Inverse Problems and Imaging doi:10.3934/ipi.2020023
Volume 14, No. 3, 2020, 489–510

UNIQUENESS IN INVERSE ACOUSTIC AND ELECTROMAGNETIC SCATTERING WITH PHASELESS NEAR-FIELD DATA AT A FIXED FREQUENCY

Xiaoxu Xu

Beijing Computational Science Research Center
Beijing 100193, China

Bo Zhang

LSEC, NCMIS and Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, China
School of Mathematical Sciences, University of Chinese Academy of Sciences
Beijing 100049, China

Haiwen Zhang*

NCMIS and Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, China
Institute for Numerical and Applied Mathematics, University of Göttingen
Lotzestr. 16-18, 37083 Göttingen, Germany

(Communicated by Hongyu Liu)

Abstract. This paper is concerned with uniqueness results in inverse acoustic and electromagnetic scattering problems with phaseless total-field data at a fixed frequency. We use superpositions of two point sources as the incident fields at a fixed frequency and measure the modulus of the acoustic total-field (called phaseless acoustic near-field data) on two spheres containing the scatterers generated by such incident fields on the two spheres. Based on this idea, we prove that the impenetrable bounded obstacle or the index of refraction of an inhomogeneous medium can be uniquely determined from the phaseless acoustic near-field data at a fixed frequency. Moreover, the idea is also applied to the electromagnetic case, and it is proved that the impenetrable bounded obstacle or the index of refraction of an inhomogeneous medium can be uniquely determined by the phaseless electric near-field data at a fixed frequency, that is, the modulus of the tangential component with the orientations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, of the electric total-field measured on a sphere enclosing the scatters and generated by superpositions of two electric dipoles at a fixed frequency located on the measurement sphere and another bigger sphere with the polarization vectors $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively. As far as we know, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data.

2010 *Mathematics Subject Classification.* Primary: 78A46; Secondary: 35P25.

Key words and phrases. Uniqueness, inverse acoustic scattering, inverse electromagnetic scattering, phaseless near-field, obstacle, inhomogeneous medium.

This work is partly supported by the NNSF of China grants 91630309, 11871466 and 11571355, the NSAF grant U1930402, and the China Postdoctoral Science Foundation grant 2019TQ0023.

* Corresponding author: Haiwen Zhang.

1. **Introduction.** Inverse scattering problems occur in many applications such as radar, remote sensing, geophysics, medical imaging and nondestructive testing. These problems aim at reconstructing the unknown scatterers from the measurement data of the scattered waves. In the past decades, inverse acoustic and electromagnetic scattering problems with phased data have been extensively studied mathematically and numerically. A comprehensive account of these studies can be found in the monographs [10, 14].

In many practical applications, it is much harder to obtain data with accurate phase information compared with just measuring the intensity (or the modulus) of the data, and thus it is often desirable to study inverse scattering with phaseless data (see, e.g., [10, Chapter 8] and the references quoted there). In fact, inverse scattering problems with phaseless data have also been widely studied numerically over the past decades (see, e.g. [2, 3, 10–13, 18, 26, 29, 34, 39, 41, 46–48] and the references quoted there).

Recently, uniqueness and stability results have also been established for inverse scattering with phaseless data (see, e.g. [1, 19, 20, 24, 25, 27, 31–33, 37, 42, 44, 45, 49]). For example, for point source incidence uniqueness results have been established in [24, 25] for inverse potential and acoustic medium scattering with the phaseless near-field data generated by point sources placed on a sphere enclosing the scatterer and measured in a small ball centered at each source position for an interval of frequencies, and in [32] for inverse acoustic medium scattering with the phaseless near-field data measured on an annulus surrounding the scatterer at fixed frequency.

The purpose of this paper is to propose a new approach to establish uniqueness results for inverse acoustic scattering problems with phaseless total-field data at a fixed frequency. Our approach is based on using superpositions of two point sources at a fixed frequency as the incident fields and making use of two spheres, which contain the scatterers, as the locations of such incident fields and the measurement surfaces of the modulus of the acoustic total-field (the sum of the incident field and the scattered field). In fact, many phase retrieval algorithms have been developed for inverse scattering problems with phaseless near-field data measured on two surfaces to ensure the reliability of the near-field phase reconstruction algorithms (see, e.g. [17, 35, 36]). Based on this idea, we prove that the impenetrable bounded obstacle or the index of refraction of the inhomogeneous medium can be uniquely determined from the phaseless total-field data at a fixed frequency. Note that the superposition of two point sources was also used in [37] as the incident field to study uniqueness for phaseless inverse scattering problems. Some related uniqueness results can be found in [40, 50, 51].

The idea is also applied to phaseless inverse electromagnetic scattering which is more complicated than the acoustic case. In this case, the electric total field is a complex vector-valued function, so we need to define the phaseless data used in this paper. In many applications (see, e.g. [5, 34, 38]), the phaseless near-field data are based on the measurement of the modulus of the tangential component of the electric total field on the measurement surface. Further, it has been elaborated in [16] that the measurement data are based on two tangential components of the electric field on the measurement sphere (see [16, p.100]). Therefore, the phaseless near-field data used is the modulus of the tangential component in the orientations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, of the electric total field measured on a sphere enclosing the scatters and generated by superpositions of two electric dipoles at a fixed frequency located on the measurement sphere and another bigger sphere with the polarizations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively. Following a similar idea as in the acoustic case, we prove that the impenetrable bounded obstacle or the refractive index of the inhomogeneous medium (under the condition that the magnetic permeability is a positive constant) can be uniquely determined by the phaseless total-field data at a fixed frequency. To the best of our knowledge, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data. It should be mentioned that our uniqueness results in this paper are based on parts of the PhD thesis [43].

The outline of this paper is as follows. The acoustic and electromagnetic scattering models considered are given in Section 2. Sections 3 and 4 are devoted to the uniqueness results for phaseless inverse acoustic and electromagnetic scattering problems, respectively. Conclusions are given in Section 5.

2. The direct scattering problems.

We will introduce the acoustic and electromagnetic scattering models considered in this paper. To this end, assume that D is an open and bounded domain in $\mathbb{R}^3$ with a C^2-boundary ∂D such that the exterior $\mathbb{R}^3\backslash\overline{D}$ is connected. Assume further that $\overline{D}\subset B_{R_1}$, where B_{R_1} is a ball centered at the origin with radius $R_1>0$ large enough.

2.1. The acoustic case.

In this paper, we consider the problem of acoustic scattering by an impenetrable obstacle or an inhomogeneous medium in $\mathbb{R}^3$. We need the following fundamental solution to the three-dimensional Helmholtz equation $\Delta w+k^2w=0$ in $\mathbb{R}^3$ with $k>0$:

$$\Phi_k(x,y):=\frac{e^{ik|x-y|}}{4\pi|x-y|},\quad x,y\in\mathbb{R}^3,\ \ x\neq y.$$

For arbitrarily fixed $y\in\mathbb{R}^3\backslash\overline{D}$ consider the time-harmonic ($e^{-i\omega t}$ time dependence) point source

$$w^i:=w^i(x,y)=\Phi_k(x,y),\quad x\in\mathbb{R}^3\backslash\overline{D},$$

which is incident on the obstacle D from the unbounded part $\mathbb{R}^3\backslash\overline{D}$, where $k=\omega/c>0$ is the wave number, ω and c are the wave frequency and speed in the homogeneous medium in the whole space. Then the problem of scattering of the point source w^i by the impenetrable obstacle D is formulated as the exterior boundary value problem:

$$\Delta_x w^s(x,y)+k^2w^s(x,y)=0,\qquad x\in\mathbb{R}^3\backslash\overline{D},\tag{1}$$

$$\mathscr{B}w=0\qquad \text{on }\ \partial D,\tag{2}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w^s}{\partial r}-ikw^s\right)=0,\qquad r=|x|,\tag{3}$$

where w^s is the scattered field, $w:=w^i+w^s$ is the total field, and (3) is the Sommerfeld radiation condition imposed on the scattered field w^s. The boundary condition $\mathscr{B}$ in (2) depends on the physical property of the obstacles D:

$$\begin{cases}\mathscr{B}w:=w \text{ on } \partial D & \text{if } D \text{ is a sound-soft obstacle,}\\ \mathscr{B}w:=\partial w/\partial\nu+\eta w \text{ on } \partial D & \text{if } D \text{ is an impedance obstacle,}\\ \mathscr{B}w:=w \text{ on } \Gamma_D,\ \mathscr{B}w:=\partial w/\partial\nu+\eta w \text{ on } \Gamma_I & \text{if } D \text{ is a partially coated obstacle,}\end{cases}$$

where ν is the unit outward normal to the boundary ∂D and η is the impedance function on ∂D satisfying that $\mathrm{Im}[\eta(x)]\geq 0$ for all $x\in\partial D$ or $x\in\Gamma_I$. We assume that $\eta\in C(\partial D)$ or $\eta\in C(\Gamma_I)$, that is, η is continuous on ∂D or Γ_I. When $\eta=0$, the impedance boundary condition becomes the Neumann boundary condition (a sound-hard obstacle). For a partially coated obstacle, we assume that the boundary ∂D has a Lipschitz dissection $\partial D = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of ∂D and having Π as their common boundary in ∂D (see, e.g., [8]).

The problem of scattering of the point source w^i by an inhomogeneous medium is modeled as follows:

$$\Delta_x w^s(x,y) + k^2 n(x) w^s(x,y) = k^2(1-n(x))w^i(x,y), \qquad x \in \mathbb{R}^3, \tag{4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial w^s}{\partial r} - ikw^s\right) = 0, \qquad r = |x|, \tag{5}$$

where w^s is the scattered field and n in (4) is the refractive index characterizing the inhomogeneous medium. We assume that $n-1$ has compact support $\overline{D}$, $n \in L^\infty(D)$, $\mathrm{Re}[n(x)] \geq n_{min} > 0$ for a constant n_{min} in D and $\mathrm{Im}[n(x)] \geq 0$ in D.

The existence of a unique (variational) solution to the problems (1)-(3) and (4)-(5) has been proved in [7, 14, 21, 22]. In particular, the scattered-field w^s has the asymptotic behavior:

$$w^s(x,y) = \frac{e^{ik|x|}}{|x|}\left\{w^\infty(\hat{x},y) + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty$$

uniformly for all observation directions $\hat{x} = x/|x| \in \mathbb{S}^2$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and $w^\infty(\hat{x},y)$ is the far-field pattern of w^s which is an analytic function of $\hat{x} \in \mathbb{S}^2$ for each $y \in \mathbb{R}^3 \setminus \overline{D}$ (see, e.g., [14, (2.13)]).

In this paper, we also consider the superposition of two point sources

$$w^i = w^i(x; y_1, y_2) = w^i(x, y_1) + w^i(x, y_2) = \Phi_k(x, y_1) + \Phi_k(x, y_2) \tag{6}$$

as the incident field, where $y_1, y_2 \in \mathbb{R}^3 \setminus \overline{D}$ are the locations of the two point sources. It then follows by the linear superposition principle that the corresponding scattered field

$$w^s(x; y_1, y_2) = w^s(x, y_1) + w^s(x, y_2) \tag{7}$$

and the corresponding total field

$$w(x; y_1, y_2) = w(x, y_1) + w(x, y_2), \tag{8}$$

where $w^s(x, y_j)$ and $w(x, y_j)$ are the scattered field and the total field corresponding to the incident point source $w^i(x, y_j)$, respectively, $j = 1, 2$.

The *inverse acoustic obstacle (or medium) scattering problem* we consider in this paper is to reconstruct the obstacle D and its physical property (or the index of refraction n of the inhomogeneous medium) from the phaseless total field $|w(x; y_1, y_2)|$ for x, y_1, y_2 on some spheres enclosing D and the inhomogeneous medium.

