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A Second Order Energy Stable BDF Numerical Scheme for the Swift–Hohenberg Equation

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Abstract
In this paper, we propose and analyze a second-order energy stable numerical scheme for the Swift–Hohenberg equation, with a mixed finite element approximation in space. We employ second-order backward differentiation formula scheme with a second-order stabilized term, which guarantees the long time energy stability. We prove that our two-step scheme is unconditionally energy stable and uniquely solvable. Furthermore, we present an optimal error estimate for the scheme. In the end, several numerical experiments are presented to support our theoretical analysis.

Keyword Swift–Hohenberg equation · Optimal convergence analysis · Mixed finite element · Energy stability

1 Introduction

The Swift–Hohenberg (SH) equation has been widely used as a model for the study of pattern formation [12,36] and in complex fluids and biological materials [22,23,33]. It is an L^2-gradient flow for the following free energy functional [31]

$$E(\phi) := \int_\Omega \left(\frac{1}{4}\phi^4 + \frac{1-\epsilon}{2}\phi^2 - |\nabla\phi|^2 + \frac{1}{2}(\triangle\phi)^2 \right) \mathrm{d}x \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^d (d = 1, 2, 3)$, ϕ is the density field and $\epsilon > 0$ is a constant with physical significance, and ∇ and $\triangle$ are the gradient and Laplacian operators, respectively. The phase field crystal (PFC) model is the H^{-1} gradient flow in terms of the same free energy functional. The SH equation is given by

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$$\frac{\partial \phi}{\partial t} = -\frac{\delta E}{\delta \phi} = -(\phi^3 + (1-\epsilon)\phi + 2\triangle\phi + \triangle^2\phi), \tag{1.2}$$

where $\delta E/\delta\phi$ denotes the variational derivative of E with respect to ϕ. This equation models the convection of a thin layer of fluid heated from below for which you can think of the ϕ as representing the temperature of the fluid in the mid plane. The parameter ϵ measures how far the temperature is above the minimum temperature required for convection: for $\epsilon < 0$, the heating is too small to cause convection, while for $\epsilon > 0$, convection occurs. Note that if the $\frac{\partial\phi}{\partial t} = \epsilon\phi$ were the whole equation, then we would just observe exponential growth (for $\epsilon > 0$) or decay (for $\epsilon < 0$). The dissipation term acts to smooth out sharp edges in pattern. The phase field crystal equation is thus the conserved counterpart of the Swift–Hohenberg equation. This relationship is completely analogous to that between the Cahn-Hilliard equation and the Allen-Cahn equation. Here, we study the numerical scheme of SH equation (1.2) with boundary condition $\partial_n\phi = \partial_n(\triangle\phi) = 0$. The energy functional (1.1) is decreasing in time:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\Omega \frac{\delta E}{\delta\phi}\frac{\partial\phi}{\partial t}\mathrm{d}x = -\int_\Omega \left(\frac{\partial\phi}{\partial t}\right)^2 \mathrm{d}x \le 0.$$

The SH equation is a fourth-order nonlinear partial differential equation and cannot generally be solved analytically. Therefore, accurate and efficient numerical methods are desirable in understanding of nonequilibrium processing. The primary challenge is associated with a proper discretization of the nonlinear term and the long time energy stability. It is well known that the standard explicit Euler method is unstable for time step $\triangle t$ above a threshold proportional to $(\triangle x)^4$, where $\triangle x$ is grid spacing. In order to alleviate the time step restriction, various computational algorithms have been developed. A first-order semi-implicit finite difference method was proposed in [11] which splits the linear terms into backward and forward pieces while treating the nonlinear term explicitly. In [15] the authors proposed a first-order semi-implicit method which adds an extra stabilizing term to improve the stability while preserving the simplicity. A second-order semi-implicit method which is based on the Crank–Nicolson method was introduced in [18], where the authors use Newton's method to solve the nonlinear system at every time step to reduce the nonlinear residual to a specified tolerance, but the optimal error estimate for the scheme was not given. In [26], the author presented first- and second-order semi-analytical Fourier spectral methods as an accurate and efficient approach for solving the SH equation, which were based on the operator splitting method, while the theoretical justification of the energy stability and convergence analysis was not given. A first- and second-order accurate methods for the SH equation with quadratic-cubic nonlinearity was proposed in [27], where the author used the Fourier spectral method for the spatial discretization. In [43] the authors proposed and analyzed a large time-stepping numerical method for the SH equation based on the finite difference method. In [28], the author introduced a new mass conservative SH equation and proposed its mass conservative first- and second-order operator splitting methods, he presented several numerical results to illustrate the effectiveness of his numerical schemes, but no convergence or error analyses were presented. There have also been extensive works of energy stable and convergent numerical schemes for the phase field crystal (PFC) equation [14,21,30,40], the modified phase field crystal (MPFC) equation [1,2,38], and the square phase field crystal (SPFC) equation [10], using the implicit treatment for the nonlinear terms, such as the convex splitting approach [2,14,21]. Both the first and second order convex splitting schemes [2,14,21], both the finite difference [1,21,30,38,40]and Fourier pseudo-spectral spatial approximations [41,42], both the two-dimensional and three-dimensional numerical implementation [30,41], have been extensively reported. However, to the best of our knowledge, optimal error estimates for the fully discrete finite element schemes for the SH equation or PFC equation are lacking in the existing literature.

The work presented in this paper on the SH equation is unique in the following sense. We are able to prove unique solvability, unconditional energy stability and optimal error estimates for a fully discrete finite element scheme. In the past few years, many efforts have been devoted to develop second order accurate, energy stable schemes for numerically solving the phase-field equations. For example, in [35] a second-order convex splitting scheme, in the modified Crank–Nicolson version, is proposed and discussed. There have also been existing works for the modified BDF2 scheme applied to the standard Cahn–Hilliard equation, with both energy stability and optimal rate convergence analysis [13], with either mixed finite element [25,34]or long stencil fourth order finite difference spatial approximation [8]. And also, there have been extensive works on the artificial regularization to various gradient flows, such as the epitaxial thin film model, either with or without slope selection [4,29,39]. Both the second order BDF2 method for the epitaxial thin film equation with slope selection [16], the square phase field crystal equation [10], stabilized second order exponential time differencing (ETD) multistep method for no-slope-selection thin film growth model [24], a third order exponential time differencing numerical scheme for no-slope-selection epitaxial thin film model [9], and energy stable higher order linear ETD multi-step methods for gradient flows [7], have been extensively reported. In all these works, the artificial regularization term has played an essential role in the energy stability analysis. In particular, there has been a recent work on artificial regularization parameter analysis for the no-slope-selection epitaxial thin film model [6], in which the effect of the parameter on the numerical schemes has been analyzed in details. In this work, we combine the mixed finite element scheme with a second order BDF temporal discretization and provide a theoretical proof of the optimal convergence rate, $O(h^{q+1} + \triangle t^2)$, for this scheme. Comparing to the primal formulation which requires H^2 elements in the discretization, the mixed formulation only needs to use H^1 elements. Similar results for the thin film epitaxial growth model have been presented in [29,39]. In order to improve the energy stability, a second order accurate Douglas–Dupont regularization term is added in the numerical scheme. Numerical experiments are presented to validate the accuracy and energy stability of the proposed numerical strategy.

The rest of the paper is organized as follows. In Sect. 2, we propose the semidiscrete mixed finite element scheme for the SH equation and give the corresponding error estimate. In Sect. 3, we apply a modified BDF2 algorithm to carry out the time discretization and prove the unique solvability, the energy stability and the optimal error estimate. Numerical results are presented in Sect. 4. We conclude the paper in Sect. 5.

2 The Semidiscrete Scheme

In this section, we define the weak formulation of the problem (1.2) and then derive the corresponding error estimate.

We denote by $W^{m,p}(\Omega)$ the Sobolev spaces, $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ are the standard norm and semi-norm respectively. Let $H^m(\Omega)$ denote $W^{m,2}(\Omega)$. We omit the subscript when $m = 0$ and write $\|\cdot\|_{0,2}$ and $|\cdot|_{0,2}$ as $\|\cdot\|$ and $|\cdot|$ for simplicity. Moreover, we use $(\cdot,\cdot)$ to represent the L^2 inner product.

We define $X = H^1(\Omega)$ and let $\mathcal{T}_h$ be a quasi-uniform partition of Ω with mesh grid size h and the finite element space X_h is defined as

$$X_h = \{v \in C^0(\Omega) \mid v|_K \in \mathcal{P}_q(K),\ \forall K \in \mathcal{T}_h\},$$

where $\mathcal{P}_q(K)$ is the standard space of all polynomial functions of degree not greater than q on K. In the semidiscrete problem, we also need to introduce Bochner space

$$L^2(0,T;X_h) = \left\{ v : (0,T) \to X_h,\ \|v\|_{L^2(0,T;X_h)} = \left(\int_0^T \|v(t)\|^2_{X_h} \mathrm{d}t \right)^{1/2} < \infty \right\}.$$

We focus on the mixed finite element methods in this paper. Let $\omega = -\triangle\phi$. The mixed form of the problem (1.2) is:

$$\begin{cases} \partial_t\phi + \phi^3 + (1-\epsilon)\phi + 2\triangle\phi - \triangle\omega = 0, \text{ in } \Omega \times (0,T], \\ \omega + \triangle\phi = 0, \text{ in } \Omega \times (0,T], \\ \partial_n\phi = \partial_n\omega = 0, \text{ in } \partial\Omega \times (0,T], \\ \phi(x,0) = \phi_0, \text{ in } \Omega, \end{cases} \tag{2.1}$$

The corresponding weak form of system (2.1) is

$$\begin{cases} (\partial_t\phi, \varphi) + (\nabla\omega, \nabla\varphi) - 2(\nabla\phi, \nabla\varphi) + (\phi^3 + (1-\epsilon)\phi, \varphi) = 0, \quad \forall t \in (0,T] \\ (\omega, v) - (\nabla\phi, \nabla v) = 0, \quad \forall t \in (0,T] \\ (\phi(x,0), \varphi) = (\phi_0, \varphi). \end{cases} \tag{2.2}$$

The corresponding finite element form of system (2.2) turns out to be: find $(\phi_h, \omega_h) \in L^\infty(0,T;X_h) \times L^2(0,T;X_h)$ and $\partial_t\phi_h \in L^2(0,T;X_h)$, such that for any $(\varphi_h, v_h) \in X_h \times X_h$

$$\begin{cases} (\partial_t\phi_h, \varphi_h) + (\nabla\omega_h, \nabla\varphi_h) - 2(\nabla\phi_h, \nabla\varphi_h) + (\phi_h^3 + (1-\epsilon)\phi_h, \varphi_h) = 0, \quad \forall t \in (0,T] \\ (\omega_h, v_h) - (\nabla\phi_h, \nabla v_h) = 0, \quad \forall t \in (0,T] \\ (\phi_h(x,0), \varphi_h) = (\phi_0, \varphi_h). \end{cases} \tag{2.3}$$

