IMA Journal of Applied Mathematics (2020) **85**, 823–839
doi:10.1093/imamat/hxaa024
Advance Access publication on 10 August 2020

Uniqueness in inverse electromagnetic scattering problem with phaseless far-field data at a fixed frequency

Xiaoxu Xu
Beijing Computational Science Research Center, Beijing 100193, China

Bo Zhang
LSEC, NCMIS and Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

and

Haiwen Zhang*
NCMIS and Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
*Corresponding author: zhanghaiwen@amss.ac.cn

[Received on 1 October 2019; revised on 22 May 2020; accepted on 1 July 2020]

This paper is concerned with uniqueness in inverse electromagnetic scattering with phaseless far-field pattern at a fixed frequency. In our previous work (2018,SIAM J. Appl. Math. **78**, 3024–3039), by adding a known reference ball into the acoustic scattering system, it was proved that the impenetrable obstacle and the index of refraction of an inhomogeneous medium can be uniquely determined by the acoustic phaseless far-field patterns generated by infinitely many sets of superpositions of two plane waves with different directions at a fixed frequency. In this paper, we extend these uniqueness results to the inverse electromagnetic scattering case. The phaseless far-field data are the modulus of the tangential component in the orientations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, of the electric far-field pattern measured on the unit sphere and generated by infinitely many sets of superpositions of two electromagnetic plane waves with different directions and polarizations. Our proof is mainly based on Rellich's lemma and the Stratton–Chu formula for radiating solutions to the Maxwell equations.

Keywords: uniqueness, inverse electromagnetic scattering, phaseless far-field pattern, impenetrable obstacle, inhomogeneous medium.

1. Introduction

Inverse scattering theory has wide applications in such fields as radar, sonar, geophysics, medical imaging and non-destructive testing (see, e.g. Colton & Kress, 2013; Kirsch & Grinberg, 2008). This paper is concerned with inverse electromagnetic scattering by a bounded obstacles or an inhomogeneous medium from phaseless far-field data, associated with incident plane waves at a fixed frequency.

Inverse scattering problems with phased data have been extensively studied both mathematically and numerically in the past several decades (see, e.g. Chen, 2018; Colton & Kress, 2013; Kirsch & Grinberg, 2008). However, in many applications, it is difficult to measure the phase of the wave field accurately, compared with the modulus of the wave field. Therefore, it is desirable to reconstruct the scatterers from the phaseless near-field or far-field data (i.e. the intensity of the near field or far field), which is called the *phaseless inverse scattering problem*.

The main difficulty of inverse scattering problems with phaseless far-field data is the so-called *translation invariance* property of the phaseless far-field pattern, i.e. the modulus of the far-field pattern generated by one plane wave is invariant under the translation of the scatterers. This implies that it is impossible to recover the location of the scatterer from the phaseless far-field data with one plane wave as the incident field. Several iterative methods have been proposed in Ivanyshyn (2007); Ivanyshyn & Kress (2010, 2011); Kress & Rundell (1997) to reconstruct the shape of the scatterer. Under *a priori* condition that the sound-soft scatterer is a ball or disk, it was proved in Liu & Zhang (2010) that the radius of the scatterer can be uniquely determined by a single phaseless far-field datum. It was proved in Majda (1976) that the shape of a general, sound-soft, strictly convex obstacle can be uniquely determined by the phaseless far-field data generated by one plane wave at a high frequency. However, there is no translation invariance property for phaseless near-field data. Therefore, many numerical algorithms for inverse scattering problems with phaseless near-field data have been developed (see, e.g. Chen, 2018; Chen *et al.*, 2017; Chen & Huang, 2017; Klibanov & Romanov, 2016; Shin, 2016; Xu *et al.*, 2019, for the acoustic case and Chen & Huang, 2016, for the electromagnetic case). Uniqueness results and stability have also been established for inverse scattering problems with phaseless near-field data (see Klibanov, 2014, 2017; Klibanov & Romanov, 2017; Maretzke & Hohage, 2017; Novikov, 2015, 2016; Romanov & Yamamoto, 2018; Xu *et al.*, 2020; Zhang *et al.*, 2020a,b, for the acoustic and potential scattering case and Romanov, 2017; Xu *et al.*, 2020, for the electromagnetic scattering case).

Recently in Zhang & Zhang (2017b), it was proved that the translation invariance property of the phaseless far-field pattern can be broken by using superpositions of two plane waves as the incident fields with an interval of frequencies. Following this idea, several algorithms have been developed for inverse acoustic scattering problems with phaseless far-field data, based on using the superposition of two plane waves as the incident field (see Zhang & Zhang, 2017a,b, 2018). Further, by using the spectral properties of the far-field operator, rigorous uniqueness results have also been established in Xu *et al.* (2018a) for inverse acoustic scattering problems with phaseless far-field data generated by infinitely many sets of superpositions of two plane waves with different directions at a fixed frequency, under certain *a priori* assumptions on the property of the scatterers. In Xu *et al.* (2018b), by adding a known reference ball into the acoustic scattering system, it was shown that the uniqueness results obtained in Xu *et al.* (2018a) remain true without the *a priori* assumptions on the property of the scatterers. Note that the idea of adding a known reference ball to the scattering system was first applied in Li *et al.* (2010) to numerically enhance the reconstruction results of the linear sampling method and also used in Zhang & Guo (2018) to prove uniqueness in inverse acoustic scattering problems with phaseless far-field data. Recently, the reference ball technique has also been used to numerically reconstruct the unknown scatterers from phaseless far-field data (see Dong *et al.*, 2019b, for the inverse acoustic scattering problem, Dong *et al.*, 2019a, for the inverse elastic scattering problem and Dong *et al.*, 2020, for the inverse acoustic–elastic interaction problem). On the other hand, by adding a reference point scatterer into the scattering system, two direct sampling algorithms have been proposed to recover acoustic obstacles in Ji *et al.* (2019c) and acoustic sources in Ji *et al.* (2019b) from phaseless far-field data generated with incident plane waves, while, in Ji *et al.* (2019a), the authors developed two direct sampling algorithms to recover acoustic obstacles from phaseless far-field measurements generated by superpositions of plane waves and point sources with fixed source location and different scattering strengths. It should be remarked that there are certain studies on uniqueness for phaseless inverse scattering problems with using superpositions of two point sources as the incident fields (see Romanov & Yamamoto, 2018; Sun *et al.*, 2019; Xu *et al.*, 2020; Zhang *et al.*, 2020a,b).

It is worth mentioning that some inversion algorithms have been developed for array imaging problems with phaseless data. For the acoustic case, we refer to Novikov *et al.* (2015) for a coherent imaging method and Moscoso *et al.* (2017) for a holography-based approach using the time reversal operator. For the electromagnetic case, we refer to Bardsley *et al.* (2018) for imaging the polarizability of polarizable scatterers from the Stokes parameters (which are phaseless data in electromagnetism).

The purpose of this paper is to establish uniqueness results in inverse electromagnetic scattering problems with phaseless far-field data at a fixed frequency, extending the uniqueness results in Xu *et al.* (2018b) for the acoustic case to the electromagnetic case. Different from the acoustic case considered in Xu *et al.* (2018b), the electric far-field pattern is a complex-valued vector function, so the measurement of the phaseless electric far-field pattern is more complicated. In practice, one usually makes measurement of the modulus of each tangential component of the electric total-field or electric far-field pattern on the measurement surface (see, e.g. Hansen, 1988; Pan *et al.*, 2011; Schmidt *et al.*, 2010; Zhang & Wang, 2019). Motivated by this and the idea in Xu *et al.* (2018a,b), we make use of superpositions of two electromagnetic plane waves with different directions and polarizations as the incident fields and consider the modulus of the tangential component in the orientations $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$, respectively, of the corresponding electric far-field pattern measured on the unit sphere as the measurement data (called the phaseless electric far-field data). We then prove that, by adding a known reference ball into the electromagnetic scattering system, the impenetrable obstacle or the refractive index of the inhomogeneous medium (under the condition that the magnetic permeability is a positive constant) can be uniquely determined by the phaseless electric far-field data at a fixed frequency. Our proof is mainly based on Rellich's lemma and the Stratton–Chu formula for radiating solutions to the Maxwell equations, which is possible due to the introduction of the reference ball into the scattering system.

The rest of this paper is organized as follows. In Section 2, we introduce the electromagnetic scattering problems considered. The uniqueness results for inverse obstacle and medium electromagnetic scattering with phaseless electric far-field data are presented in Sections 3 and 4, respectively. Conclusions are given in Section 5.

