\hat{x} \in \mathbb{S}_+^2$ for the locally rough surface scattering problems. From the linear superposition principle it follows that

$$u^s(x; d_1, d_2) = u^s(x, d_1) + u^s(x, d_2)$$

and

$$u^\infty(\hat{x}; d_1, d_2) = u^\infty(\hat{x}, d_1) + u^\infty(\hat{x}, d_2),$$

where $u^s(x, d_j)$ and $u^\infty(\hat{x}, d_j)$ are the scattered field and its far-field pattern corresponding to the incident plane wave $u^i(x, d_j)$, respectively, $j = 1, 2$.

The *inverse obstacle or medium scattering problem* we consider in this paper is to reconstruct the obstacle D and its physical property or the index of refraction n of the inhomogeneous medium from the phaseless far-field pattern $|u^\infty(\hat{x}; d_1, d_2)|$ for $\hat{x}, d_j \in \mathbb{S}^2$, $j = 1, 2$, while the *inverse rough surface scattering problem* considered is to recover the locally rough surface Γ from the phaseless far-field pattern $|u^\infty(\hat{x}; d_1, d_2)|$ for $\hat{x} \in \mathbb{S}^2_+$, $d_j \in \mathbb{S}^2_-$, $j = 1, 2$. The purpose of this paper is to prove uniqueness results for these inverse problems.

3. Uniqueness for inverse obstacle scattering. Denote by u_j^s and u_j^∞ the scattered field and its far-field pattern, respectively, associated with the obstacle $D_j \cup B$ and corresponding to the incident field u^i , $j = 1, 2$, where B is a given, sound-soft reference ball. See Figure 3.1 for the geometry of the scattering problem. Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B , where k^2 is called a Dirichlet eigenvalue of $-\Delta$ in B if the interior Dirichlet boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases}$$

has a nontrivial solution u . Note that if the radius ρ of the ball B satisfies that $k\rho < \pi$, then k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B (see [11, Corollary 5.3]).

THEOREM 3.1. *Assume that B is a given sound-soft reference ball such that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B . Suppose D_1 and D_2 are two obstacles with $\overline{D_1} \cup \overline{D_2} \subset B_R$, where B_R is a ball of radius R and centered at the origin and satisfies that $\overline{B} \cap \overline{B_R} = \emptyset$. If the corresponding far-field patterns satisfy that*

$$(3.1) \quad |u_1^\infty(\hat{x}, d)| = |u_2^\infty(\hat{x}, d)| \quad \forall \hat{x}, d \in \mathbb{S}^2,$$

$$(3.2) \quad |u_1^\infty(\hat{x}; d, d_0)| = |u_2^\infty(\hat{x}; d, d_0)| \quad \forall \hat{x}, d \in \mathbb{S}^2$$

for an arbitrarily fixed $d_0 \in \mathbb{S}^2$, then $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.

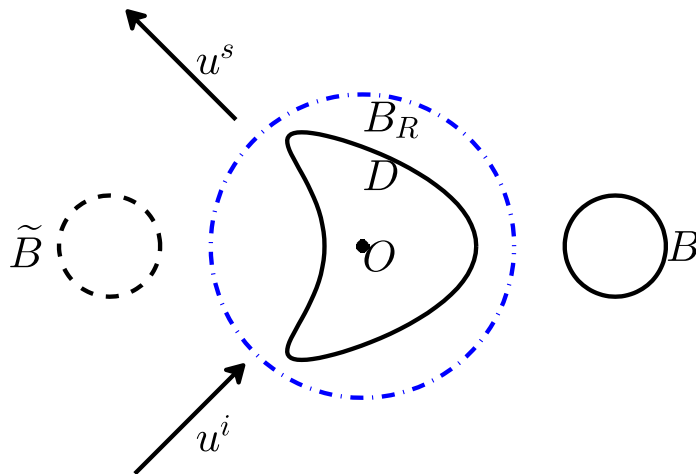


FIG. 3.1. Scattering by a bounded obstacle.

Proof. Arguing similarly as in the proof of (4.9) and (4.10) in [37] (see the proof of Theorem 2.2 in [37]), we can obtain from (3.1) and (3.2) that either

$$(3.3) \quad u_1^\infty(\hat{x}, d) = e^{i\alpha} u_2^\infty(\hat{x}, d) \quad \forall \hat{x}, d \in \mathbb{S}^2$$

or

$$(3.4) \quad u_1^\infty(\hat{x}, d) = e^{i\beta} \overline{u_2^\infty(\hat{x}, d)} \quad \forall \hat{x}, d \in \mathbb{S}^2$$

holds for some real constants α, β which are independent of \hat{x} and d .