In order to obtain an optimal error estimate, we define the Ritz projection $R_h : X \to X_h$ as

$$(\nabla R_h u, \nabla v_h) = (\nabla u, \nabla v_h), \quad \forall v_h \in X_h, \tag{2.4}$$

and the L^2 projection $P_h : X \to X_h$:

$$(P_h u, v_h) = (u, v_h), \quad \forall v_h \in X_h.$$

Moreover, the discrete Laplacian $\triangle_h : X_h \cap L_0^2 \to X_h \cap L_0^2$ is introduced as in [37]: for any $\psi_h \in X_h \cap L_0^2$, let $\triangle_h\psi_h$ be the unique solution to

$$(\triangle_h\psi_h, \chi_h) = -(\nabla\psi_h, \nabla\chi_h), \quad \forall \chi_h \in X_h. \tag{2.5}$$

One has $\triangle_h R_h = P_h\triangle$ as shown in [37]. Also, we recall the optimal $W^{1,p}$ estimate for the Ritz projection [17]:

Lemma 2.1 *Assume that Ω is a convex polygon and $\mathcal{T}_h$ is a quasi-uniform regular triangulation. Let $0 \le s \le q$ and $1 \le p \le \infty$ (when $q = 1$, then $2 \le p < \infty$). There exists a constant $C > 0$, independent of h, such that the projection R_h satisfies the following error estimate:*

$$\|v - R_h v\|_{0,p} + h|v - R_h v|_{1,p} \le Ch^{1+s}\|v\|_{s+1,p}, \quad \forall v \in W^{s+1,p}(\Omega).$$

Let ϕ and ω be the exact solution pair to the (2.2). Define

$$e_\phi := \rho_\phi + \sigma_\phi, \qquad e_\omega := \rho_\omega + \sigma_\omega \tag{2.6}$$

with $\rho_\phi = \phi - R_h\phi,\ \sigma_\phi = R_h\phi - \phi_h,\ \rho_\omega = \omega - R_h\omega,\ \sigma_\omega = R_h\omega - \omega_h$. Then we get the following error equations:

$$(\partial_t\sigma_\phi, \varphi_h) + (\nabla\sigma_\omega, \nabla\varphi_h) = -(\partial_t\rho_\phi, \varphi_h) + 2(\nabla\sigma_\phi, \nabla\varphi_h)$$
$$-(\phi^3 - \phi_h^3, \varphi_h) - (1-\epsilon)(\rho_\phi + \sigma_\phi, \varphi_h), \tag{2.7}$$
$$(\sigma_\omega, v_h) - (\nabla\sigma_\phi, \nabla v_h) = -(\rho_\omega, v_h). \tag{2.8}$$

In order to establish the error estimate, we need an additional auxiliary technique about the super-closeness property between the Ritz projection of the continuous solution and the discrete solution. Its proof is referred to [5].

Lemma 2.2 *Given a real-valued function $a(\mathbf{x}) \in W^{1,\infty}(\Omega)$ (or $W^{1,\infty}(\Omega)^{2\times2}$). Then ρ_ϕ and σ_ϕ satisfy*

$$(\nabla\rho_\phi, a(\mathbf{x})\nabla\sigma_\phi) \le C_1\|\triangle_h\sigma_\phi\|^2 + \frac{Ch^{2(q+1)}}{C_1}\|\phi\|_{q+1}^2,$$

in which C_1 is an arbitrary positive constant.

Before further investigation, we introduce the Gronwall lemma [32] and a discrete Gronwall inequality [20].

Lemma 2.3 *(Gronwall Lemma) Let $f \in L^1(t_0, T)$ be a non-negative function, g and φ be continuous functions on $[t_0, T]$. If φ satisfies*

$$\varphi(t) \le g(t) + \int_{t_0}^t f(\tau)\varphi(\tau)\mathrm{d}\tau, \quad \forall t \in [t_0, T],$$

then

$$\varphi(t) \le g(t) + \int_{t_0}^t f(s)g(s)\exp\left(\int_s^t f(\tau)\mathrm{d}\tau\right)\mathrm{d}s \quad \forall t \in [t_0, T].$$

If moreover g is non-decreasing, then

$$\varphi(t) \le g(t)\exp\left(\int_{t_0}^t f(\tau)\mathrm{d}\tau\right) \quad t \in [t_0, T].$$

Lemma 2.4 *(Discrete Gronwall Inequality)Let k, B, a_n, b_n, c_n, α_n be non-negative numbers for integers $n \ge 1$ and let the inequality*

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le B + k\sum_{n=1}^{N+1} c_n + k\sum_{n=1}^{N+1}\alpha_n a_n \quad for\ N \ge 0$$

hold. If $k\alpha_n < 1$ for all $n = 1, 2, \cdots, N+1$, then

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le \exp\left(k\sum_{n=1}^{N+1}\frac{\alpha_n}{1-k\alpha_n}\right)\left(B + k\sum_{n=1}^{N+1} c_n\right) \quad for\ N \ge 0.$$

If the inequality

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le B + k\sum_{n=1}^{N+1} c_n + k\sum_{n=1}^{N} \alpha_n a_n \quad for\ N \ge 0$$

is given, then it holds

$$a_{N+1} + k\sum_{n=1}^{N+1} b_n \le \exp\left(k\sum_{n=1}^{N+1} \alpha_n\right)\left(B + k\sum_{n=1}^{N+1} c_n\right) \quad for\ N \ge 0.$$

We denote by (ϕ, ω) the exact solution pair to the original Eq. (2.2), then we say that the solution pair is of the regularity class $\mathcal{C}$ if and only if

$$\begin{aligned}
\phi \in & H^1(0, T; H^{q+1}) \cap L^2(0, T; W^{q+1,6}) \cap L^\infty(0, T; W^{2,\infty}),\\
\omega \in & L^2(0, T; H^{q+1}),
\end{aligned}$$

and furthermore, the solution pair is of the regularity class $\mathcal{C}_1$ if and only if

$$\begin{aligned}
\phi \in & L^\infty(0, T; W^{2,\infty}) \cap L^\infty(0, T; W^{q+1,6}) \cap H^1(0, T; H^{q+1}) \cap W^{2,\infty}(0, T; L^2)\\
& \cap W^{1,\infty}(0, T; H^2) \cap H^3(0, T; L^2) \cap H^2(0, T; H^1),\\
\omega \in & L^\infty(0, T; H^{q+1}) \cap H^1(0, T; H^{q+1}).
\end{aligned}$$

Next, we provide an optimal error estimate for the semidiscrete scheme.

Theorem 2.1 *Let (ϕ, ω) be the solution of (2.2). Then the finite element approximation (ϕ_h, ω_h) of (2.3) with $\phi_h(x, 0) = R_h\phi(x, 0)$ has the following error estimate*

$$\|\phi(x, T) - \phi_h(x, T)\|^2 + \int_0^T \|\omega - \omega_h\|^2 \mathrm{d}t \le C(\epsilon, T)h^{2q+2}, \tag{2.9}$$

where $C(\epsilon, T)$ is a constant that only depends on ϵ and T.

Proof Let $\varphi_h = \sigma_\phi$ in (2.7), $v_h = \triangle_h\sigma_\phi$ in (2.8) and add up the two equations, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma_\phi\|^2 + \|\triangle_h\sigma_\phi\|^2 =& -(\partial_t\rho_\phi, \sigma_\phi) - (\rho_\omega, \triangle_h\sigma_\phi) + 2\|\nabla\sigma_\phi\|^2\\
& -(1-\epsilon)(\rho_\phi, \sigma_\phi) - (1-\epsilon)\|\sigma_\phi\|^2 + \mathcal{N}_1 + \mathcal{N}_2,
\end{aligned} \tag{2.10}$$

where

$$\mathcal{N}_1 = ((\phi_h)^3 - (R_h\phi)^3, \sigma_\phi), \quad \mathcal{N}_2 = ((R_h\phi)^3 - \phi^3, \sigma_\phi).$$

Using the Young's inequality for $\mathcal{N}_1$, it follows that

$$\begin{aligned}
\mathcal{N}_1 =& ((\phi_h)^3, R_h\phi - \phi_h) - ((R_h\phi)^3, R_h\phi - \phi_h)\\
=& -((R_h\phi)^3, R_h\phi) + ((R_h\phi)^2, R_h\phi\phi_h) + ((\phi_h)^2, \phi_h R_h\phi) - ((\phi_h)^3, \phi_h)\\
\le& -\|\phi_h\|_{0,4}^4 - \|R_h\phi\|_{0,4}^4 + \frac{1}{2}\|\phi_h\|_{0,4}^4 + \frac{1}{2}(\phi_h^2, (R_h\phi)^2) + \frac{1}{2}\|R_h\phi\|_{0,4}^4 + \frac{1}{2}((R_h\phi)^2, \phi_h^2)\\
=& -\frac{1}{2}\|R_h\phi\|_{0,4}^4 - \frac{1}{2}\|\phi_h\|_{0,4}^4 + (|R_h\phi|^2, |\phi_h|^2)\\
\le& 0.
\end{aligned} \tag{2.11}$$

In order to estimate $\mathcal{N}_2$, we first split it into two parts

$$\begin{aligned}\mathcal{N}_2 &= ((R_h\phi)^3 - R_h\phi(\phi)^2 + R_h\phi\phi^2 - \phi^3, \sigma_\phi)\\ &= ((R_h\phi - \phi)\phi^2, \sigma_\phi) + (((R_h\phi)^2 - \phi^2)R_h\phi, \sigma_\phi)\\ &:= \Pi_1 + \Pi_2. \end{aligned} \tag{2.12}$$

Using Lemma 2.1, we get

$$\begin{aligned}|\Pi_1| &= \|\phi\|_{0,\infty}^2 |(\rho_\phi, \sigma_\phi)| \le \|\phi\|_{0,\infty}^2 \left(\frac{C_3}{2}\|\rho_\phi\|^2 + \frac{1}{2C_3}\|\sigma_\phi\|^2\right)\\ &\le \frac{C^2C_3\|\phi\|_{0,\infty}^2 h^{2q+2}}{2}\|\phi\|_{q+1}^2 + \frac{\|\phi\|_{0,\infty}^2}{2C_3}\|\sigma_\phi\|^2. \end{aligned} \tag{2.13}$$

Then, split Π_2 into three parts:

$$\begin{aligned}\Pi_2 &= (((R_h\phi)^2 - \phi^2)R_h\phi, \sigma_\phi)\\ &= (((\phi - \rho_\phi)^2 - \phi^2)(\phi - \rho_\phi), \sigma_\phi)\\ &= (\rho_\phi(2\phi - \rho_\phi)(\phi - \rho_\phi), \sigma_\phi)\\ &= (-2\phi^2\rho_\phi + 3\phi\rho_\phi^2 - \rho_\phi^3, \sigma_\phi)\\ &= -2(\phi^2\rho_\phi, \sigma_\phi) + 3(\phi\rho_\phi^2, \sigma_\phi) - (\rho_\phi^3, \sigma_\phi)\\ &:= A_1 + A_2 + A_3. \end{aligned} \tag{2.14}$$

