§9.3 The phase plane; linear system

The xy plane is referred to as the phase plane.

In this section we will continue our discussion of the autonomous system

$$\frac{dx}{dt} = F(x,y), \qquad \frac{dy}{dt} = G(x,y). \tag{9.3.1}$$

In particular we will be concerned with the behavior of the trajectories of the system (9.3.1) in the neighborhood of a critical point (x^*, y^*). It is convenient to choose the critical point to be at the origin of the phase plane: This involves no loss of generality, since if $x^* \neq 0$, $y^* \neq 0$, it is always possible to make the substitution $x = x^* + u$, $y = y^* + v$ in Equation (9.3.1), so that u and v will satisfy an autonomous system of equations with a critical point at the origin. For example, the $(0, \varepsilon_2/\alpha_2)$ is a critical point for the competing species model

$$\begin{cases} \frac{dx}{dt} = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y), \\ \frac{dy}{dt} = y(\varepsilon_2 - \sigma_2 x - \alpha_2 y). \end{cases}$$

The substitution $x = u$, $y = v + \varepsilon/\alpha_2$ takes the model to the form of

$$\begin{cases} \frac{du}{dt} = (\varepsilon_1 - \frac{\alpha_1 \varepsilon_2}{\alpha_2})u - \sigma_1 u^2 - \alpha_1 uv), \\ \frac{dv}{dt} = -\frac{\sigma_2 \varepsilon_2}{\alpha_2}u - \varepsilon_2 v + -\sigma_2 uv - \alpha_2 v^2). \end{cases}$$

We will further assume that the origin is an isolated critical point of the system (9.3.1); that is, we assume that there is some circle about the critical point, inside which there are no other critical points. Finally, we assume that in the neighborhood of (0,0), the system (9.3.1) has the the form

$$\frac{dx}{dt} = ax + by + F_1(x,y), \qquad \frac{dy}{dt} = cx + dy + G_1(x,y), \tag{9.3.2}$$

where $ad - bc \neq 0$. Note that there are no constant terms in Eqs (9.3.2) since $F(0,0) = 0$ and $G(0,0) = 0$. Also, we require that the function $F_1(x,y)$ and $G_1(x,y)$ be continuous, have continuous first partial derivatives, and be small in the sense that

$$\lim_{r \to 0} \frac{F_1(rcox\theta, rsin\theta)}{r} = 0, \quad \lim_{r \to 0} \frac{G_1(rcox\theta, rsin\theta)}{r} = 0, \tag{9.3.3}$$

where $r = \sqrt{x^2 + y^2}$. Such a system is often referred as an almost linear system in the neighborhood of the critical point (0,0). Essentially, Eqs. (9.3.2) and (9.3.3) say that for (x,y) near (0,0) the functions F and G are well approximated by the linear functions $ax + by$ and $cx + dy$, respectively.

The conditions on F and G are satisfied by many functions of two variables. For example,

$$\begin{cases} \frac{dx}{dt} = -y + lx^2 + mxy + ny^2, \\ \frac{dy}{dt} = x + \varepsilon y + ax^2 + bxy, \end{cases}$$
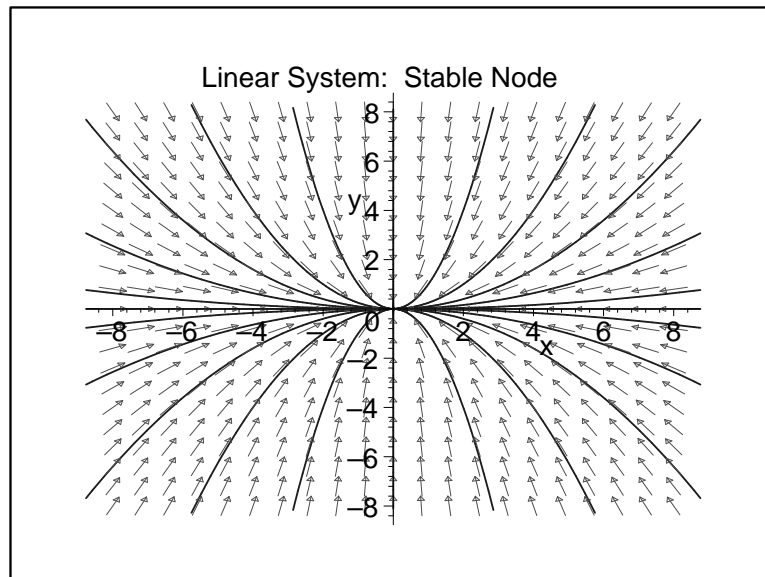
Figure 9-3-1 Some orbits and direction field for the system $x' = -x, \ \ y' = -2y$

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -\frac{g}{l}x - \frac{g}{l}(sinx - x), \end{cases}$$

are almost linear systems in the neighborhood of the (0,0).