2. The electromagnetic scattering problems

In this section, we introduce the electromagnetic scattering problems considered in this paper. To give a precise description of the scattering problems, we assume that D is an open and bounded domain in $\mathbb{R}^3$ with C^2-boundary ∂D satisfying that the exterior $\mathbb{R}^3 \setminus \overline{D}$ of D is connected. Note that D may not be connected and thus may consist of several (finitely many) connected components. We consider the time-harmonic ($e^{-i\omega t}$ time dependence) incident electromagnetic plane waves described by the matrices $E^i(x,d)$ and $H^i(x,d)$ defined by

$$E^i(x,d)p := \frac{i}{k}\operatorname{curl}\operatorname{curl} pe^{ikx\cdot d} = ik(d\times p)\times de^{ikx\cdot d}, \quad x\in\mathbb{R}^3, \tag{2.1}$$

$$H^i(x,d)p := \operatorname{curl} pe^{ikx\cdot d} = ikd\times pe^{ikx\cdot d}, \quad x\in\mathbb{R}^3, \tag{2.2}$$

where $d\in\mathbb{S}^2$ is the incident direction with $\mathbb{S}^2$ being the unit sphere, $p\in\mathbb{R}^3$ is the polarization vector, $k=\omega/\sqrt{\varepsilon_0\mu_0}$ is the wave number, ω is the frequency and ε_0 and μ_0 are the electric permittivity and magnetic permeability of a homogeneous medium, respectively. From (2.1) and (2.2) it can be seen that the incident plane waves $E^i = E^i(x,d)p$ and $H^i = H^i(x,d)p$ satisfy the homogeneous Maxwell equations:

$$\operatorname{curl} E^i - ikH^i = 0 \qquad \text{in } \mathbb{R}^3,$$
$$\operatorname{curl} H^i + ikE^i = 0 \qquad \text{in } \mathbb{R}^3.$$

When D is an impenetrable obstacle, then the scattering problem can be modeled by the exterior boundary value problem:

$$\operatorname{curl} E - ikH = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.3a}$$

$$\operatorname{curl} H + ikE = 0 \qquad \text{in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.3b}$$

$$\mathscr{B}E = 0 \qquad \text{on } \partial D, \tag{2.3c}$$

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0, \qquad r = |x|, \tag{2.3d}$$

where (E^s, H^s) is the scattered field, $E := E^i + E^s$ and $H := H^i + H^s$ are the electric total-field and the magnetic total-field, respectively, the equations (2.3a)–(2.3b) are the Maxwell equations and (2.3d) is the Silver–Müller radiation condition. The boundary condition $\mathscr{B}$ in (2.3c) depends on the physical property of the obstacle D, i.e. $\mathscr{B}E = \nu \times E$ if D is a perfectly conducting obstacle, $\mathscr{B}E = \nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = 0$ if D is an impedance obstacle, and $\mathscr{B}E = \nu \times E$ on Γ_D, $\mathscr{B}E = \nu \times \operatorname{curl} E - i\lambda(\nu \times E) \times \nu = 0$ on Γ_I if D is a partially coated obstacle, where ν is the unit outward normal vector on the boundary ∂D. Here, for the case when D is an impedance obstacle, we assume that λ is the impedance function on ∂D with $\lambda \in C(\partial D)$ and $\lambda(x) \geq 0$ for all $x \in \partial D$. Further, for the case when D is a partially coated obstacle, we assume that ∂D has a Lipschitz dissection $\partial D = \Gamma_D \cup \Pi \cup \Gamma_I$ with Γ_D and Γ_I being disjoint and relatively open subsets of ∂D and having Π as their common boundary in ∂D (see, e.g. McLean, 2000) and λ is the impedance function on Γ_I with $\lambda \in C(\Gamma_I)$ and $\lambda(x) \geq 0$ for all $x \in \Gamma_I$.

When D is an inhomogeneous medium, we assume that the magnetic permeability $\mu = \mu_0$ is a positive constant in the whole space. Then the scattering problem is modeled by the medium scattering problem

$$\operatorname{curl} E - ikH = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.4a}$$

$$\operatorname{curl} H + iknE = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.4b}$$

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0, \qquad r = |x|, \tag{2.4c}$$

where (E^s, H^s) is the scattered field and $E := E^i + E^s$ and $H := H^i + H^s$ are the electric total field and the magnetic total field, respectively. The refractive index n in (2.4b) is given by

$$n(x) := \frac{1}{\varepsilon_0}\left(\varepsilon(x) + i\frac{\sigma(x)}{\omega}\right),$$

where $\varepsilon(x)$ is the electric permittivity with $\varepsilon(x) \geq \varepsilon_{min}$ in $\mathbb{R}^3$ for a constant $\varepsilon_{min} > 0$ and $\sigma(x)$ is the electric conductivity with $\sigma(x) \geq 0$ in $\mathbb{R}^3$. We assume further that $n - 1$ has a compact support $\overline{D}$ and $n \in C^{2,\gamma}(\mathbb{R}^3)$ for $0 < \gamma < 1$. From the above assumptions, it can be seen that $\mathrm{Re}[n(x)] \geq n_{min} := \varepsilon_{min}/\varepsilon_0 > 0$ and $\mathrm{Im}[n(x)] \geq 0$ for all $x \in \mathbb{R}^3$.

The existence of a unique (variational) solution to the problems (2.3a)–(2.3d) and (2.4a)–(2.4c) has been proved in Cakoni *et al.* (2004, 2011); Colton & Kress (1981, 2013); McLean (2000) (see Theorem 6.21 in Colton & Kress, 2013, and Theorem 10.8 in McLean, 2000, for scattering by a perfectly conducting obstacle or an impedance obstacle with $\lambda \equiv 0$ on ∂D, Theorems 6.11 and 9.11 in Colton & Kress, 2013, for scattering by an impedance obstacle with constant impedance function, Theorems 2.1 and 3.3 in Colton & Kress, 1981, for scattering by an impedance obstacle with $\lambda \in C^{0,\gamma}(\partial D)$, Theorem 3.5 in Cakoni *et al.*, 2011, (see also Theorem 2.7 in Cakoni *et al.*, 2004) for scattering by a partly coated obstacle or an impedance obstacle and Theorem 5.5 in Kirsch & Grinberg, 2008 (see also Theorem 9.5 in Colton & Kress, 2013) for scattering by an inhomogeneous medium). In particular, it is well known from Colton & Kress (2013) that the electric and magnetic scattered fields E^s and H^s have the asymptotic behavior

$$E^s(x,d)p = \frac{e^{ik|x|}}{|x|}\left\{E^\infty(\hat{x},d)p + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty,$$

$$H^s(x,d)p = \frac{e^{ik|x|}}{|x|}\left\{H^\infty(\hat{x},d)p + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty$$

uniformly for all observation directions $\hat{x} = x/|x| \in \mathbb{S}^2$, where $E^\infty(\hat{x},d)p$ is the electric far-field pattern of $E^s(x,d)p$ and $H^\infty(\hat{x},d)p$ is the magnetic far-field pattern of $H^s(x,d)p$ for any $p \in \mathbb{R}^3$, satisfying that (see the formula (6.24) in Colton & Kress, 2013)

$$H^\infty(\hat{x},d)p = \hat{x} \times E^\infty(\hat{x},d)p, \quad \hat{x} \cdot E^\infty(\hat{x},d)p = \hat{x} \cdot H^\infty(\hat{x},d)p = 0. \tag{2.5}$$

Because of the linearity of the direct scattering problem with respect to the incident field, the scattered waves, the total fields and the corresponding far-field patterns can be represented by matrices $E^s(x,d)$ and $H^s(x,d)$, $E(x,d)$ and $H(x,d)$ and $E^\infty(\hat{x},d)$ and $H^\infty(\hat{x},d)$, respectively. Each component of the matrices $E^\infty(\hat{x},d)$ and $H^\infty(\hat{x},d)$ is an analytic function of $\hat{x} \in \mathbb{S}^2$ for each $d \in \mathbb{S}^2$ and of $d \in \mathbb{S}^2$ for each $\hat{x} \in \mathbb{S}^2$ (see, e.g. Colton & Kress, 2013).