2.2. **The electromagnetic case.** In this paper, we consider two electromagnetic scattering models, that is, scattering by an impenetrable obstacle and scattering by an inhomogeneous medium. We will consider the time-harmonic ($e^{-i\omega t}$ time dependence) incident electric dipole located at $y \in \mathbb{R}^3 \setminus \overline{D}$ and described by the matrices $E^i(x, y)$ and $H^i(x, y)$ defined by

$$E^i(x,y)p := \frac{i}{k}\mathrm{curl}_x\mathrm{curl}_x[p\Phi_k(x,y)], \quad H^i(x,y)p := \mathrm{curl}_x[p\Phi_k(x,y)], \quad x \neq y,$$

for $x \in \mathbb{R}^3$, where $p \in \mathbb{R}^3 \backslash \{(0,0,0)\}$ is the polarization vector, $k := \omega/c > 0$ is the wave number, ω and $c := 1/\sqrt{\varepsilon_0 \mu_0}$ are the wave frequency and speed in the homogeneous medium in $\mathbb{R}^3 \setminus \overline{D}$, respectively, and ε_0 and μ_0 are the electric permittivity and the magnetic permeability of the homogeneous medium, respectively. A direct calculation shows that for $x \neq y$,

$$
\begin{aligned}
(9) \quad E^i(x,y) &= ik\Phi_k(x,y)I + \frac{i}{k}\nabla_x\nabla_x\Phi_k(x,y) \\
&= \frac{i}{k}\left\{\left[k^2 + \left(ik - \frac{1}{|x-y|}\right)\frac{1}{|x-y|}\right] I + \widehat{x-y}\cdot\widehat{x-y}^{\top} f(|x-y|)\right\}\Phi_k(x,y),
\end{aligned}
$$

where I is a 3×3 identity matrix, $\nabla_x\nabla_x := (\partial_{x_i}\partial_{x_j})_{3\times 3}$, $\widehat{x-y} = (x-y)/|x-y|$ and $f(r) := 3/r^2 - 3ik/r - k^2$. Then the problem of scattering of the electric dipole E^i and H^i by the impenetrable obstacle D can be modeled as the exterior boundary value problem:

$$\mathrm{curl}_x E^s - ikH^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{10}$$

$$\mathrm{curl}_x H^s + ikE^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{11}$$

$$\mathscr{B}E = 0 \quad \text{on } \partial D, \tag{12}$$

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0, \quad r = |x|, \tag{13}$$

where (E^s, H^s) is the scattered field, $E := E^i + E^s$ and $H := H^i + H^s$ are the electric total field and the magnetic total field, respectively, and (13) is the Silver–Müller radiation condition which holds uniformly for all $\hat{x} \in \mathbb{S}^2$ and ensures the uniqueness of the scattered field. The boundary condition $\mathscr{B}$ in (12) depends on the physical property of the obstacle D, that is, $\mathscr{B}E := \nu \times E$ on ∂D (called as the PEC condition) if D is a perfect conductor, where ν is the unit outward normal to the boundary ∂D, $\mathscr{B}E := \nu \times \mathrm{curl}E - i\lambda(\nu \times E) \times \nu$ on ∂D if D is an impedance obstacle, where λ is the impedance function on ∂D, and

$$\mathscr{B}E := \nu \times E \ \text{ on } \Gamma_D, \quad \mathscr{B}E := \nu \times \mathrm{curl}E - i\lambda(\nu \times E) \times \nu \ \text{ on } \Gamma_I$$

if D is a partially coated obstacle, where ∂D has a Lipschitz dissection $\partial D = \Gamma_D \cup \Pi \cup \Gamma_I$ with Γ_D and Γ_I being disjoint and relatively open subsets of ∂D and having Π as their common boundary in ∂D and λ is the impedance function on Γ_I. We assume throughout this paper that $\lambda \in C(\partial D)$ with $\lambda(x) \geq 0$ for all $x \in \partial D$ or $\lambda \in C(\Gamma_I)$ with $\lambda(x) \geq 0$ for all $x \in \Gamma_I$.

The problem of scattering of an electric dipole by an inhomogeneous medium is modeled as the medium scattering problem:

$$\mathrm{curl}_x E^s - ikH^s = 0 \quad \text{in } \mathbb{R}^3, \tag{14}$$

$$\mathrm{curl}_x H^s + ikn(x)E^s = ik(1-n(x))E^i \quad \text{in } \mathbb{R}^3, \tag{15}$$

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0, \quad r = |x|, \tag{16}$$

where (E^s, H^s) is the scattered field and $(E, H) := (E^i, H^i) + (E^s, H^s)$ is the total field. The refractive index $n(x)$ in (15) is given by

$$n(x) := \frac{1}{\varepsilon_0}\left(\varepsilon(x) + i\frac{\sigma(x)}{\omega}\right),$$

where $\varepsilon(x)$ and $\sigma(x)$ are the electric permittivity and electric conductivity in $\mathbb{R}^3$, respectively. In this paper, we assume the magnetic permeability $\mu = \mu_0$ to be a positive constant in the whole space. We assume further that $n - 1$ has a compact support $\overline{D}$, $n \in C^{2,\gamma}(\mathbb{R}^3)$ for $0 < \gamma < 1$, $\mathrm{Re}[n(x)] \geq n_{min} > 0$ in D for a constant n_{min} and $\mathrm{Im}[n(x)] \geq 0$ in D.

The existence of a unique (variational) solution to the problems (10)–(13) and (14)–(16) has been established in [8, 9, 14]. In particular, it is well known that the electromagnetic scattered field E^s has the asymptotic behavior:

$$E^s(x,y)p = \frac{e^{ik|x|}}{|x|}\left\{E^\infty(\hat{x},y)p + \mathcal{O}\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty$$

uniformly for all observation directions $\hat{x} = x/|x| \in \mathbb{S}^2$, where $E^\infty(\hat{x},y)$ is the electric far-field pattern of E^s which is an analytic function of $\hat{x} \in \mathbb{S}^2$ for each $y \in \mathbb{R}^3 \setminus \overline{D}$ (see, e.g., [14, (6.23)]). Because of the linearity of the direct scattering problem with respect to the incident field, we can express the scattered waves by matrices $E^s(x,y)$ and $H^s(x,y)$, the total waves by matrices $E(x,y)$ and $H(x,y)$, and the far-field patterns by $E^\infty(\hat{x},y)$ and $H^\infty(\hat{x},y)$, respectively.

We will also consider the following superposition of two electric dipoles as the incident field:

$$\begin{aligned} E^i &= E^i(x,y_1)p_1 + E^i(x,y_2)p_2 = \frac{i}{k}\mathrm{curl}_x\mathrm{curl}_x[p_1\Phi_k(x,y_1) + p_2\Phi_k(x,y_2)], \\ H^i &= H^i(x,y_1)p_1 + H^i(x,y_2)p_2 = \mathrm{curl}_x[p_1\Phi_k(x,y_1) + p_2\Phi_k(x,y_2)], \end{aligned}$$

where $x \in \mathbb{R}^3$, $y_1, y_2 \in \mathbb{R}^3 \setminus \overline{D}$, $x \neq y_1$, $x \neq y_2$ and $p_1, p_2 \in \mathbb{R}^3 \backslash \{(0,0,0)\}$. For convenience, we define the following incident field:

$$\begin{aligned} E^i &= E^i(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2) := \tau_1 E^i(x,y_1)p_1 + \tau_2 E^i(x,y_2)p_2 \\ &= \frac{i}{k}\mathrm{curl}_x\mathrm{curl}_x[\tau_1 p_1\Phi_k(x,y_1) + \tau_2 p_2\Phi_k(x,y_2)], \\ H^i &= H^i(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2) := \tau_1 H^i(x,y_1)p_1 + \tau_2 H^i(x,y_2)p_2 \\ &= \mathrm{curl}_x[\tau_1 p_1\Phi_k(x,y_1) + \tau_2 p_2\Phi_k(x,y_2)], \end{aligned}$$

with $x \in \mathbb{R}^3$, $y_1, y_2 \in \mathbb{R}^3 \setminus \overline{D}$, $x \neq y_1$, $x \neq y_2$, $p_1, p_2 \in \mathbb{R}^3\backslash\{(0,0,0)\}$ and $(\tau_1,\tau_2) \in \{(1,0),(0,1),(1,1)\}$. By the linear superposition principle, the electric scattered field and total field corresponding to the incident field $E^i(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2)$, $H^i(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2)$ satisfy

$$E^s(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2) := \tau_1 E^s(x,y_1)p_1 + \tau_2 E^s(x,y_2)p_2$$

and

$$E(x,y_1,p_1,\tau_1,y_2,p_2,\tau_2) := \tau_1 E(x,y_1)p_1 + \tau_2 E(x,y_2)p_2, \tag{17}$$

where $E^s(x,y_j)p_j$ and $E(x,y_j)p_j$ are the electric scattered field and the electric total field corresponding to the incident electric field $E^i(x,y_j)p_j$, respectively, $j = 1,2$.

Following [16, 34, 38], we measure the modulus of the tangential component of the electric total field on a sphere ∂B_r centered at the origin with radius $r > 0$. To represent the tangential components, we introduce the following spherical coordinate

$$\begin{cases} x_1 = r\sin\theta\cos\phi, \\ x_2 = r\sin\theta\sin\phi, \\ x_3 = r\cos\theta, \end{cases}$$

with $x := (x_1,x_2,x_3) \in \mathbb{R}^3$ and $(r,\theta,\phi) \in [0,+\infty)\times[0,\pi]\times[0,2\pi)$. For any $x \in \partial\mathring{B}_r$, the spherical coordinate gives an one-to-one correspondence between x and (r,ϕ,θ).

Here, $\partial\mathring{B}_r := \partial B_r \setminus \{N_r, S_r\}$ with $N_r := (0,0,r)$ and $S_r := (0,0,-r)$ denoting the north and south poles of ∂B_r, respectively. If we define

$$\boldsymbol{e}_\phi(x) := (-\sin\phi, \cos\phi, 0), \quad \boldsymbol{e}_\theta(x) := (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),$$

then $\boldsymbol{e}_\phi(x)$ and $\boldsymbol{e}_\theta(x)$ are two orthonormal tangential vectors of ∂B_r at $x \in \partial\mathring{B}_r$. Now, we can represent our phaseless measurement data by

$$|\boldsymbol{e}_m(x)\cdot E(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)|$$

with $x, y_1, y_2 \in \partial\mathring{B}_r$, $x \neq y_1$, $x \neq y_2$, $m, n, l \in \{\phi, \theta\}$ and $(\tau_1, \tau_2) \in \{(1,0), (0,1), (1,1)\}$.

The *inverse electromagnetic obstacle or medium scattering problem* we consider in this paper is to reconstruct the obstacle D and its physical property or the index of refraction n of the inhomogeneous medium from the modulus of the tangential component of the electric total field, $|\boldsymbol{e}_m(x)\cdot E(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)|$, for all x, y_1, y_2 in some spheres enclosing D or the inhomogeneous medium, $m, n, l \in \{\phi, \theta\}$ and $(\tau_1, \tau_2) \in \{(1,0), (0,1), (1,1)\}$. The purpose of this paper is to prove uniqueness results for the above inverse acoustic and electromagnetic scattering problems.

3. Inverse acoustic scattering with phaseless total-field data.

This section is devoted to the uniqueness results for inverse acoustic scattering with phaseless total-field data at a fixed frequency measured on two spheres enclosing the scatterers (see Figure 1).

Denote by w_j^s and w_j the scattered field and the total field, respectively, associated with the impenetrable obstacle D_j (or the refractive index n_j) and corresponding to the incident field w^i, $j = 1, 2$. Let B_{R_2} denote the ball centered at the origin with radius $R_2 > R_1 > 0$ with ∂B_{R_2} denoting the boundary of B_{R_2}. By appropriately choosing $R_2 > R_1 > 0$, it can be ensured that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2} \setminus \overline{B_{R_1}}$. Here, k^2 is called a Dirichlet eigenvalue of $-\Delta$ in a bounded domain V if the the interior Dirichlet boundary value problem

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } V, \\ u = 0 & \text{on } \partial V \end{cases}$$

has a nontrivial solution u. The above assumption on R_1 and R_2 can be easily satisfied since the Dirichlet eigenvalues of $-\Delta$ in a bounded domain are discrete and satisfy the strong monotonicity property [28, Theorem 4.7] (see also the arguments in the proof of [14, Theorem 5.2]). Let G denote the unbounded component of the complement of $D_1 \cup D_2$. Then we have the following global uniqueness results for the phaseless inverse scattering problems.

