Numerical Algorithms (2021) 88:1137–1182
https://doi.org/10.1007/s11075-021-01070-4

ORIGINAL PAPER

A second-order decoupled algorithm with different subdomain time steps for the non-stationary Stokes/Darcy model

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Received: 24 August 2020 / Accepted: 7 January 2021 / Published online: 17 February 2021

Abstract
In this paper, we propose and analyze a second-order decoupled algorithm with different subdomain time steps for the non-stationary Stokes/Darcy model. It is based on the second-order spectral deferred correction method in time and the finite element method in space. We provide the stability and convergence results of our decoupled scheme. Last, some numerical experiments are given to illustrate the accuracy and effectiveness of our decoupled scheme.

Keywords Stokes/Darcy model · Decoupled algorithm · Different time steps · Spectral deferred correction

1 Introduction

There are various applications involving different physical processes in different regions of a simulation domain, such as the interaction between the surface and subsurface flows and blood motion in vessels. We are interested in the non-stationary mixed Stokes/Darcy model [1, 3, 12, 20, 26] for the coupled fluid flow and porous media flow, which are coupled with certain interface conditions. In order to solve this kind multi-modeling problems, one can solve the coupled problem directly, otherwise one can use the decoupled methods to solve these models individually. In [25], the authors pointed out the appealing reasons of using the decoupled numerical methods. As a matter of fact, there are many decoupled methods for the Stokes/Darcy

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equations, such as the domain decomposition [8, 9, 17], the two-grid method [10, 18, 25, 34], and the Lagrange multiplier approach [27].

Now, we focus on the decoupled scheme with the multiple-time-step technique for the non-stationary mixed Stokes/Darcy model. The reason this technique can be implemented is that the flows in the fluid region and the porous media region own different physical processes and characteristic speeds. More precisely, the fluid velocity from the porous media flow is much smaller than that from the fluid flow. So compared to the fluid region, we can choose a large time step in the porous media region to improve its efficiency. In fact, this kind decoupled method, which was partitioned time stepping, was first given by Connors in [6] for the atmosphere-ocean coupling. Since then, this kind of multiple-time-step method is further analyzed and applied, [7, 29–32]. This multirate decoupled scheme for the non-stationary Stokes/Darcy model is presented in [31], which allows different time steps in the fluid and porous domains. In [29], the authors developed a mass conservative multirate time integration method, which is an improved version and owns the long time stability. But, no matter which is specified, the multiple-time-step decoupled method is proved to be effective.

In order to construct a second-order algorithm for the Stokes/Darcy model, we will use the spectral deferred correction (SDC) method, which is one kind of high-order numerical methods for differential equations. It was first proposed for the stiff ordinary differential equations in [11]. After that this method is further developed in different variants [4, 5, 22–24] and widely applied in different problems [13–15, 19, 21, 28, 33]. The main advantage of SDC method compared with other high-order numerical methods is that one can use a lower order numerical method to get a numerical solution with high order accuracy by solving a series of deferred correction equations during each time step. What's more, since SDC method starts with the approximation of the equivalent Picard equation not the differential equations directly, it avoids the instabilities associated with repeated differentiation. In this sense, the SDC method is more stable than other high-order methods.

In this paper, combining the multiple-time-step technique and SDC method, we will construct a second-order algorithm for the non-stationary Stokes/Darcy model. In the meantime, we will update the interface terms explicitly to decouple this model. In this case, our decoupled scheme will need two different size time steps: the small time step size Δt for the Stokes equations and the large time step size Δs with the integer ration $r = \Delta s/\Delta t$ for the Darcy equations. What's more, in order to apply the second-order SDC method, the first step we need to do is applying the first-order coupled backward Euler method with the small time step size for the Stokes/Darcy model to obtain the temporary numerical solutions over each large time interval. It is the only time we need to solve the coupled scheme and this step is to prepare for the next decoupled scheme. Then, let us start our decoupled scheme. It is worth noting that these approximations for the sub-problems are based on the second-order SDC method, and the detailed idea of SDC method based on Euler method can be found in [11, 22]. Since we want to construct a second-order scheme here, we will use the trapezoid formula to achieve the numerical quadrature approximation. Next, we are going to pay more attention to the way of how to express the interface term explicitly to decouple the Stokes/Darcy model. Over each large time interval, we first need to approximate the Stokes problem at each small time step, whereby utilizing the second-order solutions at the previous large time level to approximate the interface terms explicitly. Next, we will approximate the Darcy equations at the large time step, and we explicitly update the interface term by computing the average fluid velocity. Then, let us repeat the above decoupled steps to get the numerical solutions over the next time interval.

The remainder of this paper is organized as follows. Some preliminary notations and results are presented in Section 2. Section 3 provides the second-order decoupled scheme with different subdomain time steps for the non-stationary mixed Stokes/Darcy model, as well as the second-order coupled scheme with the same time steps. In Section 6, we will compare the numerical results of the two schemes to verify the efficiency of our decoupled scheme. The stability and convergence results of the decoupled scheme are provided in Sections 4 and 5, respectively. Numerical experiments in Section 6 are reported to support our theoretical analysis results. In the final section some conclusions are drawn.

2 Preliminaries

Let us consider a fluid flow in Ω_f coupled with a porous media flow in Ω_p, where $\Omega_f, \Omega_p \subset \mathbb{R}^d$, $d = 2$ or 3 are bounded domains, $\Omega_f \cap \Omega_p = \emptyset$, $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. Denote by $\boldsymbol{n}_f$ and $\boldsymbol{n}_p$ the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, respectively. Note that $\boldsymbol{n}_f = -\boldsymbol{n}_p$ on Γ. And $\boldsymbol{\tau}_i,\ i = 1, \ldots, d-1$, is the unit tangential vectors on the interface Γ (see Fig. 1).