Throughout this paper, we assume that the wave number k is arbitrarily fixed, i.e. the frequency ω is arbitrarily fixed. Following Xu *et al.* (2018a,b); Zhang & Zhang (2017a,b), we make use of the following superposition of two plane waves as the incident (electric) field:

$$E^i := E^i(x,d_1,d_2,p_1,p_2) = E^i(x,d_1)p_1 + E^i(x,d_2)p_2 = \frac{i}{k}\mathrm{curl}\,\mathrm{curl}\,p_1 e^{ikx\cdot d_1} + \frac{i}{k}\mathrm{curl}\,\mathrm{curl}\,p_2 e^{ikx\cdot d_2},$$

where $d_1, d_2 \in \mathbb{S}^2$ and $p_1, p_2 \in \mathbb{R}^3$. Then the (electric) scattered field E^s has the asymptotic behavior

$$E^s(x,d_1,d_2,p_1,p_2) = \frac{e^{ik|x|}}{|x|}\left\{E^\infty(\hat{x},d_1,d_2,p_1,p_2) + O\left(\frac{1}{|x|}\right)\right\}, \ |x| \to \infty$$

uniformly for all observation directions $\hat{x} \in \mathbb{S}^2$. From the linear superposition principle it follows that

$$E^s(x, d_1, d_2, p_1, p_2) = E^s(x, d_1)p_1 + E^s(x, d_2)p_2$$

and

$$E^\infty(\hat{x}, d_1, d_2, p_1, p_2) = E^\infty(\hat{x}, d_1)p_1 + E^\infty(\hat{x}, d_2)p_2, \tag{2.6}$$

where $E^s(x, d_j)p_j$ and $E^\infty(\hat{x}, d_j)p_j$ are the (electric) scattered field and its far-field pattern corresponding to the incident electric field $E^i(x, d_j)p_j$, respectively, $j = 1, 2$.

Following the idea in Hansen (1988); Pan *et al.* (2011); Schmidt *et al.* (2010); Zhang & Wang (2019), we measure the modulus of the tangential component of the electric far-field pattern on the unit sphere $\mathbb{S}^2$. To present the tangential components, we introduce the spherical coordinates

$$\begin{cases} \hat{x}_1 = \sin\theta\cos\phi, \\ \hat{x}_2 = \sin\theta\sin\phi, \\ \hat{x}_3 = \cos\theta, \end{cases}$$

with $\hat{x} := (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{S}^2$ and $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. For any $\hat{x} \in \mathbb{S}^2 \backslash \{N, S\}$, the spherical coordinates give a one-to-one correspondence between $\hat{x}$ and (ϕ, θ), where $N := (0, 0, 1)$ and $S := (0, 0, -1)$ denote the north and south poles of $\mathbb{S}^2$, respectively. Define

$$\boldsymbol{e}_\phi(\hat{x}) := (-\sin\phi, \cos\phi, 0), \quad \boldsymbol{e}_\theta(\hat{x}) := (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta).$$

Then $\boldsymbol{e}_\phi(\hat{x})$ and $\boldsymbol{e}_\theta(\hat{x})$ are two orthonormal tangential vectors of $\mathbb{S}^2$ at $\hat{x} \notin \{N, S\}$. Thus, the phaseless far-field data we use are $|\boldsymbol{e}_m(\hat{x}) \cdot E^\infty(\hat{x}, d_1, d_2, p_1, p_2)|$, $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $m \in \{\phi, \theta\}$, $d_j \in \mathbb{S}^2$ and $p_j \in \mathbb{R}^3$ such that $d_j \perp p_j$, $j = 1, 2$.

The inverse electromagnetic obstacle (or medium) scattering problem we consider in this paper is to reconstruct the obstacle D and its physical property (or the refractive index n of the inhomogeneous medium) from the phaseless far-field data $|\boldsymbol{e}_m(\hat{x}) \cdot E^\infty(\hat{x}, d_1, d_2, p_1, p_2)|$, $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $m \in \{\phi, \theta\}$, $d_j \in \mathbb{S}^2$ and $p_j \in \mathbb{R}^3$ such that $d_j \perp p_j$, $j = 1, 2$. The purpose of this paper is to establish the uniqueness results for these inverse problems.

3. Uniqueness for inverse electromagnetic obstacle scattering

This section is devoted to establishing the uniqueness result in the inverse electromagnetic obstacle scattering problem. As discussed in the Introduction, uniqueness results have been established in Xu *et al.* (2018a,b) for inverse acoustic scattering with phaseless far-field data. Note that, in Xu *et al.* (2018a), the uniqueness results were proved by establishing the spectral properties of the acoustic far-field operator, which is essential in our proof. However, we do not yet know how to establish the desired spectral properties of the electromagnetic far-field operator. Thus, in the present paper, we follow the strategy in our previous acoustic work (Xu *et al.*, 2018b). Namely, we modify the electromagnetic scattering system by adding a known reference perfectly conducting ball to the scattering system, so we consider the measurement of the phaseless far-field data associated with the obstacles plus a reference

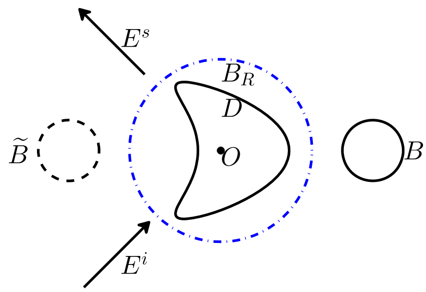

FIG. 1. Scattering by a bounded obstacle.

perfectly conducting ball. The introduction of the reference ball allows us to use the Rellich's lemma and the Stratten–Chu formula to prove our uniqueness result.

Let B be a given, perfectly conducting ball and let us assume that k is not a Maxwell eigenvalue in B. Here, k is called a Maxwell eigenvalue in B if the electromagnetic interior boundary value problem

$$\operatorname{curl}\widetilde{E} - ik\widetilde{H} = 0 \quad \text{in } B, \tag{3.1}$$

$$\operatorname{curl}\widetilde{H} + ik\widetilde{E} = 0 \quad \text{in } B, \tag{3.2}$$

$$\nu \times \widetilde{E} = 0 \quad \text{on } \partial B \tag{3.3}$$

has a non-trivial solution $(\widetilde{E}, \widetilde{H})$.

Now, denote by E_j^s, H_j^s, E_j^∞ and H_j^∞ the electric scattered field, the magnetic scattered field, the electric far-field pattern and the magnetic far-field pattern, respectively, associated with the obstacle $D_j \cup B$ and corresponding to the incident electromagnetic waves E^i and H^i, $j = 1, 2$. The geometry of the scattering problem is given in Fig. 1. Then we have the following uniqueness result for the inverse electromagnetic obstacle problem.

THEOREM 3.1 Assume that B is a given perfectly conducting reference ball such that k is not a Maxwell eigenvalue in B. Suppose D_1 and D_2 are two obstacles with $\overline{D_1 \cup D_2} \subset B_R$, where B_R is a ball of radius R and centered at the origin satisfying that $\overline{B} \cap \overline{B_R} = \emptyset$. If the corresponding electric far-field patterns satisfy that

$$|\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d_1, d_2, p_1, p_2)| = |\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d_1, d_2, p_1, p_2)| \tag{3.4}$$

for all $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d_1, d_2 \in \mathbb{S}^2$, $m \in \{\phi, \theta\}$ and $p_1, p_2 \in \mathbb{R}^3$ satisfying that $d_1 \perp p_1$ and $d_2 \perp p_2$, then $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$.

To simplify the proof of Theorem 3.1 we need the following lemma.

LEMMA 3.1 Under the assumptions of Theorem 3.1, the following equation does not hold:

$$E_1^\infty(\hat{x}, d_0)\boldsymbol{e}_m(d_0) = e^{i\beta}\overline{E_2^\infty(\hat{x}, d_0)}\boldsymbol{e}_m(d_0) \quad \forall \hat{x} \in \mathbb{S}^2, \tag{3.5}$$

where $m \in \{\phi, \theta\}$ and $d_0 \in \mathbb{S}^2 \setminus \{N, S\}$ are arbitrarily fixed and β is a real constant.

Proof. Assume to the contrary that (5) holds. Then, by using the Stratton–Chu formula (see Theorem 6.7 in Colton & Kress, 2013), we have that the electric scattered field $E_2^s(x,d_0)\boldsymbol{e}_m(d_0)$ satisfies that

$$
\begin{aligned}
E_2^s(x,d_0)\boldsymbol{e}_m(d_0) \;=\; & \operatorname{curl}\int_{\partial B\cup\partial B_R} \nu(y)\times[E_2^s(y,d_0)\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y) \\
& -\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial B\cup\partial B_R} \nu(y)\times[H_2^s(y,d_0)\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y),\;\; x\in\mathbb{R}^3\setminus\overline{B\cup B_R},
\end{aligned}
$$

where $\nu(y)$ is the unit normal vector at $y\in\partial B$ or $y\in\partial B_R$ directed into the exterior of B or B_R and $\Phi(x,y)$ is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$ given by

$$
\Phi(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\quad x\neq y.
$$

Then it follows from the formula (6.25) in Colton & Kress (2013) that the corresponding far-field pattern $E_2^\infty(\hat{x},d_0)\boldsymbol{e}_m(d_0)$ is given as

$$
\begin{aligned}
& E_2^\infty(\hat{x},d_0)\boldsymbol{e}_m(d_0) \\
&= \frac{ik}{4\pi}\hat{x}\times\int_{\partial B\cup\partial B_R}\left\{\nu(y)\times[E_2^s(y,d_0)\boldsymbol{e}_m(d_0)]+\left[\nu(y)\times[H_2^s(y,d_0)\boldsymbol{e}_m(d_0)]\right]\times\hat{x}\right\}e^{-ik\hat{x}\cdot y}\mathrm{d}s(y),\;\;\hat{x}\in\mathbb{S}^2.
\end{aligned}
$$