Using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}|A_1| &\le \|\phi\|_{0,\infty}^2 \left(C_3\|\rho_\phi\|^2 + \frac{1}{C_3}\|\sigma_\phi\|^2\right)\\ &\le C^2C_3\|\phi\|_{0,\infty}^2 h^{2q+2}\|\phi\|_{q+1}^2 + \frac{1}{C_3}\|\phi\|_{0,\infty}^2\|\sigma_\phi\|^2. \end{aligned} \tag{2.15}$$

$$\begin{aligned}|A_2| &\le \frac{3}{2}\|\phi\|_{0,\infty}\left(C_3\|\rho_\phi\|_{0,4}^4 + \frac{1}{C_3}\|\sigma_\phi\|^2\right)\\ &\le \frac{3}{2}C_3\|\phi\|_{0,\infty}\|\sigma_\phi\|^2 + \frac{3}{2C_3}\|\phi\|_{0,\infty}C^4h^{4q+4}\|\phi\|_{q+1,4}^4. \end{aligned} \tag{2.16}$$

$$\begin{aligned}|A_3| &= |(\rho_\phi^3, \sigma_\phi)| \le \|\rho_\phi^3\|\|\sigma_\phi\|\\ &\le \frac{1}{2}\left(C_3\|\rho_\phi\|_{0,6}^6 + \frac{1}{C_3}\|\sigma_\phi\|^2\right)\\ &\le \frac{1}{2}C_3C^6h^{6q+6}\|\phi\|_{q+1,6}^6 + \frac{1}{2C_3}\|\sigma_\phi\|^2. \end{aligned} \tag{2.17}$$

Therefore, we get

$$\begin{aligned}|\Pi_2| \le &\frac{1}{C_3}\left(\frac{1}{2} + \frac{3}{2}\|\phi\|_{0,\infty} + \|\phi\|_{0,\infty}^2\right)\|\sigma_\phi\|^2 + C^2C_3\|\phi\|_{0,\infty}^2 h^{2q+2}\|\phi\|_{q+1}^2\\ &+ \frac{3}{2}\|\phi\|_{0,\infty}C^4C_3h^{4q+4}\|\phi\|_{q+1,4}^4 + \frac{1}{2}C_3C^6h^{6q+6}\|\phi\|_{q+1,6}^6. \end{aligned} \tag{2.18}$$

and hence

$$
\begin{aligned}
\mathcal{N}_2 \le |\Pi_1| + |\Pi_2| \le \frac{1}{C_3}\left(\frac{1}{2} + \frac{3}{2}\|\phi\|_{0,\infty} + \frac{3}{2}\|\phi\|_{0,\infty}^2\right)\|\sigma_\phi\|^2 \\
+ \frac{3}{2}C^2C_3\|\phi\|_{0,\infty}^2 h^{2q+2}\|\phi\|_{q+1}^2 + \frac{3}{2}\|\phi\|_{0,\infty}C^4C_3h^{4q+4}\|\phi\|_{q+1,4}^4 \\
+ \frac{1}{2}C_3C^6h^{6q+6}\|\phi\|_{q+1,6}^6.
\end{aligned}
\tag{2.19}
$$

In addition, we have

$$
\begin{aligned}
-(\partial_t\rho_\phi, \sigma_\phi) &\le \frac{1}{2}\|\partial_t\rho_\phi\|^2 + \frac{1}{2}\|\sigma_\phi\|^2 \\
&\le \frac{1}{2}C^2h^{2q+2}\|\partial_t\phi\|_{q+1}^2 + \frac{1}{2}\|\sigma_\phi\|^2.
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
&-(1-\epsilon)(\rho_\phi + \sigma_\phi, \sigma_\phi) \\
&\le \frac{C_3|1-\epsilon|}{2}\|\rho_\phi + \sigma_\phi\|^2 + \frac{|1-\epsilon|}{2C_3}\|\sigma_\phi\|^2 \\
&= \frac{C_3|1-\epsilon|}{2}(\|\rho_\phi\|^2 + \|\sigma_\phi\|^2 + 2(\rho_\phi, \sigma_\phi)) + \frac{|1-\epsilon|}{2C_3}\|\sigma_\phi\|^2 \\
&\le \frac{C_3|1-\epsilon|}{2}\Bigg(\|\rho_\phi\|^2 + \|\sigma_\phi\|^2 + \frac{1}{\frac{1}{C_3^2}-1}\|\rho_\phi\|^2 \\
&\quad +(\frac{1}{C_3^2}-1)\|\sigma_\phi\|^2\Bigg) + \frac{|1-\epsilon|}{2C_3}\|\sigma_\phi\|^2 \\
&\le \frac{C_3|1-\epsilon|}{2(1-C_3^2)}\|\rho_\phi\|^2 + \frac{|1-\epsilon|}{C_3}\|\sigma_\phi\|^2 \\
&\le \frac{C_3C^2|1-\epsilon|}{2(1-C_3^2)}h^{2q+2}\|\phi\|_{q+1}^2 + \frac{|1-\epsilon|}{C_3}\|\sigma_\phi\|^2.
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
-(\rho_\omega, \triangle_h\sigma_\phi) &\le \|\rho_\omega\|^2 + \frac{1}{4}\|\triangle_h\sigma_\phi\|^2 \\
&\le C^2h^{2q+2}\|\omega\|_{q+1}^2 + \frac{1}{4}\|\triangle_h\sigma_\phi\|^2.
\end{aligned}
\tag{2.22}
$$

Also notice that

$$
2\|\nabla\sigma_\phi\|^2 \le \frac{1}{4}\|\triangle_h\sigma_\phi\|^2 + 4\|\sigma_\phi\|^2. \tag{2.23}
$$

A substitution of (2.11–2.23) into (2.10) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma_\phi\|^2 + \frac{1}{2}\|\triangle_h\sigma_\phi\|^2 \\
&\quad\le C_1\|\sigma_\phi\|^2 + C_2\Big(h^{2q+2}(\|\partial_t\phi\|_{q+1}^2 + \|\phi\|_{q+1} \\
&\qquad +\|\omega\|_{q+1}^2) + h^{4q+4}\|\phi\|_{q+1,4}^4 + h^{6q+6}\|\phi\|_{q+1,6}^6\Big).
\end{aligned}
\tag{2.24}
$$

where

$$C_1 = \frac{9}{2} + \frac{|1-\epsilon|}{C_3} + \frac{1}{2C_3} + \frac{3}{2C_3}\|\phi\|_{0,\infty} + \frac{3}{2C_3}\|\phi\|_{0,\infty}^2$$
$$C_2 = \max\left\{C^2, \frac{C_3|1-\epsilon|}{2(1-C_3^2)}C^2 + \frac{3}{2}C_3C^2\|\phi\|_{0,\infty}^2, \frac{3}{2}\|\phi\|_{0,\infty}C^4C_3, \frac{1}{2}C^6C_3\right\}.$$

Integrating (2.24) over $[0, T]$, we get

$$\begin{aligned}\|\sigma_\phi(x,T)\|^2 &+ \int_0^T \|\triangle_h\sigma_\phi\|^2 \mathrm{d}t \\ &\le 2C_1\int_0^T \|\sigma_\phi\|^2\mathrm{d}t + \widetilde{C_2}h^{2q+2} + o(h^{2q+2}) \qquad (2.25)\end{aligned}$$

where

$$\widetilde{C_2} = C_2\int_0^T (\|\partial_t\phi\|_{q+1}^2 + \|\phi\|_{q+1}^2 + \|\omega\|_{q+1}^2)\mathrm{d}t.$$

From $\|\sigma_\phi(x,T)\|^2 \le 2C_1\int_0^T \|\sigma_\phi\|^2\mathrm{d}t + \widetilde{C_2}h^{2q+2} + o(h^{2q+2})$ and the Gronwall lemma, we get

$$\|\sigma_\phi(x,T)\|^2 \le \widetilde{C_2}e^{2C_1T}h^{2q+2} + o(h^{2q+2}). \qquad (2.26)$$

which means that, for any $t > 0$, we have $\|\sigma_\phi(x,t)\|^2 \le \widetilde{C_2}e^{2C_1t}h^{2q+2} + o(h^{2q+2})$.

Therefore,

$$\begin{aligned}\int_0^T \|\triangle_h\sigma_\phi\|^2\mathrm{d}t &\le 2C_1\int_0^T \|\sigma_\phi\|^2\mathrm{d}t + \widetilde{C_2}h^{2q+2} + o(h^{2q+2}) \\ &\le 2C_1\int_0^T e^{2C_1t}\mathrm{d}t\widetilde{C_2}h^{2q+2} + \widetilde{C_2}h^{2q+2} + o(h^{2q+2}) \\ &= \widetilde{C_2}e^{2C_1T}h^{2q+2} + o(h^{2q+2}). \qquad (2.27)\end{aligned}$$

It follows that

$$\begin{aligned}\|\sigma_\phi(x,T)\|^2 &+ \int_0^T \|\triangle_h\sigma_\phi\|^2\mathrm{d}t \\ &\le 2\widetilde{C_2}e^{2C_1T}h^{2q+2} + o(h^{2q+2}). \qquad (2.28)\end{aligned}$$

Note that $\phi_h(x,0) = R_h\phi(x,0)$ has been used to eliminate the term $\sigma_\phi(x,0)$, then we arrive at the estimate for e_ϕ.

$$\|\phi(x,T) - \phi_h(x,T)\|^2 = \|e_\phi(x,T)\|^2 = \|\rho_\phi + \sigma_\phi\|^2 \le 2(\|\rho_\phi\|^2 + \|\sigma_\phi\|^2) \le C(\epsilon,T)h^{2q+2}.$$

As for e_ω, by the second equation of (2.2) and (2.3) and the relationship between P_h and R_h, i.e., $\triangle_hR_h = P_h\triangle$, one gets,

$$\begin{aligned}\|e_\omega\| &= \|\omega - \omega_h\| \\ &= \|\omega - P_h\omega + P_h\omega - \omega_h\| \\ &= \|(I-P_h)\omega - P_h\triangle\phi + \triangle_h\phi_h\| \\ &= \|(I-P_h)\omega - \triangle_h(R_h\phi - \phi_h)\| \\ &= \|(I-P_h)\omega - \triangle_h\sigma_\phi\| \\ &\le \|(I-P_h)\omega\| + \|\triangle_h\sigma_\phi\|.\end{aligned}$$

The estimate for e_ω then follows from (2.28) and the approximation property of the L^2-orthogonal projection, which completes the proof. □

Remark 2.1 The key point of this proof is to establish a super-closeness estimate between the Ritz projection of the continuous solution and the discrete solution. Similar techniques have also been used in [5,39].

3 The Fully Discrete Scheme

In this section, we carry out the temporal discretization over the time interval $[0, T]$ by using BDF2 algorithm. For a given positive integer N, let $\triangle t = T/N$ be the uniform time step size and the nodes are denoted by $t_n = n\triangle t, 0 \le n \le N$. Then we provide a fully discrete error estimate. Moreover, by constructing a modified energy functional, the energy decay property of the fully discrete scheme is proved.