9.3.1 Phase portraits of special linear system
In this subsection we draw phase portraits for some special linear systems.

Example 9.3.1
Plot the phase portrait of the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = -2y. \tag{9.3.4}$$

Solution
The solution of (9.3.4) with the initial condition $x(0) = x_0$, $y(0) = y_0$ is $x = x_0 e^{-t}$, $y = y_0 e^{-2t}$. From the explicit expression of the solution we see that the trajectory is $y = \frac{y_0}{x_0^2}x^2$. Then we can obtain the phase portrait of (9.3.4)(See Figure 9-3-1).
We can also plot the phase portrait by following Maple commands.

```
With(DEtools):
DE931 :=[diff(x(t),t)=-x(t),
diff(y(t),t)=-2*y(t)];
DEplot( DE931, [x(t),y(t)],t=-10..10,
```

```
[[x(0)=0,y(0)=2], [x(0)=0,y(0)=-2], [x(0)=-2,y(0)=0],
[x(0)=2,y(0)=0],  [x(0)=1,y(0)=0.01], [x(0)=1,y(0)=0.04],
[x(0)=1,y(0)=0.1], [x(0)=1,y(0)=0.25],[x(0)=1,y(0)=1],
[x(0)=1,y(0)=-0.01], [x(0)=1,y(0)=-0.04], [x(0)=1,y(0)=-0.1],
[x(0)=1,y(0)=-0.25],[x(0)=1,y(0)=-1],[x(0)=-1,y(0)=0.01],
[x(0)=-1,y(0)=0.04], [x(0)=-1,y(0)=0.1], [x(0)=-1,y(0)=0.25],
[x(0)=-1,y(0)=1],[x(0)=-1,y(0)=-0.01], [x(0)=-1,y(0)=-0.04],
[x(0)=-1,y(0)=-0.1], [x(0)=-1,y(0)=-0.25],[x(0)=-1,y(0)=-1]],
x=-8..8,y=-8..8,stepsize=0.05, dirgrid=[21,21], color=red,
linecolor=blue,axes=BOXED,
title=" Linear System:  Stable Node ",
arrows=SLIM);
```

Example 9.3.2

Plot the phase portrait of the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = 2y. \qquad (9.3.5)$$

Solution

The solution of (9.3.5) with the initial condition $x(0) = x_0$, $y(0) = y_0$ is $x = x_0 e^{-t}$, $y = y_0 e^{2t}$. From the explicit expression of the solution we see that the trajectory is $yx^2 = y_0 x_0^2$. Then we can obtain the phase portrait of (9.3.4)(See Figure 9-3-2).

We can also plot the phase portrait by following Maple commands. We can also draw the phase portrait by the Maple command.

```
With(DEtools): DE932 :=[diff(x(t),t)=-x(t),
diff(y(t),t)=2*y(t)];
DEplot( DE932, [x(t),y(t)],t=-10..10,
 [[x(0)=0,y(0)=2], [x(0)=0,y(0)=-2], [x(0)=-2,y(0)=0],
 [x(0)=2,y(0)=0],  [x(0)=1,y(0)=1], [x(0)=2,y(0)=2],
 [x(0)=3,y(0)=3], [x(0)=4,y(0)=4.2],[x(0)=5,y(0)=6],
 [x(0)=1,y(0)=-1], [x(0)=2,y(0)=-2], [x(0)=3,y(0)=-3],
 [x(0)=4,y(0)=-4.2],[x(0)=5,y(0)=-6],[x(0)=-1,y(0)=1],
 [x(0)=-2,y(0)=2], [x(0)=-3,y(0)=3], [x(0)=-4,y(0)=4.2],
 [x(0)=-5,y(0)=6],[x(0)=-1,y(0)=-1], [x(0)=-2,y(0)=-2],
 [x(0)=-3,y(0)=-3], [x(0)=-4,y(0)=-4.2],[x(0)=-5,y(0)=-6] ],
 x=-8..8,y=-8..8,stepsize=0.05, dirgrid=[21,21], color=red,
 linecolor=blue,axes=BOXED,
 title="Linear System : Saddle(UnstableEquilibrium)",
  arrows=SLIM);
```
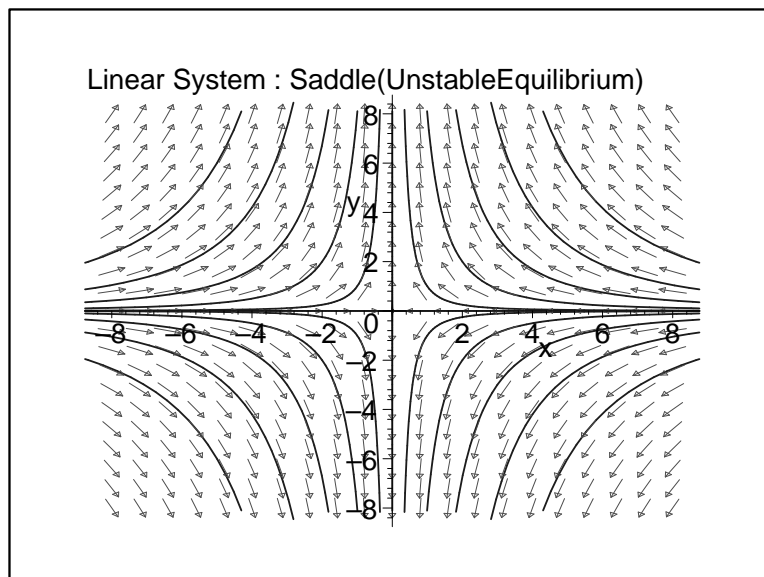
Figure 9-3-2 Some orbits and direction field for the system $x' = -x, \quad y' = 2y$