We now show that (3.4) does not hold. In fact, suppose (3.4) holds. By Green's representation for the scattering solution u_2^s in $\mathbb{R}^3 \setminus (\overline{B \cup B_R})$ (see [11, Theorem 2.5]) we obtain that for any fixed $d \in \mathbb{S}^2$,

$$u_2^s(x, d) = \int_{\partial B \cup \partial B_R} \left\{ u_2^s(y, d) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u_2^s(y, d)}{\partial \nu(y)} \Phi(x, y) \right\} ds(y),$$

$$x \in \mathbb{R}^3 \setminus \overline{(B \cup B_R)},$$

where $\nu(y)$ is the unit normal vector at $y \in \partial B$ or $y \in \partial B_R$ directed into the exterior of B or B_R and $\Phi(x, y)$ is the fundamental solution to the Helmholtz equation in \mathbb{R}^3 given by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

and thus the far-field pattern of u_2^s is given as follows (see [11, (2.14)]):

$$u_2^\infty(\hat{x}, d) = \frac{1}{4\pi} \int_{\partial B \cup \partial B_R} \left\{ u_2^s(y, d) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial u_2^s(y, d)}{\partial \nu} e^{-ik\hat{x}\cdot y} \right\} ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

From this and (3.4) it follows that for any $\hat{x} \in \mathbb{S}^2$,

$$\begin{aligned} u_1^\infty(\hat{x}, d) &= e^{i\beta} \overline{u_2^\infty(\hat{x}, d)} \\ &= \frac{e^{i\beta}}{4\pi} \int_{\partial B \cup \partial B_R} \left\{ \overline{u_2^s(y, d)} \frac{\partial e^{ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial \overline{u_2^s}(y, d)}{\partial \nu} e^{ik\hat{x}\cdot y} \right\} ds(y) \\ &= \frac{e^{i\beta}}{4\pi} \int_{\partial \tilde{B} \cup \partial B_R} \left\{ \overline{u_2^s(-y, d)} \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial \overline{u_2^s}(-y, d)}{\partial \nu} e^{-ik\hat{x}\cdot y} \right\} ds(y), \end{aligned}$$

where $\tilde{B} := \{x \in \mathbb{R}^3 : -x \in B\}$. Rellich's lemma gives

$$(3.5) \quad u_1^s(x, d) = e^{i\beta} \int_{\partial \tilde{B} \cup \partial B_R} \left\{ \overline{u_2^s(-y, d)} \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \overline{u_2^s}(-y, d)}{\partial \nu} \Phi(x, y) \right\} ds(y),$$

$$x \in \mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}.$$

This means that $u_1^s(\cdot, d)$ can be analytically extended into $\mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$ and satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$. On the other hand, $u_1^s(\cdot, d)$ also satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B \cup B_R}$. Since $\overline{B} \cap \overline{B_R} = \emptyset$, then the origin $O \notin B$ and $B \cap \tilde{B} = \emptyset$. Therefore, $u_1^s(\cdot, d)$ satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \overline{B_R}$. Noting that the total field $u_1 := u^i + u_1^s$ satisfies the sound-soft boundary condition on ∂B , we have

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0 & \text{in } B, \\ u_1 = 0 & \text{on } \partial B. \end{cases}$$

Since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B , we have $u_1 \equiv 0$ in B , which, together with the analyticity of the total field u_1 in $\mathbb{R}^3 \setminus \overline{B_R}$, implies that $u_1 \equiv 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$. This is a contradiction, so (3.4) does not hold. Therefore, (3.3) holds.

Now, by (3.3) and Rellich's lemma we obtain that

$$u_1^s(x, d) = e^{i\alpha} u_2^s(x, d) \quad \forall x \in G, d \in \mathbb{S}^2,$$

where G denotes the unbounded component of the complement of $B \cup D_1 \cup D_2$. By the sound-soft boundary condition on ∂B , we have $u_j^s(x, d) = -e^{ikx \cdot d}$ for all $x \in \partial B$, $j = 1, 2$, which implies that

$$-e^{ikx \cdot d} = -e^{i\alpha} e^{ikx \cdot d} \quad \forall x \in \partial B.$$

Hence $e^{i\alpha} = 1$, which means that

$$u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d) \quad \forall \hat{x}, d \in \mathbb{S}^2.$$

This, together with [11, Theorem 5.6], implies that $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$. The proof is thus complete. \square

Remark 3.2. Theorem 3.1 remains true for the two-dimensional case, and the proof is similar.