By this and (3.5) we deduce that for any $\hat{x}\in\mathbb{S}^2$,

$$
\begin{aligned}
& e^{-i\beta}E_1^\infty(\hat{x},d_0)\boldsymbol{e}_m(d_0) = \overline{E_2^\infty(\hat{x},d_0)}\boldsymbol{e}_m(d_0) \\
&= -\frac{ik}{4\pi}\hat{x}\times\int_{\partial B\cup\partial B_R}\left\{\nu(y)\times[\overline{E_2^s(y,d_0)}\boldsymbol{e}_m(d_0)]+\left[\nu(y)\times[\overline{H_2^s(y,d_0)}\boldsymbol{e}_m(d_0)]\right]\times\hat{x}\right\}e^{ik\hat{x}\cdot y}\mathrm{d}s(y) \\
&= -\frac{ik}{4\pi}\hat{x}\times\int_{\partial\widetilde{B}\cup\partial B_R}\left\{-\nu(y)\times[\overline{E_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]+\left[-\nu(y)\times[\overline{H_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\right]\times\hat{x}\right\}e^{-ik\hat{x}\cdot y}\mathrm{d}s(y) \\
&= \frac{ik}{4\pi}\hat{x}\times\int_{\partial\widetilde{B}\cup\partial B_R}\left\{\nu(y)\times[\overline{E_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]+\left[\nu(y)\times[\overline{H_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\right]\times\hat{x}\right\}e^{-ik\hat{x}\cdot y}\mathrm{d}s(y),
\end{aligned}
$$

where $\widetilde{B} := \{x\in\mathbb{R}^3 : -x\in B\}$. This, together with the formulas (6.26) and (6.27) in Colton & Kress (2013), implies that $e^{-i\beta}E_1^\infty(\hat{x},d_0)\boldsymbol{e}_m(d_0)$ is the far-field pattern of $\widetilde{E}^s$ given by

$$
\begin{aligned}
\widetilde{E}^s(x) := & \operatorname{curl}\int_{\partial\widetilde{B}\cup\partial B_R}\nu(y)\times[\overline{E_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y) \\
& -\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial\widetilde{B}\cup\partial B_R}\nu(y)\times[\overline{H_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y),\;\; x\in\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}.
\end{aligned}
$$

It can be seen that $\widetilde{E}^s(x)$ is an analytic function in $x\in\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}$. On the other hand, it is known that $e^{-i\beta}E_1^\infty(\hat{x},d_0)\boldsymbol{e}_m(d_0)$ is the far-field pattern of $e^{-i\beta}E_1^s(x,d_0)\boldsymbol{e}_m(d_0)$, which is an analytic function in $x\in\mathbb{R}^3\setminus\overline{B\cup B_R}$. Since the well-known Rellich's lemma establishes a one-to-one correspondence between the electric scattered field and its far-field pattern (cf. Theorem 6.10 in Colton & Kress, 2013), we obtain that $e^{-i\beta}E_1^s(x,d_0)\boldsymbol{e}_m(d_0)=\widetilde{E}^s(x)$ in $x\in\mathbb{R}^3\setminus\overline{B\cup\widetilde{B}\cup B_R}$. Then using the analyticity of $\widetilde{E}^s(x)$ in $x\in\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}$ again, we get that $E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ can be analytically extended into $\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}$ and

$$e^{-i\beta}E_1^s(x,d_0)\boldsymbol{e}_m(d_0)=\widetilde{E}^s(x)=\operatorname{curl}\int_{\partial\widetilde{B}\cup\partial B_R}\nu(y)\times[\overline{E_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y)$$
$$-\frac{1}{ik}\operatorname{curl}\operatorname{curl}\int_{\partial\widetilde{B}\cup\partial B_R}\nu(y)\times[\overline{H_2^s(-y,d_0)}\boldsymbol{e}_m(d_0)]\Phi(x,y)\mathrm{d}s(y),\ \ x\in\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}. \tag{3.6}$$

Since

$$H_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)=\frac{1}{ik}\operatorname{curl}[E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)],$$

then it follows from (3.6) that $E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ and $H_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ satisfy the Maxwell equations (2.3a)–(2.3b) in $\mathbb{R}^3\setminus\overline{\widetilde{B}\cup B_R}$. On the other hand, by the definitions of E_1^s and H_1^s, it is known that $E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ and $H_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ also satisfy the Maxwell equations (2.3a)–(2.3b) in $\mathbb{R}^3\setminus\overline{B\cup B_R}$. Since $\overline{B}\cap\overline{B_R}=\emptyset$, then the origin $0\notin B$ and $B\cap\widetilde{B}=\emptyset$. Thus, it is concluded that $E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ and $H_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ satisfy the Maxwell equations (2.3a)–(2.3b) in $\mathbb{R}^3\setminus\overline{B_R}$. Since the electric total field $E_1:=E_1(\cdot,d_0)\boldsymbol{e}_m(d_0)=E_1^i(\cdot,d_0)\boldsymbol{e}_m(d_0)+E_1^s(\cdot,d_0)\boldsymbol{e}_m(d_0)$ and the magnetic total field $H_1:=(1/ik)\operatorname{curl}E_1$ satisfy the perfectly conducting boundary condition on ∂B, then (E_1,H_1) satisfies the problem (1)–(3). By the fact that k is not a Maxwell eigenvalue in B we have that $E_1\equiv 0$ in B, which, together with the analyticity of the electric total field E_1 in $\mathbb{R}^3\setminus\overline{B_R}$, implies that $E_1\equiv 0$ in $\mathbb{R}^3\setminus\overline{B_R}$. This is a contradiction, and so (5) does not hold. The proof is complete. □

We are now ready to prove Theorem 3.1.

Proof. of Theorem 3.1 Using (2.6) and (3.4), we have

$$|\boldsymbol{e}_m(\hat{x})\cdot[E_1^\infty(\hat{x},d_1)p_1+E_1^\infty(\hat{x},d_2)p_2]|=|\boldsymbol{e}_m(\hat{x})\cdot[E_2^\infty(\hat{x},d_1)p_1+E_2^\infty(\hat{x},d_2)p_2| \tag{3.7}$$

for all $\hat{x}\in\mathbb{S}^2\setminus\{N,S\}$, $d_1,d_2\in\mathbb{S}^2$, $m\in\{\phi,\theta\}$ and $p_1,p_2\in\mathbb{R}^3$ satisfying that $d_1\perp p_1$ and $d_2\perp p_2$. By (2.1) we know that $E^i(x,d)d=0$ for all $x\in\mathbb{R}^3$ and $d\in\mathbb{S}^2$, and so, from the wellposedness of the scattering problem it follows that

$$E_j^\infty(\hat{x},d)d=0\quad\forall\hat{x},d\in\mathbb{S}^2. \tag{3.8}$$

Thus, (3.7) is equivalent to the condition

$$|\boldsymbol{e}_m(\hat{x})\cdot[E_1^\infty(\hat{x},d_1)p_1+E_1^\infty(\hat{x},d_2)p_2]|=|\boldsymbol{e}_m(\hat{x})\cdot[E_2^\infty(\hat{x},d_1)p_1+E_2^\infty(\hat{x},d_2)p_2| \tag{3.9}$$

for all $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d_1, d_2 \in \mathbb{S}^2$, $m \in \{\phi, \theta\}$ and $p_1, p_2 \in \mathbb{R}^3$. This implies that

$$\mathrm{Re}\left\{[\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d_1)p_1]\overline{[\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d_2)p_2]}\right\} = \mathrm{Re}\left\{[\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d_1)p_1]\overline{[\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d_2)p_2]}\right\} \tag{3.10}$$

for all $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d_1, d_2 \in \mathbb{S}^2$, $m \in \{\phi, \theta\}$ and $p_1, p_2 \in \mathbb{R}^3$.

For $d, q \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$ define $r_j(\hat{x}, d, q, p) := |q \cdot E_j^\infty(\hat{x}, d)p|$, $j = 1, 2$. Then, by setting $d_1 = d_2 =: d$ and $p_1 = p_2 =: p$ in (3.9), we have

$$r_1(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) = r_2(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) =: r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \quad \forall \hat{x} \in \mathbb{S}^2 \setminus \{N, S\},\ d \in \mathbb{S}^2,\ m \in \{\phi, \theta\},\ p \in \mathbb{R}^3. \tag{3.11}$$

Thus, we know that

$$\boldsymbol{e}_m(\hat{x}) \cdot E_j^\infty(\hat{x}, d)p = r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p)e^{i\vartheta_j^{(m)}(\hat{x}, d, p)} \quad \forall \hat{x} \in \mathbb{S}^2 \setminus \{N, S\},\ d \in \mathbb{S}^2,\ m \in \{\phi, \theta\},\ p \in \mathbb{R}^3, j = 1, 2,$$

where $\vartheta_j^{(m)}$ is a real-valued function, $j = 1, 2$.