Example 9.3.3

Plot the phase portrait of the system

$$\frac{dx}{dt} = -x/10 - y, \qquad \frac{dy}{dt} = x - y/10. \tag{9.3.5}$$

Solution

By introducing the new variable $x = r\cos\theta$, $y = r\sin\theta$ we can obtain the differential equations for r and θ:

$$\frac{dr}{dt} = -r/10, \qquad \frac{d\theta}{dt} = 1.$$

The new system has the solution $r = r_0 e^{-t/10}$, $\theta = t + \theta_0$. From the explicit expression of the solution we see that the trajectory is spiral. The phase portrait is sketched in Figure 9-3-3.

We can also plot the phase portrait by following Maple commands.

```
With(DEtools): DE933 :=[diff(x(t),t)=-x(t)/10-y(t),
diff(y(t),t)=x(t)-y(t)/10];
DEplot( DE933, [x(t),y(t)],t=-10..40,
 [[x(0)=8,y(0)=0],[x(0)=6,y(0)=0]],
 x=-8..8,y=-8..8,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Linear System : Stable Focus(Spiral) ",
 arrows=SLIM);
```
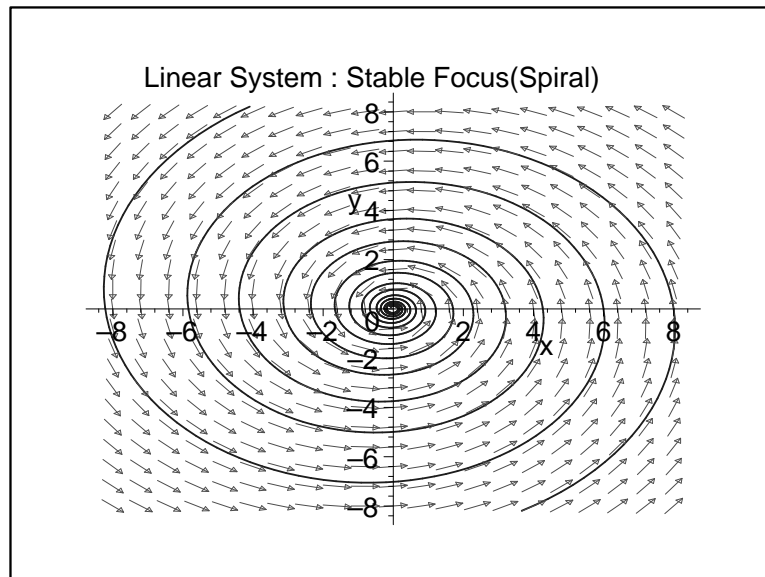
Figure 9-3-3 Some orbits and direction field for the system
$$x' = -x/10 - y,, \quad y' = x - y/10$$

Example 9.3.4
Plot the phase portrait of the system

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x. \tag{9.3.6}$$

Solution

From the given equations we get $\frac{dy}{dx} = -\frac{x}{y}$. And it is easy to see that the trajectories of the system is circle centered at the origin: $x^2 + y^2 = x_0^2 + y_0^2$. The phase portrait is sketched in Figure 9-3-4.