4. Uniqueness for inverse medium scattering. Assume that B is a given homogeneous medium ball characterized by the positive constant refractive index $n_0 \neq 1$. Assume further that $n_1, n_2 \in L^\infty(\mathbb{R}^3)$ are the refractive indices of two inhomogeneous media with $m_j := n_j - 1$ supported in $\overline{D_j}, j = 1, 2$. Denote by u_j^s and u_j^∞ the scattered field and its far-field pattern, respectively, associated with the inhomogeneous medium with the refractive index \tilde{n}_j and corresponding to the incident field $u^i, j = 1, 2$. Here, the refractive index \tilde{n}_j is given by

$$\tilde{n}_j(x) := \begin{cases} n_0, & x \in B, \\ n_j(x), & x \in \mathbb{R}^3 \setminus \overline{B}, \end{cases}$$

for $j = 1, 2$. See Figure 4.1 for a geometric description of the scattering problem. Suppose k^2 is not an interior transmission eigenvalue of $-\Delta$ in B . Here, k^2 is called an interior transmission eigenvalue of $-\Delta$ in B if the interior transmission problem

$$\begin{cases} \Delta w + k^2 n_0 w = 0 & \text{in } B, \\ \Delta v + k^2 v = 0 & \text{in } B, \\ w = v, \quad \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \partial B \end{cases}$$

has a nontrivial solution (w, v) . Note that if the radius ρ of the ball B satisfies that $\rho < \pi/[2k(\sqrt{n_0} + 1)]$, then k^2 is not an interior transmission eigenvalue of $-\Delta$ in B (see [41, Theorem 4.1]).

THEOREM 4.1. *Assume that B is a given homogeneous medium ball with the constant refractive index $n_0 \neq 1$ such that k^2 is not an interior transmission eigenvalue of $-\Delta$ in B . Assume further that $n_1, n_2 \in L^\infty(\mathbb{R}^3)$ are the refractive indices of two inhomogeneous media with $m_j := n_j - 1$ supported in $\overline{D_j}, j = 1, 2$. Suppose $\overline{D_1} \cup \overline{D_2} \subset B_R$, where B_R is a ball of radius R and centered at the origin and satisfies that $\overline{B} \cap \overline{B_R} = \emptyset$. If the corresponding far field patterns satisfy (3.1) and (3.2) for an arbitrarily fixed $d_0 \in \mathbb{S}^2$, then $n_1 = n_2$.*

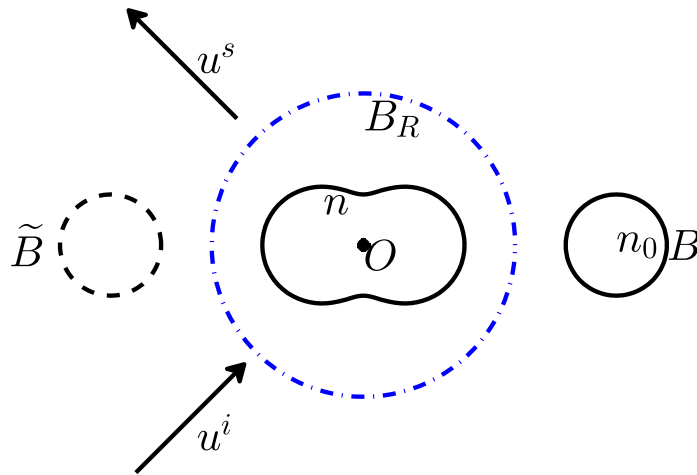


FIG. 4.1. Scattering by an inhomogeneous medium.

Proof. We first claim that for any fixed $d_0 \in \mathbb{S}^2$ we have $u_j^\infty(\cdot, d_0) \not\equiv 0$ in \mathbb{S}^2 , $j = 1, 2$. Otherwise, $u_j^\infty(\hat{x}, d_0) \equiv 0$ for any $\hat{x} \in \mathbb{S}^2$, and so, by Rellich's lemma, $u_j^s(x, d_0) \equiv 0$ for any $x \in \mathbb{R}^3 \setminus (\overline{B \cup D_j})$, $j = 1, 2$. Thus, $\Delta u_j + k^2 n_0 u_j = 0$ in B , where $u_j = u^i + u_j^s$ is the total field satisfying that $u_j = u^i$, $\partial u_j / \partial \nu = \partial u^i / \partial \nu$ on ∂B . Note that $\Delta u^i + k^2 u^i = 0$ in B . Since k^2 is not an interior transmission eigenvalue of $-\Delta$ in B , we have $u_j = u^i \equiv 0$ in B . This is a contradiction, implying that our claim is true. Thus, by arguing similarly as in the beginning of the proof of Theorem 3.1 it follows from (3.1) and (3.2) that either (3.3) or (3.4) holds.