We propose the following fully discrete numerical scheme: for $n \ge 1$, given $(\phi_h^n, \omega_h^n) \in X_h \times X_h$, find $(\phi_h^{n+1}, \omega_h^{n+1}) \in X_h \times X_h$ such that for arbitrary $(\varphi_h, v_h) \in X_h \times X_h$

$$\begin{cases} \left(\frac{3\phi_h^{n+1}-4\phi_h^n+\phi_h^{n-1}}{2\triangle t}, \varphi_h\right) + (\nabla\omega_h^{n+1}, \nabla\varphi_h) - 2(\nabla(2\phi_h^n - \phi_h^{n-1}), \nabla\varphi_h) + ((\phi_h^{n+1})^3, \varphi_h) \\ +(1-\epsilon)(\phi_h^{n+1}, \varphi_h) + A\triangle t(\nabla(\omega_h^{n+1} - \omega_h^n), \nabla\varphi_h) = 0, \\ (\omega_h^{n+1}, v_h) - (\nabla\phi_h^{n+1}, \nabla v_h) = 0. \end{cases} \tag{3.1}$$

where A is a given constant. The scheme requires an initialization step. Let $\phi_h^0 = R_h\phi_0$, we set

$$\begin{cases} \left(\frac{\phi_h^1-\phi_h^0}{\triangle t}, \varphi_h\right) + (\nabla\omega_h^1, \nabla\varphi_h) - 2(\nabla\phi_h^0, \nabla\varphi_h) + ((\phi_h^1)^3, \varphi_h) + (1-\epsilon)(\phi_h^1, \varphi_h) = 0, \\ (\omega_h^1, v_h) - (\nabla\phi_h^1, \nabla v_h) = 0. \end{cases} \tag{3.2}$$

3.1 Unique Solvability

Theorem 3.1 *The fully discrete scheme (3.1) and (3.2) has a unique solution.*

Proof Taking the test function as $v_h = -\triangle_h\varphi_h$ in the second equation in our mixed scheme (3.1), we obtain

$$(\omega_h^{n+1}, -\triangle_h\varphi_h) = (\nabla\phi_h^{n+1}, -\nabla\triangle_h\varphi_h),$$

by using (2.5), the above equation can be written as

$$(\nabla\omega_h^{n+1}, \nabla\varphi_h) = (\triangle_h\phi_h^{n+1}, \triangle_h\varphi_h),$$

Thus the scheme (3.1) becomes

$$\begin{aligned} &\left(\frac{3\phi_h^{n+1} - 4\phi_h^n + \phi_h^{n-1}}{2\triangle t}, \varphi_h\right) + (\triangle_h\phi_h^{n+1}, \triangle_h\varphi_h) \\ &- 2(\nabla(2\phi_h^n - \phi_h^{n-1}), \nabla\varphi_h) + ((\phi_h^{n+1})^3, \varphi_h) \\ &+ (1-\epsilon)(\phi_h^{n+1}, \varphi_h) + A\triangle t(\triangle_h(\phi_h^{n+1} - \phi_h^n), \triangle_h\varphi_h) = 0. \end{aligned}$$

By rearranging the above equation, we get, for every $\varphi_h \in X_h$,

$$\left(\frac{3}{2\triangle t} + 1 - \epsilon\right)(\phi_h^{n+1}, \varphi_h) + (1 + A\triangle t)(\triangle_h \phi_h^{n+1}, \triangle_h \varphi_h) \\ +((\phi_h^{n+1})^3, \varphi_h) = f[\phi_h^n, \phi_h^{n-1}](\varphi_h). \tag{3.3}$$

where $f[\phi_h^n, \phi_h^{n-1}]$ is a bounded linear functional involving the previous time iterations. Based on the scheme (3.3), we define the following functional:

$$J(\phi_h) = \left(\frac{3}{4\triangle t} + \frac{1-\epsilon}{2}\right)\|\phi_h\|^2 + \frac{1 + A\triangle t}{2}\|\triangle_h \phi_h\|^2 \\ +\frac{1}{4}\|\phi_h\|_{0,4}^4 - f[\phi_h^n, \phi_h^{n-1}](\phi_h)$$

It may be shown that ϕ_h^{n+1} is the unique minimizer of $J(\phi_h)$ if and only if it solves, for any φ_h,

$$\frac{\mathrm{d}}{\mathrm{d}s} J(\phi_h + s\varphi_h)|_{s=0} = \left(\frac{3}{2\triangle t} + 1 - \epsilon\right)(\phi_h, \varphi_h) \\ +(1 + A\triangle t)(\triangle_h \phi_h, \triangle_h \varphi_h) + (\phi_h^3, \varphi_h) - f[\phi_h^n, \phi_h^{n-1}](\varphi_h) = 0.$$

Since

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} J(\phi_h + s\varphi_h)|_{s=0} = \left(\frac{3}{2\triangle t} + 1 - \epsilon\right)\|\varphi_h\|^2 \\ +(1 + A\triangle t)\|\triangle_h \varphi_h\|^2 + 3\|\phi_h \varphi_h\|^2 > 0,$$

the corresponding functional $J(\phi_h)$ is a strictly convex functional and the uniqueness of the solution of scheme (3.1) is proved. The unique solvability of the initialization scheme (3.2) is similar. □

3.2 Energy stability

We introduce a discrete energy which is consistent with the continuous space energy as $h \to 0$:

$$E(\phi_h^{n+1}, \omega_h^{n+1}) = \frac{1}{4}\|\phi_h^{n+1}\|_{0,4}^4 + \frac{1-\epsilon}{2}\|\phi_h^{n+1}\|^2 - \|\nabla \phi_h^{n+1}\|^2 + \frac{1}{2}\|\omega_h^{n+1}\|^2. \tag{3.4}$$

We first consider the energy stability for the initial step. Taking $\varphi_h = \phi_h^1 - \phi_h^0$ and $\upsilon_h = -\triangle_h(\phi_h^1 - \phi_h^0)$ in (3.2), we obtain

$$\begin{cases} \frac{1}{\triangle t}\|\phi_h^1 - \phi_h^0\|^2 + (\nabla \omega_h^1, \nabla(\phi_h^1 - \phi_h^0)) - 2(\nabla \phi_h^0, \nabla(\phi_h^1 - \phi_h^0)) + ((\phi_h^1)^3, \phi_h^1 - \phi_h^0) \\ +(1-\epsilon)(\phi_h^1, \phi_h^1 - \phi_h^0) = 0, \\ (\omega_h^1, -\triangle_h(\phi_h^1 - \phi_h^0)) - (\nabla \phi_h^1, -\nabla \triangle_h(\phi_h^1 - \phi_h^0)) = 0. \end{cases} \tag{3.5}$$

According to the second equation of (3.5), we have

$$
\begin{aligned}
&(\nabla\omega_h^1, \nabla(\phi_h^1-\phi_h^0)) = (\triangle_h\phi_h^1, \triangle_h\phi_h^1-\triangle_h\phi_h^0)\\
&= \frac{1}{2}(\|\triangle_h\phi_h^1\|^2-\|\triangle_h\phi_h^0\|^2+\|\triangle_h\phi_h^1-\triangle_h\phi_h^0\|^2)\\
&\geq \frac{1}{2}\|\triangle_h\phi_h^1\|^2-\frac{1}{2}\|\triangle_h\phi_h^0\|^2 = \frac{1}{2}\|\omega_h^1\|^2-\frac{1}{2}\|\omega_h^0\|^2. - 2(\nabla\phi_h^0, \nabla\phi_h^1-\nabla\phi_h^0)\\
&= 2(\nabla\phi_h^0, \nabla\phi_h^0-\nabla\phi_h^1)\\
&= \|\nabla\phi_h^0\|^2-\|\nabla\phi_h^1\|^2+\|\nabla\phi_h^0-\nabla\phi_h^1\|^2\\
&\geq \|\nabla\phi_h^0\|^2-\|\nabla\phi_h^1\|^2. \qquad (3.6)\\
&((\phi_h^1)^3, \phi_h^1-\phi_h^0) = ((\phi_h^1)^3, \phi_h^1)-((\phi_h^1)^3, \phi_h^0)\\
&= \|\phi_h^1\|_{0,4}^4-((\phi_h^1)^2, \phi_h^1\phi_h^0)\\
&= \|\phi_h^1\|_{0,4}^4-\frac{1}{2}\|\phi_h^1\|_{0,4}^4-\frac{1}{2}\|\phi_h^1\phi_h^0\|^2\\
&\geq \frac{1}{2}\|\phi_h^1\|_{0,4}^4-\frac{1}{4}\|\phi_h^1\|_{0,4}^4-\frac{1}{4}\|\phi_h^0\|_{0,4}^4\\
&= \frac{1}{4}\|\phi_h^1\|_{0,4}^4-\frac{1}{4}\|\phi_h^0\|_{0,4}^4. \qquad (3.7)\\
&(1-\epsilon)(\phi_h^1, \phi_h^1-\phi_h^0) = \frac{1-\epsilon}{2}(\|\phi_h^1\|^2-\|\phi_h^0\|^2+\|\phi_h^1-\phi_h^0\|^2)\\
&\geq \frac{1-\epsilon}{2}\|\phi_h^1\|^2-\frac{1-\epsilon}{2}\|\phi_h^0\|^2 \qquad (3.8)
\end{aligned}
$$

Substituting (3.6), (3.6), (3.7) and (3.8) into the first equation of (3.5), we have

$$
\frac{1}{\triangle t}\|\phi_h^1-\phi_h^0\|^2+E(\phi_h^1, \omega_h^1)-E(\phi_h^0, \omega_h^0)\leq 0.
$$

Therefore, initial energy decay, $E(\phi_h^1, \omega_h^1)\leq E(\phi_h^0, \omega_h^0)$ is proved. But such a property is not available for $n\geq 1$, we define a modified energy for the analysis:

$$
\widetilde{E}(\phi_h^{n+1}, \omega_h^{n+1}) = E(\phi_h^{n+1}, \omega_h^{n+1})+\frac{1}{4\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2+\|\nabla(\phi_h^{n+1}-\phi_h^n)\|^2. \qquad (3.9)
$$

Theorem 3.2 *The discrete energy $E(\phi_h^n, \omega_h^n)$ decays at the initial step. And the modified energy $\widetilde{E}(\phi_h^n, \omega_h^n)$ has the following decay property:*

$$
\widetilde{E}(\phi_h^{n+1}, \omega_h^{n+1})\leq \widetilde{E}(\phi_h^n, \omega_h^n), \quad \forall n\geq 1, \qquad (3.10)
$$

provided that $A\geq \frac{1}{4}$.