We can also plot the phase portrait by following Maple commands.

```
With(DEtools): DE934 :=[diff(x(t),t)=-y(t),
diff(y(t),t)=x(t)];
DEplot( DE934, [x(t),y(t)],t=-10..10,
 [[x(0)=1,y(0)=0],[x(0)=2,y(0)=0],[x(0)=3,y(0)=0],
 [x(0)=4,y(0)=0],[x(0)=5,y(0)=0],[x(0)=6,y(0)=0],
 [x(0)=7,y(0)=0],[x(0)=8,y(0)=0]],
 x=-8..8,y=-8..8,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Linear System : Center ",
arrows=SLIM);
```
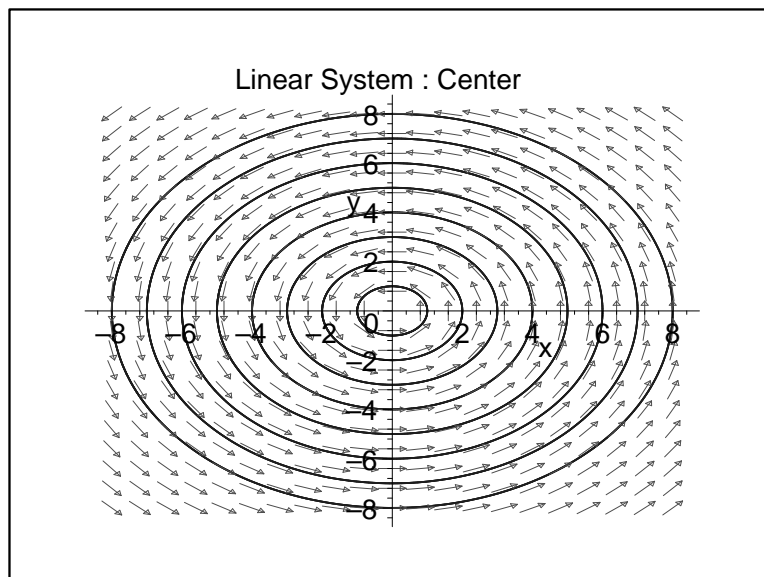
Figure 9-3-4 Some orbits and direction field for the system $x' = -y, \quad y' = x$

Example 9.3.5

Plot the phase portrait of the system

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = 5x - y. \tag{9.3.7}$$

Solution

The solution of (9.3.7) with the initial condition $x(0) = x_0$, $y(0) = y_0$ is $x = x_0 e^{-t}$, $y = y_0 e^{-t} + 5x_0 t e^{-t}$. We can plot the phase portrait by following Maple commands.

```
With(DEtools): DE935 :=[diff(x(t),t)=-x(t),
               diff(y(t),t)=5*x(t)-y(t)];
DEplot( DE935, [x(t),y(t)],t=-10..10,
 [[x(0)=0,y(0)=2], [x(0)=0,y(0)=-2],
 [x(0)=2,y(0)=-8], [x(0)=4,y(0)=-8],
 [x(0)=6,y(0)=-8], [x(0)=8,y(0)=-8],
 [x(0)=8,y(0)=5], [x(0)=-3,y(0)=-8],
 [x(0)=-4,y(0)=13],
 [x(0)=-8,y(0)=-8],[x(0)=-8,y(0)=5]],
 x=-8..8,y=-18..18,
stepsize=0.05, dirgrid=[21,21], color=red, linecolor=blue,
axes=BOXED, title="Linear System : Improper Node ",
arrows=SLIM);
```
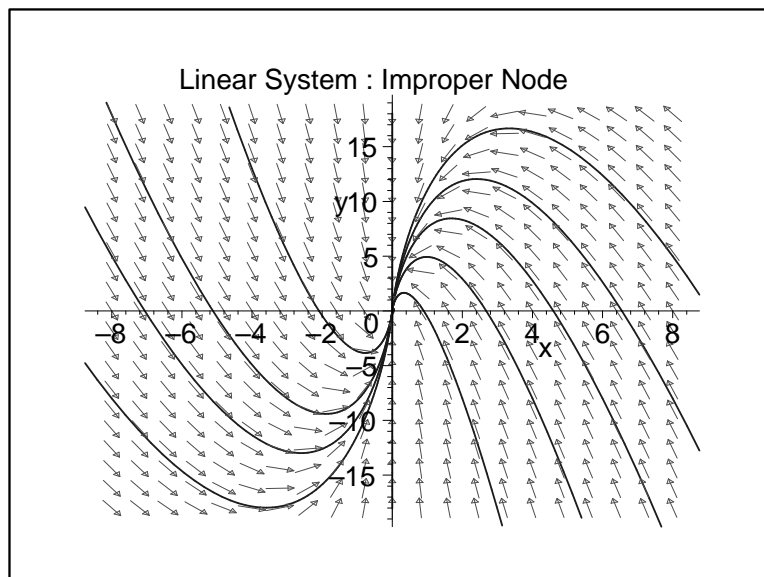
Figure 9-3-5 Some orbits and direction field for the system $x' = -x, \;\; y' = 5x - y$

9.3.2 Phase portrait for the general linear system

In this subsection we present a complete picture of all orbits of the linear differential equation

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x} = \left(\begin{array}{c} x \\ y \end{array} \right), \quad A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right). \tag{9.3.8}$$

This picture is called a phase portrait, and it depends almost completely on the eigenvalues of matrix A.