Again we can prove that (3.4) does not hold. Suppose (3.4) holds to the contrary. Then, by a similar argument as in the proof of (3.5) we can obtain that

$$(4.1) \quad u_1^s(x, d) = e^{i\beta} \int_{\partial \tilde{B} \cup \partial B_R} \left\{ \overline{u_2^s(-y, d)} \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \overline{u_2^s}}{\partial \nu}(-y, d) \Phi(x, y) \right\} ds(y),$$

$$x \in \mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}.$$

Since $\overline{B} \cap \overline{B_R} = \emptyset$, then $B \cap \tilde{B} = \emptyset$ and $B \subset \mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$. Thus, by (4.1) we know that the total field $u_1 := u^i + u_1^s$ satisfies that $\Delta u_1 + k^2 u_1 = 0$ in B . On the other hand, by the definition of u_1 we have that $\Delta u_1 + k^2 n_0 u_1 = 0$ in B . Then, we obtain that $u_1 \equiv 0$ in B . By (4.1) again it is known that u_1^s (and thus u_1) is analytic in $\mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$. Thus, and on noting that $B \subset \mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$, we obtain by the analytic continuation that $u_1 \equiv 0$ in $\mathbb{R}^3 \setminus \overline{\tilde{B} \cup B_R}$. This is a contradiction, implying that (3.4) does not hold. Hence, (3.3) holds.

By (3.3) and Rellich's lemma it follows that

$$(4.2) \quad u_1^s(x, d) = e^{i\alpha} u_2^s(x, d) \quad \forall x \in G, d \in \mathbb{S}^2,$$

where G denotes the unbounded component of the complement of $B \cup D_1 \cup D_2$. For any fixed $d \in \mathbb{S}^2$, define $u := u_1 - e^{i\alpha} u_2$ and $v := (1 - e^{i\alpha}) u^i$ in \mathbb{R}^3 . Then, by (4.2) it is seen that $u = v$ in G , and so, $u = v$ and $\partial u / \partial \nu = \partial v / \partial \nu$ on ∂B . On the other

hand, by the definition of u_j , $j = 1, 2$, we have $\Delta u + k^2 n_0 u = 0$ in B . As a result, we obtain that

$$\begin{cases} \Delta u + k^2 n_0 u = 0 & \text{in } B, \\ \Delta v + k^2 v = 0 & \text{in } B, \\ u = v, \quad \partial u / \partial \nu = \partial v / \partial \nu & \text{on } \partial B. \end{cases}$$

Since k^2 is not an interior transmission eigenvalue of $-\Delta$ in B , we must have $e^{i\alpha} = 1$, and so, (3.3) becomes

$$u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d) \quad \forall \hat{x}, d \in \mathbb{S}^2.$$

By this and [16, Theorem 6.26] we have $n_1 = n_2$. The proof is thus complete. \square

Remark 4.2. Theorem 4.1 also holds in two dimensions if the assumption $n_j \in L^\infty(\mathbb{R}^3)$ is replaced by the condition that n_j is piecewise in $W^{1,p}(D_j)$ with $p > 2$, $j = 1, 2$. In this case, the proof is similar except that we need Bukhgeim’s result in [6] (see also the theorem in section 4.1 in [5]) instead of [16, Theorem 6.26] in the proof.

5. Uniqueness for inverse scattering by locally rough surfaces. Suppose $h_j \in C^2(\mathbb{R}^2)$ has a compact support in \mathbb{R}^2 , $j = 1, 2$. Let us denote by $\Gamma_j = \{(x_1, x_2, x_3) : x_3 = h_j(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ the locally rough surface and by $D_{j,+} = \{(x_1, x_2, x_3) : x_3 > h_j(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ the half-space above the locally rough surface Γ_j , $j = 1, 2$. Let $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ be a ball with radius R large enough so that the local perturbation $\Gamma_{jp} := \Gamma_j \setminus \Gamma_0 = \{(x_1, x_2, h_j(x_1, x_2)) : (x_1, x_2) \in \text{supp}(h_j)\} \subset B_R$, $j = 1, 2$, where $\Gamma_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$ is the infinite plane. Then $\Gamma_{jR} := \Gamma_j \cap B_R$ represents the part of Γ_j containing the local perturbation Γ_{jp} of the infinite plane Γ_0 , $j = 1, 2$. Let $B \subset D_{1,+} \cap D_{2,+}$ be a given, sound-soft reference ball (that is, above the locally rough surfaces Γ_1 and Γ_2) satisfying that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B . See Figure 5.1 for the geometry of the scattering problem.

Denote by u_j^s and u_j^∞ the scattered field and its far-field pattern, respectively, associated with $\{\Gamma_j, B\}$ and corresponding to the incident plane wave u^i , $j = 1, 2$. Then we have the following theorem on uniqueness in inverse scattering by locally rough surfaces with phaseless far-field data.