Proof For $n\geq 1$, taking $\varphi_h = \phi_h^{n+1}-\phi_h^n$ in the first equation of (3.1) yields

$$
\begin{aligned}
0 = &\left(\frac{3\phi_h^{n+1}-4\phi_h^n+\phi_h^{n-1}}{2\triangle t}, \phi_h^{n+1}-\phi_h^n\right)\\
&+(\nabla\omega_h^{n+1}, \nabla\phi_h^{n+1}-\nabla\phi_h^n)-2(2\nabla\phi_h^n-\nabla\phi_h^{n-1}, \nabla\phi_h^{n+1}-\nabla\phi_h^n)\\
&+((\phi_h^{n+1})^3, \phi_h^{n+1}-\phi_h^n)+(1-\epsilon)(\phi_h^{n+1}, \phi_h^{n+1}-\phi_h^n)\\
&+A\triangle t(\nabla(\omega_h^{n+1}-\omega_h^n), \nabla(\phi_h^{n+1}-\phi_h^n))\\
:= &I_1+I_2+I_3+I_4+I_5+I_6. \qquad (3.11)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} I_1 &= \frac{1}{2\triangle t}(3(\phi_h^{n+1}-\phi_h^n)-(\phi_h^n-\phi_h^{n-1}),\phi_h^{n+1}-\phi_h^n) \\ &= \frac{3}{2\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2-\frac{1}{2\triangle t}(\phi_h^n-\phi_h^{n-1},\phi_h^{n+1}-\phi_h^n) \\ &\geq \frac{3}{2\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2-\frac{1}{2\triangle t}\left(\frac{1}{2}\|\phi_h^n-\phi_h^{n-1}\|^2+\frac{1}{2}\|\phi_h^{n+1}-\phi_h^n\|^2\right) \\ &= \frac{1}{\triangle t}\left(\frac{5}{4}\|\phi_h^{n+1}-\phi_h^n\|^2-\frac{1}{4}\|\phi_h^n-\phi_h^{n-1}\|^2\right) \\ &= \frac{1}{\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2+\frac{1}{4\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2-\frac{1}{4\triangle t}\|\phi_h^n-\phi_h^{n-1}\|^2. \end{aligned} \tag{3.12}$$

Likewise, I_3 and I_4 have the following lower bounds:

$$I_4 = ((\phi_h^{n+1})^3,\phi_h^{n+1}-\phi_h^n) = \|\phi_h^{n+1}\|_{0,4}^4-((\phi_h^{n+1})^2,\phi_h^{n+1}\phi_h^n) \geq \frac{1}{4}\|\phi_h^{n+1}\|_{0,4}^4-\frac{1}{4}\|\phi_h^n\|_{0,4}^4. \tag{3.13}$$

$$\begin{aligned} I_3 &= -2(2\nabla\phi_h^n-\nabla\phi_h^{n-1},\nabla\phi_h^{n+1}-\nabla\phi_h^n) \\ &= 2(\nabla\phi_h^n,\nabla\phi_h^n-\nabla\phi_h^{n+1})-2(\nabla\phi_h^n-\nabla\phi_h^{n-1},\nabla\phi_h^{n+1}-\nabla\phi_h^n) \\ &\geq \|\nabla\phi_h^n\|^2-\|\nabla\phi_h^{n+1}\|^2+\|\nabla\phi_h^n-\nabla\phi_h^{n+1}\|^2-\|\nabla\phi_h^n-\nabla\phi_h^{n-1}\|^2-\|\nabla\phi_h^{n+1}-\nabla\phi_h^n\|^2 \\ &= -\|\nabla\phi_h^{n+1}\|^2+\|\nabla\phi_h^n\|^2-\|\nabla\phi_h^n-\nabla\phi_h^{n-1}\|^2. \end{aligned} \tag{3.14}$$

For I_2, we employ the second part of (3.1) as well as the Cauchy-Schwarz inequality

$$\begin{aligned} I_2 &= (\nabla\omega_h^{n+1},\nabla(\phi_h^{n+1}-\phi_h^n)) = -(\omega_h^{n+1},\triangle_h(\phi_h^{n+1}-\phi_h^n)) \\ &= -(\nabla\phi_h^{n+1},\nabla\triangle_h(\phi_h^{n+1}-\phi_h^n)) = (\triangle_h\phi_h^{n+1},\triangle_h\phi_h^{n+1}-\triangle_h\phi_h^n) \\ &= \frac{1}{2}(\|\triangle_h\phi_h^{n+1}\|^2-\|\triangle_h\phi_h^n\|^2+\|\triangle_h\phi_h^{n+1}-\triangle_h\phi_h^n\|) \\ &\geq \frac{1}{2}\|\triangle_h\phi_h^{n+1}\|^2-\frac{1}{2}\|\triangle_h\phi_h^n\|^2. \end{aligned} \tag{3.15}$$

$$\begin{aligned} I_5 &= (1-\epsilon)(\phi_h^{n+1},\phi_h^{n+1}-\phi_h^n) = \frac{1-\epsilon}{2}(\|\phi_h^{n+1}\|^2-\|\phi_h^n\|^2+\|\phi_h^{n+1}-\phi_h^n\|^2) \\ &\geq \frac{1-\epsilon}{2}\|\phi_h^{n+1}\|^2-\frac{1-\epsilon}{2}\|\phi_h^n\|^2. \end{aligned} \tag{3.16}$$

In addition, making use of (2.5), the artificial term can be handled in the same manner

$$\begin{aligned} &I_6+\frac{1}{\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2 \\ &= A\triangle t(\nabla(\omega_h^{n+1}-\omega_h^n),\nabla(\phi_h^{n+1}-\phi_h^n))+\frac{1}{\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2 \\ &= A\triangle t\|\triangle_h(\phi_h^{n+1}-\phi_h^n)\|^2+\frac{1}{\triangle t}\|\phi_h^{n+1}-\phi_h^n\|^2 \\ &\geq 2|(\sqrt{A}\triangle_h(\phi_h^{n+1}-\phi_h^n),\phi_h^{n+1}-\phi_h^n)| \\ &= 2\sqrt{A}|(\nabla(\phi_h^{n+1}-\phi_h^n),\nabla(\phi_h^{n+1}-\phi_h^n))| \\ &= 2\sqrt{A}\|\nabla(\phi_h^{n+1}-\phi_h^n)\|^2 \end{aligned} \tag{3.17}$$

Therefore, a combination of (3.11–3.17) results in

$$\widetilde{E}(\phi_h^{n+1}, \omega_n^{n+1}) - \widetilde{E}(\phi_h^n, \omega_n^n) + (2\sqrt{A} - 1)\|\nabla(\phi_h^{n+1} - \phi_h^n)\|^2 \le 0.$$

provided that $A \ge \frac{1}{4}$. This completes the proof. □

3.3 The Optimal Error Estimate

In this subsection, we derive the optimal error estimate of the fully discrete system (3.1) and (3.2).

Theorem 3.3 *Let (ϕ^n, ω^n) and (ϕ_h^n, ω_h^n) be the solution of (2.2) in the regularity class $\mathcal{C}_1$ and (3.1–3.2) at time t_n, respectively, then we have the following error estimate*

$$\|\phi^n - \phi_h^n\| + \left(\triangle t \sum_{m=1}^{n} \|\omega^m - \omega_h^m\|^2\right)^{\frac{1}{2}} \le C_{\epsilon,T}(h^{q+1} + \triangle t^2), \tag{3.18}$$

for any $1 \le n \le N$, where $C_{\epsilon,T}$ is a constant that only depends on ϵ and T.

Proof The corresponding error equations for $n \ge 1$ become

$$\begin{aligned}(\delta_{\triangle t}^{n+1}\sigma_\phi, \varphi_h) &+ (\nabla\sigma_\omega^{n+1}, \nabla\varphi_h) + A\triangle t(\nabla(\sigma_\omega^{n+1} \\ &- \sigma_\omega^n), \nabla\varphi_h) = -(\delta_{\triangle t}^{n+1}\rho_\phi, \varphi_h) + 2(2\nabla\sigma_\phi^n - \nabla\sigma_\phi^{n-1}, \nabla\varphi_h) - (1-\epsilon)(\rho_\phi^{n+1} + \sigma_\phi^{n+1}, \varphi_h) \\ &+ (\mathcal{R}_1^{n+1}, \varphi_h) + A\triangle t(\mathcal{R}_2^{n+1}, \nabla\varphi_h) + 2(\mathcal{R}_3^{n+1}, \nabla\varphi_h) + (\mathcal{N}_1^{n+1}, \varphi_h) + (\mathcal{N}_2^{n+1}, \varphi_h),\end{aligned} \tag{3.19}$$

$$(\sigma_\omega^{n+1}, v_h) - (\nabla\sigma_\phi^{n+1}, \nabla v_h) = -(\rho_\omega^{n+1}, v_h). \tag{3.20}$$

where

$$\begin{aligned}\delta_{\triangle t}^{n+1}u &= \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\triangle t}, \quad \mathcal{R}_1^{n+1} = \delta_{\triangle t}^{n+1}\phi - \phi_t^{n+1}, \\ \mathcal{R}_2^{n+1} &= \nabla(\omega^{n+1} - \omega^n), \quad \mathcal{R}_3^{n+1} = \nabla(\phi^{n+1} - 2\phi^n + \phi^{n-1}), \\ \mathcal{N}_1^{n+1} &= (\phi_h^{n+1})^2\phi_h^{n+1} - |R_h\phi^{n+1}|^2 R_h\phi^{n+1}, \\ \mathcal{N}_2^{n+1} &= |R_h\phi^{n+1}|^2 R_h\phi^{n+1} - |\phi^{n+1}|^2\phi^{n+1}.\end{aligned}$$

And for $n = 0$, one has

$$\begin{aligned}&\left(\frac{\sigma_\phi^1 - \sigma_\phi^0}{\triangle t}, \varphi_h\right) + (\nabla\sigma_\omega^1, \nabla\varphi_h) - 2(\nabla\sigma_\phi^0, \nabla\varphi_h) \\ &- 2(\nabla\phi^1 - \nabla\phi^0, \nabla\varphi_h) + (1-\epsilon)(\rho_\phi^1 + \sigma_\phi^1, \varphi_h) \\ &= -\left(\frac{\rho_\phi^1 - \rho_\phi^0}{\triangle t}, \varphi_h\right) + \left(\frac{\phi^1 - \phi^0}{\triangle t} - \phi_t^1, \varphi_h\right) + (\mathcal{N}_1^1, \varphi_h) + (\mathcal{N}_2^1, \varphi_h),\end{aligned} \tag{3.21}$$

$$(\sigma_\omega^1, v_h) - (\nabla\sigma_\phi^1, \nabla v_h) = -(\rho_\omega^1, v_h). \tag{3.22}$$

with

$$\begin{aligned}\mathcal{N}_1^1 &= (\phi_h^1)^2\phi_h^1 - |R_h\phi^1|^2 R_h\phi^1, \\ \mathcal{N}_2^1 &= |R_h\phi^1|^2 R_h\phi^1 - |\phi^1|^2\phi^1.\end{aligned}$$