Theorem 3.1. *Let D_1, D_2 be two bounded domains and let $R_2 > R_1 > 0$ be large enough so that $\overline{D_1 \cup D_2} \subset B_{R_1}$. Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2} \setminus \overline{B_{R_1}}$.*

(a) Assume that D_1 and D_2 are two impenetrable obstacles with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. If the corresponding total fields satisfy

$$|w_1(x,y)| = |w_2(x,y)|, \ \forall (x,y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times (\{y_0\} \cup \partial B_{R_2})), \quad x \neq y \tag{18}$$

and

$$|w_1(x; y, y_0)| = |w_2(x; y, y_0)|, \ \forall (x,y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2}), \quad x \neq y, \ y_0 \tag{19}$$

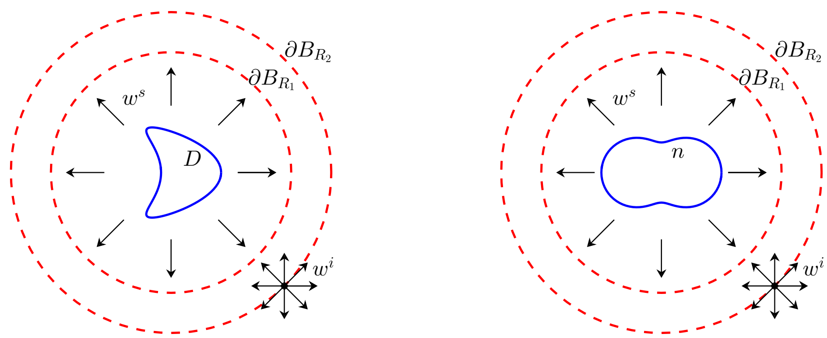

FIGURE 1. Acoustic scattering by an obstacle (left) or a medium (right).

for an arbitrarily fixed $y_0 \in \partial B_{R_1}$*, then* $D_1 = D_2$ *and* $\mathscr{B}_1 = \mathscr{B}_2$*.*

(b) Assume that $n_1, n_2 \in L^\infty(\mathbb{R}^3)$ *are the indices of refraction of two inhomogeneous media with* $n_j - 1$ *supported in* $\overline{D_j}$*,* $j = 1, 2$*. If the corresponding total fields satisfy (18) and (19), then* $n_1 = n_2$*.*

To prove Theorem 3.1, we need the following lemmas on the property of the total field.

Lemma 3.2. *Let* $R_2 > R_1 > 0$ *and let* D *be a bounded domain such that* $\overline{D} \subset B_{R_1}$*. Suppose* $w(x, y)$ *is the total field of the obstacle scattering problem (1)-(3) or the medium scattering problem (4)-(5) associated with the point source* $w^i(x, y)$*. Then, for any fixed* $y_0 \in \partial B_{R_1}$ *we have*

$$w(x, y_0) \not\equiv 0, \quad x \in \partial B_{R_1}, \quad x \neq y_0, \tag{20}$$

$$w(x, y_0) \not\equiv 0, \quad x \in \partial B_{R_2}, \tag{21}$$

$$w(x, y) \not\equiv 0, \quad (x, y) \in \partial B_{R_2} \times \partial B_{R_2}, \quad x \neq y. \tag{22}$$

Proof. Since $w(x, y)$ is singular at $x = y_0$ or y, we know that (20) and (22) are true.

We now prove (21). Assume to the contrary that $w(x, y_0) \equiv 0$ for $x \in \partial B_{R_2}$, that is, $w^s(x, y_0) = -\Phi_k(x, y_0)$ for $x \in \partial B_{R_2}$. Then, by the uniqueness of the exterior Dirichlet problem it follows that $w^s(x, y_0) = -\Phi_k(x, y_0)$ for all $x \in \mathbb{R}^3 \setminus \overline{B_{R_2}}$. Since the scattered field $w^s(x, y_0)$ is analytic for $x \in \mathbb{R}^3 \setminus \overline{D}$ and $\Phi_k(x, y_0)$ is analytic for $x \in \mathbb{R}^3 \setminus \{y_0\}$, we have $w^s(x, y_0) = -\Phi_k(x, y_0)$ for all $x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{y_0\})$. This is a contradiction since $\Phi_k(x, y_0)$ has a singularity at $x = y_0 \in \partial B_{R_1}$ and $w^s(x, y_0)$ is analytic when x is in a neighbourhood of y_0. Thus, (21) is true. □

Lemma 3.3. *Under the assumption of Lemma 3.2, we have the following results.*

(i) There exist two open sets $U_1, U_2 \subset \partial B_{R_1}$ *such that* $U_1 \cap U_2 = \emptyset$ *and* $w(x, y) \neq 0$ *for all* $(x, y) \in U_1 \times U_2$*.*

(ii) There exist two open sets $U_1', U_2' \subset \partial B_{R_2}$ *such that* $U_1' \cap U_2' = \emptyset$ *and* $w(x, y) \neq 0$ *for all* $(x, y) \in U_1' \times (U_2' \cup \{y_0\})$*, where* $y_0 \in \partial B_{R_1}$*.*

Proof. We only prove (ii). The proof of (i) is similar.

By (21) we know that for $y_0 \in \partial B_{R_1}$ there exists $x_0 \in \partial B_{R_2}$ such that $w(x_0, y_0) \neq 0$. Since $w(x, y)$ is continuous for $x, y \in \mathbb{R}^3 \setminus \overline{D}$ with $x \neq y$, there exists a neighbourhood $U' \subset \partial B_{R_2}$ of x_0 such that $w(x, y_0) \neq 0$ for all $x \in U'$. Further, since $w(x, y)$ is analytic with respect to $x \in \partial B_{R_2}$ and $y \in \partial B_{R_2}$, respectively, when

$x \neq y$, then it follows from (22) that there exist two points $x_1 \in U'$ and $x_2 \in \partial B_{R_2}$ such that $w(x_1, x_2) \neq 0$ with $x_1 \neq x_2$. Finally, again by the continuity of $w(x, y)$ for $x, y \in \mathbb{R}^3 \setminus \overline{D}$ with $x \neq y$, there exists a neighbourhood $U_1' \subset U'$ of x_1 and a neighbourhood $U_2' \subset \partial B_{R_1}$ of x_2 such that $U_1' \cap U_2' = \emptyset$ and $w(x, y) \neq 0$ for all $(x, y) \in U_1' \times U_2'$. Thus, $w(x, y) \neq 0$ for all $(x, y) \in U_1' \times (U_2' \cup \{y_0\})$. This completes the proof. □

Proof of Theorem 3.1. From (8) it is easy to see that (19) is equivalent to the equation

$$|w_1(x, y) + w_1(x, y_0)| = |w_2(x, y) + w_2(x, y_0)|$$

for all $(x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2})$ with $x \neq y, y_0$. This, together with (18), implies that

$$\mathrm{Re}\{w_1(x, y)\overline{w_1(x, y_0)}\} = \mathrm{Re}\{w_2(x, y)\overline{w_2(x, y_0)}\} \tag{23}$$

for all $(x, y) \in (\partial B_{R_1} \times \partial B_{R_1}) \cup (\partial B_{R_2} \times \partial B_{R_2})$ with $x \neq y, y_0$. Define $r_j(x, y) := |w_j(x, y)|$, $j = 1, 2$. Then it follows from (18) that $r_1(x, y) = r_2(x, y) =: r(x, y)$, for all $x \in \partial B_{R_1}$, $y \in \partial B_{R_1} \cup \partial B_{R_2}$ with $x \neq y$, so we can write

$$w_j(x, y) = r(x, y)e^{i\vartheta_j(x,y)}, \quad \forall x, y \in \partial B_{R_1} \cup \partial B_{R_2}, \ \ x \neq y, \ \ j = 1, 2$$

with real-valued functions $\vartheta_j(x, y)$, $j = 1, 2$.

Case 1. (23) holds with $(x, y) \in \partial B_{R_1} \times \partial B_{R_1}$, $x \neq y$.

Since $w_j^s(x, y)$, $j = 1, 2$, are analytic functions of $x \in \partial B_{R_1}$ and $y \in \partial B_{R_1}$, respectively, and $\Phi_k(x, y)$ has a singularity at $x = y$, then, by Lemma 3.3 we can choose two open sets $U_1, U_2 \subset \partial B_{R_1}$ small enough so that $U_1 \cap U_2 = \emptyset$, $r(x, y) \neq 0$ for all $(x, y) \in U_1 \times (U_2 \cup y_0)$, and $\vartheta_j(x, y)$, $j = 1, 2$, are analytic with respect to $x \in U_1$ and $y \in U_2$, respectively.

Now, by (23) we have

$$\cos[\vartheta_1(x, y) - \vartheta_1(x, y_0)] = \cos[\vartheta_2(x, y) - \vartheta_2(x, y_0)] \tag{24}$$

for all $(x, y) \in U_1 \times U_2$. Since $\vartheta_j(x, y)$, $j = 1, 2$, are real-valued analytic functions of $x \in U_1$ and $y \in U_2$, respectively, we have either

$$\vartheta_1(x, y) - \vartheta_1(x, y_0) = \vartheta_2(x, y) - \vartheta_2(x, y_0) + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2 \tag{25}$$

or

$$\vartheta_1(x, y) - \vartheta_1(x, y_0) = -[\vartheta_2(x, y) - \vartheta_2(x, y_0)] + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2, \tag{26}$$

where $q \in \mathbb{Z}$.

For the case when (25) holds, we have

$$\vartheta_1(x, y) - \vartheta_2(x, y) = \vartheta_1(x, y_0) - \vartheta_2(x, y_0) + 2q\pi, \quad \forall (x, y) \in U_1 \times U_2.$$

This implies that $\alpha(x) := \vartheta_1(x, y) - \vartheta_2(x, y) = \vartheta_1(x, y_0) - \vartheta_2(x, y_0) + 2q\pi$ depends only on $x \in U_1$. Then it follows that

$$w_1(x, y) = r(x, y)e^{i\vartheta_1(x,y)} = r(x, y)e^{i\alpha(x) + i\vartheta_2(x,y)} = e^{i\alpha(x)}w_2(x, y)$$

for all $x \in U_1$ and $y \in U_2 \cup \{y_0\}$. By the analyticity of $w_1(x, y) - e^{i\alpha(x)}w_2(x, y)$ in $y \in \partial B_{R_1}$ with $y \neq x$, we get

$$w_1(x, y) = e^{i\alpha(x)}w_2(x, y), \quad \forall x \in U_1, y \in \partial B_{R_1}, \quad x \neq y. \tag{27}$$

Changing the variables $x \to y$ and $y \to x$ in (27) gives

$$w_1(y, x) = e^{i\alpha(y)}w_2(y, x), \quad \forall x \in \partial B_{R_1}, y \in U_1, \quad x \neq y. \tag{28}$$

Use (27), (28) and the reciprocity relation that $w_j^s(x,y) = w_j^s(y,x)$ for all $x, y \in \partial B_{R_1}$, $j = 1,2$ (see [14, Theorem 3.17]) to give

$$e^{i\alpha(x)} w_2(x,y) = e^{i\alpha(y)} w_2(x,y), \quad \forall x, y \in U_1 \text{ with } x \neq y. \tag{29}$$

Since $w_j(x,y)$ has a singularity at $x = y$, and by (29) and the analyticity of $w_j(x,y)$ $(j = 1,2)$ with respect to $x \in \partial B_{R_1}$ and $y \in \partial B_{R_1}$, respectively, with $x \neq y$, it follows that $e^{i\alpha(x)} = e^{i\alpha(y)}$ for all $x, y \in U_1$ with $x \neq y$. This means that $e^{i\alpha(x)} \equiv e^{i\alpha}$ for all $x \in U_1$, where α is a real constant. Substituting this formula into (27) gives that $w_1(x,y) = e^{i\alpha} w_2(x,y)$ for all $x \in U_1, y \in \partial B_{R_1}$ with $x \neq y$. Again, by the analyticity of $w_j(x,y)$ $(j = 1,2)$ with respect to $x \in \partial B_{R_1}$ with $x \neq y$ we have

$$w_1(x,y) = e^{i\alpha} w_2(x,y), \ \forall x, y \in \partial B_{R_1} \text{ with } x \neq y, \tag{30}$$

which gives

$$w_1^s(x,y) - e^{i\alpha} w_2^s(x,y) = (e^{i\alpha} - 1)\Phi_k(x,y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y. \tag{31}$$

Since $w_j^s(x,y)$, $j = 1,2$, are analytic for $x \in G$ and $y \in G$, respectively, and $\Phi_k(x,y)$ has a singularity at $x = y$, then passing the limit $y \to x$ in (31) gives that $e^{i\alpha} = 1$, so

$$w_1^s(x,y) = w_2^s(x,y), \quad \forall x, y \in \partial B_{R_1}. \tag{32}$$

For the case when (26) holds, a similar argument as above gives

$$w_1(x,y) = e^{i\beta} \overline{w_2(x,y)}, \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y \tag{33}$$

for a real constant β, that is,

$$w_1^s(x,y) - e^{i\beta} \overline{w_2^s(x,y)} = e^{i\beta} \overline{\Phi_k(x,y)} - \Phi_k(x,y), \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y.$$

Since $w_j^s(x,y)$, $j = 1,2$, are analytic for $x \in G$ and $y \in G$, respectively, $\mathrm{Re}[\Phi_k(x,y)]$ has a singularity at $x = y$ and $\mathrm{Im}[\Phi_k(x,y)]$ is analytic for all $x, y \in \mathbb{R}^3$, then $e^{i\beta} = 1$. Thus, it follows from (33) that

$$w_1(x,y) = \overline{w_2(x,y)}, \quad \forall x, y \in \partial B_{R_1} \text{ with } x \neq y. \tag{34}$$

Case 2. (23) holds with $(x,y) \in \partial B_{R_2} \times \partial B_{R_2}$, $x \neq y$.