Let $T > 0$ be a finite time. The fluid flow in the fluid region Ω_f is governed by the Stokes equations for the fluid velocity $\boldsymbol{u}$ and kinematic pressure p:

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}_1, & \text{in } \Omega_f \times (0, T], \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega_f \times (0, T], \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}^0, & \text{in } \Omega_f, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega_f \backslash \Gamma \times (0, T]. \end{cases} \tag{1}$$

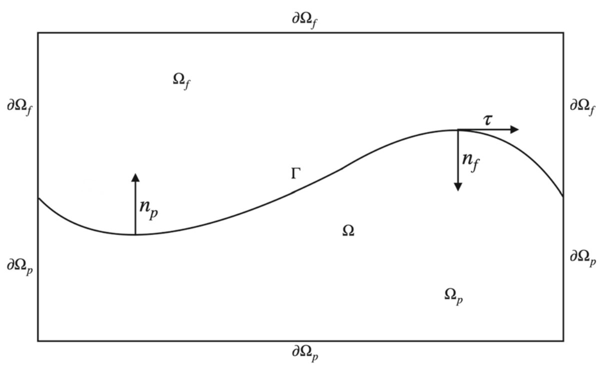

Fig. 1 The global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p separated by the interface Γ

Here $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}_1$ is the external force.

The porous media flow in the porous media region Ω_p is governed by the Darcy equations for the piezometric head φ:

$$\begin{cases} S_0\varphi_t - \nabla \cdot (\boldsymbol{K}\nabla\varphi) = f_2, & \text{in } \Omega_p \times (0, T], \\ \varphi(x, 0) = \varphi^0, & \text{in } \Omega_p, \\ \varphi = 0, & \text{on } \partial\Omega_p \backslash \Gamma \times (0, T]. \end{cases} \tag{2}$$

Here S_0 is the specific mass storativity coefficient and f_2 is the source term. We assume the hydraulic conductivity tensor $\boldsymbol{K}$ is a symmetric positive definite matrix, $\boldsymbol{K} = \text{diag}(K, \ldots, K)$ with $K > 0$, and it is uniformly bounded in Ω_p: there are $k_{min} > 0$ and $k_{max} > 0$ such that

$$k_{min}|x|^2 \leq \boldsymbol{K}x \cdot x \leq k_{max}|x|^2 \quad a.e.\ \ x \in \Omega_p. \tag{3}$$

On the interface Γ, we impose the following interface coupling conditions:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n}_f - \boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_p = 0, & \text{on } \Gamma \times (0, T], \\ p - \nu n_f \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_f} = g\varphi, & \text{on } \Gamma \times (0, T], \\ -\nu \boldsymbol{\tau}_i \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}_f} = \alpha\sqrt{\frac{\nu g}{\text{tr}(\boldsymbol{K})}}(\boldsymbol{u} \cdot \boldsymbol{\tau}_i), \quad i = 1, \ldots, d-1 & \text{on } \Gamma \times (0, T]. \end{cases} \tag{4}$$

Here g is the gravitational acceleration, α is a positive parameter depending on the properties of the porous medium. These three interface conditions represent the mass conservation, the balance of normal forces and the Beavers-Joseph-Saffman law, respectively.

We define the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{W}_f &= \{\boldsymbol{v} \in (H^1(\Omega_f))^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_f \backslash \Gamma\}, \\ W_p &= \{\psi \in H^1(\Omega_p) : \psi = 0 \text{ on } \partial\Omega_p \backslash \Gamma\}, \\ \boldsymbol{W} &= \boldsymbol{W}_f \times W_p, \quad Q = L^2(\Omega_f). \end{aligned}$$

The space $\boldsymbol{W}$ is equipped with the following norms: for all $\underline{\boldsymbol{w}} = (\boldsymbol{u}, \varphi) \in \boldsymbol{W}$,

$$\begin{aligned} \|\underline{\boldsymbol{w}}\|_0 &= \sqrt{(\boldsymbol{u}, \boldsymbol{u})_{\Omega_f} + gS_0(\varphi, \varphi)_{\Omega_p}}, \\ \|\underline{\boldsymbol{w}}\|_W &= \sqrt{\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{u})_{\Omega_f} + g(\boldsymbol{K}\nabla\varphi, \nabla\varphi)_{\Omega_p}}, \end{aligned} \tag{5}$$

where $(\cdot, \cdot)_D$ refers to the scalar product in the corresponding domain D for $D = \Omega_f$ or Ω_p.