We first consider the case for $n \geq 1$. Taking $\varphi_h = \sigma_\phi^{n+1}$ in (3.19), $\upsilon_h = \triangle_h \sigma_\phi^{n+1}$ in (3.20) and adding up the two equations lead to

$$\begin{aligned}
&(\delta_{\triangle t}^{n+1}\sigma_\phi, \sigma_\phi^{n+1}) + \|\triangle_h \sigma_\phi^{n+1}\|^2 + A\triangle t(\nabla(\sigma_\omega^{n+1} - \sigma_\omega^n), \nabla \sigma_\phi^{n+1}) \\
&= -(\delta_{\triangle t}^{n+1}\rho_\phi, \sigma_\phi^{n+1}) + 2(\nabla(2\sigma_\phi^n \\
&\quad -\sigma_\phi^{n-1}), \nabla\sigma_\phi^{n+1}) - (1-\epsilon)(\rho_\phi^{n+1} + \sigma_\phi^{n+1}, \sigma_\phi^{n+1}) + (\mathcal{R}_1^{n+1}, \sigma_\phi^{n+1}) \\
&\quad + A\triangle t(\mathcal{R}_2^{n+1}, \nabla\sigma_\phi^{n+1}) + 2(\mathcal{R}_3^{n+1}, \nabla\sigma_\phi^{n+1}) \\
&\quad + (\mathcal{N}_1^{n+1}, \sigma_\phi^{n+1}) + (\mathcal{N}_2^{n+1}, \sigma_\phi^{n+1}) - (\rho_\omega^{n+1}, \triangle_h\sigma_\phi^{n+1})
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&(\mathcal{N}_1^{n+1}, \sigma_\phi^{n+1}) \\
&= ((\phi_h^{n+1})^2\phi_h^{n+1} - |R_h\phi^{n+1}|^2 R_h\phi^{n+1}, R_h\phi^{n+1} - \phi_h^{n+1}) \\
&= ((\phi_h^{n+1})^2\phi_h^{n+1}, R_h\phi^{n+1} - \phi_h^{n+1}) - (|R_h\phi^{n+1}|^2 R_h\phi^{n+1}, R_h\phi^{n+1} - \phi_h^{n+1}) \\
&= -\|\phi_h^{n+1}\|_{0,4}^4 - \|R_h\phi^{n+1}\|_{0,4}^4 + ((\phi_h^{n+1})^2\phi_h^{n+1}, R_h\phi^{n+1}) + (|R_h\phi^{n+1}|^2 R_h\phi^{n+1}, \phi_h^{n+1}) \\
&\le -\frac{1}{2}\|\phi_h^{n+1}\|_{0,4}^4 - \frac{1}{2}\|R_h\phi^{n+1}\|_{0,4}^4 \\
&\quad + (|R_h\phi^{n+1}|^2, |\phi_h^{n+1}|^2) \\
&\le 0.
\end{aligned} \tag{3.24}$$

Besides, the estimate (2.12) implies that

$$\begin{aligned}
|(\mathcal{N}_2^{n+1}, \sigma_\phi^{n+1})| &\le \frac{1}{C_3}\left(\frac{1}{2} + \frac{3}{2}\|\phi^{n+1}\|_{0,\infty} + \frac{3}{2}\|\phi^{n+1}\|_{0,\infty}^2\right)\|\sigma_\phi^{n+1}\|^2 \\
&\quad + \frac{3}{2}C^2C_3\|\phi^{n+1}\|_{0,\infty}^2 h^{2q+2}\|\phi^{n+1}\|_{q+1}^2 \\
&\quad + \frac{3}{2}\|\phi^{n+1}\|_{0,\infty}C^4C_3h^{4q+4}\|\phi^{n+1}\|_{q+1,4}^4 + \frac{1}{2}C^6C_3h^{6q+6}\|\phi^{n+1}\|_{q+1,6}^6.
\end{aligned} \tag{3.25}$$

From (2.5) and (3.20), together with the inequality

$$\begin{aligned}
\|\rho_\omega^{n+1} - \rho_\omega^n\|^2 &= \|\int_{t_n}^{t_{n+1}} \partial_t\rho_\omega \mathrm{d}t\|^2 \le \left(\int_{t_n}^{t_{n+1}}\|\rho_{\omega_t}\|\mathrm{d}t\right)^2 \le \int_{t_n}^{t_{n+1}}\|\rho_{\omega_t}\|^2\mathrm{d}t\cdot\triangle t \\
&\le C^2h^{2q+2}\triangle t\int_{t_n}^{t_{n+1}}\|\omega_t\|_{q+1}^2\mathrm{d}t,
\end{aligned}$$

one gets

$$\begin{aligned}
A\triangle t(\nabla(\sigma_\omega^{n+1} - \sigma_\omega^n), \nabla\sigma_\phi^{n+1}) &= -A\triangle t(\nabla(\sigma_\phi^{n+1} - \sigma_\phi^n), \nabla\triangle_h\sigma_\phi^{n+1}) + A\triangle t(\rho_\omega^{n+1} \\
&\quad - \rho_\omega^n, \triangle_h\sigma_\phi^{n+1}) \\
&\ge \frac{A\triangle t}{2}(\|\triangle_h\sigma_\phi^{n+1}\|^2 - \|\triangle_h\sigma_\phi^n\|^2) - \frac{1}{2C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2 \\
&\quad - \frac{1}{2}C^2A^2C_4(\triangle t)^3h^{2q+2}\int_{t_n}^{t_{n+1}}\|\omega_t\|_{q+1}^2\mathrm{d}t.
\end{aligned} \tag{3.26}$$

Applying (2.5) and the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
2(\nabla(2\sigma_\phi^n-\sigma_\phi^{n-1}),\nabla\sigma_\phi^{n+1}) &\le 2|(2\sigma_\phi^n-\sigma_\phi^{n-1},\triangle_h\sigma_\phi^{n+1})|\\
&\le \frac{1}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+4C_4\|2\sigma_\phi^n-\sigma_\phi^{n-1}\|^2\\
&\le \frac{1}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+4C_4(4\|\sigma_\phi^n\|^2+\|\sigma_\phi^{n-1}\|^2-2(\sigma_\phi^n,\sigma_\phi^{n-1}))\\
&\le \frac{1}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+4C_4(4\|\sigma_\phi^n\|^2+\|\sigma_\phi^{n-1}\|^2+\|\sigma_\phi^n\|^2+\|\sigma_\phi^{n-1}\|^2)\\
&\le \frac{1}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+20C_4(\|\sigma_\phi^n\|^2+\|\sigma_\phi^{n-1}\|^2). \qquad (3.27)
\end{aligned}
$$

To analyze the other several terms, we resort to the Cauchy–Schwarz inequality and the Taylor expansion:

$$
\begin{aligned}
-(\delta_{\triangle t}^{n+1}\rho_\phi,\sigma_\phi^{n+1}) &\le \frac{1}{2C_3}\|\sigma_\phi^{n+1}\|^2+\frac{C_3}{2}\|\delta_{\triangle t}^{n+1}\rho_\phi\|^2\\
&\le \frac{1}{2C_3}\|\sigma_\phi^{n+1}\|^2+\frac{9C_3h^{2q+2}}{4\triangle t}\int_{t_{n-1}}^{t_{n+1}}\|\phi_t\|_{q+1}^2\mathrm{d}t, \qquad (3.28)\\
(\mathcal{R}_1^{n+1},\sigma_\phi^{n+1}) &\le \frac{1}{2C_3}\|\sigma_\phi^{n+1}\|^2+\frac{C_3(\triangle t)^3}{2}\int_{t_{n-1}}^{t_{n+1}}\|\phi_{ttt}\|^2\mathrm{d}t+\mathcal{O}(\triangle t^4), \qquad (3.29)\\
A\triangle t(\mathcal{R}_2^{n+1},\nabla\sigma_\phi^{n+1}) &\le \frac{\widehat{C_3}}{2}\|\nabla\sigma_\phi^{n+1}\|^2+\frac{A^2(\triangle t)^3}{2\widehat{C_3}}\int_{t_n}^{t_{n+1}}\|\nabla\omega_t\|^2\mathrm{d}t, \qquad (3.30)\\
2(\mathcal{R}_3^{n+1},\nabla\sigma_\phi^{n+1}) &\le \widehat{C_3}\|\nabla\sigma_\phi^{n+1}\|^2+\frac{2(\triangle t)^3}{\widehat{C_3}}\int_{t_{n-1}}^{t_{n+1}}\|\nabla\phi_{tt}\|^2\mathrm{d}t+\mathcal{O}(\triangle t^4), \qquad (3.31)\\
-(\rho_\omega^{n+1},\triangle_h\sigma_\phi^{n+1}) &\le \frac{1}{2C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+\frac{C_4C^2h^{2q+2}}{2}\|\omega^{n+1}\|_{q+1}^2, \qquad (3.32)\\
-(1-\epsilon)(\rho_\phi^{n+1}+\sigma_\phi^{n+1},\sigma_\phi^{n+1}) &\le \frac{|1-\epsilon|}{C_3}\|\sigma_\phi^{n+1}\|^2+\frac{C_3|1-\epsilon|C^2}{2(1-C_3^2)}h^{2q+2}\|\phi^{n+1}\|_{q+1}^2. \qquad (3.33)
\end{aligned}
$$

Recall the G-norm introduced in [3]. Denote $\mathbf{p}^{k+1}=[\sigma_\phi^k,\sigma_\phi^{k+1}]^T$, and define $\|\mathbf{p}^{k+1}\|_{\mathbf{G}}^2:=(\mathbf{p}^{k+1},\mathbf{G}\mathbf{p}^{k+1})$ where $\mathbf{G}=\begin{pmatrix}\frac{1}{2} & -1\\ -1 & \frac{5}{2}\end{pmatrix}$ is a positive definite symmetric matrix. Simple calculation gives

$$
\begin{aligned}
(\delta_{\triangle t}^{n+1}\sigma_\phi,\sigma_\phi^{n+1}) &= \frac{1}{2\triangle t}(\|\mathbf{p}^{n+1}\|_{\mathbf{G}}^2-\|\mathbf{p}^n\|_{\mathbf{G}}^2)\\
&\quad+\frac{1}{4\triangle t}\|\sigma_\phi^{n+1}-2\sigma_\phi^n+\sigma_\phi^{n-1}\|^2. \qquad (3.34)
\end{aligned}
$$