By a similar argument as in Case 1, it can be obtained that there holds either

$$w_1^s(x,y) = w_2^s(x,y), \quad \forall x \in \partial B_{R_2}, \ y \in \partial B_{R_2} \cup \{y_0\} \tag{35}$$

or

$$w_1(x,y) = \overline{w_2(x,y)}, \quad \forall x \in \partial B_{R_2}, \ y \in \partial B_{R_2} \cup \{y_0\} \text{ with } x \neq y. \tag{36}$$

We now prove that both (34) and (36) can not hold simultaneously. Suppose this is not the case. Then define $v(x) := w_1(x, y_0) - \overline{w_2(x, y_0)}$ for $x \in G$ with $x \neq y_0$. Since $\Phi_k(x,y) - \overline{\Phi_k(x,y)} = i \sin(k|x-y|)/(2\pi|x-y|)$ is analytic for all $x, y \in \mathbb{R}^3$, then, by the analyticity of $w_j^s(x,y)$ $(j = 1,2)$ with respect to $x \in G$ it follows that v can be extended as an analytic function of $x \in G$, denoted by v again. Further, since $i \sin(k|x-y|)/(2\pi|x-y|)$ and $w_j^s(x,y)$ $(j = 1,2)$ as functions of x satisfy the Helmholtz equation $\Delta u + k^2 u = 0$ in G, we have by (34) and (36) that v satisfies the Dirichlet boundary value problem:

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } B_{R_2} \backslash \overline{B_{R_1}}, \\ v = 0 & \text{on } \partial B_{R_1} \cup \partial B_{R_2}. \end{cases}$$

From the assumption that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in $B_{R_2}\backslash\overline{B_{R_1}}$, it is known that $v(x) = 0$ for any $x \in B_{R_2}\backslash\overline{B_{R_1}}$. Thus $w_1(x, y_0) = \overline{w_2(x, y_0)}$ for all $x \in B_{R_2}\backslash\overline{B_{R_1}}$ with $x \neq y_0$. By the analyticity of $w_j(x, y_0)$ $(j = 1, 2)$ with respect to $x \in G$ with $x \neq y_0$, we obtain

$$w_1(x, y_0) = \overline{w_2(x, y_0)}, \quad \forall x \in G,\ x \neq y_0, \tag{37}$$

which contradicts to the fact that $w_j(x, y_0) = \Phi_k(x, y_0) + w_j^s(x, y_0),\ j = 1, 2$, satisfy the Sommerfeld radiation condition. We then conclude that both (34) and (36) can not hold simultaneously. This means that at least one of the formulas (34) and (36) does not hold.

If (34) does not hold, then it follows that (32) holds. By the reciprocity relation, the well-posedness of the exterior Dirichlet problem and the analyticity of $w_j^s(x, y)$ $(j = 1, 2)$ with respect to $x \in G$ and $y \in G$, respectively, it is easily derived from (32) that

$$w_1^s(x, y) = w_2^s(x, y), \quad \forall x, y \in G. \tag{38}$$

Then, by [14, Theorem 2.13] and the mixed reciprocity relation $4\pi w_j^\infty(-d, z) = u_j^s(z, d)$ for all $d \in \mathbb{S}^2$ and $z \in G,\ j = 1, 2$ (see [14, Theorem 3.16]) it is obtained on applying (38) that

$$u_1^\infty(\hat{x}, d) = u_2^\infty(\hat{x}, d), \quad \forall \hat{x}, d \in \mathbb{S}^2, \tag{39}$$

where u_j^∞ is the far-field pattern associated with the obstacle D_j (or the refractive index n_j) and corresponding to the incident field $u^i(x, d) = e^{ikx\cdot d},\ j = 1, 2$. Similarly, if (36) does not hold, then (35) holds and thus we can also show that (39) holds.

Finally, for the case with two impenetrable obstacles D_1 and D_2, it follows from (39), [14, Theorem 5.6] and [30, Theorem 3.7] that $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$, while for the case with two refractive indices n_1 and n_2, we have by (39) and [21, Theorem 6.26] that $n_1 = n_2$. Theorem 3.1 is thus proved. □

Remark 1. (i) Theorem 3.1 (a) remains true for the two-dimensional case, and the proof is similar.

(ii) Theorem 3.1 (b) also holds in two dimensions if the assumption $n_1, n_2 \in L^\infty(\mathbb{R}^3)$ is replaced by the condition that n_j is piecewise in $W^{1,p}(D_j)$ with $p > 2,\ j = 1, 2$. In this case, the proof is similar except that we need Bukhgeim's result in [6] (see also the theorem in Section 4.1 in [4]) instead of [21, Theorem 6.26] in the proof.

(iii) Theorem 3.1 (b) generalizes the uniqueness results in [24,25,27,32] substantially in the sense that our uniqueness results only need the measurement data of the modulus of the total-field on two spheres enclosing the inhomogeneous medium at a fixed frequency, under no smoothness assumption on the refractive index, instead of the measurement data in a ball for each point source in a sphere for an interval of frequencies as used in [24, 25, 27] or in an open domain for each point source in another open domain at a fixed frequency as used in [32].

4. Inverse electromagnetic scattering with phaseless electric total field data.

In this section, we extend the uniqueness results in Section 3 for the acoustic case to the case of inverse electromagnetic scattering problems with phaseless electric total-field data at a fixed frequency. In this case, we consider the measurement of the modulus of the tangential component of the electric total-field on two spheres enclosing the scatterers, generated by superpositions of two electric dipoles located also on the two spheres. Denote by E_j, E_j^s, H_j^s and H_j the electric scattered-field, electric total-field, magnetic scattered-field and magnetic total-field, respectively, associated with the obstacle D_j (or the refractive index n_j) and corresponding to the incident electric field E^i, $j = 1, 2$. Let B_{R_2} denote the ball centered at the origin with radius $R_2 > R_1 > 0$ with ∂B_{R_2} denoting the boundary of B_{R_2} and let G denote the unbounded component of the complement of $D_1 \cup D_2$. Denote by N_{R_j} and S_{R_j} the north and south poles of ∂B_{R_j}, respectively, $j = 1, 2$. Define $\partial \mathring{B}_{R_j} := \partial B_{R_j} \setminus \{N_{R_j}, S_{R_j}\}$, $j = 1, 2$. See Figure 2 for the geometry of the electromagnetic scattering problem.

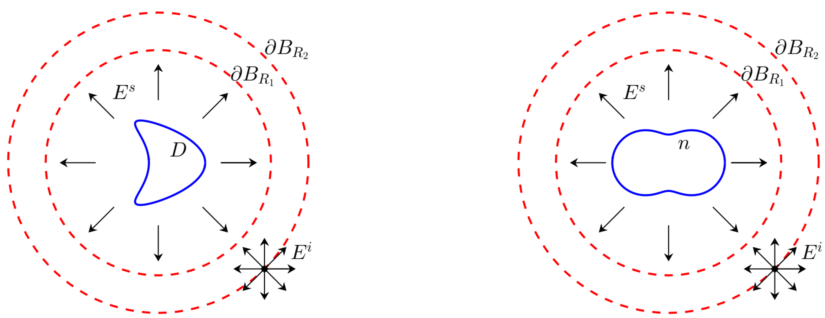

FIGURE 2. Electromagnetic scattering by an obstacle (left) or a medium (right).

By choosing appropriate R_1 and R_2 (see Lemma 4.1), it can be ensured that k^2 is not a Maxwell eigenvalue in $B_{R_2} \setminus \overline{B_{R_1}}$. Here, k^2 is called a Maxwell eigenvalue in a bounded domain V if the interior Maxwell problem

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \ V \\ \operatorname{curl} H + ikE = 0 & \text{in } \ V \\ \nu \times E = 0 & \text{on } \ \partial V \end{cases}$$

has a nontrivial solution (E, H).

Lemma 4.1. *k^2 is not a Maxwell eigenvalue in $B_{R_2} \setminus \overline{B_{R_1}}$ if and only if*

$$\begin{cases} \begin{vmatrix} j_n(kR_1) & y_n(kR_1) \\ j_n(kR_2) & y_n(kR_2) \end{vmatrix} \neq 0, \\ \begin{vmatrix} j_n(kR_1) + kR_1 j_n'(kR_1) & y_n(kR_1) + kR_1 y_n'(kR_1) \\ j_n(kR_2) + kR_2 j_n'(kR_2) & y_n(kR_2) + kR_2 y_n'(kR_2) \end{vmatrix} \neq 0 \end{cases} \tag{40}$$

for all $n = 1, 2, \cdots$, where j_n and y_n are the spherical Bessel functions and spherical Neumann functions of order n, respectively.

Proof. Assume that (E, H) solves the interior Maxwell problem

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \ B_{R_2} \setminus \overline{B_{R_1}} \\ \operatorname{curl} H + ikE = 0 & \text{in } \ B_{R_2} \setminus \overline{B_{R_1}} \\ \nu \times E = 0 & \text{on } \ \partial B_{R_2} \cup \partial B_{R_1}. \end{cases} \tag{41}$$

A similar argument as in the proof of [23, Theorems 2.48 and 2.50] gives the following expansion in the spherical vector harmonics of the electric field E in $\overline{B_{R_2}}\setminus B_{R_1}$ as a uniformly convergent series:

$$E(x)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[a_n^m\operatorname{curl}\{xj_n(k|x|)Y_n^m(\hat{x})\}+b_n^m\operatorname{curl}\operatorname{curl}\{xj_n(k|x|)Y_n^m(\hat{x})\}\Big]$$
$$+\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[c_n^m\operatorname{curl}\{xy_n(k|x|)Y_n^m(\hat{x})\}+d_n^m\operatorname{curl}\operatorname{curl}\{xy_n(k|x|)Y_n^m(\hat{x})\}\Big],$$
$$x\in\overline{B_{R_2}}\setminus B_{R_1},$$

where Y_n^m, $m=-n,\dots,n$, $n=0,1,2,\dots$, are the spherical harmonics. By [14, (6.71) and (6.72)], we have that for any $x\in\partial B_r$ with $r\in[R_1,R_2]$,

$$\hat{x}\times E(x)$$
$$=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[a_n^m j_n(kr)\mathrm{Grad}Y_n^m(\hat{x})+b_n^m\frac{1}{r}\{j_n(kr)+krj_n'(kr)\}\,\hat{x}\times\mathrm{Grad}Y_n^m(\hat{x})\Big]$$
$$+\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\Big[c_n^m y_n(kr)\mathrm{Grad}Y_n^m(\hat{x})+d_n^m\frac{1}{r}\{y_n(kr)+kry_n'(kr)\}\,\hat{x}\times\mathrm{Grad}Y_n^m(\hat{x})\Big].$$

By the perfectly conducting boundary condition on $\partial B_{R_2}\cup\partial B_{R_1}$ we have

$$\begin{pmatrix} j_n(kR_1) & y_n(kR_1)\\ j_n(kR_2) & y_n(kR_2)\end{pmatrix}\begin{pmatrix} a_n^m\\ c_n^m\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} j_n(kR_1)+kR_1j_n'(kR_1) & y_n(kR_1)+kR_1y_n'(kR_1)\\ j_n(kR_2)+kR_2j_n'(kR_2) & y_n(kR_2)+kR_2y_n'(kR_2)\end{pmatrix}\begin{pmatrix} b_n^m\\ d_n^m\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{43}$$

for all $n=1,2,\cdots$, $m=-n,\cdots,n$. By (40) we have $a_n^m=b_n^m=c_n^m=d_n^m=0$ for all $n=1,2,\cdots$, $m=-n,\cdots,n$, and so k^2 is not a Maxwell eigenvalue in $B_{R_2}\setminus\overline{B_{R_1}}$.