Let $m \in \{\phi, \theta\}$ be arbitrarily fixed. We then prove that

$$E_1^\infty(\hat{x}, d)\boldsymbol{e}_m(d) = E_2^\infty(\hat{x}, d)\boldsymbol{e}_m(d) \quad \forall \hat{x} \in \mathbb{S}^2,\ d \in \mathbb{S}^2 \setminus \{N, S\}. \tag{3.12}$$

To do this, we distinguish between the following two cases.

Case 1. $r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \not\equiv 0$ for $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$.

In this case, by the analyticity of $\boldsymbol{e}_m(\hat{x}) \cdot E_j^\infty(\hat{x}, d)p$ with respect to $\hat{x}$, d and p, respectively, $j = 1, 2$, and the continuity of $\boldsymbol{e}_m(\hat{x}) \cdot E_j^\infty(\hat{x}, d)p$ in $(\hat{x}, d, p) \in (\mathbb{S}^2 \setminus \{N, S\}) \times \mathbb{S}^2 \times \mathbb{R}^3$, it can be seen that there exist open sets $U_1 \subset \mathbb{S}^2 \setminus \{N, S\}$, $U_2 \subset \mathbb{S}^2$ and $V \subset \mathbb{R}^3$ small enough such that (i) $r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \neq 0$ for all $\hat{x} \in U_1$, $d \in U_2$ and $p \in V$, and (ii) $\vartheta_j^{(m)}(\hat{x}, d, p)$ is analytic with respect to $\hat{x} \in U_1$, $d \in U_2$ and $p \in V$, respectively, $j = 1, 2$. Then, and by (3.10), we obtain that

$$\cos[\vartheta_1^{(m)}(\hat{x}, d_1, p_1) - \vartheta_1^{(m)}(\hat{x}, d_2, p_2)] = \cos[\vartheta_2^{(m)}(\hat{x}, d_1, p_1) - \vartheta_2^{(m)}(\hat{x}, d_2, p_2)] \tag{3.13}$$

for all $\hat{x} \in U_1$, $d_1, d_2 \in U_2$ and $p_1, p_2 \in V$. From (3.13) and the fact that $\vartheta_j^{(m)}(\hat{x}, d, p)$ is a real-valued analytic function of $\hat{x} \in U_1$, $d \in U_2$ and $p \in V$, respectively, $j = 1, 2$, it is derived that there holds either

$$\vartheta_1^{(m)}(\hat{x}, d_1, p_1) - \vartheta_1^{(m)}(\hat{x}, d_2, p_2) = \vartheta_2^{(m)}(\hat{x}, d_1, p_1) - \vartheta_2^{(m)}(\hat{x}, d_2, p_2) + 2l\pi \tag{3.14}$$

or

$$\vartheta_1^{(m)}(\hat{x}, d_1, p_1) - \vartheta_1^{(m)}(\hat{x}, d_2, p_2) = -[\vartheta_2^{(m)}(\hat{x}, d_1, p_1) - \vartheta_2^{(m)}(\hat{x}, d_2, p_2)] + 2l\pi \tag{3.15}$$

for some $l \in \mathbb{Z}$ and for all $\hat{x} \in U_1$, $d_1, d_2 \in U_2$ and $p_1, p_2 \in V$.

For the case when (14) holds, we have

$$\vartheta_1^{(m)}(\hat{x}, d_1, p_1) - \vartheta_2^{(m)}(\hat{x}, d_1, p_1) = \vartheta_1^{(m)}(\hat{x}, d_2, p_2) - \vartheta_2^{(m)}(\hat{x}, d_2, p_2) + 2l\pi \quad \forall \hat{x} \in U_1, d_1, d_2 \in U_2, p_1, p_2 \in V. \tag{3.16}$$

Fix $d_2 \in U_2, p_2 \in V$ and define

$$\alpha^{(m)}(\hat{x}) := \vartheta_1^{(m)}(\hat{x}, d_2, p_2) - \vartheta_2^{(m)}(\hat{x}, d_2, p_2) \quad \forall \hat{x} \in U_1. \tag{3.17}$$

Then, by (3.16), we get

$$\begin{aligned}
\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d)p &= r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p)e^{i\vartheta_1^{(m)}(\hat{x},d,p)} \\
&= r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p)e^{i\alpha^{(m)}(\hat{x})+i\vartheta_2^{(m)}(\hat{x},d,p)} \\
&= e^{i\alpha^{(m)}(\hat{x})}\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d)p
\end{aligned}$$

for all $\hat{x} \in U_1, d \in U_2$ and $p \in V$. By the analyticity of $\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d)p - e^{i\alpha^{(m)}(\hat{x})}\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d)p$ in $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$, respectively, it is deduced that

$$\boldsymbol{e}_m(\hat{x}) \cdot E_1^\infty(\hat{x}, d)p = e^{i\alpha^{(m)}(\hat{x})}\boldsymbol{e}_m(\hat{x}) \cdot E_2^\infty(\hat{x}, d)p \quad \forall \hat{x} \in U_1,\ d \in \mathbb{S}^2,\ p \in \mathbb{R}^3. \tag{3.18}$$

Changing the variables $\hat{x} \to -d$ and $d \to -\hat{x}$ in (3.18) gives

$$\boldsymbol{e}_m(-d) \cdot E_1^\infty(-d, -\hat{x})p = e^{i\alpha^{(m)}(-d)}\boldsymbol{e}_m(-d) \cdot E_2^\infty(-d, -\hat{x})p \quad \forall -d \in U_1,\ \hat{x} \in \mathbb{S}^2,\ p \in \mathbb{R}^3.$$

The reciprocity relation $E_j^\infty(\hat{x}, d) = [E_j^\infty(-d, -\hat{x})]^\top$ for all $\hat{x},\ d \in \mathbb{S}^2$ $(j = 1, 2)$ (see Theorem 6.30 in Colton & Kress, 2013) leads to the result

$$p \cdot E_1^\infty(\hat{x}, d)\boldsymbol{e}_m(-d) = e^{i\alpha^{(m)}(-d)}p \cdot E_2^\infty(\hat{x}, d)\boldsymbol{e}_m(-d) \quad \forall -d \in U_1,\ \hat{x} \in \mathbb{S}^2,\ p \in \mathbb{R}^3.$$

Since $\boldsymbol{e}_\phi(d) = -\boldsymbol{e}_\phi(-d)$ and $\boldsymbol{e}_\theta(d) = \boldsymbol{e}_\theta(-d)$, we have

$$E_1^\infty(\hat{x}, d)\boldsymbol{e}_m(d) = e^{i\alpha^{(m)}(-d)}E_2^\infty(\hat{x}, d)\boldsymbol{e}_m(d) \quad \forall -d \in U_1,\ \hat{x} \in \mathbb{S}^2. \tag{3.19}$$

Now, by Rellich's lemma (cf. Theorem 6.10 in Colton & Kress, 2013), we obtain that

$$E_1^s(x, d)\boldsymbol{e}_m(d) = e^{i\alpha^{(m)}(-d)}E_2^s(x, d)\boldsymbol{e}_m(d) \quad \forall x \in G,\ -d \in U_1, \tag{3.20}$$

where G denotes the unbounded component of the complement of $B \cup D_1 \cup D_2$. The perfectly conducting boundary condition on ∂B gives that $\nu \times [E_j^s(\cdot, d)\boldsymbol{e}_m(d)] = -\nu \times [E^i(\cdot, d)\boldsymbol{e}_m(d)]$ on ∂B $(j = 1, 2)$, which, together with (3.20), implies that

$$-\nu \times [E^i(\cdot, d)\boldsymbol{e}_m(d)] = -e^{i\alpha^{(m)}(-d)}\nu \times [E^i(\cdot, d)\boldsymbol{e}_m(d)] \quad \text{on} \quad \partial B \tag{3.21}$$

for all $-d \in U_1$. For arbitrarily fixed $-d \in U_1$, define $\widetilde{E} := (1 - e^{i\alpha^{(m)}(-d)})E^i(\cdot, d)\boldsymbol{e}_m(d)$ and $\widetilde{H} := (1/ik)\mathrm{curl}\,\widetilde{E}$. Then, by (3.21), it follows that $(\widetilde{E}, \widetilde{H})$ satisfies the electromagnetic interior boundary value problem

$$\begin{cases} \mathrm{curl}\,\widetilde{E} - ik\widetilde{H} = 0 & \text{in} \quad B, \\ \mathrm{curl}\,\widetilde{H} + ik\widetilde{E} = 0 & \text{in} \quad B, \\ \nu \times \widetilde{E} = 0 & \text{on} \quad \partial B. \end{cases}$$

Since k is not a Maxwell eigenvalue in B and $E^i(\cdot, d)\boldsymbol{e}_m(d) \not\equiv 0$ in B, we have $e^{i\alpha^{(m)}(-d)} = 1$ for all $-d \in U_1$. Thus, it follows from (3.19) that

$$E_1^\infty(\hat{x}, d)\boldsymbol{e}_m(d) = E_2^\infty(\hat{x}, d)\boldsymbol{e}_m(d) \quad \forall -d \in U_1,\ \hat{x} \in \mathbb{S}^2. \tag{3.22}$$

By the analyticity of $E_j^\infty(\hat{x}, d)\boldsymbol{e}_m(d)$ in $d \in \mathbb{S}^2 \setminus \{N, S\}, j = 1, 2$, the required equation (3.12) follows.