The system of equations (9.3.8) can be solved by the characteristic method. The characteristic polynomial of (9.3.8) is

$$\left| \begin{array}{cc} a - \lambda & b \\ c & d - \lambda \end{array} \right| = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - p\lambda + q = 0, \tag{9.3.9}$$

where $p = a + d$, $q = ad - bc$. The roots of (9.3.8) are

$$\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}), \qquad \lambda_2 = \frac{1}{2}(p - \sqrt{\Delta}), \tag{9.3.9}$$

where $\Delta = p^2 - 4q$.

We are now in a position to derive the phase portraits of (9.3.8). We distinguish the following cases.

Case 1. Real unequal roots of the same sign: $q > 0, \; \Delta > 0$

The linear transformation

$$\begin{cases} x_1 = (d - \lambda_1)x - by, \\ y_1 = (d - \lambda_2)x - by, \end{cases} \qquad \begin{cases} x = -\frac{1}{\sqrt{\Delta}}(x_1 - y_1), \\ y = \frac{1}{\sqrt{\Delta}}[(d - \lambda_2)x_1 - (d - \lambda_1)y_1], \end{cases} \qquad (9.3.10)$$

can take (9.3.8) to the form of

$$\frac{dx_1}{dt} = \lambda_1 x_1, \qquad \frac{dy_1}{dt} = \lambda_2 y_1. \qquad (9.3.11)$$

In fact,

$$\begin{aligned} \frac{dx_1}{dt} &= (d - \lambda_1)\frac{dx}{dt} - b\frac{dy}{dt} = (d - \lambda_1)(ax + by) - b(cx + dy) \\ &= [a(d - \lambda_1) - bc]x + [b(d - \lambda_1) - bd]y = (ad - bc - a\lambda_1)x - b\lambda_1 y \\ &= [ad - bc - (a + d)\lambda_1 + \lambda_1^2 + \lambda_1(d - \lambda_1)]x - b\lambda_1 y = \lambda_1 x_1, \\ \frac{dy_1}{dt} &= (d - \lambda_2)\frac{dx}{dt} - b\frac{dy}{dt} = (d - \lambda_2)(ax + by) - b(cx + dy) \\ &= [a(d - \lambda_2) - bc]x + [b(d - \lambda_2) - bd]y = (ad - bc - a\lambda_2)x - b\lambda_2 y \\ &= [ad - bc - (a + d)\lambda_2 + \lambda_2^2 + \lambda_2(d - \lambda_2)]x - b\lambda_2 y = \lambda_2 y x_2. \end{aligned}$$

The solution of (9.3.11) is

$$x_1 = x_1^0 e^{\lambda_1 t}, \qquad y_1 = y_1^0 e^{\lambda_2 t}.$$

If $\lambda_2 < \lambda_1 < 0$, then

$$\lim_{t \to +\infty} x_1(t) = 0, \qquad \lim_{t \to +\infty} y_1(t) = 0.$$

and

$$\lim_{t \to +\infty} \frac{y_1(t)}{x_1(t)} = \lim_{t \to +\infty} \frac{y_1^0}{x_1^0} e^{(\lambda_2 - \lambda_1)t} = 0.$$

$$\lim_{t \to -\infty} \frac{y_1(t)}{x_1(t)} = \lim_{t \to -\infty} \frac{y_1^0}{x_1^0} e^{(\lambda_2 - \lambda_1)t} = \infty.$$

The qualitative diagram of the orbits appears in Figure 9-3-1. And the phase portrait for the original system can be easily obtained from the transformation(See Figure 9-3-6). The critical point in this case is called a stable node. The case with two positive eigenvalues λ_1 and λ_2 is similar.

Main feature of a stable node:

(1) Two solution line L_1 and L_2;

(2) $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = 0$;