On the other hand, if

$$\begin{vmatrix} j_n(kR_1) & y_n(kR_1)\\ j_n(kR_2) & y_n(kR_2)\end{vmatrix}=0$$

or

$$\begin{vmatrix} j_n(kR_1)+kR_1j_n'(kR_1) & y_n(kR_1)+kR_1y_n'(kR_1)\\ j_n(kR_2)+kR_2j_n'(kR_2) & y_n(kR_2)+kR_2y_n'(kR_2)\end{vmatrix}=0$$

for some $n\in\mathbb{N}^+$, then (42) or (43) has non-zero solutions. Thus there exists a nontrivial solution to the interior Maxwell problem (41), and so k^2 is a Maxwell eigenvalue in $B_{R_2}\setminus\overline{B_{R_1}}$. The proof is thus complete. □

We have the following uniqueness results for the phaseless inverse electromagnetic scattering problems.

Theorem 4.2. *Let D_1,D_2 be two bounded domains and let $R_2>R_1>0$ be large enough such that $\overline{D_1\cup D_2}\subset B_{R_1}$ and k^2 is not a Maxwell eigenvalue in $B_{R_2}\setminus\overline{B_{R_1}}$.*

(a) Assume that D_1 and D_2 are two impenetrable obstacles with boundary conditions $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively. If the corresponding electric total fields satisfy

$$|\boldsymbol{e}_m(x)\cdot E_1(x,y_1,\boldsymbol{e}_\phi(y_1),\tau_1,y_2,\boldsymbol{e}_\theta(y_2),\tau_2)|$$
$$=|\boldsymbol{e}_m(x)\cdot E_2(x,y_1,\boldsymbol{e}_\phi(y_1),\tau_1,y_2,\boldsymbol{e}_\theta(y_2),\tau_2)| \tag{44}$$

for all $x, y_1, y_2 \in \partial\mathring{B}_{R_1}$ *with* $x \neq y_1, y_2$, $(\tau_1, \tau_2) \in \{(1,0),(0,1),(1,1)\}$, $m \in \{\phi, \theta\}$ *and*

$$\begin{aligned}&|\boldsymbol{e}_m(x)\cdot E_1(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)|\\ &\qquad = |\boldsymbol{e}_m(x)\cdot E_2(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)| \end{aligned}\tag{45}$$

for all $x, y_1 \in \partial\mathring{B}_{R_1}$ *with* $x \neq y_1$, $y_2 \in \partial\mathring{B}_{R_2}$, $(\tau_1, \tau_2) \in \{(0,1),(1,1)\}$, $m, n, l \in \{\phi, \theta\}$, *then* $D_1 = D_2$ *and* $\mathscr{B}_1 = \mathscr{B}_2$.

(b) Assume that $n_1, n_2 \in C^{2,\gamma}(\mathbb{R}^3)$ *with* $\gamma > 0$ *are the refractive indices of two inhomogeneous media with* $n_j - 1$ *supported in* $\overline{D_j}$ $(j = 1, 2)$. *If the corresponding electric total fields satisfy* (44) *and* (45), *then* $n_1 = n_2$.

Remark 2. Since $E^i_j(x, y) = [E^i_j(y, x)]^\top$, and by the reciprocity relation $E^s_j(x, y) = [E^s_j(y, x)]^\top$ for all $x, y \in G$ (see [14, Theorem 6.32]), $j = 1, 2$, we know that (44) with $m = \phi$ and $(\tau_1, \tau_2) = (0, 1)$ is equivalent to (44) with $m = \theta$ and $(\tau_1, \tau_2) = (1, 0)$.

To prove Theorem 4.2, we need some results on the phaseless electric total-fields measured on ∂B_{R_1}.

Lemma 4.3. *Assume that the assumptions of Theorem 4.2 are satisfied. If for any fixed* $m \in \{\phi, \theta\}$ *there holds*

$$\begin{aligned}&|\boldsymbol{e}_m(x)\cdot E_1(x, y_1, \boldsymbol{e}_\phi(y_1), \tau_1, y_2, \boldsymbol{e}_\theta(y_2), \tau_2)|\\ &\qquad = |\boldsymbol{e}_m(x)\cdot E_2(x, y_1, \boldsymbol{e}_\phi(y_1), \tau_1, y_2, \boldsymbol{e}_\theta(y_2), \tau_2)| \end{aligned}\tag{46}$$

for all $x, y_1, y_2 \in \partial\mathring{B}_{R_1}$ *with* $x \neq y_1, y_2$, $(\tau_1, \tau_2) \in \{(1,0),(0,1),(1,1)\}$, *then we have either*

$$\boldsymbol{e}_m(x)\cdot E_1(x, y_1)\boldsymbol{e}_\phi(y_1) = \boldsymbol{e}_m(x)\cdot E_2(x, y_1)\boldsymbol{e}_\phi(y_1),\ \forall x, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1,\tag{47}$$

$$\boldsymbol{e}_m(x)\cdot E_1(x, y_2)\boldsymbol{e}_\theta(y_2) = \boldsymbol{e}_m(x)\cdot E_2(x, y_2)\boldsymbol{e}_\theta(y_2),\ \forall x, y_2 \in \partial\mathring{B}_{R_1}, x \neq y_2\tag{48}$$

or

$$\boldsymbol{e}_m(x)\cdot E_1(x, y_1)\boldsymbol{e}_\phi(y_1) = -\boldsymbol{e}_m(x)\cdot \overline{E_2(x, y_1)}\boldsymbol{e}_\phi(y_1),\ \forall x, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1,\tag{49}$$

$$\boldsymbol{e}_m(x)\cdot E_1(x, y_2)\boldsymbol{e}_\theta(y_2) = -\boldsymbol{e}_m(x)\cdot \overline{E_2(x, y_2)}\boldsymbol{e}_\theta(y_2),\ \forall x, y_2 \in \partial\mathring{B}_{R_1}, x \neq y_2.\tag{50}$$

Proof. We only consider the case $m = \phi$ since the case $m = \theta$ can be proved similarly.

Using (17) and (46) and arguing similarly as in the proof of Theorem 3.1 give

$$\begin{aligned}&\mathrm{Re}\{[\boldsymbol{e}_\phi(x)\cdot E_1(x, y_1)\boldsymbol{e}_\phi(y_1)] \times [\overline{\boldsymbol{e}_\phi(x)\cdot E_1(x, y_2)\boldsymbol{e}_\theta(y_2)}]\}\\ &\qquad = \mathrm{Re}\{[\boldsymbol{e}_\phi(x)\cdot E_2(x, y_1)\boldsymbol{e}_\phi(y_1)] \times [\overline{\boldsymbol{e}_\phi(x)\cdot E_2(x, y_2)\boldsymbol{e}_\theta(y_2)}]\}\end{aligned}\tag{51}$$

for all $x, y_1, y_2 \in \partial\mathring{B}_{R_1}$, $x \neq y_1, y_2$. For $x, y \in \partial\mathring{B}_{R_1}$, $x \neq y$, define

$$r_j^{(\phi\phi)}(x, y) := |\boldsymbol{e}_\phi(x)\cdot E_j(x, y)\boldsymbol{e}_\phi(y)|,\ r_j^{(\phi\theta)}(x, y) := |\boldsymbol{e}_\phi(x)\cdot E_j(x, y)\boldsymbol{e}_\theta(y)|,\ j = 1, 2.$$

It then follows from (46) with $(\tau_1, \tau_2) = (1, 0)$ and $(\tau_1, \tau_2) = (0, 1)$ that

$$r_1^{(\phi\phi)}(x, y) = r_2^{(\phi\phi)}(x, y) =: r^{(\phi\phi)}(x, y),\ r_1^{(\phi\theta)}(x, y) = r_2^{(\phi\theta)}(x, y) =: r^{(\phi\theta)}(x, y)$$

for all $x, y \in \partial\mathring{B}_{R_1}$, $x \neq y$. Therefore we can write

$$\begin{aligned}&\boldsymbol{e}_\phi(x)\cdot E_j(x, y_1)\boldsymbol{e}_\phi(y_1) := r^{(\phi\phi)}(x, y_1)e^{i\vartheta_j^{(\phi\phi)}(x, y_1)},\ \ \forall x, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1,\\ &\boldsymbol{e}_\phi(x)\cdot E_j(x, y_1)\boldsymbol{e}_\theta(y_2) := r^{(\phi\theta)}(x, y_2)e^{i\vartheta_j^{(\phi\theta)}(x, y_2)},\ \ \forall x, y_2 \in \partial\mathring{B}_{R_1}, x \neq y_2,\end{aligned}$$

where $\vartheta_j^{(\phi\phi)}$ and $\vartheta_j^{(\phi\theta)}$, $j = 1, 2$ are real-valued functions.

We now prove that $r^{(\phi\phi)}(x, y_1) \not\equiv 0$, $x, y_1 \in \partial\mathring{B}_{R_1}$, $x \neq y_1$, and $r^{(\phi\theta)}(x, y_2) \not\equiv 0$, $x, y_2 \in \partial\mathring{B}_{R_1}$, $x \neq y_2$. In fact, fix $y_1 \in \partial\mathring{B}_{R_1}$ and define the circle $C_{\boldsymbol{e}_\phi(y_1)} := \{x \in \partial B_{R_1} : (x - y_1) \cdot \boldsymbol{e}_\phi(y_1) = 0\}$, which is the intersection of the sphere ∂B_{R_1} with the plane whose normal vector is $\boldsymbol{e}_\phi(y_1)$ at y_1. When x tends to y_1 along the circle $C_{\boldsymbol{e}_\phi(y_1)}$, we have $\widehat{x - y_1}^\top \boldsymbol{e}_\phi(y_1) \to 0$ and $\boldsymbol{e}_\phi(x) \cdot \boldsymbol{e}_\phi(y_1) \to 1$. Thus, by (9) it is known that

$$(52)\quad \boldsymbol{e}_\phi(x) \cdot E^i(x, y_1)\boldsymbol{e}_\phi(y_1) \sim \frac{i}{k}\left[k^2 + \left(ik - \frac{1}{|x - y_1|}\right)\frac{1}{|x - y_1|}\right]\Phi_k(x, y_1)$$

as x goes to y_1 along the circle $C_{\boldsymbol{e}_\phi(y_1)}$. The singularity in (52) implies that $r^{(\phi\phi)}(x, y_1) \not\equiv 0$ for $x, y_1 \in \partial\mathring{B}_{R_1}$ with $x \neq y_1$ since $E_j^s(x, y_1)$ is analytic with respect to $x, y_1 \in \partial\mathring{B}_{R_1}$ with $x \neq y_1$, respectively $(j = 1, 2)$. Further, fix $y_2 \in \partial\mathring{B}_{R_1}$ and define the circle

$$C_{\boldsymbol{e}_\phi(y_2)+\boldsymbol{e}_\theta(y_2)} := \{x \in \partial B_{R_1} : (x - y_2) \cdot (\boldsymbol{e}_\phi(y_2) + \boldsymbol{e}_\theta(y_2)) = 0\}.$$

Then, on letting x tend to y_2 along $C_{\boldsymbol{e}_\phi(y_2)+\boldsymbol{e}_\theta(y_2)}$ we have $\boldsymbol{e}_\phi(x) \cdot \boldsymbol{e}_\theta(y_2) \to 0$ and $\boldsymbol{e}_\phi(x) \cdot \left[\widehat{x - y_2} \cdot \widehat{x - y_2}^\top \boldsymbol{e}_\theta(y_2)\right] \to c_1$ for a non-zero constant c_1. Thus it follows from (9) that

$$(53)\qquad \boldsymbol{e}_\phi(x) \cdot E^i(x, y_2)\boldsymbol{e}_\theta(y_2) = \frac{1}{|x - y_2|^2}\Phi_k(x, y_2)\,[c_2 + o(1)]$$

as $x \to y_2$ along $C_{\boldsymbol{e}_\phi(y_2)+\boldsymbol{e}_\theta(y_2)}$, where c_2 is a non-zero constant. Therefore the singularity in (53) implies that $r^{(\phi\theta)}(x, y_2) \not\equiv 0$ for $x, y_2 \in \partial\mathring{B}_{R_1}$ with $x \neq y_2$ since $E_j^s(x, y_2)$ is analytic with respect to $x, y_2 \in \partial\mathring{B}_{R_1}$, $x \neq y_2$, respectively $(j = 1, 2)$. Then, similarly as in the proof of Theorem 3.1, we can show that there are three small enough open sets $U, U_1, U_2 \subset \partial\mathring{B}_{R_1}$ such that U, U_1 and U_2 are disjoint, $r^{(\phi\phi)}(x, y_1) \neq 0$ and $r^{(\phi\theta)}(x, y_2) \neq 0$ for all $x \in U$, $y_1 \in U_1$ and $y_2 \in U_2$, and $\vartheta_j^{(\phi\phi)}(x, y_1)$ and $\vartheta_j^{(\phi\theta)}(x, y_2)$ are analytic with respect to $x \in U$, $y_1 \in U_1$, $y_2 \in U_2$, respectively, $j = 1, 2$.