Assume that

$$\boldsymbol{f}_1 \in L^2\left(0, T; L^2(\Omega_f)^d\right), \quad f_2 \in L^2\left(0, T; L^2(\Omega_p)\right), \quad \boldsymbol{K} \in L^\infty(\Omega_p)^{d\times d},$$

then the weak formulation of the non-stationary mixed Stokes/Darcy equations reads: find $\underline{\boldsymbol{w}} = (\boldsymbol{u}, \varphi) \in (L^2(0, T; \boldsymbol{W}_f) \cap L^\infty(0, T; L^2(\Omega_f))^d) \times (L^2(0, T; W_p) \cap L^\infty(0, T; L^2(\Omega_p)))$ and $p \in L^2(0, T; Q)$, such that

$$\begin{cases} [\underline{\boldsymbol{w}}_t, \underline{z}] + a(\underline{\boldsymbol{w}}, \underline{z}) + b(\underline{z}, p) = (\boldsymbol{f}, \underline{z}), & \forall \underline{z} = (\boldsymbol{v}, \psi) \in \boldsymbol{W}, \\ b(\underline{\boldsymbol{w}}, q) = 0, & \forall q \in Q, \\ \underline{\boldsymbol{w}}(0) = \underline{\boldsymbol{w}}^0, \end{cases} \tag{6}$$

where

$$
\begin{aligned}
&[\underline{\boldsymbol{w}}_t, \underline{z}] = (\boldsymbol{u}_t, \boldsymbol{v})_{\Omega_f} + gS_0(\varphi_t, \psi)_{\Omega_p},\\
&a(\underline{\boldsymbol{w}}, \underline{z}) = a_f(\boldsymbol{u}, \boldsymbol{v}) + a_p(\varphi, \psi) + a_\Gamma(\underline{\boldsymbol{w}}, \underline{z}),\\
&a_f(\boldsymbol{u}, \boldsymbol{v}) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega_f} + \sum_{i=1}^{d-1} \int_\Gamma \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} (\boldsymbol{u}\cdot\boldsymbol{\tau}_i)(\boldsymbol{v}\cdot\boldsymbol{\tau}_i),\\
&a_p(\varphi, \psi) = g(\boldsymbol{K}\nabla\varphi, \nabla\psi)_{\Omega_p},\\
&a_\Gamma(\underline{\boldsymbol{w}}, \underline{z}) = g\int_\Gamma (\varphi \boldsymbol{v}\cdot n_f - \psi \boldsymbol{u}\cdot n_f),\\
&b(\underline{z}, q) = -(q, \nabla\cdot\boldsymbol{v})_{\Omega_f},\\
&(\boldsymbol{f}, \underline{z}) = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_f} + g(f_2, \psi)_{\Omega_p}.
\end{aligned} \tag{7}
$$

In addition, the interface term $a_\Gamma(\cdot,\cdot)$ satisfies the anti-symmetric properties:

$$
a_\Gamma(\underline{\boldsymbol{w}}, \underline{z}) = a_\Gamma(\underline{z}, \underline{\boldsymbol{w}}), \quad a_\Gamma(\underline{z}, \underline{z}) = 0, \quad \forall \underline{\boldsymbol{w}}, \underline{z} \in \boldsymbol{W}. \tag{8}
$$

Let us consider the qusiuniform triangulation T^h of the global domain $\overline{\Omega}$, depending on a positive parameter $h > 0$. And we assume the triangulations are compatible on the interface. Based on the above triangulations, the spaces $\boldsymbol{W}_{fh} \subset \boldsymbol{W}_f$, $W_{ph} \subset W_p$ and $Q_h \subset Q$ are chosen to be the finite element spaces with the $k_1 \ge 1$, $k_2 \ge 1$ and $k_3 \ge 1$ order accuracy, respectively. And define $\boldsymbol{W}_h = \boldsymbol{W}_{fh} \times W_{ph}$. Furthermore, the finite element spaces $\boldsymbol{W}_{fh}$ and Q_h are assumed to satisfy the well-known discrete inf-sup condition: there exists a positive constant β independent of h, such that $\forall \underline{z}_h = (\boldsymbol{v}_h, 0) \in \boldsymbol{W}_h$, $\boldsymbol{v}_h \neq 0$ and $q_h \in Q_h$,

$$
b(\underline{z}_h, q_h) \ge \beta \|\underline{z}_h\|_W \|q_h\|_{L^2(\Omega_f)}. \tag{9}
$$

Next, we provide the Poincaré, trace and inverse inequalities: there exist constants C_p, C_t, C_I which only depend on the region Ω_f, and $\tilde{C}_p, \tilde{C}_t, \tilde{C}_I$ which only depend on the region Ω_p, such that for all $\boldsymbol{v} \in \boldsymbol{W_f}$, $\psi \in W_p$, $\boldsymbol{v}_h \in \boldsymbol{W}_{fh}$ and $\psi_h \in W_{ph}$,

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{L^2(\Omega_f)} \le C_p \|\nabla \boldsymbol{v}\|_{L^2(\Omega_f)}, && \|\psi\|_{L^2(\Omega_p)} \le \tilde{C}_p \|\nabla\psi\|_{L^2(\Omega_p)},\\
&\|\boldsymbol{v}\|_{L^2(\Gamma)} \le C_t \|\boldsymbol{v}\|^{\frac{1}{2}}_{L^2(\Omega_f)} \|\nabla \boldsymbol{v}\|^{\frac{1}{2}}_{L^2(\Omega_f)}, && \|\psi\|_{L^2(\Gamma)} \le \tilde{C}_t \|\psi\|^{\frac{1}{2}}_{L^2(\Omega_p)} \|\nabla\psi\|^{\frac{1}{2}}_{L^2(\Omega_p)},\\
&\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega_f)} \le C_I h^{-1} \|\boldsymbol{v}_h\|_{L^2(\Omega_f)}, && \|\nabla\psi_h\|_{L^2(\Omega_p)} \le \tilde{C}_I h^{-1} \|\psi_h\|_{L^2(\Omega_p)}.
\end{aligned} \tag{10}
$$

Hereinafter, we denote $\|\cdot\|_D = \|\cdot\|_{L^2(D)}$ with $D = \Omega_f$ or Ω_p as L^2-norm.