For the case when (3.15) holds, a similar argument as above gives the result

$$E_1^\infty(\hat{x}, d)\boldsymbol{e}_m(d) = e^{i\beta^{(m)}(-d)}\overline{E_2^\infty(\hat{x}, d)}\boldsymbol{e}_m(d) \quad \forall \hat{x} \in \mathbb{S}^2, -d \in U_1, \tag{3.23}$$

where $\beta^{(m)}$ is a real-valued function defined by

$$\beta^{(m)}(\hat{x}) := \vartheta_1^{(m)}(\hat{x}, d_2, p_2) + \vartheta_2^{(m)}(\hat{x}, d_2, p_2) \tag{3.24}$$

for all $\hat{x} \in U_1$ and for some fixed $d_2 \in U_2, p_2 \in V$. However, by Lemma 3.1, (3.23) does not hold.

Case 2. $r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \equiv 0$ for $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$. In this case, it is easily seen that (3.12) holds.

Finally, by (3.8), (3.12) and the linear combination of $\boldsymbol{e}_\phi(d), \boldsymbol{e}_\theta(d), d$, and noting the arbitrariness of $m \in \{\phi, \theta\}$ in (3.12) and the analyticity of $E_j^\infty(\hat{x}, d)$ in $d \in \mathbb{S}^2, j = 1, 2$, we deduce that

$$E_1^\infty(\hat{x}, d) = E_2^\infty(\hat{x}, d) \quad \forall \hat{x}, d \in \mathbb{S}^2. \tag{3.25}$$

This, together with Theorem 7.1 in Colton & Kress (2013), Theorem 1 in Kress (2002) and Theorem 3.1 in Cakoni *et al.* (2011), implies that $D_1 = D_2$ and $\mathscr{B}_1 = \mathscr{B}_2$. The proof is thus complete. □

REMARK 3.1 Let r be the radius of the ball B. Then it is known that if r is chosen such that $j_n(kr) \neq 0$ and $j_n(kr) + krj_n'(kr) \neq 0$ for $n = 0, 1, \ldots$, then k is not a Maxwell eigenvalue in B, where j_n denotes the spherical Bessel function of order n (see page 252 in Colton & Kress, 2013). This may give a practical criterion on how to choose the ball B such that k is not a Maxwell eigenvalue in B. Further, by using the same argument as above, it can be proved that Theorem 3.1 still holds true if the reference ball B is replaced by any other domain with smooth boundary satisfying all the other conditions in Theorem 3.1. However, if we choose B to be a domain that is not a ball, then usually there is no practical criterion on how to choose B such that k is not a Maxwell eigenvalue in B. Due to this reason, for simplicity, we choose the domain B to be a ball in this paper.

REMARK 3.2 In the proof of Theorem 3.1, we have used a simple identity $2\mathrm{Re}(a\overline{b}) = |a+b|^2 - |a|^2 - |b|^2$, $a, b \in \mathbb{C}$, to obtain (10). Note that a similar identity was used in a similar context of phaseless

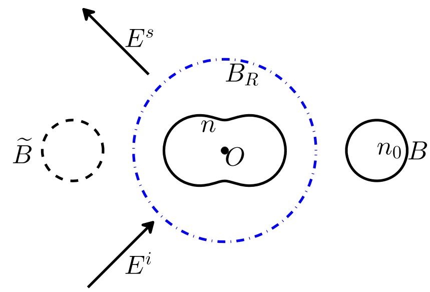

FIG. 2. Scattering by an inhomogeneous medium.

inverse scattering problems in Novikov *et al.* (2015) to recover the phase from the measurements of the intensity illuminations obtained with two incident waves (see the polarization identity (3.4) in Novikov *et al.* (2015)).

4. Uniqueness for inverse electromagnetic medium scattering

This section is concerned with the uniqueness result in the inverse electromagnetic medium scattering problem. Similarly as in Section 3, we will establish the uniqueness result by adding a reference medium into the scattering system. To be more specific, assume that B is the given reference ball and that $n_0 \in C^{2,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, is the refractive index of a given inhomogeneous medium with the support of $n_0 - 1$ in $\overline{B}$. Assume further that $n_1, n_2 \in C^{2,\gamma}(\mathbb{R}^3)$ are the refractive indices of two inhomogeneous media with $m_j := n_j - 1$ supported in $\overline{D_j}$, $j = 1, 2$. Denote by E_j^s, H_j^s, E_j^∞ and H_j^∞ the electric scattered field, the magnetic scattered field, the electric far-field pattern and the magnetic far-field pattern, respectively, associated with the inhomogeneous medium with the refractive index $\widetilde{n}_j$ and corresponding to the incident electromagnetic waves E^i and H^i, $j = 1, 2$. Here, the refractive index $\widetilde{n}_j$ is given by

$$\widetilde{n}_j(x) := \begin{cases} n_0(x), & x \in B, \\ n_j(x), & x \in \mathbb{R}^3 \setminus \overline{B} \end{cases}$$

for $j = 1, 2$. It is noticed that if $\overline{D_j} \cap \overline{B} = \emptyset$ then $\widetilde{n}_j \in C^{2,\gamma}(\mathbb{R}^3)$. See Fig. 2 for the geometric description of the scattering problem. Suppose k is not an electromagnetic interior transmission eigenvalue in B. Here, k is called an electromagnetic interior transmission eigenvalue in B if the homogeneous electromagnetic interior transmission problem

$$\begin{cases} \operatorname{curl} \widetilde{E} - ik\widetilde{H} = 0, \quad \operatorname{curl} \widetilde{H} + ikn_0\widetilde{E} = 0 & \text{in } B, \\ \operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 & \text{in } B, \\ \nu \times (\widetilde{E} - E_0) = 0, \quad \nu \times (\widetilde{H} - H_0) = 0 & \text{on } \partial B \end{cases} \tag{4.1}$$

has a non-trivial solution $(\widetilde{E}, \widetilde{H}, E_0, H_0)$.

THEOREM 4.1 Assume that B is a given ball filled with the inhomogeneous medium of the refractive index $n_0 \in C^{2,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$, such that the support of $n_0 - 1$ is $\overline{B}$ and k is not an electromagnetic interior transmission eigenvalue in B. Assume further that $n_1, n_2 \in C^{2,\gamma}(\mathbb{R}^3)$ are the refractive indices of two inhomogeneous media with $m_j := n_j - 1$ supported in $\overline{D_j}$, $j = 1, 2$. Suppose $\overline{D_1 \cup D_2} \subset B_R$, where B_R is a ball of radius R and centered at the origin and satisfies that $\overline{B} \cap \overline{B_R} = \emptyset$. If the corresponding electric far-field patterns satisfy (3.4) for all $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d_1, d_2 \in \mathbb{S}^2$, $m \in \{\phi, \theta\}$ and $p_1, p_2 \in \mathbb{R}^3$ satisfying that $d_1 \perp p_1$ and $d_2 \perp p_2$, then $n_1 = n_2$.

To simplify the proof of Theorem 4.1, we need the following lemma that is similar to Lemma 3.1.

Lemma 4.1 Under the assumptions of Theorem 4.1, there does not hold the equation

$$E_1^\infty(\hat{x}, d_0)\boldsymbol{e}_m(d_0) = e^{i\beta}\overline{E_2^\infty(\hat{x}, d_0)\boldsymbol{e}_m(d_0)} \quad \forall \hat{x} \in \mathbb{S}^2, \tag{4.2}$$

where $m \in \{\phi, \theta\}$ and $d_0 \in \mathbb{S}^2$ are arbitrarily fixed and β is a real constant.