Now, by (51) we have

$$(54)\qquad \cos[\vartheta_1^{(\phi\phi)}(x, y_1) - \vartheta_1^{(\phi\theta)}(x, y_2)] = \cos[\vartheta_2^{(\phi\phi)}(x, y_1) - \vartheta_2^{(\phi\theta)}(x, y_2)]$$

for all $(x, y_1, y_2) \in U \times U_1 \times U_2$. Since $\vartheta_j^{(\phi\phi)}(x, y_1)$ and $\vartheta_j^{(\phi\theta)}(x, y_2)$ are analytic functions of $x \in U$, $y_1 \in U_1$ and $y_2 \in U_2$, respectively $(j = 1, 2)$, we obtain that there holds either

$$(55)\qquad \vartheta_1^{(\phi\phi)}(x, y_1) - \vartheta_1^{(\phi\theta)}(x, y_2) = \vartheta_2^{(\phi\phi)}(x, y_1) - \vartheta_2^{(\phi\theta)}(x, y_2) + 2q\pi$$

or

$$(56)\qquad \vartheta_1^{(\phi\phi)}(x, y_1) - \vartheta_1^{(\phi\theta)}(x, y_2) = -[\vartheta_2^{(\phi\phi)}(x, y_1) - \vartheta_2^{(\phi\theta)}(x, y_2)] + 2q\pi$$

for all $(x, y_1, y_2) \in U \times U_1 \times U_2$, where $q \in \mathbb{Z}$.

For the case when (55) holds, we have

$$\alpha(x) := \vartheta_1^{(\phi\phi)}(x, y_1) - \vartheta_2^{(\phi\phi)}(x, y_1) = \vartheta_1^{(\phi\theta)}(x, y_2) - \vartheta_2^{(\phi\theta)}(x, y_2) + 2q\pi$$

depends only on x, which is a real-valued analytic function in $x \in U$. Thus

$$\begin{aligned} \boldsymbol{e}_\phi(x)\cdot E_1(x,y_1)\boldsymbol{e}_\phi(y_1) &= r^{(\phi\phi)}(x,y_1)e^{i\vartheta_1^{(\phi\phi)}(x,y_1)} \\ &= r^{(\phi\phi)}(x,y_1)e^{i\alpha(x)+i\vartheta_2^{(\phi\phi)}(x,y_1)} \\ &= e^{i\alpha(x)}\boldsymbol{e}_\phi(x)\cdot E_2(x,y_1)\boldsymbol{e}_\phi(y_1), \\ \boldsymbol{e}_\phi(x)\cdot E_1(x,y_2)\boldsymbol{e}_\theta(y_2) &= r^{(\phi\theta)}(x,y_2)e^{i\vartheta_1^{(\phi\theta)}(x,y_2)} \\ &= r^{(\phi\theta)}(x,y_2)e^{i\alpha(x)+i\vartheta_2^{(\phi\theta)}(x,y_2)} \\ &= e^{i\alpha(x)}\boldsymbol{e}_\phi(x)\cdot E_2(x,y_2)\boldsymbol{e}_\theta(y_2) \end{aligned}$$

for all $(x,y_1,y_2)\in U\times U_1\times U_2$. By the analyticity of $E_1(x,y)-e^{i\alpha(x)}E_2(x,y)$ in $y\in\partial B_{R_1}$ for $y\neq x$, we obtain

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_1)\boldsymbol{e}_\phi(y_1) = e^{i\alpha(x)}\boldsymbol{e}_\phi\cdot E_2(x,y_1)\boldsymbol{e}_\phi(y_1),\ \forall x\in U, y_1\in\partial\mathring{B}_{R_1}, x\neq y_1, \tag{57}$$

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_2)\boldsymbol{e}_\theta(y_2) = e^{i\alpha(x)}\boldsymbol{e}_\phi\cdot E_2(x,y_2)\boldsymbol{e}_\theta(y_2), \forall x\in U, y_2\in\partial\mathring{B}_{R_1}, x\neq y_2. \tag{58}$$

From (57) it follows that

$$\begin{aligned} &\boldsymbol{e}_\phi(x)\cdot[E_1^s(x,y_1)\boldsymbol{e}_\phi(y_1)-e^{i\alpha(x)}E_2^s(x,y_1)\boldsymbol{e}_\phi(y_1)] \\ &\qquad = [e^{i\alpha(x)}-1]\boldsymbol{e}_\phi(x)\cdot E^i(x,y_1)\boldsymbol{e}_\phi(y_1) \end{aligned} \tag{59}$$

for all $x\in U$ and $y_1\in\partial\mathring{B}_{R_1}$ with $x\neq y_1$. For arbitrarily fixed $y_1\in U$, the left-hand side of (59) is analytic in $x\in U$, while, by (52) the right-hand side of (59) is singular when x is close to y_1 along the circle $C_{\boldsymbol{e}_\phi(y_1)}$. Therefore, $e^{i\alpha(y_1)}=1$. Since $y_1\in U$ is arbitrary, we have $e^{i\alpha(x)}=1$ for all $x\in U$, and so (57) and (58) become

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_1)\boldsymbol{e}_\phi(y_1) = \boldsymbol{e}_\phi\cdot E_2(x,y_1)\boldsymbol{e}_\phi(y_1),\ \forall x\in U, y_1\in\partial\mathring{B}_{R_1}, x\neq y_1, \tag{60}$$

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_2)\boldsymbol{e}_\theta(y_2) = \boldsymbol{e}_\phi\cdot E_2(x,y_2)\boldsymbol{e}_\theta(y_2),\ \forall x\in U, y_2\in\partial\mathring{B}_{R_1}, x\neq y_2. \tag{61}$$

This, together with the analyticity of $E_j(x,y)$ $(j=1,2)$ in $x\in\partial B_{R_1}$ with $x\neq y$, gives (47) and (48).

Similarly, for the case when (56) holds, we can deduce

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_1)\boldsymbol{e}_\phi(y_1) = e^{i\beta(x)}\boldsymbol{e}_\phi(x)\cdot\overline{E_2(x,y_1)}\boldsymbol{e}_\phi(y_1),\ \forall x\in U, y_1\in\partial\mathring{B}_{R_1}, x\neq y_1, \tag{62}$$

$$\boldsymbol{e}_\phi(x)\cdot E_1(x,y_2)\boldsymbol{e}_\theta(y_2) = e^{i\beta(x)}\boldsymbol{e}_\phi(x)\cdot\overline{E_2(x,y_2)}\boldsymbol{e}_\theta(y_2),\ \forall x\in U, y_2\in\partial\mathring{B}_{R_1}, x\neq y_2, \tag{63}$$

where β is a real-valued analytic function of $x\in U$. From (62) it is easy to derive that

$$\begin{aligned} &\boldsymbol{e}_\phi(x)\cdot[E_1^s(x,y_1)-e^{i\beta(x)}\overline{E_2^s(x,y_1)}]\boldsymbol{e}_\phi(y_1) \\ &\qquad = \boldsymbol{e}_\phi(x)\cdot[e^{i\beta(x)}\overline{E^i(x,y_1)}-E^i(x,y_1)]\boldsymbol{e}_\phi(y_1) \end{aligned} \tag{64}$$

for all $x\in U$, $y_1\in\partial B_{R_1}$, $x\neq y_1$. For arbitrarily fixed $y_1\in U$, the left-hand side of (64) is analytic in $x\in U$, but, by (9) and a direct calculation, the right-hand side of (64) has a singularity at $x=y_1$ unless $e^{i\beta(x)}=-1$ for $x\in C_{\boldsymbol{e}_\phi(y_1)}$ near y_1. This means that $e^{i\beta(y_1)}=-1$. By the arbitrariness of $y_1\in U$, we have $e^{i\beta(x)}=-1$ for all $x \in U$, and so

$$\begin{aligned} e^{i\beta(x)}\overline{E^i(x,y)} - E^i(x,y) &= -\overline{E^i(x,y)} - E^i(x,y) \\ &= (k^2 I + \nabla_x\nabla_x)\frac{i}{k}\left[\overline{\Phi_k(x,y)} - \Phi_k(x,y)\right] \end{aligned} \tag{65}$$

is analytic in $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$, respectively, since $\overline{\Phi_k(x,y)} - \Phi_k(x,y)$ is analytic in $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$, respectively. Thus (62) and (63) are reduced to

(66)
$$\boldsymbol{e}_\phi(x) \cdot E_1(x,y_1)\boldsymbol{e}_\phi(y_1) = -\boldsymbol{e}_\phi(x) \cdot \overline{E_2(x,y_1)}\boldsymbol{e}_\phi(y_1), \quad \forall x \in U, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1,$$

(67)
$$\boldsymbol{e}_\phi(x) \cdot E_1(x,y_2)\boldsymbol{e}_\theta(y_2) = -\boldsymbol{e}_\phi(x) \cdot \overline{E_2(x,y_2)}\boldsymbol{e}_\theta(y_2), \quad \forall x \in U, y_2 \in \partial\mathring{B}_{R_1}, x \neq y_2.$$

Both (49) and (50) then follow from the analyticity of $E_j(x,y)$ $(j = 1,2)$ in $x \in \partial B_{R_1}$ for $x \neq y$. The proof is thus complete. $\square$

Lemma 4.4. *Assume that the assumptions of Theorem 4.2 are satisfied. If for any fixed $m, n, l \in \{\phi, \theta\}$ there holds*

$$\begin{aligned} &|\boldsymbol{e}_m(x) \cdot E_1(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)| \\ &\qquad = |\boldsymbol{e}_m(x) \cdot E_2(x, y_1, \boldsymbol{e}_n(y_1), \tau_1, y_2, \boldsymbol{e}_l(y_2), \tau_2)| \end{aligned} \tag{68}$$

for all $x, y_1 \in \partial\mathring{B}_{R_1}$ with $x \neq y_1$, $y_2 \in \partial\mathring{B}_{R_2}$, $(\tau_1, \tau_2) \in \{(1,0),(0,1),(1,1)\}$, then we have either

(69)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_1)\boldsymbol{e}_n(y_1) = \boldsymbol{e}_m(x) \cdot E_2(x,y_1)\boldsymbol{e}_n(y_1), \ \forall x, y_1 \in \partial\mathring{B}_{R_1} \text{ with } x \neq y_1,$$

(70)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2) = \boldsymbol{e}_m(x) \cdot E_2(x,y_2)\boldsymbol{e}_l(y_2), \ \forall x \in \partial\mathring{B}_{R_1}, y_2 \in \partial\mathring{B}_{R_2}$$

or

(71)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_1)\boldsymbol{e}_n(y_1) = -\boldsymbol{e}_m(x) \cdot \overline{E_2(x,y_1)}\boldsymbol{e}_n(y_1), \ \forall x, y_1 \in \partial\mathring{B}_{R_1} \text{ with } x \neq y_1,$$

(72)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2) = -\boldsymbol{e}_m(x) \cdot \overline{E_2(x,y_2)}\boldsymbol{e}_l(y_2), \ \forall x \in \partial\mathring{B}_{R_1}, y_2 \in \partial\mathring{B}_{R_2}.$$

Proof. Since $|\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2)|$ is analytic in $x \in \partial\mathring{B}_{R_1}$ and $y_2 \in \partial\mathring{B}_{R_2}$, respectively, we only need to distinguish between two cases:

$$\textbf{A)} \ \ |\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2)| \not\equiv 0, \ \ \forall (x,y_2) \in \partial\mathring{B}_{R_1} \times \partial\mathring{B}_{R_2},$$
$$\textbf{B)} \ \ |\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2)| \equiv 0, \ \ \forall (x,y_2) \in \partial\mathring{B}_{R_1} \times \partial\mathring{B}_{R_2}.$$