Following [26], we define a projection operator

$$
P_h : (\underline{\boldsymbol{w}}(t), p(t)) \in \boldsymbol{W} \times Q \mapsto (P_h^w \underline{\boldsymbol{w}}(t), P_h^p p(t)) \in \boldsymbol{W}_h \times Q_h, \ \forall t \in [0, T]
$$

by requiring

$$
\begin{cases}
a(P_h^w \underline{\boldsymbol{w}}(t), \underline{z}_h) + b(\underline{z}_h, P_h^p p(t)) = a(\underline{\boldsymbol{w}}(t), \underline{z}_h) + b(\underline{z}_h, p(t)), & \forall \underline{z}_h \in \boldsymbol{W}_h,\\
b(P_h^w \underline{\boldsymbol{w}}(t), q_h) = 0, & \forall q_h \in Q_h.
\end{cases} \tag{11}
$$

In addition, for any $t \in [0, T]$, if we assume the exact solutions of Stokes/Darcy equations satisfy $\underline{\boldsymbol{w}}(t) \in H^l(\Omega_f)^d \times H^s(\Omega_p)$ and $p(t) \in H^r(\Omega_f)$ $(l, s > 1, r > 0)$, then the following approximation properties hold:

$$\begin{aligned} &\|\underline{\boldsymbol{w}}(t) - P_h^w \underline{\boldsymbol{w}}(t)\|_0 \le Ch^l \|\boldsymbol{u}(t)\|_{H^l(\Omega_f)} + Ch^s \|\varphi(t)\|_{H^s(\Omega_p)}, \\ &\|\underline{\boldsymbol{w}}(t) - P_h^w \underline{\boldsymbol{w}}(t)\|_W \le Ch^{l-1} \|\boldsymbol{u}(t)\|_{H^l(\Omega_f)} + Ch^{s-1} \|\varphi(t)\|_{H^s(\Omega_p)}, \\ &\|p(t) - P_h^p p(t))\|_{L^2(\Omega_f)} \le Ch^r \|p(t)\|_{H^r(\Omega_f)}, \end{aligned} \tag{12}$$

where C is a positive constant which is different in different places but independent of mesh size and time step.

The following lemmas will be utilized in our analysis.

Lemma 1 [31] *For all $\underline{\boldsymbol{w}}$, $\underline{\boldsymbol{z}} \in \boldsymbol{W}$, there exist constants $C_1 = C_t^2 \tilde{C}_t^2$ and $C_2 = C_p \tilde{C}_p$ such that $\forall \varepsilon > 0$,*

$$|a_\Gamma(\underline{\boldsymbol{w}}, \underline{\boldsymbol{z}})| \le \varepsilon \|\underline{\boldsymbol{w}}\|_W^2 + \frac{gC_1C_2}{4\varepsilon \nu k_{min}} \|\underline{\boldsymbol{z}}\|_W^2, \tag{13}$$

$$|a_\Gamma(\underline{\boldsymbol{w}}, \underline{\boldsymbol{z}})| \le \frac{\varepsilon}{2}(\|\underline{\boldsymbol{w}}\|_W^2 + \|\underline{\boldsymbol{z}}\|_W^2) + \frac{gC_1}{8\varepsilon\sqrt{\nu S_0 k_{min}}}(\|\underline{\boldsymbol{w}}\|_0^2 + \|\underline{\boldsymbol{z}}\|_0^2). \tag{14}$$

Furthermore, if the finite element spaces satisfy the inverse inequality, we have, $\forall \underline{\boldsymbol{w}}_h$, $\underline{\boldsymbol{z}}_h \in \boldsymbol{W}_h$,

$$|a_\Gamma(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{z}}_h)| \le \varepsilon \|\underline{\boldsymbol{w}}_h\|_W^2 + \frac{gC_1}{4\varepsilon h} \max\left\{\frac{\tilde{C}_p C_I}{k_{min}}, \frac{C_p \tilde{C}_I}{\nu S_0}\right\} \|\underline{\boldsymbol{z}}_h\|_0^2. \tag{15}$$

Lemma 2 [16] *(The discrete Gronwall's lemma) Suppose that n and N are non-negative integers, $n \le N$. The real numbers a_n, b_n, c_n, κ_n, Δt, C are non-negative and satisfy that*

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le \Delta t \sum_{n=0}^{N} \kappa_n a_n + \Delta t \sum_{n=0}^{N} c_n + C.$$

If $\Delta t \kappa_n < 1$ for each n, then

$$a_N + \Delta t \sum_{n=0}^{N} b_n \le exp\left(\Delta t \sum_{n=0}^{N} \frac{\kappa_n}{1 - \Delta t \kappa_n}\right)\left(\Delta t \sum_{n=0}^{N} c_n + C\right). \tag{16}$$

3 Numerical algorithms

In this part, we provide the second-order decoupled scheme with different subdomain time steps for the non-stationary Stokes/Darcy equations, based on the second-order spectral deferred correction method in time and the finite element method in space. For further comparison, we also present the second-order coupled scheme with the same time steps for the Stokes/Darcy equations.