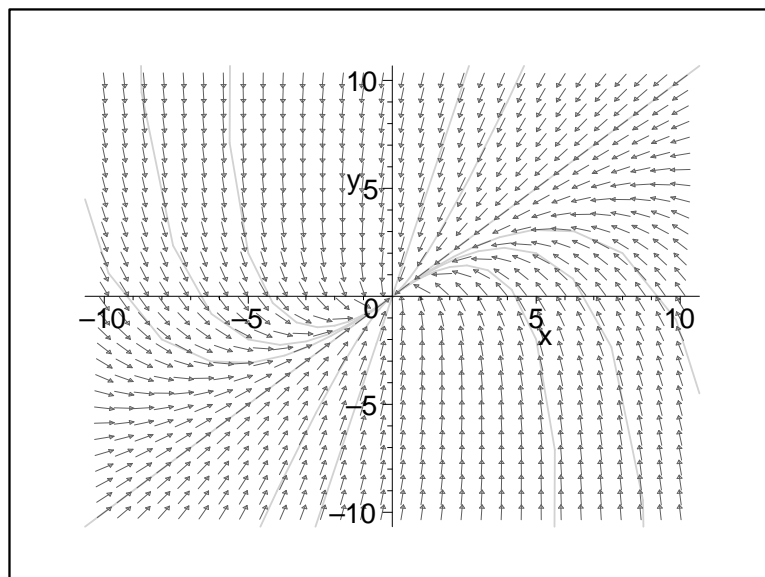(3) All the trajectories approach the origin tangent to L_1 or L_2.

Figure 9-3-6 Some orbits and direction field for the system
$$x' = -2x - y, \quad y' = 4x - 7y$$

Example 9.3.6 Plot the phase portrait of the following system

$$\frac{dx}{dt} = -2 - y, \qquad \frac{dy}{dt} = 4x - 7y.$$

Solution

$p = 9 > 0$, $q = 18 > 0$, $\Delta = 81 - 72 = 9 > 0$, the critical point $O(0,0)$ is a stable node. In order to determine the solution line, we solve the equation

$$k = \frac{dy}{dx} = \frac{y}{x} = \frac{4 - 7k}{-2 - k}$$

and obtain $k_1 = 1$, $k_2 = 4$. for the point on the positive x-axis the trajectory of the system paths it upward. The phase portrait of this system is shown in Figure 9-3-6.

Case 2. Real roots of opposite sign: $q < 0$

We assume that $\lambda_1 < 0 < \lambda_2$. The linear transformation (9.3.10) can take (9.3.8) to the form of (9.3.11). The solution of (9.3.11) $x_1 = x_1^0 e^{\lambda_1 t}$, $y_1 = y_1^0 e^{\lambda_2 t}$ has the property that

$$\lim_{t \to +\infty} x_1(t) = 0, \qquad \lim_{t \to +\infty} y_1(t) = \infty,$$

and

$$\lim_{t \to +\infty} \frac{y_1(t)}{x_1(t)} = \lim_{t \to +\infty} \frac{y_1^0}{x_1^0} e^{(\lambda_2 - \lambda_1)t} = \infty,$$

$$\lim_{t \to -\infty} \frac{y_1(t)}{x_1(t)} = \lim_{t \to -\infty} \frac{y_1^0}{x_1^0} e^{(\lambda_2 - \lambda_1)t} = 0.$$

The trajectories of (9.3.11) are sketched in Figure 9.3, and the trajectories of the original system (9.3.8) are sketched in Figure 9-3-7.

In this case the critical point is called a saddle point. Main features of a saddle point:

(1) Two solution line L_1 and L_2;

(2) $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = \infty$;

(3) All the trajectories approach infinity tangent to L_1 or L_2 as t tends to infinity.

Example 9.3.7 Draw the phase portrait of the linear equations

$$\frac{dx}{dt} = x - 3y, \qquad \frac{dy}{dt} = -3x + y.$$

Solution

It is easy to see that $p = 2 > 0$, $q = -8 < 0$. Hence the critical point $O(0,0)$ is a saddle. In order to determine the solution line, we solve the equation

$$k = \frac{dy}{dx} = \frac{y}{x} = \frac{-3 + k}{1 - 3k}$$

and obtain $3k^2 = 3$, and $k_1 = 1$, $k_2 = -1$. The phase portrait of this system is shown in Figure 9-3-7.