For the case when **A**) holds, by arguing similarly as in the proof of Lemma 4.3 it can be deduced from (68) that we have either

(73)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_1)\boldsymbol{e}_n(y_1) = e^{i\alpha(x)}\boldsymbol{e}_m(x) \cdot E_2(x,y_1)\boldsymbol{e}_n(y_1), \ \forall x \in U, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1,$$

(74)
$$\boldsymbol{e}_m(x) \cdot E_1(x,y_2)\boldsymbol{e}_l(y_2) = e^{i\alpha(x)}\boldsymbol{e}_m(x) \cdot E_2(x,y_2)\boldsymbol{e}_l(y_2), \ \forall x \in U, y_2 \in \partial\mathring{B}_{R_2}$$

or

$$\boldsymbol{e}_m(x)\cdot E_1(x,y_1)\boldsymbol{e}_n(y_1)=e^{i\beta(x)}\boldsymbol{e}_m(x)\cdot\overline{E_2(x,y_1)}\boldsymbol{e}_n(y_1),\ \forall x\in U, y_1\in\partial\mathring{B}_{R_1}, x\neq y_1, \tag{75}$$

$$\boldsymbol{e}_m(x)\cdot E_1(x,y_2)\boldsymbol{e}_l(y_2)=e^{i\beta(x)}\boldsymbol{e}_m(x)\cdot\overline{E_2(x,y_2)}\boldsymbol{e}_l(y_2),\ \forall x\in U, y_2\in\partial\mathring{B}_{R_2}, \tag{76}$$

where U is some small open subset of $\partial\mathring{B}_{R_1}$, and $\alpha(x)$ and $\beta(x)$ are real-valued functions of x. By (47) and (49) in Lemma 4.3 it follows easily that $e^{i\alpha(x)}=1$ and $e^{i\beta(x)}=-1$. This, together with (73)-(76) and the analyticity of the total fields $E_j(x,y)$, $j=1,2$, in x for $x\neq y$, implies that either (69) and (70) hold or (71) and (72) hold.

For the case when **B**) holds, it follows from (68) that

$$|\boldsymbol{e}_m(x)\cdot E_2(x,y_2)\boldsymbol{e}_l(y_2)|\equiv 0,\ \ \forall(x,y_2)\in\partial\mathring{B}_{R_1}\times\partial\mathring{B}_{R_2}.$$

Therefore, both (70) and (72) hold. Further, by Lemma 4.3 we have that either (69) or (71) holds. The proof is thus complete. ☐

Using Lemmas 4.3 and 4.4 we can prove the following lemma.

Lemma 4.5. *Assume that the assumptions of Theorem 4.2 are satisfied. If (46) and (68) hold for all $m,n,l\in\{\phi,\theta\}$, then we have*

$$E_1(x,y)=E_2(x,y),\quad \forall x,y\in G,\ x\neq y. \tag{77}$$

Proof. We first show that for any fixed $m\in\{\phi,\theta\}$,

$$\begin{aligned}\boldsymbol{e}_m(x)\cdot E_1(x,y_1)\boldsymbol{e}_n(y_1)&=\boldsymbol{e}_m(x)\cdot E_2(x,y_1)\boldsymbol{e}_n(y_1),\\ &\forall x,y_1\in\partial\mathring{B}_{R_1}, x\neq y_1,\ \forall n\in\{\phi,\theta\}.\end{aligned}\tag{78}$$

To this end, for any fixed $m\in\{\phi,\theta\}$ we need to distinguish between the following two cases.

Case 1. $\mathrm{Re}[\boldsymbol{e}_m(x)\cdot E_1(x,y_1)\boldsymbol{e}_l(y_1)]=0$ for all $x,y_1\in\partial\mathring{B}_{R_1}$ with $x\neq y_1$ and for all $l\in\{\phi,\theta\}$.

In this case, by Lemma 4.3 it follows that $\mathrm{Re}[\boldsymbol{e}_m(x)\cdot E_2(x,y_1)\boldsymbol{e}_l(y_1)]=0$ for all $x,y_1\in\partial\mathring{B}_{R_1}$ with $x\neq y_1$ and for all $l\in\{\phi,\theta\}$. By Lemma 4.3 again we have (78).

Case 2. $\mathrm{Re}[\boldsymbol{e}_m(x)\cdot E_1(x,y_1)\boldsymbol{e}_l(y_1)]\neq 0$ for some $x,y_1\in\partial\mathring{B}_{R_1}$ with $x\neq y_1$, $l\in\{\phi,\theta\}$. Here, we only consider the case with $l=\phi$. The case $l=\theta$ can be treated similarly.

In this case, by Lemma 4.3 we have that either both (47) and (48) hold or both (49) and (50) hold. We can prove that both (49) and (50) can not hold simultaneously. Suppose this is not the case. Then we have

$$\begin{aligned}\boldsymbol{e}_m(x)\cdot[E_1(x,y_1)\boldsymbol{e}_n(y_1)]&=-\boldsymbol{e}_m(x)\cdot[\overline{E_2(x,y_1)}\boldsymbol{e}_n(y_1)],\\ &\forall x,y_1\in\partial\mathring{B}_{R_1},\ x\neq y_1,\ \forall n\in\{\phi,\theta\}.\end{aligned}\tag{79}$$

This, together with Lemmas 4.3 and 4.4, implies that

$$\begin{aligned}\boldsymbol{e}_m(x)\cdot[E_1(x,y_2)\boldsymbol{e}_n(y_2)]&=-\boldsymbol{e}_m(x)\cdot[\overline{E_2(x,y_2)}\boldsymbol{e}_n(y_2)],\\ &\forall x\in\partial\mathring{B}_{R_1}, y_2\in\partial\mathring{B}_{R_2},\forall n\in\{\phi,\theta\}.\end{aligned}\tag{80}$$

We now show that both (79) and (80) can not hold simultaneously. In fact, by the reciprocity relation $E_j(x,y) = [E_j(y,x)]^\top$ for all $x, y \in G$ $(j = 1,2)$, we deduce from (79) and (80) that

$$\begin{aligned} \boldsymbol{e}_n(y_1)\cdot[E_1(y_1,x)\boldsymbol{e}_m(x)] &= -\boldsymbol{e}_n(y_1)\cdot[\overline{E_2(y_1,x)}\boldsymbol{e}_m(x)], \\ &\qquad \forall x, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1, \forall n \in \{\phi,\theta\}, \end{aligned} \tag{81}$$

$$\begin{aligned} \boldsymbol{e}_n(y_2)\cdot[E_1(y_2,x)\boldsymbol{e}_m(x)] &= -\boldsymbol{e}_n(y_2)\cdot[\overline{E_2(y_2,x)}\boldsymbol{e}_m(x)], \\ &\qquad \forall x \in \partial\mathring{B}_{R_1}, y_2 \in \partial\mathring{B}_{R_2}, \forall n \in \{\phi,\theta\}. \end{aligned} \tag{82}$$

This, together with the linear combination of $\boldsymbol{e}_\phi(y_j)$ and $\boldsymbol{e}_\theta(y_j)$ $(j = 1,2)$, gives that

$$\nu(y_1)\times[E_1(y_1,x)\boldsymbol{e}_m(x)] = -\nu(y_1)\times[\overline{E_2(y_1,x)}\boldsymbol{e}_m(x)],\ \forall x, y_1 \in \partial\mathring{B}_{R_1}, x \neq y_1, \tag{83}$$

$$\nu(y_2)\times[E_1(y_2,x)\boldsymbol{e}_m(x)] = -\nu(y_2)\times[\overline{E_2(y_2,x)}\boldsymbol{e}_m(x)],\ \forall x \in \partial\mathring{B}_{R_1}, y_2 \in \partial\mathring{B}_{R_2}. \tag{84}$$

For any fixed $x \in \partial\mathring{B}_{R_1}$ and $m \in \{\phi,\theta\}$, define $\widetilde{E}(y) := E_1(y,x)\boldsymbol{e}_m(x) + \overline{E_2(y,x)}$ $\boldsymbol{e}_m(x)$, $y \neq x$. Since $2\mathrm{Re}[E^i(y,x)] := E^i(y,x) + \overline{E^i(y,x)}$ is analyticity for all $x, y \in \mathbb{R}^3$ (see (65)), then, by the analyticity of $E_j^s(y,x)$ with respect to $y \in G$ $(j = 1,2)$, it follows that $\widetilde{E}$ can be extended as an analytic function of $y \in G$, which we denote by $\widetilde{E}$ again. Define $\widetilde{H}(y) := [1/(ik)]\mathrm{curl}_y\widetilde{E}(y)$. Then $(\mathrm{Re}[E^i(y,x)]\boldsymbol{e}_m(x), \mathrm{Im}[H^i(y,x)]$ $\boldsymbol{e}_m(x))$ and $(E_j^s(y,x)\boldsymbol{e}_m(x), H_j^s(y,x)\boldsymbol{e}_m(x))$ satisfy the Maxwell equations for $x \in G$, $j = 1,2$. Thus it follows by (83), (84) and the analyticity of $E_j(y,x)$ in $y \in G$ with $y \neq x$ $(j = 1,2)$ that $(\widetilde{E}, \widetilde{H})$ satisfies the interior Maxwell problem

$$\begin{cases} \mathrm{curl}\,\widetilde{E} - ik\widetilde{H} = 0 & \text{in } B_{R_2}\setminus\overline{B_{R_1}}, \\ \mathrm{curl}\,\widetilde{H} + ik\widetilde{E} = 0 & \text{in } B_{R_2}\setminus\overline{B_{R_1}}, \\ \nu\times\widetilde{E} = 0 & \text{on } \partial B_{R_1}\cup\partial B_{R_2}. \end{cases}$$

Since k^2 is not a Maxwell eigenvalue in $B_{R_2}\setminus\overline{B_{R_1}}$, then $\widetilde{E} = 0$ in $B_{R_2}\setminus\overline{B_{R_1}}$. Thus, and by the analyticity of $E_j(y,x)$ in $y \in G$ with $y \neq x$ $(j = 1,2)$, we have $E_1(y,x)\boldsymbol{e}_\phi(x) = -\overline{E_2(y,x)}\boldsymbol{e}_\phi(x)$ for all $y \in G$, $y \neq x$. This contradicts to the fact that $E_j(y,x)\boldsymbol{e}_m(x) = E^i(y,x)\boldsymbol{e}_m(x) + E_j^s(y,x)\boldsymbol{e}_m(x)$, $j = 1,2$, satisfy the Silver-Müller radiation condition. Therefore, (79) and (80) can not be true simultaneously, which means that both (49) and (50) can not hold simultaneously. This then implies that both (47) and (48) are true, and so (78) holds.