Proof. Assume to the contrary that (4.2) holds. Then, by a similar argument as in the proof of Lemma 3.1, it can be derived that $E_1^s(\cdot, d_0)\boldsymbol{e}_m(d_0)$ and $H_1^s(\cdot, d_0)\boldsymbol{e}_m(d_0)$ can be analytically extended into $\mathbb{R}^3 \setminus \overline{B_R}$ and satisfy the Maxwell equations (2.3a)–(2.3b) in $\mathbb{R}^3 \setminus \overline{B_R}$. Noting that the incident waves $E^i(\cdot, d_0)\boldsymbol{e}_m(d_0)$ and $H^i(\cdot, d_0)\boldsymbol{e}_m(d_0)$ satisfy the Maxwell equations (2.3a)–(2.3b) in $\mathbb{R}^3$, we obtain that the total fields $E_1 := E_1(\cdot, d_0)\boldsymbol{e}_m(d_0) = E_1^i(\cdot, d_0)\boldsymbol{e}_m(d_0) + E_1^s(\cdot, d_0)\boldsymbol{e}_m(d_0)$ and $H_1 := (1/ik)\text{curl}\, E_1$ satisfy the Maxwell equations (2.3a)–(2.3b) in B.

On the other hand, from the definition of $\widetilde{n}_1$ and the electromagnetic medium scattering problem, it follows that the total fields E_1 and H_1 also satisfy the first two Maxwell equations in (4.1) in B. Thus, $(\widetilde{E}, \widetilde{H}, E_0, H_0) := (E_1, H_1, E_1, H_1)$ satisfies the problem (4.1). Since k is not an electromagnetic interior transmission eigenvalue in B, it follows that $E_1 \equiv 0$ in B. It follows from the analyticity of the incident field $E^i(\cdot, d_0)\boldsymbol{e}_m(d_0)$ and the electric scattered field $E_1^s(\cdot, d_0)\boldsymbol{e}_m(d_0)$ in $\mathbb{R}^3 \setminus \overline{B_R}$ that E_1 is also analytic in $\mathbb{R}^3 \setminus \overline{B_R}$, and thus we have that $E_1 \equiv 0$ in $\mathbb{R}^3 \setminus \overline{B_R}$. This is a contradiction, implying that (4.2) does not hold. The proof is then complete. □

With the aid of Lemma 4.1, we can now prove Theorem 4.1.

Proof of Theorem 4.1. Our proof follows similar arguments as for the case of inverse obstacle scattering in Section 3. Using the same argument as in the proof of Theorem 3.1, we can obtain (3.11). We now want to prove (3.12) for arbitrarily fixed $m \in \{\phi, \theta\}$. First, if $r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \equiv 0$ for $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$, then it is obvious that (3.12) holds.

We now consider the case $r(\hat{x}, d, \boldsymbol{e}_m(\hat{x}), p) \not\equiv 0$ for $\hat{x} \in \mathbb{S}^2 \setminus \{N, S\}$, $d \in \mathbb{S}^2$ and $p \in \mathbb{R}^3$. It is noticed that the scattering problem (2.4a)–(2.4c) is well posed (see, e.g. Theorem 9.5 in Colton & Kress, 2013) and the reciprocity relation $E_j^\infty(\hat{x}, d) = [E_j^\infty(-d, -\hat{x})]^\top$ for all $\hat{x},\ d \in \mathbb{S}^2$ $(j = 1, 2)$ still holds true when D is an inhomogeneous medium (see, e.g. Theorem 6.30 in Colton & Kress, 2013). Thus, by a similar argument as in the proof of Theorem 3.1, it can be shown that either (3.19) or (3.23) holds for some open set $U_1 \subset \mathbb{S}^2 \setminus \{N, S\}$, where $\alpha^{(m)}$ and $\beta^{(m)}$ are defined similarly as in (3.17) and (3.24), respectively, in the proof of Theorem 3.1. But, Lemma 4.1 implies that (3.23) does not hold. Thus, we only need to consider the case when (3.19) holds. By (3.19) and Rellich's lemma (cf. Theorem 6.10 in Colton & Kress, 2013), we obtain (3.20), where G is defined as above. For any fixed $-d \in U_1$, define

$$\widetilde{E} := [E^i(\cdot, d) + E_1^s(\cdot, d)]\boldsymbol{e}_m(d) - e^{i\alpha^{(m)}(-d)}[E^i(\cdot, d) + E_2^s(\cdot, d)]\boldsymbol{e}_m(d), \quad \widetilde{H} := (1/ik)\text{curl}\, \widetilde{E},$$

$$E_0 := \left(1 - e^{i\alpha^{(m)}(-d)}\right)E^i(\cdot, d)\boldsymbol{e}_m(d), \quad H_0 := (1/ik)\text{curl}\, E_0.$$

Then, by (3.20), we have $\widetilde{E} = E_0$ in G, and so $(\widetilde{E}, \widetilde{H}, E_0, H_0)$ satisfies the boundary conditions on ∂B in the problem (4.1). Further, by the definition of $\widetilde{E}$, $\widetilde{H}$, E_0 and H_0, it is known that $(\widetilde{E}, \widetilde{H}, E_0, H_0)$ satisfies the problem (4.1). Since k is not an electromagnetic interior transmission eigenvalue in B, we obtain that $e^{i\alpha^{(m)}(-d)} = 1$ for all $-d \in U_1$, which means that (3.22) holds. By this and the analyticity of $E_j^\infty(\hat{x}, d)\boldsymbol{e}_m(d)$ in $d \in \mathbb{S}^2 \setminus \{N, S\}$, $j = 1, 2$, it follows that (3.12) is true. On the other hand, the well posedness of the medium scattering problem (2.4a)–(2.4c) and the fact that $E^i(x, d)d = 0$ for $x \in \mathbb{R}^3$ and $d \in \mathbb{S}^2$ imply (3.8). Then, by the same argument as in the proof of Theorem 3.1, it follows from (3.8) and (3.12) that (3.25) holds. Since $\overline{D_1 \cup D_2} \subset B_R$ and $\overline{B} \cap \overline{B_R} = \emptyset$ then $\widetilde{n}_j \in C^{2,\gamma}(\mathbb{R}^3)$, $j = 1, 2$. Therefore, by (3.25) and Theorem 4.9 in Hähner (1998), we obtain that $n_1 = n_2$. The proof is then complete. □

REMARK 4.1 It is worth noting that if n_0 is chosen so that $\mathrm{Im}\,[n_0(x_0)] > 0$ for some $x_0 \in B$ then k is not an electromagnetic interior transmission eigenvalue in B (see the discussion in the proof of Theorem 9.8 in Colton & Kress, 2013). Moreover, by a similar argument as above, it can be proved that Theorem 4.1 still holds true if the reference ball B is replaced by any other domain with smooth boundary satisfying all the other conditions in Theorem 4.1.

REMARK 4.2 In Theorem 4.1, we assume that $n_0 \in C^{2,\gamma}(\mathbb{R}^3)$, $0 < \gamma < 1$. This assumption is necessary since we need Theorem 4.9 in Hähner (1998).

5. Conclusion

In this paper, by adding a given reference ball into the electromagnetic scattering system, we established uniqueness results for inverse electromagnetic obstacle and medium scattering with phaseless electric far-field data generated by infinitely many sets of superpositions of two electromagnetic plane waves with different directions and polarizations at a fixed frequency for the first time. These uniqueness results extend our previous results in Xu *et al.* (2018b) for the acoustic case to the electromagnetic case. Our method is based on a simple technique of using Rellich's lemma and the Stratton–Chu formula for the radiating solutions to the Maxwell equations. In the future, we hope to show the same uniqueness results without using the reference ball, which are more challenging.

Funding

National Natural Science Foundation of China (91630309, 11871466 , 11571355 and NSAF U1930402) China Postdoctoral Science Foundation (2019TQ0023).

Acknowledgements

We thank Professor Xudong Chen at the National University of Singapore for helpful and constructive discussions on the measurement technique of electromagnetic waves. We also thank the reviewers for their insightful comments, which helped improve the presentation of the paper.

REFERENCES

BARDSLEY, P., CASSIER, M. & GUEVARA VASQUEZ, F. (2018) Imaging small polarizable scatterers with polarization data. *Inverse Probl.*, **34**, 104002, 40.

CAKONI, F., COLTON, D. & MONK, P. (2004) The electromagnetic inverse-scattering problem for partly coated Lipschitz domains. *Proc. Roy. Soc. Edinburgh Sect. A*, **134**, 661–682.