The time interval $[0, T]$ is uniformly divided into N subintervals $[t^m, t^{m+1}]$, $t^m = m\Delta t$, $m = 0, \ldots, N-1$, $\Delta t = T/N$. We first provide the first-order implicit coupling scheme based on the backward Euler method for the temporal discretization and the finite element method for the spatial discretization. Let $(\underline{\boldsymbol{w}}_{1,h}^{m+1}, q_{1,h}^{m+1})$ denote the numerical approximation to the exact solution $(\underline{\boldsymbol{w}}^{m+1}, q^{m+1}) = (\underline{\boldsymbol{w}}(t^{m+1}), q(t^{m+1}))$. The first-order coupled scheme for Stokes/Darcy model reads: find $\underline{\boldsymbol{w}}_{1,h}^{m+1} = (\boldsymbol{u}_{1,h}^{m+1}, \varphi_{1,h}^{m+1}) \in \boldsymbol{W}_h$ and $p_{1,h}^{m+1} \in Q_h$ with $m = 0, 1, \ldots, N-1$, such that

$$\begin{cases} \left[\frac{\underline{\boldsymbol{w}}_{1,h}^{m+1}-\underline{\boldsymbol{w}}_{1,h}^{m}}{\Delta t}, \underline{z}_h\right] + a(\underline{\boldsymbol{w}}_{1,h}^{m+1}, \underline{z}_h) + b(\underline{z}_h, p_{1,h}^{m+1}) = (\boldsymbol{f}^{m+1}, \underline{z}_h), & \forall \underline{z}_h = (\boldsymbol{v}_h, \psi_h) \in \boldsymbol{W}_h, \\ b(\underline{\boldsymbol{w}}_{1,h}^{m+1}, q_h) = 0, & \forall q_h \in Q_h, \\ \underline{\boldsymbol{w}}_{1,h}^{0} = P_h^w \underline{\boldsymbol{w}}^0. \end{cases} \tag{17}$$

In order to construct a decoupled scheme with different subdomain time steps, let us assume a large time step $\Delta s \geq \Delta t$ such that there exists a time level t^{m_k} in the porous media region Ω_p. For simplicity, we assume the uniform time levels, that is

$$t^{m_k} = k\Delta s, \;\; k = 0, \ldots, M, \quad \Delta s = r\Delta t, \quad t^m = m\Delta t, \; m = 0, \ldots, N,$$

where $\Delta s = T/M$ and $\Delta t = T/N$, which means $m_k = kr$. For $t^m, t^{m_k} \in [0, T]$, we will denote $(\boldsymbol{u}_{2,h}^m, p_{2,h}^m, \varphi_{2,h}^{m_k})$ as the final second-order approximate solution by the following decoupled scheme.

Algorithm 1 The decoupled spectral deferred correction scheme for the Stokes/Darcy equations.

---

1. Find $(\boldsymbol{u}_{2,h}^{m+1}, p_{2,h}^{m+1}) \in (\boldsymbol{W}_{fh}, Q_h)$ with $m = m_k, m_k+1, \ldots, m_{k+1}-1$, such that $\forall(\boldsymbol{v}_h, q_h) \in (\boldsymbol{W}_{fh}, Q_h)$,

$$\begin{cases} \left(\frac{\boldsymbol{u}_{2,h}^{m+1}-\boldsymbol{u}_{2,h}^{m}}{\Delta t}, \boldsymbol{v}_h\right) + a_f(\boldsymbol{u}_{2,h}^{m+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_{2,h}^{m+1}) = -g\int_\Gamma \varphi_{2,h}^{m_k}\boldsymbol{v}_h\cdot\boldsymbol{n}_f \\ \quad +g\int_\Gamma(\varphi_{1,h}^{m_k}-\varphi_{1,h}^{m+1})\boldsymbol{v}_h\cdot\boldsymbol{n}_f + a_f\left(\frac{\boldsymbol{u}_{1,h}^{m+1}-\boldsymbol{u}_{1,h}^{m}}{2}, \boldsymbol{v}_h\right) + g\int_\Gamma\left(\frac{\varphi_{1,h}^{m+1}-\varphi_{1,h}^{m}}{2}\right)\boldsymbol{v}_h\cdot\boldsymbol{n}_f \\ \quad +b\left(\boldsymbol{v}_h, \frac{p_{1,h}^{m+1}-p_{1,h}^{m}}{2}\right) + \left(\frac{\boldsymbol{f}_1^{m+1}+\boldsymbol{f}_1^{m}}{2}, \boldsymbol{v}_h\right)_{\Omega_f}, \\ b(\boldsymbol{u}_{2,h}^{m+1}, q_h) = 0, \\ \boldsymbol{u}_{2,h}^{0} = P_h^w\boldsymbol{u}^0. \end{cases} \tag{18}$$

2. Define $S^{m_k} = \frac{1}{r}\sum_{i=m_k}^{m_{k+1}-1}\boldsymbol{u}_{2,h}^{i}$.
3. Fine $\varphi_{2,h}^{m_{k+1}} \in W_{ph}$ such that $\forall\psi_h \in W_{ph}$

$$\begin{cases} gS_0\left(\frac{\varphi_{2,h}^{m_{k+1}}-\varphi_{2,h}^{m_k}}{\Delta s}, \psi_h\right) + a_p(\varphi_{2,h}^{m_{k+1}}, \psi_h) = g\int_\Gamma \psi_h S^{m_k}\cdot\boldsymbol{n}_f \\ \quad +\frac{g}{r}\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma\psi_h(\boldsymbol{u}_{1,h}^{m_{k+1}}-\boldsymbol{u}_{1,h}^{i})\cdot\boldsymbol{n}_f + a_p\left(\frac{\varphi_{1,h}^{m_{k+1}}-\varphi_{1,h}^{m_k}}{2}, \psi_h\right) \\ \quad -g\int_\Gamma\psi_h\left(\frac{\boldsymbol{u}_{1,h}^{m_{k+1}}-\boldsymbol{u}_{1,h}^{m_k}}{2}\right)\cdot\boldsymbol{n}_f + g\left(\frac{f_2^{m_{k+1}}+f_2^{m_k}}{2}, \psi_h\right)_{\Omega_p}, \\ \varphi_{2,h}^{m_0} = P_h^w\varphi^0. \end{cases} \tag{19}$$