Finally, by (78) and the linear combination of $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$ we obtain that for arbitrarily fixed $y_1 \in \partial\mathring{B}_{R_1}$ and $n \in \{\phi,\theta\}$,

$$\nu(x)\times[E_1^s(x,y_1)\boldsymbol{e}_n(y_1)] = \nu(x)\times[E_2^s(x,y_1)\boldsymbol{e}_n(y_1)],\ \ \forall x \in \partial\mathring{B}_{R_1}.$$

By the well-posedness of the exterior Maxwell problem in $\mathbb{R}^3\setminus B_{R_1}$ with the PEC condition on ∂B_{R_1} it is deduced that for arbitrarily fixed $y_1 \in \partial\mathring{B}_{R_1}$,

$$E_1^s(x,y_1)\boldsymbol{e}_n(y_1) = E_2^s(x,y_1)\boldsymbol{e}_n(y_1),\quad \forall\, x \in \mathbb{R}^3\setminus B_{R_1},\ \ \forall\, n \in \{\phi,\theta\}.$$

This, together with the reciprocity relation $E_j^s(x,y) = [E_j^s(y,x)]^\top$ for all $x, y \in G$, $j = 1,2$, implies that for any fixed $x \in \mathbb{R}^3\setminus B_{R_1}$,

$$\nu(y)\times E_1^s(y,x) = \nu(y)\times E_2^s(y,x),\ \ \forall y \in \partial B_{R_1}.$$

Again, by the well-posedness of the exterior Maxwell problem in $\mathbb{R}^3 \setminus B_{R_1}$ with the PEC condition on ∂B_{R_1} it is derived that for any fixed $x \in \mathbb{R}^3 \setminus B_{R_1}$,

$$E_1^s(y,x) = E_2^s(y,x), \quad \forall y \in \mathbb{R}^3 \setminus B_{R_1}.$$

The required result (77) then follows from this, the reciprocity relation and the analyticity of $E_j^s(x,y)$ $(j=1,2)$ in $x \in G$ and $y \in G$, respectively. □

Proof of Theorem 4.2. By Lemma 4.5 it follows from (44) and (45) that (77) holds. For $j=1,2$, denote by $E_j^\infty(\hat{x},y)$ the far-field pattern of $E_j^s(x,y)$, $x,y \in G$, and by $E_j^s(x,d)$ and $E_j^\infty(\hat{x},d)$ the electric scattered field and its far-field pattern associated with the obstacle D_j (or the refractive index n_j) and corresponding to the incident electromagnetic plane waves described by the matrices $E^i(x,d)$, $H^i(x,d)$ defined by

$$E^i(x,d)p := \frac{i}{k}\mathrm{curl}\,\mathrm{curl}\, p e^{ikx\cdot d} = ik(d\times p)\times d e^{ikx\cdot d},$$
$$H^i(x,d)p := \mathrm{curl}\, p e^{ikx\cdot d} = ikd \times p e^{ikx\cdot d},$$

where $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3 \backslash \{(0,0,0)\}$ denote the incident direction and polarization vector, respectively, and $x \in \mathbb{R}^3$. Then, by (77) in Lemma 4.5 and the mixed reciprocity relation that $4\pi E_j^\infty(-d,x) = [E_j^s(x,d)]^\top$ for all $x \in G$, $d \in \mathbb{S}^2$ and $j=1,2$ (see [14, Theorem 6.31]), we obtain that $E_1^s(x,d) = E_2^s(x,d)$ for all $x \in G$ and all $d \in \mathbb{S}^2$ or $E_1^\infty(\hat{x},d) = E_2^\infty(\hat{x},d)$ for all $\hat{x},\ d \in \mathbb{S}^2$. By the uniqueness result for inverse electromagnetic scattering with full far-field data (see [14, Theorem 7.1] and [9, Theorem 3.1] for the obstacle case, and [15, Theorem 4.9] for the inhomogeneous medium case) it follows easily that the uniqueness statements (a) and (b) of Theorem 4.2 are true. The theorem is thus proved. □

5. Conclusions.

This paper proposed a new approach to prove uniqueness results for inverse acoustic and electromagnetic scattering for obstacles and inhomogeneous media with phaseless near-field data at a fixed frequency. The idea is to use superpositions of two point sources at a fixed frequency as the incident fields and, as the phaseless near-field data, to measure the modulus of the acoustic total-field on two spheres enclosing the scatterers generated by such incident fields located on the two spheres, in the acoustic case. For the electromagnetic case, the idea is to utilize superpositions of two electric dipoles at a fixed frequency with the polarization vectors $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, as the incident fields and, as the phaseless near-field data, to measure the modulus of the tangential component with the orientations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, of the electric total-field on a sphere enclosing the scatterers and generated by such incident fields located on the measurement sphere and another bigger sphere. As far as we know, this is the first uniqueness result for three-dimensional inverse electromagnetic scattering with phaseless near-field data.

Acknowledgments. The authors thank Professor Xudong Chen at the National University of Singapore for helpful and constructive discussions on the measurement technique of electromagnetic waves.

REFERENCES

[1] H. Ammari, Y. T. Chow and J. Zou, Phased and phaseless domain reconstructions in the inverse scattering problem via scattering coefficients, *SIAM J. Appl. Math.*, **76** (2016), 1000–1030.

[2] G. Bao, P. Li and J. Lv, Numerical solution of an inverse diffraction grating problem from phaseless data, *J. Opt. Soc. Am. A*, **30** (2013), 293–299.

[3] G. Bao and L. Zhang, Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data, *Inverse Problems*, **32** (2016), 085002, 16pp.

[4] E. Blåsten, *The Inverse Problem of the Schrödinger Equation in the Plane*, Licentiate thesis, University of Helsinki, 2011, `arXiv:1103.6200v1`.

[5] T. Brown, I. Jeffrey and P. Mojabi, Multiplicatively regularized source reconstruction method for phaseless planar near-field antenna measurements, *IEEE Trans. Antennas and Propagation*, **65** (2017), 2020–2031.

[6] A. L. Bukhgeim, Recovering a potential from Cauchy data in the two-dimensional case, *J. Inverse Ill-Posed Probl.*, **16** (2008), 19–33.

[7] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, Applied Mathematical Sciences, Vol. 188, Springer, New York, 2014.

[8] F. Cakoni, D. Colton and P. Monk, The electromagnetic inverse-scattering problem for partly coated Lipschitz domains, *Proc. Roy. Soc. Edinburgh Sect. A*, **134** (2004), 661–682.

[9] F. Cakoni, D. Colton and P. Monk, *The Linear Sampling Method in Inverse Electromagnetic Scattering*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 80, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[10] X. Chen, *Computational Methods for Electromagnetic Inverse Scattering*, John Wiley & Sons, Singapore, 2018.

[11] Z. Chen, S. Fang and G. Huang, A direct imaging method for the half-space inverse scattering problem with phaseless data, *Inverse Probl. Imaging*, **11** (2017), 901–916.

[12] Z. Chen and G. Huang, A direct imaging method for electromagnetic scattering data without phase information, *SIAM J. Imaging Sci.*, **9** (2016), 1273–1297.

[13] Z. Chen and G. Huang, Phaseless imaging by reverse time migration: Acoustic waves, *Numer. Math. Theory Methods Appl.*, **10** (2017), 1–21.

[14] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3^{rd} edition, Applied Mathematical Sciences, Vol. 93, Springer, New York, 2013.

[15] P. Hähner, *On Acoustic, Electromagnetic, and Elastic Scattering Problems in Inhomogeneous Media*, Habilitation thesis, Göttingen, 1998.

[16] J. E. Hansen, *Spherical Near-Field Antenna Measurements*, IET Electromagnetic Waves Series, Vol. 26, IET Press, London, 1988.

[17] T. Isernia, G. Leone and R. Pierri, Radiation pattern evaluation from near-field intensities on planes, *IEEE Trans. on Antennas and Propagation*, **44** (1996), 701–710.

[18] O. Ivanyshyn and R. Kress, Inverse scattering for surface impedance from phase-less far field data, *J. Comput. Phys.*, **230** (2011), 3443–3452.

[19] X. Ji, X. Liu and B. Zhang, Target reconstruction with a reference point scatterer using phaseless far field patterns, *SIAM J. Imaging Sci.*, **12** (2019), 372–391.

[20] X. Ji, X. Liu and B. Zhang, Phaseless inverse source scattering problem: Phase retrieval, uniqueness and direct sampling methods, *Journal of Computational Physics: X*, **1** (2019), 100003.

[21] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, 2^nd edition, Applied Mathematical Sciences, Vol. 120, Springer, New York, 2011.

[22] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford Lecture Series in Mathematics and its Applications, Vol. 36, Oxford University Press, Oxford, 2008.

[23] A. Kirsch and F. Hettlich, *The Mathematical Theory of Time-Harmonic Maxwell's Equations*, Expansion-, Integral-, and Variational Methods, Applied Mathematical Sciences, Vol. 190, Springer, Cham, 2015.

[24] M. V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, **74** (2014), 392–410.

[25] M. V. Klibanov, A phaseless inverse scattering problem for the 3-D Helmholtz equation, *Inverse Probl. Imaging*, **11** (2017), 263–276.

[26] M. V. Klibanov and V. G. Romanov, Reconstruction procedures for two inverse scattering problems without the phase information, *SIAM J. Appl. Math.*, **76** (2016), 178–196.

[27] M. V. Klibanov and V. G. Romanov, Uniqueness of a 3-D coefficient inverse scattering problem without the phase information, *Inverse Problems*, **33** (2017), 095007, 10pp.

[28] R. Leis, *Initial-Boundary Value Problems in Mathematical Physics*, B. G. Teubner, Stuttgart, John Wiley & Sons, Ltd., Chichester, 1986.

[29] J. Li, H. Liu and Y. Wang, Recovering an electromagnetic obstacle by a few phaseless backscattering measurements, *Inverse Problems*, **33** (2017), 035011, 20pp.

[30] X. Liu, B. Zhang and G. Hu, Uniqueness in the inverse scattering problem in a piecewise homogeneous medium, *Inverse Problems*, **26** (2010), 015002, 14pp.

[31] S. Maretzke and T. Hohage, Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging, *SIAM J. Appl. Math.*, **77** (2017), 384–408.

[32] R. G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bull. Sci. Math.*, **139** (2015), 923–936.

[33] R. G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, *J. Geom. Anal.*, **26** (2016), 346–359.

[34] L. Pan, Y. Zhong, X. Chen and S. P. Yeo, Subspace-based optimization method for inverse scattering problems utilizing phaseless data, *IEEE Trans. Geoscience and Remote Sensing*, **49** (2011), 981–987.

[35] R. Pierri, G. D'Elia and F. Soldovieri, A two probes scanning phaseless near-field far-field transformation technique, *IEEE Trans. Antennas and Propagation*, **47** (1999), 792–802.

[36] S. F. Razavi and Y. Rahmat-Samii, Phaseless measurements over nonrectangular planar near-field systems without probe corotation, *IEEE Trans. Antennas and Propagation*, **61** (2013), 143–152.

[37] V. G. Romanov and M. Yamamoto, Phaseless inverse problems with interference waves, *J. Inverse Ill-Posed Probl.*, **26** (2018), 681–688.

[38] C. H. Schmidt, S. F. Razavi, T. F. Eibert and Y. Rahmat-Samii, Phaseless spherical near-field antenna measurements for low and medium gain antennas, *Adv. Radio Sci.*, **8** (2010), 43–48.

[39] J. Shin, Inverse obstacle backscattering problems with phaseless data, *European J. Appl. Math.*, **27** (2016), 111–130.

[40] F. Sun, D. Zhang and Y. Guo, Uniqueness in phaseless inverse scattering problems with known superposition of incident point sources, *Inverse Problems*, **35** (2019), 105007.

[41] T. Takenaka, D. J. N. Wall, H. Harada and M. Tanaka, Reconstruction algorithm of the refractive index of a cylindrical object from the intensity measurements of the total field, *Microwave and Optical Technology Letters*, **14** (1998), 182–188.

[42] X. Xu, B. Zhang and H. Zhang, Uniqueness and direct imaging method for inverse scattering by locally rough surfaces with phaseless near-field data, *SIAM J. Imaging Sci.*, **12** (2019), 119–152.

[43] X. Xu, *Uniqueness in Phaseless Inverse Acoustic and Electromagnetic Scattering Problems*, Ph.D thesis, University of Chinese Academy of Sciences, Beijing, 2019.

[44] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency, *SIAM J. Appl. Math.*, **78** (2018), 1737–1753.

[45] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency. II, *SIAM J. Appl. Math.*, **78** (2018), 3024–3039.

[46] B. Zhang and H. Zhang, Imaging of locally rough surfaces from intensity-only far-field or near-field data, *Inverse Problems*, **33** (2017), 055001, 28pp.

[47] B. Zhang and H. Zhang, Recovering scattering obstacles by multi-frequency phaseless far-field data, *J. Comput. Phys.*, **345** (2017), 58–73.

[48] B. Zhang and H. Zhang, Fast imaging of scattering obstacles from phaseless far-field measurements at a fixed frequency, *Inverse Problems*, **34** (2018), 104005, 24pp.

[49] D. Zhang and Y. Guo, Uniqueness results on phaseless inverse acoustic scattering with a reference ball, *Inverse Problems*, **34** (2018), 085002, 12pp.

[50] D. Zhang, F. Sun, Y. Guo and H. Liu, Uniqueness in inverse acoustic scattering with phaseless near-field measurements, preprint, 2019, `arXiv:1905.08242v1`.

[51] D. Zhang, Y. Wang, Y. Guo and J. Li, Uniqueness in inverse cavity scattering problems with phaseless near-field data, *Inverse Problems*, **36** (2020), 025004, 10pp.

Received July 2019; revised November 2019.

E-mail address: xuxiaoxu@csrc.ac.cn
E-mail address: b.zhang@amt.ac.cn
E-mail address: zhanghaiwen@amss.ac.cn