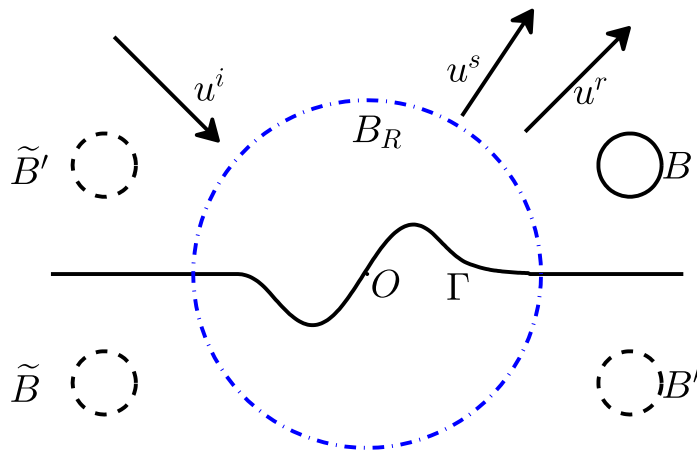


FIG. 5.1. Scattering by a locally rough surface.

THEOREM 5.1. *Suppose Γ_j , $j = 1, 2$, are two sound-soft, locally rough surfaces. Assume that $\overline{B} \cap (\overline{B_R} \cup \{(0, 0, x_3) : x_3 \in \mathbb{R}\}) = \emptyset$. If the corresponding far-field patterns satisfy that*

$$(5.1) \quad |u_1^\infty(\hat{x}, d)| = |u_2^\infty(\hat{x}, d)| \quad \forall \hat{x} \in \mathbb{S}_+^2, d \in \mathbb{S}_-^2,$$

$$(5.2) \quad |u_1^\infty(\hat{x}; d, d_0)| = |u_2^\infty(\hat{x}; d, d_0)| \quad \forall \hat{x} \in \mathbb{S}_+^2, d \in \mathbb{S}_-^2,$$

for an arbitrarily fixed $d_0 \in \mathbb{S}_-^2$, then $\Gamma_1 = \Gamma_2$.

5.1. Some lemmas. To prove Theorem 5.1 we need some lemmas. In this subsection, we assume that $h \in C^2(\mathbb{R}^2)$ has a compact support in \mathbb{R}^2 . Denote by $\Gamma := \{(x_1, x_2, x_3) : x_3 = h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ and $D_+ = \{(x_1, x_2, x_3) : x_3 > h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2\}$ the locally rough surface and the half-space above the locally rough surface Γ , respectively. Assume that the local perturbation $\Gamma_p := \Gamma \setminus \Gamma_0 = \{(x_1, x_2, h(x_1, x_2)) : (x_1, x_2) \in \text{suppt}(h)\} \subset B_R$. Then $\Gamma_R := \Gamma \cap B_R$ represents the part of Γ containing the local perturbation Γ_p of the infinite plane Γ_0 . Assume further that the sound-soft ball $B \subset D_+$ with $\overline{B} \cap \overline{B_R} = \emptyset$.

LEMMA 5.2. *For any fixed $d_0 \in \mathbb{S}_-$, let $u^\infty(\hat{x}, d_0)$ denote the far-field pattern corresponding to $\{\Gamma, B\}$. Then $u^\infty(\cdot, d_0) \not\equiv 0$ in \mathbb{S}_+^2 .*

Proof. Suppose to the contrary that $u^\infty(\hat{x}, d_0) \equiv 0$ for all $\hat{x} \in \mathbb{S}_+^2$. Then, by Rellich's lemma (see [36, Theorem 2.2]) we have

$$u^s(x, d_0) = 0 \quad \forall x \in D_+ \setminus \overline{B}.$$

The Dirichlet boundary condition implies that

$$u(x, d_0) := u^i(x, d_0) + u^r(x, d_0) + u^s(x, d_0) = u^i(x, d_0) + u^r(x, d_0) = 0 \quad \text{on } \Gamma \cup \partial B.$$

Noting that $\Delta u + k^2 u = 0$ in B and that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B , we obtain that $u(x, d_0) \equiv 0$ or $u^i(x, d_0) \equiv -u^r(x, d_0)$ for $x \in B$. This is a contradiction, and the proof is thus complete. \square

LEMMA 5.3. *Assume further that v is a solution to the Helmholtz equation $\Delta v + k^2 v = 0$ in $D_+ \setminus \overline{B}$ with the boundary condition $v = 0$ on $\Gamma \setminus \overline{B_R}$. Let \tilde{v} be the extension of v into $D_- \setminus (\overline{B_R} \cup B')$ by reflection, that is,*

$$\tilde{v}(x) = \begin{cases} v(x), & x \in D_+ \setminus \overline{B}, \\ -v(x'), & x \in D_- \setminus (\overline{B_R} \cup B'), \end{cases}$$

where $D_- := \{x \in \mathbb{R}^3 : x_3 < h(x_1, x_2)\}$, $B' := \{x \in \mathbb{R}^3 : x' \in B\}$, and $x' = (x_1, x_2, -x_3)$. Then \tilde{v} satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus (\overline{B_R} \cup B \cup B')$.