4. Let $k = k+1$. If $k = M$, then stop. Otherwise, go to 1.

In Algorithm 1, we utilize the solutions of (17) as the prediction solutions which own only first-order convergence result in time. In this sense, Algorithm 1 is the correction step in the classic spectral deferred correction method. Next, we give another correction scheme which is the coupled scheme for Stokes/Darcy equations. It is still second-order time convergence and will be used to compare with Algorithm 1 to verify the efficiency of decoupled scheme.

Algorithm 2 The coupled spectral deferred correction scheme for the Stokes/Darcy equations.

Find $(\hat{\underline{\boldsymbol{w}}}_{2,h}^{m+1}, \hat{p}_{2,h}^{m+1}) \in (\boldsymbol{W}_h, Q_h)$ for $m = 0, 1, \ldots, N-1$, such that $\forall(\underline{\boldsymbol{z}}_h, q_h) \in (\boldsymbol{W}_h, Q_h)$,

$$\begin{cases} \left[\frac{\hat{\underline{\boldsymbol{w}}}_{2,h}^{m+1}-\hat{\underline{\boldsymbol{w}}}_{2,h}^{m}}{\Delta t}, \underline{\boldsymbol{z}}_h\right] + a(\hat{\underline{\boldsymbol{w}}}_{2,h}^{m+1}, \underline{\boldsymbol{z}}_h) + b(\underline{\boldsymbol{z}}_h, \hat{p}_{2,h}^{m+1}) \\ \quad = a\left(\frac{\underline{\boldsymbol{w}}_{1,h}^{m+1}-\underline{\boldsymbol{w}}_{1,h}^{m}}{2}, \underline{\boldsymbol{z}}_h\right) + b\left(\underline{\boldsymbol{z}}_h, \frac{p_{1,h}^{m+1}-p_{1,h}^{m}}{2}\right) + \left(\frac{\boldsymbol{f}^{m+1}+\boldsymbol{f}^{m}}{2}, \underline{\boldsymbol{z}}_h\right), \\ b(\hat{\underline{\boldsymbol{w}}}_{2,h}^{m+1}, q_h) = 0, \\ \hat{\underline{\boldsymbol{w}}}_{2,h}^{0} = P_h^w \underline{\boldsymbol{w}}^0. \end{cases} \tag{20}$$

4 Stability analysis

In this section, we will prove the stability results of the decoupled scheme with different subdomain time steps, Algorithm 1. Due to that the first-order backward Euler scheme (17) is the prediction step of SDC method, we first give its stability result. The detailed process of proof is classical and can be found in previous literature [2], so we omit it and only show the final result.

Lemma 3 *If $\underline{\boldsymbol{w}}_{1,h}^{m}$, $m = 1, \ldots, N$ is the solution of (17), then we have the following stability result:*

$$\|\underline{\boldsymbol{w}}_{1,h}^{m}\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|\underline{\boldsymbol{w}}_{1,h}^{n+1}\|_W^2 \le \|\underline{\boldsymbol{w}}^0\|_0^2 + \frac{C_p^2}{\nu}\Delta t \sum_{n=0}^{m-1} \|\boldsymbol{f}_1^{n+1}\|_{\Omega_f}^2 + \frac{g\tilde{C}_p^2}{k_{min}}\Delta t \sum_{n=0}^{m-1} \|f_2^{n+1}\|_{\Omega_p}^2. \tag{21}$$

The following theorem shows the stability of our decoupled scheme, Algorithm 1.

Theorem 1 *Under the assumption*

$$\kappa \Delta t < 1, \quad \kappa := \frac{3gC_1}{\sqrt{\nu S_0 k_{min}}}, \tag{22}$$

we have the following stability results:

1. *The fluid velocity $\boldsymbol{u}_{2,h}^m$ in the first large time interval $[0, t^{m_1}]$, for any $0 \leq J \leq r-2$, satisfies*

$$\|\boldsymbol{u}_{2,h}^{J+1}\|_{\Omega_f}^2 + \sum_{i=0}^{J} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + \nu\Delta t \sum_{i=0}^{J} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2$$
$$\leq \widehat{C} \exp\left(\Delta t \sum_{i=0}^{J+1} \frac{\kappa}{1-\Delta t\kappa}\right)\left(\|\underline{\boldsymbol{w}}^0\|_0^2 + \Delta t \|\underline{\boldsymbol{w}}^0\|_W^2 + \Delta t \sum_{i=0}^{J+1} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 + g\Delta t \sum_{i=0}^{J+1} \|f_2^i\|_{\Omega_p}^2\right).$$

Here $\widehat{C} = C(r, d, \alpha, g, \nu, k_{min})$ denotes a positive constant depending on data $(r, d, \alpha, g, \nu, k_{min})$, which is different in different places.