The Maple commands are as follows.

```
DEtools[phaseportrait]
  ([diff(x(t),t)=x(t)-3*y(t),
 diff(y(t),t)=-3*x(t)+y(t)],
 [x(t),y(t)],
 t=-100..100,
 [[x(0)=1,y(0)=1],  [x(0)=1,y(0)=-1],
 [x(0)=-1,y(0)=-1], [x(0)=-1,y(0)=1],
 [x(0)=4,y(0)=0],  [x(0)=1,y(0)=0],
 [x(0)=-0.5,y(0)=0], [x(0)=-1,y(0)=0],
 [x(0)=0,y(0)=0.5],[x(0)=0,y(0)=1],
 [x(0)=0,y(0)=-0.5], [x(0)=0,y(0)=-1]],
 x=-1.5..1.5,y=-1.5..1.5,
 dirgrid=[30,30],
 stepsize=0.1,
 arrows=SLIM,
 axes=BOXED);
```
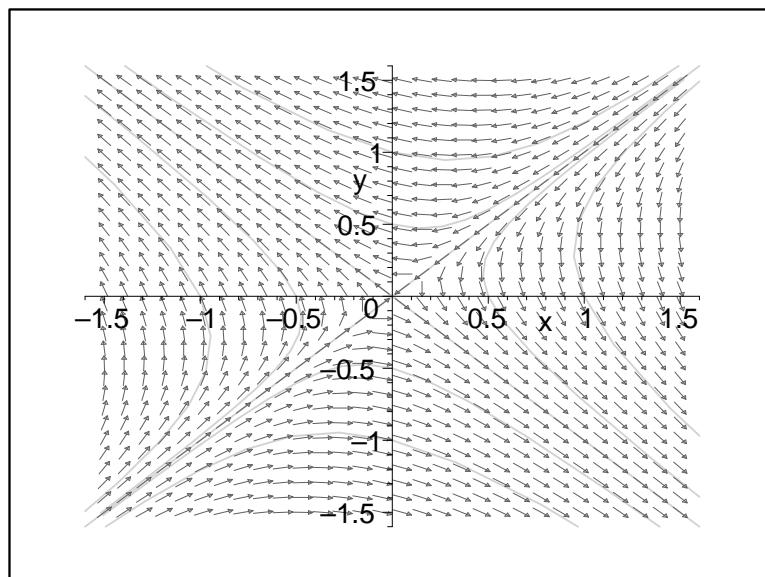
Figure 9-3-7 Some orbits and direction field for the system
$$x' = x - 3y, \quad y' = -3x + y$$

Case 3. Equal roots: $\Delta = 0$

We assume that $\lambda_1 = \lambda_2 = \lambda < 0$. The linear system (9.3.8) can be reduced to the form

$$\frac{dx_1}{dt} = \lambda x_1, \qquad \frac{dy_1}{dt} = \lambda y_1, \qquad (9.3.12)$$

or

$$\frac{dx_1}{dt} = \lambda x_1 + y_1, \qquad \frac{dy_1}{dt} = \lambda y_1. \qquad (9.3.13)$$

The solution of (9.3.12) $x_1 = x_1^0 E^{\lambda t}$, $y_1 = y_1^0 e^{\lambda t}$ has the property that

$$\lim_{t \to +\infty} x_1(t) = 0, \qquad \lim_{t \to +\infty} y_1(t) = 0,$$

and

$$\lim_{t \to +\infty} \frac{y_1(t)}{x_1(t)} = \frac{y_1^0}{x_1^0}.$$

The trajectories of (9.3.12) are sketched in Figure 9.3.10, and the trajectories of the original system (9.3.8) are similar.

In this case the critical point is called a star-shaped node. Main features of a star-shaped node:

(1) Each trajectory is a straight line;
(2) $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = 0$.

The solution of (9.3.13) $x_1 = (x_1^0 + y_1^0 t)e^{\lambda t}$, $y_1 = y_1^0 e^{\lambda t}$ has the property that

$$\lim_{t \to +\infty} x_1(t) = 0, \qquad \lim_{t \to +\infty} y_1(t) = 0,$$

and

$$\lim_{t \to +\infty} \frac{y_1(t)}{x_1(t)} = \frac{y_1^0}{x_1^0 + y_1^0 t} = 0.$$

The trajectories of (9.3.13) are sketched in Figure 9.3.11, and the trajectories of the original system (9.3.8) are similar.

In this case the critical point is called degenerate node. Main features of a star-shaped node:

(1) There exists one trajectory which is a straight line;
(2) $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = 0$;
(3) All the trajectories approach the origin tangent to the straight line.

Case 4. Complex roots: $q > 0$, $\Delta < 0$, $p \neq 0$

In this case the eigenvalues of A is $\lambda_1 = \bar{\lambda}_2 = \alpha + i\beta$, where $\alpha = -p/2$, $\beta = \sqrt{-\Delta}$. The transformation

$$\begin{cases} x_1 = -cx + (a - \alpha)y, \\ y_1 = \beta y, \end{cases} \qquad \begin{cases} x = \frac{1}{c}[-x_1 + \frac{a-\alpha}{\beta}y_1], \\ y = \frac{1}{\beta}y_1, \end{cases} \tag{9.3.14}$$

can take (9.3.8) to the form of

$$\frac{dx_1}{dt} = \alpha x_1 + \beta y_1, \qquad \frac{dy_1}{dt} = -\beta x_1 + \alpha y_1. \tag{9.3.15}$$