CAKONI, F., COLTON, D. & MONK, P. (2011) *The Linear Sampling Method in Inverse Electromagnetic Scattering*, vol. **80** of CBMS-NSF Regional Conference Series in Applied Mathematics. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

CHEN, X. (2018) *Computational Methods for Electromagnetic Inverse Scattering*. Singapore: Wiley.
CHEN, Z., FANG, S. & HUANG, G. (2017) A direct imaging method for the half-space inverse scattering problem with phaseless data. *Inverse Probl. Imaging*, **11**, 901–916.
CHEN, Z. & HUANG, G. (2016) A direct imaging method for electromagnetic scattering data without phase information. *SIAM J. Imaging Sci.*, **9**, 1273–1297.
CHEN, Z. & HUANG, G. (2017) Phaseless imaging by reverse time migration: acoustic waves. *Numer. Math. Theory Methods Appl.*, **10**, 1–21.
COLTON, D. & KRESS, R. (1981) The impedance boundary value problem for the time harmonic Maxwell equations. *Math. Methods Appl. Sci.*, **3**, 475–487.
COLTON, D. & KRESS, R. (2013) *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. **93** of Applied Mathematical Sciences, 3rd edn. New York: Springer.
DONG, H., LAI, J. & LI, P. (2019a) Inverse obstacle scattering for elastic waves with phased or phaseless far-field data. *SIAM J. Imaging Sci.*, **12**, 809–838.
DONG, H., LAI, J. & LI, P. (2020) An inverse acoustic-elastic interaction problem with phased or phaseless far-field data. *Inverse Probl.*, **36**, 035014.
DONG, H., ZHANG, D. & GUO, Y. (2019b) A reference ball based iterative algorithm for imaging acoustic obstacle from phaseless far-field data. *Inverse Probl. Imaging*, **13**, 177–195.
HÄHNER, P. (1998) On acoustic, electromagnetic, and elastic scattering problems in inhomogeneous media. *Habilitation Thesis*, University of Göttingen.
HANSEN, J. E. (1988) *Spherical Near-Field Antenna Measurements*, vol. **26** of IET Electromagnetic Waves Series. London: IET Press.
IVANYSHYN, O. (2007) Shape reconstruction of acoustic obstacles from the modulus of the far field pattern. *Inverse Probl. Imaging*, **1**, 609–622.
IVANYSHYN, O. & KRESS, R. (2010) Identification of sound-soft 3D obstacles from phaseless data. *Inverse Probl. Imaging*, **4**, 131–149.
IVANYSHYN, O. & KRESS, R. (2011) Inverse scattering for surface impedance from phase-less far field data. *J. Comput. Phys.*, **230**, 3443–3452.
JI, X., LIU, X. & ZHANG, B. (2019a) Inverse acoustic scattering with phaseless far field data: uniqueness, phase retrieval, and direct sampling methods. *SIAM J. Imaging Sci.*, **12**, 1163–1189.
JI, X., LIU, X. & ZHANG, B. (2019b) Phaseless inverse source scattering problem: phase retrieval, uniqueness and direct sampling methods. *J. Comput. Phys. X*, **1**, 100003.
JI, X., LIU, X. & ZHANG, B. (2019c) Target reconstruction with a reference point scatterer using phaseless far field patterns. *SIAM J. Imaging Sci.*, **12**, 372–391.
KIRSCH, A. & GRINBERG, N. (2008) *The Factorization Method for Inverse Problems*, vol. **36** of Oxford Lecture Series in Mathematics and Its Applications. Oxford: Oxford University Press.
KLIBANOV, M. V. (2014) Phaseless inverse scattering problems in three dimensions. *SIAM J. Appl. Math.*, **74**, 392–410.
KLIBANOV, M. V. (2017) A phaseless inverse scattering problem for the 3-D Helmholtz equation.*Inverse Probl. Imaging*, **11**, 263–276.
KLIBANOV, M. V. & ROMANOV, V. G. (2016) Reconstruction procedures for two inverse scattering problems without the phase information. *SIAM J. Appl. Math.*, **76**, 178–196.
KLIBANOV, M. V. & ROMANOV, V. G. (2017) Uniqueness of a 3-D coefficient inverse scattering problem without the phase information. *Inverse Probl.*, **33**, 095007, 10.
KRESS, R. (2002) Uniqueness in inverse obstacle scattering for electromagnetic wave. *Proceedings of the URSI General Assembly*.
KRESS, R. & RUNDELL, W. (1997) Inverse obstacle scattering with modulus of the far field pattern as data. *Inverse Problems in Medical Imaging and Nondestructive Testing (Oberwolfach, 1996)*. Vienna: Springer, pp. 75–92.
LI, J., LIU, H. & ZOU, J. (2010) Strengthened linear sampling method with a reference ball. *SIAM J. Sci. Comput.*, **31**, 4013–4040.

LIU, X. & ZHANG, B. (2010) Unique determination of a sound-soft ball by the modulus of a single far field datum. *J. Math. Anal. Appl.*, **365**, 619–624.

MAJDA, A. (1976) High frequency asymptotics for the scattering matrix and the inverse problem of acoustical scattering. *Comm. Pure Appl. Math.*, **29**, 261–291.

MARETZKE, S. & HOHAGE, T. (2017) Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging. *SIAM J. Appl. Math.*, **77**, 384–408.

MCLEAN, W. (2000) *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge: Cambridge University Press.

MOSCOSO, M., NOVIKOV, A., PAPANICOLAOU, G. & TSOGKA, C. (2017) Multifrequency interferometric imaging with intensity-only measurements. *SIAM J. Imaging Sci.*, **10**, 1005–1032.

NOVIKOV, A., MOSCOSO, M. & PAPANICOLAOU, G. (2015) Illumination strategies for intensity-only imaging. *SIAM J. Imaging Sci.*, **8**, 1547–1573.

NOVIKOV, R. G. (2015) Formulas for phase recovering from phaseless scattering data at fixed frequency. *Bull. Sci. Math.*, **139**, 923–936.

NOVIKOV, R. G. (2016) Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions. *J. Geom. Anal.*, **26**, 346–359.

PAN, L., ZHONG, Y., CHEN, X. & YEO, S. P. (2011) Subspace-based optimization method for inverse scattering problems utilizing phaseless data. *IEEE Trans. Geosci. Remote Sens.*, **49**, 981–987.

ROMANOV, V. G. (2017) The problem of recovering the permittivity coefficient from the modulus of the scattered electromagnetic field. *Siberian Math. J.*, **58**, 711–717.

ROMANOV, V. G. & YAMAMOTO, M. (2018) Phaseless inverse problems with interference waves. *J. Inverse Ill-Posed Probl.*, **26**, 681–688.

SCHMIDT, C. H., RAZAVI, S. F., EIBERT, T. F. & RAHMAT-SAMII, Y. (2010) Phaseless spherical near-field antenna measurements for low and medium gain antennas. *Adv. Radio Sci.*, **8**, 43–48.

SHIN, J. (2016) Inverse obstacle backscattering problems with phaseless data. *Eur. J. Appl. Math.*, **27**, 111–130.

SUN, F., ZHANG, D. & GUO, Y. (2019) Uniqueness in phaseless inverse scattering problems with known superposition of incident point sources. *Inverse Probl.*, **35**, 105007.

XU, X., ZHANG, B. & ZHANG, H. (2018a) Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency. *SIAM J. Appl. Math.*, **78**, 1737–1753.

XU, X., ZHANG, B. & ZHANG, H. (2018b) Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency. II. *SIAM J. Appl. Math.*, **78**, 3024–3039.

XU, X., ZHANG, B. & ZHANG, H. (2019) Uniqueness and direct imaging method for inverse scattering by locally rough surfaces with phaseless near-field data. *SIAM J. Imaging Sci.*, **12**, 119–152.

XU, X., ZHANG, B. & ZHANG, H. (2020) Uniqueness in inverse acoustic and electromagnetic scattering with phaseless near-field data at a fixed frequency. *Inverse Probl. Imaging*, **14**, 489–510.

ZHANG, B. & ZHANG, H. (2017a) Imaging of locally rough surfaces from intensity-only far-field or near-field data. *Inverse Probl.*, **33**, 055001, 28.

ZHANG, B. & ZHANG, H. (2017b) Recovering scattering obstacles by multi-frequency phaseless far-field data. *J. Comput. Phys.*, **345**, 58–73.

ZHANG, B. & ZHANG, H. (2018) Fast imaging of scattering obstacles from phaseless far-field measurements at a fixed frequency. *Inverse Probl.*, **34**, 104005.

ZHANG, D. & GUO, Y. (2018) Uniqueness results on phaseless inverse acoustic scattering with a reference ball. *Inverse Probl.*, **34**, 085002.

ZHANG, D., GUO, Y., SUN, F. & LIU, H. (2020a) Unique determinations in inverse scattering problems with phaseless near-field measurements. *Inverse Probl. Imaging*, **14**, 569–582.

ZHANG, D., WANG, Y., GUO, Y. & LI, J. (2020b) Uniqueness in inverse cavity scattering problems with phaseless near-field data. *Inverse Probl.*, **36**, 025004.

ZHANG, Y. & WANG, W. (2019) Robust array diagnosis against array mismatch using amplitude-only far field data. *IEEE Access*, **7**, 5589–5596.