Upon this, in combination with the above estimates for (3.23), one has

$$
\begin{aligned}
&\frac{1}{2\triangle t}(\|\mathbf{p}^{n+1}\|_{\mathbf{G}}^2-\|\mathbf{p}^n\|_{\mathbf{G}}^2)+\|\triangle_h\sigma_\phi^{n+1}\|^2+\frac{A\triangle t}{2}(\|\triangle_h\sigma_\phi^{n+1}\|^2-\|\triangle_h\sigma_\phi^n\|^2)\\
&=\left(\frac{3}{2C_3}+\frac{|1-\epsilon|}{C_3}+\frac{3}{2C_3}\|\phi^{n+1}\|_{0,\infty}+\frac{3}{2C_3}\|\phi^{n+1}\|_{0,\infty}^2\right)\|\sigma_\phi^{n+1}\|^2\\
&\quad+\frac{5}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2+\frac{3\widehat{C_3}}{2}\|\nabla\sigma_\phi^{n+1}\|^2+\frac{1}{2}C^6C_3h^{6q+6}\|\phi^{n+1}\|_{q+1,6}^6+\frac{3}{2}\|\phi^{n+1}\|_{0,\infty}C_3C^4h^{4q+4}\|\phi^{n+1}\|_{q+1,4}^4\\
&\quad+\frac{3}{2}C^2C_3\|\phi^{n+1}\|_{0,\infty}^2h^{2q+2}\|\phi^{n+1}\|_{q+1}^2\\
&\quad+\frac{C^2A^2C_4}{2}(\triangle t)^3h^{2q+2}\int_{t_n}^{t_{n+1}}\|\omega_t\|_{q+1}^2\mathrm{d}t+\frac{9C_3h^{2q+2}}{4\triangle t}\int_{t_{n-1}}^{t_{n+1}}\|\phi_t\|_{q+1}^2\mathrm{d}t+\frac{1}{2}C_3(\triangle t)^3\int_{t_{n-1}}^{t_{n+1}}\|\phi_{ttt}\|^2\mathrm{d}t\\
&\quad+\frac{A^2}{2\widehat{C_3}}(\triangle t)^3\int_{t_n}^{t_{n+1}}\|\nabla\omega_t\|^2\mathrm{d}t+\frac{2(\triangle t)^3}{\widehat{C_3}}\int_{t_{n-1}}^{t_{n+1}}\|\nabla\phi_{tt}\|^2\mathrm{d}t+\frac{C_4C^2}{2}h^{2q+2}\|\omega^{n+1}\|_{q+1}^2\\
&\quad+\frac{C_3|1-\epsilon|}{2(1-C_3^2)}C^2h^{2q+2}\|\phi^{n+1}\|^2+20C_4(\|\sigma_\phi^n\|^2+\|\sigma_\phi^{n-1}\|^2).
\end{aligned}\tag{3.35}
$$

Noticing that $\|\mathbf{p}^{n+1}\|_{\mathbf{G}}^2\geq\frac{1}{2}\|\sigma_\phi^{n+1}\|^2$ and $\|\mathbf{p}^1\|_{\mathbf{G}}^2=\frac{5}{2}\|\sigma_\phi^1\|^2$, take $\widehat{C}_3=\frac{1}{\sqrt{C_3C_4}}$, then $\frac{3\widehat{C_3}}{2}\leq\frac{3}{4C_3}\|\sigma_\phi^{n+1}\|^2+\frac{3}{4C_4}\|\triangle_h\sigma_\phi^{n+1}\|^2$. Multiplying equation (3.35) by $2\triangle t$, summing up for n, we obtain

$$
\begin{aligned}
&(1-C_5\triangle t)\|\sigma_\phi^{n+1}\|^2+(4-\frac{8}{C_4})\triangle t\sum_{m=1}^{n}\|\triangle_h\sigma_\phi^m\|^2\\
&\quad\leq C_{\epsilon,T}(h^{6q+6}+h^{4q+4}+h^{2q+2}+\triangle t^4)+(160C_4+C_5)\triangle t\sum_{m=1}^{n}\|\sigma_\phi^m\|^2+5\|\sigma_\phi^1\|^2,
\end{aligned}\tag{3.36}
$$

where $C_5=\frac{1}{C_3}(9+|1-\epsilon|+6\|\phi\|_{L^\infty(0,T,L^2)}+6\|\phi\|_{L^\infty(0,T,L^2)}^2)$.
Next, we turn to the case $n=0$. Taking $\varphi_h=\sigma_\phi^1$, $v_h=\triangle_h\sigma_\phi^1$ in (3.21) and (3.22) and adding the equations up, we get

$$
\begin{aligned}
&\left(\frac{\sigma_\phi^1-\sigma_\phi^0}{\triangle t},\sigma_\phi^0\right)+\|\triangle_h\sigma_\omega^1\|^2=2(\nabla\sigma_\phi^0,\nabla\sigma_\phi^1)+2(\nabla\phi^1\\
&\quad-\nabla\phi^0,\nabla\sigma_\phi^1)-(1-\epsilon)(\rho_\phi^1+\sigma_\phi^1,\sigma_\phi^1)\\
&\quad-(\rho_\omega^1,\triangle_h\sigma_\phi^1)-\left(\frac{\rho_\phi^1-\rho_\phi^0}{\triangle t},\sigma_\phi^1\right)+\left(\frac{\phi^1-\phi^0}{\triangle t}-\phi_t^1,\sigma_\phi^1\right)\\
&\quad+(\mathcal{N}_1^1,\sigma_\phi^1)+(\mathcal{N}_2^1,\sigma_\phi^1).
\end{aligned}\tag{3.37}
$$

Similarly, we have the estimate

$$
(1-C_6)\|\sigma_\phi^1\|^2+\frac{1}{2}\triangle t\|\triangle_h\sigma_\phi^1\|^2\leq C_{\epsilon,T}(h^{6q+6}+h^{4q+4}+h^{2q+2}+\triangle t^4),\tag{3.38}
$$

where $C_6=\frac{1}{C_3}(3+3|1-\epsilon|\triangle t+2\triangle t+3\triangle t\|\phi^1\|_{0,\infty}+3\triangle t\|\phi^1\|_{0,\infty}^2)$.
Combining (3.36) and (3.38) and using the discrete Gronwall inequality yields

$$
\|\sigma_\phi^{n+1}\|^2+\triangle t\sum_{m=1}^{n}\|\triangle_h\sigma_\phi^{m+1}\|^2\leq C_{\epsilon,T}(h^{2q+2}+\triangle t^4).\tag{3.39}
$$

Table 1 The errors and order of convergence at $T = 1.0E-5$ for the density field ϕ with different mesh size

	h	L^2 error	order		h	L^2 error	order
$\mathcal{P}_1$ element	1/4	1.41455E-1	–	$\mathcal{P}_2$ element	1/4	1.26626E-2	–
	1/8	3.87441E-2	1.86829		1/8	1.71664E-3	2.88291
	1/16	9.88649E-3	1.97045		1/16	2.25398E-4	2.92904
	1/32	2.48480E-3	1.99233		1/32	2.86915E-5	2.97378
	1/64	6.22093E-4	1.99793		1/64	3.61061E-6	2.99031

The time step is $\triangle t = 1.0E-7$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Table 2 The errors and order of convergence at $T = 1$ for the density field ϕ with different mesh size

	h	L^2 error	order		h	L^2 error	order
$\mathcal{P}_1$ element	1/4	1.19445E-1	–	$\mathcal{P}_2$ element	1/4	7.28731E-3	-
	1/8	3.86876E-2	1.62641		1/8	7.76787E-4	3.22980
	1/16	1.04641E-2	1.88643		1/16	8.83431E-5	3.13633
	1/32	2.67306E-3	1.96888		1/32	1.07337E-5	3.04097
	1/64	6.72073E-4	1.99180		1/64	1.33395E-6	3.00837

The time step is $\triangle t = h^2$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Table 3 The errors and order of convergence at $T = 1$ for the density field ϕ with different time step

	$\triangle t = h$	L^2 error	order		$\triangle t = h$	L^2 error	order
$\mathcal{P}_1$ element	1/4	1.25080E-1	–	$\mathcal{P}_2$ element	1/4	2.33766E-2	-
	1/8	3.56815E-2	1.80960		1/8	6.06920E-3	1.94549
	1/16	9.49042E-3	1.91063		1/16	1.50024E-3	2.01632
	1/32	2.41639E-3	1.97362		1/32	3.71666E-4	2.01311
	1/64	6.07296E-4	1.99238		1/64	9.23652E-5	2.00859
	1/128	1.52063E-4	1.99773		1/128	2.30133E-5	2.00488

The mesh size $\triangle t = h$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Using the same arguments as in the last part of Theorem 2.1 and combining with (3.39) make (3.18). □

4 Numerical Experiments

4.1 Convergence and Energy Stability Test

In this subsection we first present some numerical tests to check the theoretical convergence of the proposed scheme (3.1–3.2). We implemented the codes using the software package FreeFem++ [19]. Firstly, we set $\Omega = (0, 1)^2$ and $\epsilon = 0.5$. The exact solution is given by

$$\phi_e(x, y, t) = \cos(\pi x)\cos(2\pi y)e^{-t}. \tag{4.1}$$

Table 4 The errors and order of convergence at $T = 2$ for the density field ϕ with different time step

	h	$\triangle t$	L^2 error	order		h	$\triangle t$	L^2 error	order
$\mathcal{P}_1$ element	1/4	2/4	4.21841E-2	–	$\mathcal{P}_2$ element	1/4	2/4	3.63191E-3	–
	1/8	2/8	1.42875E-2	1.56195		1/8	2/8	4.52130E-4	3.00592
	1/16	2/16	3.87900E-3	1.88100		1/16	2/16	6.83474E-5	2.72578
	1/32	2/32	9.92028E-4	1.96723		1/32	2/32	1.31688E-5	2.37577
	1/64	2/64	2.49452E-4	1.99162		1/64	2/64	2.97177E-6	2.14772
	1/128	2/128	6.24528E-5	1.99792		1/128	2/128	7.16547E-7	2.05219

The mesh size $\triangle t = 5h$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Table 5 The errors and order of convergence at $T = 5$ for the density field ϕ with different time step

	h	$\triangle t$	L^2 error	order		h	$\triangle t$	L^2 error	order
$\mathcal{P}_1$ element	1/4	5/4	3.50417E-3	–	$\mathcal{P}_2$ element	1/4	5/4	8.82170E−4	–
	1/8	5/8	7.74662E−4	2.17743		1/8	5/8	1.12410E−4	2.97229
	1/16	5/16	2.02811E−4	1.93343		1/16	5/16	1.96682E−5	2.51483
	1/32	5/32	5.18762E−5	1.96699		1/32	5/32	4.11619E−6	2.25649
	1/64	5/64	1.30162E−5	1.99476		1/64	5/64	9.45032E−7	2.12287
	1/128	5/128	3.25376E−6	2.00013		1/128	5/128	2.26759E−7	2.05921

The mesh size $\triangle t = 2h$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Table 6 The errors and order of convergence at $T = 10$ for the density field ϕ with different time step

	h	$\triangle t$	L^2 error	order		h	$\triangle t$	L^2 error	order
$\mathcal{P}_1$ element	1/4	10/4	1.83469E-3	–	$\mathcal{P}_2$ element	1/4	10/4	2.37972E−4	–
	1/8	10/8	5.43928E−5	5.07597		1/8	10/8	5.33013E−6	5.48048
	1/16	10/16	1.76650E−6	4.94445		1/16	10/16	7.05961E−7	2.91651
	1/32	10/32	4.28626E−7	2.04311		1/32	10/32	1.28952E−7	2.45275
	1/64	10/64	1.04643E−7	2.03424		1/64	10/64	2.75001E−8	2.22933
	1/128	10/128	2.58095E−8	2.01951		1/128	10/128	6.35261E-9	2.11402

The mesh size $\triangle t = 10h$, $A = 2$ and the physical parameter is $\epsilon = 0.5$

Next, in order to satisfy the PDE (1.2) and boundary conditions $\partial_n \phi = \partial_n(\triangle\omega) = 0$, where $\omega = -\triangle\phi$, we add an artificial, time-dependent forcing term on the right hand side:

$$\partial_t \phi_e + \phi_e^3 + (1-\epsilon)\phi_e + 2\triangle\phi_e + \triangle^2\phi_e = f, \quad (x, y, t) \in \Omega \times (0, T],$$

with $f = (25\pi^4 - 10\pi^2 - \epsilon)\cos(\pi x)\cos(2\pi y)e^{-t} + \cos^3(\pi x)\cos^3(2\pi y)e^{-3t}$. The problem is solved using the scheme (3.1–3.2). Numerical tests are running on spatial meshes with characteristic size $h = 1/4$,$1/8$, $1/16$, $1/32$, $1/64$, $1/128$. Both $\mathcal{P}_1$ and $\mathcal{P}_2$ elements are used in the spatial discretization. We have verified numerically that the stabilized term constant A has no effect on the convergence order of the algorithm, so, without loss of generality, we set $A = 2$ in the following convergence order tests.