In fact,

$$\frac{dx_1}{dt} = -c\frac{dx}{dt} + (a - \alpha)\frac{dy}{dt} = -c(ax + by) + (a - \alpha)(cx + dy)$$
$$= -cax + (ad - bc - d\alpha)y = \alpha[-cx + (a - \alpha)y] + (ad - bc - d\alpha - a\alpha + \alpha^2)y$$
$$= \alpha x_1 + [\beta^2 + 2\alpha^2 - (a + d)\alpha]y = \alpha x_1 + \beta y_1,$$
$$\text{since} \quad 4\beta^2 = 4(ad - bc) - (a + d)^2, \quad \alpha = \frac{a + d}{2}.$$
$$\frac{dy_1}{dt} = \beta\frac{dy}{dt} = -\beta x_1 + (a - \alpha)y_1 + dy_1$$
$$= -\beta x_1 + \alpha y_1.$$

The new variable

$$\begin{cases} r = \sqrt{x_1^2 + y_1^2}, \\ \theta = tan^{-1}\frac{y_1}{x_1} \end{cases} \qquad \begin{cases} x_1 = r cos\theta, \\ y_1 = r sin\theta, \end{cases} \tag{9.3.14}$$

can take (9.3.14) to the form of

$$\frac{dr}{dt} = \alpha r, \qquad \frac{d\theta}{dt} = -\beta. \tag{9.3.15}$$

The solution of (9.3.15) with the initial value $r(0) = r_0$, $\theta(0) = \theta_0$ is

$$r = r_0 e^{\alpha t}, \qquad \theta = -\beta t + \theta_0.$$

Finally, eliminating t gives

$$r = r_0 e^{-(\theta - \theta_0)/\beta},$$

which represents a family of spirals, one of which is sketched in Figure 9.3.12.

The critical point is called a spiral point(focus). The more general sketch of trajectories near a spiral point is shown in Figure 9.3.13.

Main features of a spiral:

(1) Each trajectory is a spiral;

(2) $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y(t) = 0$.

Case 5. Pure imaginary roots: $q > 0$, $\Delta < 0$, $p = 0$

This is a special case of the previous one with $\alpha = 0$. It must be treated separately, since the trajectories are no longer spirals. In this case the motion is periodic in time and the trajectories are closed curves. The phase portrait is shown in Figure 9.3.5.

In this case the origin is called a center. Main features of a center:

(1) Each trajectory is a closed orbit;

(2) The trajectories neither approach nor recede from the critical point.

In each of the cases we have discussed, the trajectories of the system exhibited one of the following three types of behavior.

1. All of the trajectories approach the critical point as $t \to \infty$. This will be the case if the roots of the characteristic equation are real and negative or complex with negative real parts.

2. The trajectories neither approach the critical point nor to infinity as $t \to \infty$. this will be the case if the roots of the characteristic equation are pure imaginary.

3. At least one(possibly all) of the trajectories tends to infinity as $t \to \infty$. This will be the case if at least one of the roots of the characteristic equation is positive or if the roots have positive real parts.

These three possibilities illustrate the concepts of asymptotic stability, stability, and instability, respectively, of the critical point at the origin of the system (9.3.8). The precise definitions of these terms will be given in next section.

9.3.3 The classification of the critical point of the system (9.3.8) in parameter space
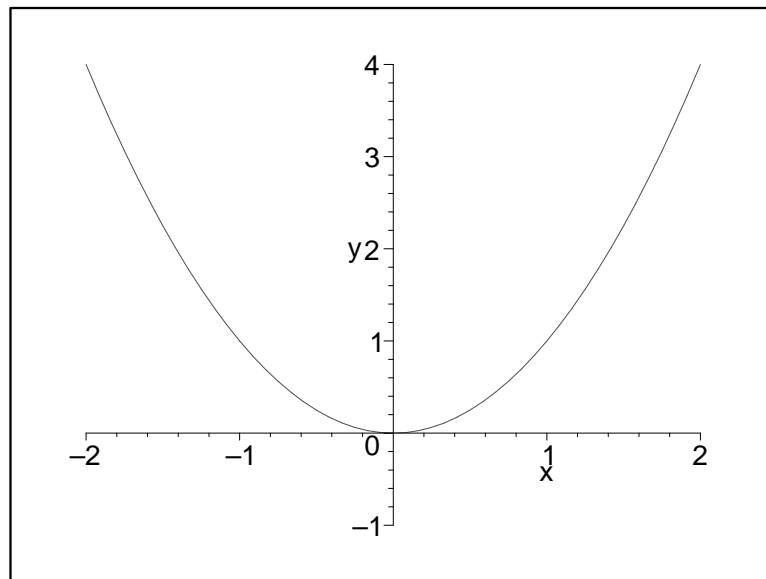
See Figure on Page 266.

Figure 9-3-9 Stability diagram of the system $x' = ax + by, \quad y' = cx + dy$