Proof. Note that the local perturbation $\Gamma_p \in B_R$ and $\overline{B} \cap \overline{B_R} = \emptyset$. By a regularity argument (see [36, Theorem 3.1]) and the reflection principle, we know that $\tilde{v} \in C^2(\mathbb{R}^3 \setminus (\overline{B_R} \cup B \cup B'))$ and satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus (\overline{B_R} \cup B \cup B')$ (cf. [17, Theorem 2.18(a)]). The proof is complete. \square

LEMMA 5.4. *For any $d \in \mathbb{S}_-^2$, let $u^\infty(\hat{x}, d)$ denote the far-field pattern corresponding to $\{\Gamma, B\}$. Then we have the reciprocity relation*

$$(5.3) \quad u^\infty(\hat{x}, d) = u^\infty(-d, -\hat{x}) \quad \forall \hat{x} \in \mathbb{S}_+^2, d \in \mathbb{S}_-^2.$$

Proof. Choose $R' > R$ large enough such that $\overline{B_R \cup B} \subset B_{R'}$. Extend the total field $u(x, d) = u^i(x, d) + u^r(x, d) + u^s(x, d)$ into $D_- \setminus \overline{B_{R'}}$ by reflection as in Lemma 5.3, which is an odd function and denoted again by $u(x, d)$. Apply Green's second formula to $(u^i + u^r)(y, d)$ and $(u^i + u^r)(y, -\hat{x})$ in $B_{R'}$ to get

$$(5.4) \quad 0 = \int_{\partial B_{R'}} \left[(u^i + u^r)(y, d) \frac{\partial(u^i + u^r)(y, -\hat{x})}{\partial \nu} - (u^i + u^r)(y, -\hat{x}) \frac{\partial(u^i + u^r)(y, d)}{\partial \nu} \right] ds(y).$$

On the other hand, applying Green's second formula to $u^s(y, d)$ and $u^s(y, -\hat{x})$ in $\mathbb{R}^3 \setminus \overline{B_{R'}}$ and making use of the Sommerfeld radiation condition for the scattered field u^s we obtain that

$$(5.5) \quad 0 = \int_{\partial B_{R'}} \left[u^s(y, d) \frac{\partial u^s(y, -\hat{x})}{\partial \nu} - u^s(y, -\hat{x}) \frac{\partial u^s(y, d)}{\partial \nu} \right] ds(y).$$

Recalling the definition of u^i and u^r it follows from [11, Theorem 2.6] that

$$\begin{aligned} 4\pi u^\infty(\hat{x}, d) &= \int_{\partial B_{R'}} \left[u^s(y, d) \frac{\partial u^i}{\partial \nu}(y, -\hat{x}) - u^i(y, -\hat{x}) \frac{\partial u^s}{\partial \nu}(y, d) \right] ds(y) \\ &= \int_{\partial B_{R'}} \left[u^s(y, d) \frac{\partial u^r}{\partial \nu}(y, -\hat{x}) - u^r(y, -\hat{x}) \frac{\partial u^s}{\partial \nu}(y, d) \right] ds(y), \end{aligned}$$

which implies that

$$(5.6) \quad 4\pi u^\infty(\hat{x}, d) = \frac{1}{2} \int_{\partial B_{R'}} \left[u^s(y, d) \frac{\partial(u^i + u^r)}{\partial \nu}(y, -\hat{x}) - (u^i + u^r)(y, -\hat{x}) \frac{\partial u^s}{\partial \nu}(y, d) \right] ds(y).$$

Interchanging the role of \hat{x} and d in the above equation gives

$$(5.7) \quad 4\pi u^\infty(-d, -\hat{x}) = \frac{1}{2} \int_{\partial B_{R'}} \left[u^s(y, -\hat{x}) \frac{\partial(u^i + u^r)}{\partial \nu}(y, d) - (u^i + u^r)(y, d) \frac{\partial u^s}{\partial \nu}(y, -\hat{x}) \right] ds(y).$$

Combining (5.4)–(5.7) leads to the result

$$4\pi[u^\infty(\hat{x}, d) - u^\infty(-d, -\hat{x})] = \frac{1}{2} \int_{\partial B_{R'}} \left[u(y, d) \frac{\partial u}{\partial \nu}(y, -\hat{x}) - u(y, -\hat{x}) \frac{\partial u}{\partial \nu}(y, d) \right] ds(y).$$