Firstly, we test the convergence order with respect to the spatial grid size h. The time step length $\triangle t$ should be chosen small enough so that the time discretization error is negligible

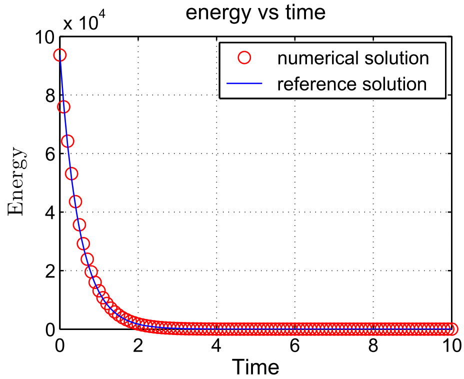

Fig. 1 The plot of energy evolution of numerical solution and exact solution

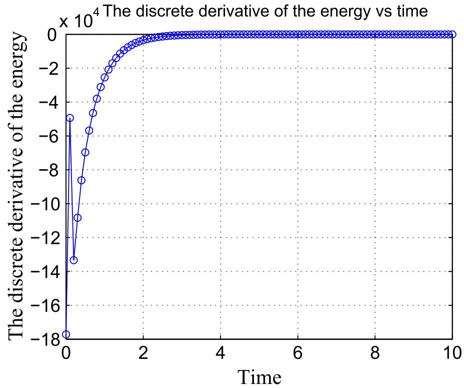

Fig. 2 The evolution of the discrete derivative of the energy of numerical solution

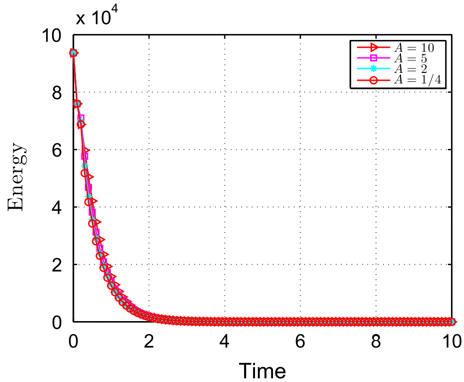

Fig. 3 The plot of energy evolution with different parameter A. where $h = 1/16$, $\epsilon = 0.5$, $\triangle t = 0.1$

compared with the spatial discretization error. Therefore, we take $\triangle t = 1 \times 10^{-7}$, the final time $T = 1 \times 10^{-5}$. The L^2 errors of the phase variable ϕ between the exact solution and the numerical solution are listed in Table 1, which shows the optimal convergence rates of $\mathcal{P}_1$ element and $\mathcal{P}_2$ element in L^2 norm. We can also set $\triangle t = h^2$ and $T = 1$, we expect theoretically a convergence rate of $O(h^{q+1} + h^4)$. In the case, the spatial approximation error dominates, and we expect convergence rates of $O(h^2)$ for $\mathcal{P}_1$ element and $O(h^3)$ for $\mathcal{P}_2$ element. Numerical results given in Table 2 agree well with the exceptions.

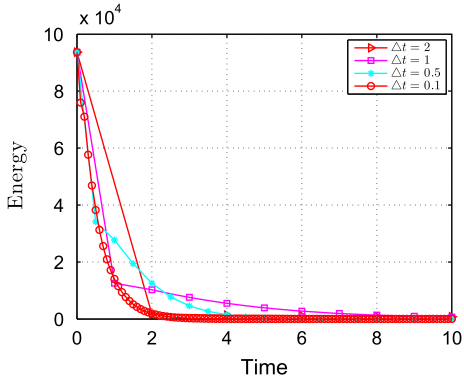

Fig. 4 The plot of energy evolution with different time step size $\triangle t$. where $h = 1/16$, $\epsilon = 0.5$, $A = 2$

Then, we test the convergence order with respect to the time step length $\triangle t$. The strategy used in testing the spatial convergence order is a straight way, but taking h too small leads to a massive increasing in computational efforts. Here, we test the rate of convergence in time in another way. We set the time step size $\triangle t = Ch$, in which C is a constant, for both the $\mathcal{P}_1$ element and the $\mathcal{P}_2$ element cases, thus at the final time, we expect error $O(\triangle t^2) + O(h^2) = O(\triangle t^2)$ and $O(\triangle t^2) + O(h^3) = O(\triangle t^2)$ as $h \to 0$. Without loss of generality, we take $\triangle t = h, \triangle t = 2h, \triangle t = 5h, \triangle t = 10h$, respectively. The corresponding results are displayed in Tables 3–6, which are consistent with our theoretical analysis.

We plot the time evolution of the energy functional of numerical solution and exact solution in Fig. 1 and the evolution of the discrete derivative of the energy of numerical solution in Fig. 2, from which we could find that the energy is non-increasing. It demonstrates that the proposed scheme is unconditionally energy stable. Figure 3 shows that the energy decay is robust with respect to the parameter A. Figure 4 shows the energy evolution with different time step size $\triangle t$.

4.2 Phase Transition Behaviors

In this subsection, we apply the proposed scheme to check the evolution from a randomly perturbed non-equilibrium state to a steady state on the computational domain $(-15, 15) \times (-15, 15)$. With the initial value condition $\phi^0 = \overline{\phi}$+rand, where $\overline{\phi} = 0.4$ and rand is a randomly chosen number between -0.02 and 0.02 at the grid points. Let the time step be $\triangle t = 0.1$, the spatial grid size be $h = 1/8$, $A = 2$ and the parameter be $\epsilon = 0.1$. Figure 5 shows the time evolution of the phase transition behavior, which validates that our proposed scheme does lead to the expected states.

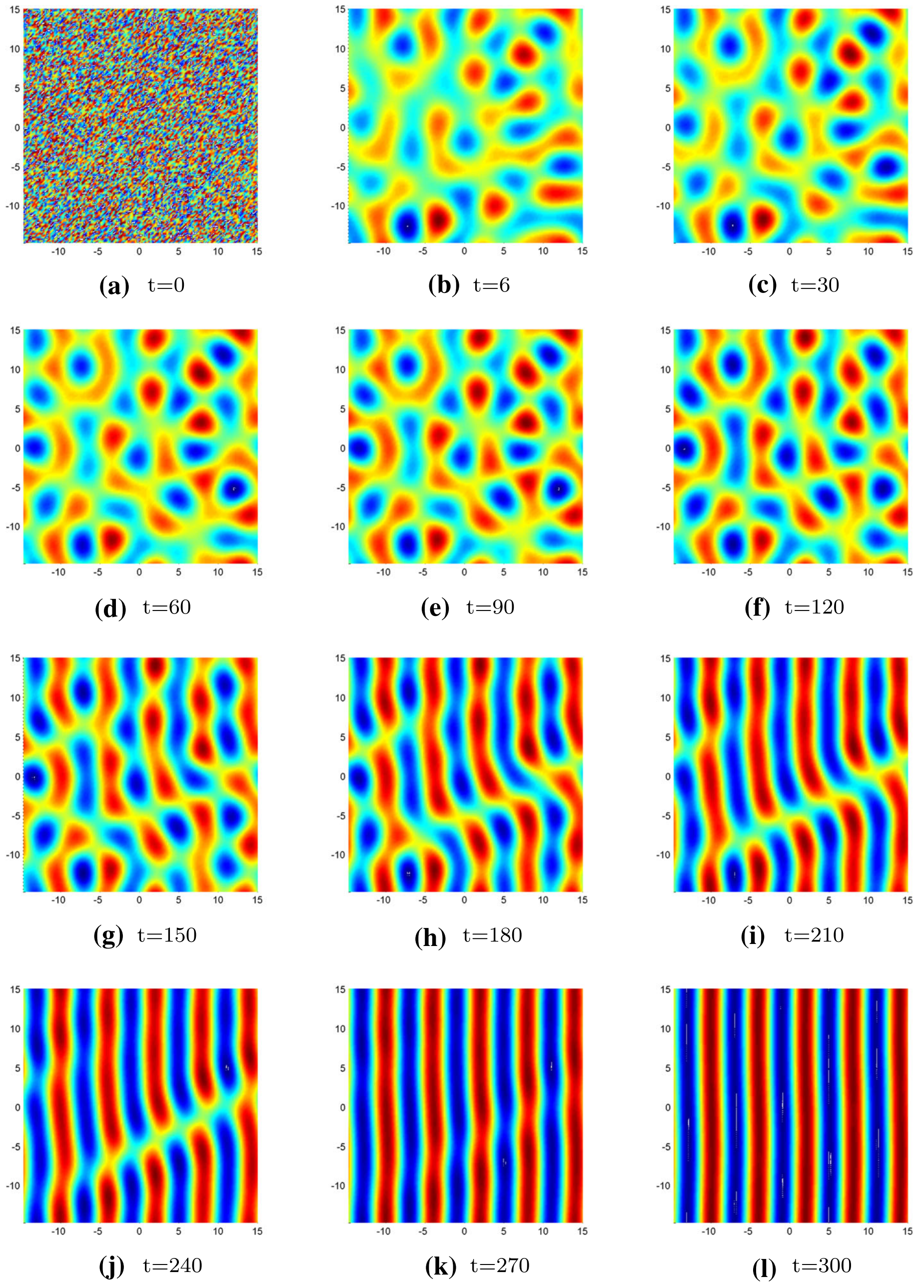

Fig. 5 The evolution of the phase transition behavior in 2D with $\overline{\phi} = 0.4$. Snapshots of the numerical approximation of the density field ϕ are taken at $t = 0, 6, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300$. The computational domain is $(-15, 15) \times (-15, 15)$. The parameters are $\epsilon = 0.1$, $A = 2$, $T = 300$. The time step is $\triangle t = 0.1$, the spatial grid size is $h = 1/8$

5 Concluding Remarks

In this paper, we have proposed and analyzed a mixed finite element method with modified second-order backward differentiation formula for solving the Swift–Hohenberg equation. The unconditional energy stability and unconditional unique solvability have been established, and an optimal convergence rate $O(h^{q+1} + \triangle t^2)$ has been proved. Furthermore, the corresponding numerical tests have been undertaken to verify the theoretical analysis.

Data availability statement The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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