2. *For the fluid velocity $\boldsymbol{u}_{2,h}^m$ and the piezometric head $\varphi_{2,h}^{m_k}$ on $[t^{m_1}, T]$, for all $0 \leq l \leq M-2$, $-1 \leq J \leq r-1$, they satisfy*

$$\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2 + \frac{2}{3}\nu\Delta t \sum_{i=0}^{m_{l+1}+J} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + gS_0\|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 + \frac{2}{3}g\Delta s \sum_{k=0}^{l} \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2$$
$$\leq \widehat{C} \exp\left(\Delta t \sum_{i=0}^{m_{l+1}+J+1} \frac{\kappa}{1-\Delta t\kappa}\right)\left(\|\underline{\boldsymbol{w}}^0\|_0^2 + \Delta t \|\underline{\boldsymbol{w}}^0\|_W^2 + \Delta t \sum_{i=0}^{m_{l+1}+J+1} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 + g\Delta t \sum_{i=0}^{m_{l+1}+J+1} \|f_2^i\|_{\Omega_p}^2\right).$$

Proof Taking $\boldsymbol{v}_h = 2\Delta t \boldsymbol{u}_{2,h}^{m+1}$ in (18) and using the divergence-free property, we have

$$\|\boldsymbol{u}_{2,h}^{m+1}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m}\|_{\Omega_f}^2 + \|\boldsymbol{u}_{2,h}^{m+1} - \boldsymbol{u}_{2,h}^{m}\|_{\Omega_f}^2 + 2\Delta t a_f(\boldsymbol{u}_{2,h}^{m+1}, \boldsymbol{u}_{2,h}^{m+1})$$
$$= -2g\Delta t \int_\Gamma \varphi_{2,h}^{m_k}\boldsymbol{u}_{2,h}^{m+1}\cdot\boldsymbol{n}_f + 2g\Delta t \int_\Gamma (\varphi_{1,h}^{m_k} - \varphi_{1,h}^{m+1})\boldsymbol{u}_{2,h}^{m+1}\cdot\boldsymbol{n}_f$$
$$+\Delta t a_f(\boldsymbol{u}_{1,h}^{m+1} - \boldsymbol{u}_{1,h}^{m}, \boldsymbol{u}_{2,h}^{m+1})$$
$$+g\Delta t \int_\Gamma (\varphi_{1,h}^{m+1} - \varphi_{1,h}^{m})\boldsymbol{u}_{2,h}^{m+1}\cdot\boldsymbol{n}_f + \Delta t(\boldsymbol{f}_1^{m+1} + \boldsymbol{f}_1^{m}, \boldsymbol{u}_{2,h}^{m+1})_{\Omega_f}.$$

Note that $2\Delta t a_f(\boldsymbol{u}_{2,h}^{m+1}, \boldsymbol{u}_{2,h}^{m+1}) = 2\nu\Delta t\|\nabla \boldsymbol{u}_{2,h}^{m+1}\|_{\Omega_f}^2 + 2\Delta t \sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|\boldsymbol{u}_{2,h}^{m+1}\cdot\tau_j\|_{L^2(\Gamma)}^2$. Then we sum it over $m = m_k, m_k+1, \ldots, m_{k+1}-1$ with $0 \leq k \leq M-1$,

$$\|\boldsymbol{u}_{2,h}^{m_{k+1}}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_f(\boldsymbol{u}_{2,h}^{m+1}, \boldsymbol{u}_{2,h}^{m+1})$$
$$= -2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma \varphi_{2,h}^{m_k}\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f + 2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{1,h}^{m_k} - \varphi_{1,h}^{i+1})\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f$$
$$+\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_f(\boldsymbol{u}_{1,h}^{i+1} - \boldsymbol{u}_{1,h}^{i}, \boldsymbol{u}_{2,h}^{i+1})$$
$$+g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{1,h}^{i+1} - \varphi_{1,h}^{i})\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f + \Delta t \sum_{i=m_k}^{m_{k+1}-1} (\boldsymbol{f}_1^{i+1} + \boldsymbol{f}_1^{i}, \boldsymbol{u}_{2,h}^{i+1})_{\Omega_f}.$$

Taking $\psi_h = 2\Delta s \varphi_{2,h}^{m_{k+1}}$ in (19) yields

$$\begin{aligned}
&gS_0(\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 - \|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2 + \|\varphi_{2,h}^{m_{k+1}} - \varphi_{2,h}^{m_k}\|_{\Omega_p}^2) + 2\Delta s a_p(\varphi_{2,h}^{m_{k+1}}, \varphi_{2,h}^{m_{k+1}}) \\
&= 2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma \varphi_{2,h}^{m_{k+1}} \boldsymbol{u}_{2,h}^i \cdot \boldsymbol{n}_f + 2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma \varphi_{2,h}^{m_{k+1}} (\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^i) \cdot \boldsymbol{n}_f \\
&\quad + \Delta s a_p(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k}, \varphi_{2,h}^{m_{k+1}}) - g\Delta s \int_\Gamma \varphi_{2,h}^{m_{k+1}} (\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{m_k}) \cdot \boldsymbol{n}_f \\
&\quad + g\Delta s (f_2^{m_{k+1}} + f_2^{m_k}, \varphi_{2,h}^{m_{k+1}})_{\Omega_p}.
\end{aligned}$$