Since u is odd, the integrand of the integral term on the right-hand side of the above equation is even and so,

$$\begin{aligned} 4\pi[u^\infty(\hat{x}, d) - u^\infty(-d, -\hat{x})] &= \int_{\partial B_{R'}^+} \left[u(y, d) \frac{\partial u}{\partial \nu}(y, -\hat{x}) - u(y, -\hat{x}) \frac{\partial u}{\partial \nu}(y, d) \right] ds(y) \\ &= \int_{\partial B \cup \Gamma_{R'}} \left[u(y, d) \frac{\partial u}{\partial \nu}(y, -\hat{x}) - u(y, -\hat{x}) \frac{\partial u}{\partial \nu}(y, d) \right] ds(y), \end{aligned}$$

where $\partial B_{R'}^+ := \{x \in \partial B_{R'} : x_3 > 0\}$, $\Gamma_{R'} := \Gamma \cap B_{R'}$ and Green's second formula is applied to $u(y, d)$ and $u(y, -\hat{x})$ in $D_+ \cap B_{R'}$ to obtain the last equality. The required equation (5.3) follows from the above equation on using the boundary condition $u(y, d) = u(y, -\hat{x}) = 0$ for $y \in \Gamma \cup \partial B$. The lemma is thus proved. \square

5.2. Proof of Theorem 5.1. By making use of Lemmas 5.2 and 5.4 and arguing similarly as in the proof of (4.9) and (4.10) in [37], it can be obtained from (5.1) and (5.2) that

$$(5.8) \quad u_1^\infty(\hat{x}, d) = e^{i\alpha} u_2^\infty(\hat{x}, d) \quad \forall \hat{x} \in \mathbb{S}_+^2, \quad d \in \mathbb{S}_-^2,$$

or

$$(5.9) \quad u_1^\infty(\hat{x}, d) = e^{i\beta} \overline{u_2^\infty(\hat{x}, d)} \quad \forall \hat{x} \in \mathbb{S}_+^2, \quad d \in \mathbb{S}_-^2,$$

where α and β are real constants independent of \hat{x} and d .

We now prove that (5.9) does not hold. To this end, we consider the extension \tilde{u}_j^s of u_j^s defined by Lemma 5.3, $j = 1, 2$. For arbitrarily fixed $d \in \mathbb{S}_-^2$, by Green's representation for the scattering solution \tilde{u}_2^s in $\mathbb{R}^3 \setminus \overline{B_R \cup B \cup B'}$ (see [11, Theorem 2.5]) we obtain that

$$\begin{aligned} \tilde{u}_2^s(x, d) &= \int_{\partial B_R \cup \partial B \cup \partial B'} \left[\tilde{u}_2^s(y, d) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \tilde{u}_2^s(y, d)}{\partial \nu(y)} \Phi(x, y) \right] ds(y) \\ &\quad \forall x \in \mathbb{R}^3 \setminus \overline{B_R \cup B \cup B'}. \end{aligned}$$

Thus the far-field pattern u_2^∞ of u_2^s is given by

$$\begin{aligned} u_2^\infty(\hat{x}, d) &= \tilde{u}_2^\infty(\hat{x}, d) = \frac{1}{4\pi} \int_{\partial B_R \cup \partial B \cup \partial B'} \left[\tilde{u}_2^s(y, d) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial \tilde{u}_2^s(y, d)}{\partial \nu} e^{-ik\hat{x}\cdot y} \right] ds(y) \\ &\quad \forall \hat{x} \in \mathbb{S}_+^2. \end{aligned}$$

From (5.9) it follows that

$$\begin{aligned} u_1^\infty(\hat{x}, d) &= e^{i\beta} \overline{u_2^\infty(\hat{x}, d)} \\ &= \frac{e^{i\beta}}{4\pi} \int_{\partial B_R \cup \partial B \cup \partial B'} \left[\overline{\tilde{u}_2^s(y, d)} \frac{\partial e^{ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial \overline{\tilde{u}_2^s(y, d)}}{\partial \nu} e^{ik\hat{x}\cdot y} \right] ds(y) \\ &= \frac{e^{i\beta}}{4\pi} \int_{\partial B_R \cup \partial \tilde{B} \cup \partial \tilde{B}'} \left[\overline{\tilde{u}_2^s(-y, d)} \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial \overline{\tilde{u}_2^s(-y, d)}}{\partial \nu} e^{-ik\hat{x}\cdot y} \right] ds(y), \end{aligned}$$

where $\tilde{B}' := \{x \in \mathbb{R}^3 : -x \in B'\}$. By Rellich's lemma (see [36, Theorem 2.2]) we have

$$\begin{aligned} u_1^s(x, d) &= e^{i\beta} \int_{\partial B_R \cup \partial \tilde{B} \cup \partial \tilde{B}'} \left[\overline{\tilde{u}_2^s(-y, d)} \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial \overline{\tilde{u}_2^s(-y, d)}}{\partial \nu} \Phi(x, y) \right] ds(y) \\ &\quad \forall x \in \mathbb{R}_+^3 \setminus \overline{B_R \cup \tilde{B} \cup \tilde{B}'}. \end{aligned}$$

This means that $u_1^s(\cdot, d)$ satisfies the Helmholtz equation in $\mathbb{R}_+^3 \setminus \overline{\tilde{B} \cup \tilde{B}'}$ and is analytic in $\mathbb{R}_+^3 \setminus \overline{\tilde{B} \cup \tilde{B}'}$. On the other hand, $u_1^s(\cdot, d)$ satisfies the Helmholtz equation in $\mathbb{R}_+^3 \setminus \overline{B \cup B_R}$. Since $\overline{B} \cap (\overline{B_R} \cup \{(0, 0, x_3) : x_3 \in \mathbb{R}\}) = \emptyset$, $(B \cup B_R) \cap \tilde{B}' = \emptyset$. Therefore,

$u_1^s(\cdot, d)$ satisfies the Helmholtz equation in $\mathbb{R}_+^3 \setminus \overline{B_R}$. Thus the total field $u_1 = u^i + u^r + u_1^s$ satisfies that

$$\begin{cases} \Delta u_1 + k^2 u_1 = 0 & \text{in } B, \\ u_1 = 0 & \text{on } \partial B. \end{cases}$$

Since k^2 is not a Dirichlet eigenvalue of $-\Delta$ in B , we have $u_1 \equiv 0$ in B , which, together with the analyticity of the total field u_1 in $\mathbb{R}_+^3 \setminus \overline{B_R}$, implies that $u_1 \equiv 0$ in $\mathbb{R}_+^3 \setminus \overline{B_R}$. This is a contradiction, leading to the fact that (5.9) is impossible. Thus, (5.8) is true.

Now, by (5.8) and Rellich’s lemma (see [36, Theorem 2.2]) we have

$$(5.10) \quad u_1^s(x, d) = e^{i\alpha} u_2^s(x, d) \quad \forall x \in \mathbb{R}_+^3 \setminus \overline{B \cup B_R}, \quad d \in \mathbb{S}_-^2.$$

From the boundary condition on ∂B it follows that $u_j^s(x, d) = -[u^i(x, d) + u^r(x, d)]$ for all $x \in \partial B$, $d \in \mathbb{S}_-^2$, $j = 1, 2$. This, together with (5.10), yields that $e^{i\alpha} = 1$, leading to the fact that

$$(5.11) \quad u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d) \quad \forall \hat{x} \in \mathbb{S}_+^2, \quad d \in \mathbb{S}_-^2.$$

By (5.11) and Theorem 4.1 in [42] it follows that $\Gamma_1 = \Gamma_2$. This completes the proof of Theorem 5.1.

Similarly, we can prove the following theorem for the Neumann boundary condition case.

THEOREM 5.5. *Assume that Γ_j , $j = 1, 2$, are two sound-hard, locally rough surfaces. Assume that $\overline{B} \cap (\overline{B_R} \cup \{(0, 0, x_3) : x_3 \in \mathbb{R}\}) = \emptyset$. If the corresponding far-field patterns satisfy that*

$$\begin{aligned} |u_1^\infty(\hat{x}; d)| &= |u_2^\infty(\hat{x}; d)| \quad \forall \hat{x} \in \mathbb{S}_+^2, \quad d \in \mathbb{S}_-^2, \\ |u_1^\infty(\hat{x}; d, d_0)| &= |u_2^\infty(\hat{x}; d, d_0)| \quad \forall \hat{x} \in \mathbb{S}_+^2, \quad d \in \mathbb{S}_-^2, \end{aligned}$$

for an arbitrarily fixed $d_0 \in \mathbb{S}_-^2$, then $\Gamma_1 = \Gamma_2$.

Remark 5.6. Theorems 5.1 and 5.5 also hold in the two-dimensional case, and the proof is similar.

6. Conclusion. In this paper, we have proved uniqueness results for inverse obstacle and medium scattering with phaseless far-field data generated by infinitely many sets of superpositions of two plane waves with different incident directions at a fixed frequency, without knowing the physical property of the obstacle and the inhomogeneous medium in advance. Our method is based on putting a known reference ball into the scattering system in conjunction with a simple technique of using Green’s representation formula for the scattering solutions and Rellich’s lemma. These results improved our previous results in [37], where similar uniqueness results have been obtained under certain a priori assumptions on the physical property of the obstacle and the inhomogeneous medium. We have further used our method to prove uniqueness in determining a sound-soft or sound-hard locally rough surface from the phaseless far-field patterns corresponding to infinitely many sets of superpositions of two plane waves with different incident directions at a fixed frequency. Our results are also valid in the two-dimensional case. It is interesting to prove the same uniqueness results without the reference ball. For the locally rough surface case, our uniqueness results require us to know the property of the surface, and our analysis relies on the reflection principle. Thus, it is interesting to establish similar results for the impedance and penetrable surface cases.

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