Here $a_p(\varphi_{2,h}^{m_{k+1}}, \varphi_{2,h}^{m_{k+1}}) = g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2$. Then combining the above two equations together, we get

$$\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{k+1}}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2 \\
&\quad + gS_0(\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 - \|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2 + \|\varphi_{2,h}^{m_{k+1}} - \varphi_{2,h}^{m_k}\|_{\Omega_p}^2) \\
&\quad + 2\nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{u}_{2,h}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 + 2g\Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 \\
&= 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{u}_{2,h}^{i+1}, \varphi_{2,h}^{m_{k+1}}; \boldsymbol{u}_{2,h}^i, \varphi_{2,h}^{m_k}) + 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{u}_{2,h}^{i+1}, \varphi_{2,h}^{m_{k+1}}; \boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^i, \varphi_{1,h}^{i+1} - \varphi_{1,h}^{m_k}) \\
&\quad + \Delta t \sum_{i=m_k}^{m_{k+1}-1} a_f(\boldsymbol{u}_{1,h}^{i+1} - \boldsymbol{u}_{1,h}^i, \boldsymbol{u}_{2,h}^{i+1}) + \Delta s a_p(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k}, \varphi_{2,h}^{m_{k+1}}) \\
&\quad + g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{1,h}^{i+1} - \varphi_{1,h}^i) \boldsymbol{u}_{2,h}^{i+1} \cdot \boldsymbol{n}_f - g\Delta s \int_\Gamma \varphi_{2,h}^{m_{k+1}} (\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{m_k}) \cdot \boldsymbol{n}_f \\
&\quad + \Delta t \sum_{i=m_k}^{m_{k+1}-1} (\boldsymbol{f}_1^{i+1} + \boldsymbol{f}_1^i, \boldsymbol{u}_{2,h}^{i+1})_{\Omega_f} + g\Delta s (f_2^{m_{k+1}} + f_2^{m_k}, \varphi_{2,h}^{m_{k+1}})_{\Omega_p}. \qquad (23)
\end{aligned}$$

For the first term on the right-hand side of (23), by using (14), we obtain

$$\begin{aligned}
&2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{u}_{2,h}^{i+1}, \varphi_{2,h}^{m_{k+1}}; \boldsymbol{u}_{2,h}^i, \varphi_{2,h}^{m_k}) \\
&\le \varepsilon\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu\|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + \nu\|\nabla \boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_k}\|_{\Omega_p}^2\right) \\
&\quad + \frac{gC_1}{4\varepsilon\sqrt{\nu S_0 k_{min}}} \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\|\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + gS_0\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + \|\boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2 + gS_0\|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2\right) \\
&\le 2\varepsilon\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu\|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_k}\|_{\Omega_p}^2\right)
\end{aligned}$$

$$+2\varepsilon\Delta t\nu\|\nabla \boldsymbol{u}_{2,h}^{m_k}\|_{\Omega_f}^2+\frac{gC_1}{2\varepsilon\sqrt{\nu S_0 k_{min}}}\Delta t\sum_{i=m_k}^{m_{k+1}}\|\boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2$$
$$+\frac{gC_1}{4\varepsilon\sqrt{\nu S_0 k_{min}}}\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(gS_0\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2+gS_0\|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2\right).$$

We bound the second term by (13),

$$2\Delta t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(\boldsymbol{u}_{2,h}^{i+1},\varphi_{2,h}^{m_{k+1}};\boldsymbol{u}_{1,h}^{m_{k+1}}-\boldsymbol{u}_{1,h}^i,\varphi_{1,h}^{i+1}-\varphi_{1,h}^{m_k})$$
$$\leq\varepsilon\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2\right)$$
$$+\frac{gC_1C_2}{\varepsilon\nu k_{min}}\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla(\boldsymbol{u}_{1,h}^{m_{k+1}}-\boldsymbol{u}_{1,h}^i)\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla(\varphi_{1,h}^{i+1}-\varphi_{1,h}^{m_k})\|_{\Omega_p}^2\right)$$
$$\leq\varepsilon\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2\right)$$
$$+\frac{rgC_1C_2}{\varepsilon\nu k_{min}}\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla(\boldsymbol{u}_{1,h}^{i+1}-\boldsymbol{u}_{1,h}^i)\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla(\varphi_{1,h}^{i+1}-\varphi_{1,h}^i)\|_{\Omega_p}^2\right)$$
$$\leq\varepsilon\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2\right)+\frac{4rgC_1C_2}{\varepsilon\nu k_{min}}\Delta t\sum_{i=m_k}^{m_{k+1}}$$
$$\times\left(\nu\|\nabla\boldsymbol{u}_{1,h}^i\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^i\|_{\Omega_p}^2\right).$$

The third and fourth terms on the right-hand side of (23) are bounded by (10), Young and Hölder inequalities,

$$\Delta t\sum_{i=m_k}^{m_{k+1}-1}a_f(\boldsymbol{u}_{1,h}^{i+1}-\boldsymbol{u}_{1,h}^i,\boldsymbol{u}_{2,h}^{i+1})$$
$$\leq\Delta t\sum_{i=m_k}^{m_{k+1}-1}\Bigg(\nu\|\nabla(\boldsymbol{u}_{1,h}^{i+1}-\boldsymbol{u}_{1,h}^i)\|_{\Omega_f}\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|(\boldsymbol{u}_{1,h}^{i+1}-\boldsymbol{u}_{1,h}^i)$$
$$\cdot\tau_j\|_{L^2(\Gamma)}\|\boldsymbol{u}_{2,h}^{i+1}\cdot\tau_j\|_{L^2(\Gamma)}\Bigg)$$
$$\leq\varepsilon\nu\Delta t\sum_{i=m_k}^{m_{k+1}-1}\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2+\frac{\nu}{4\varepsilon}\Delta t\sum_{i=m_k}^{m_{k+1}-1}\|\nabla(\boldsymbol{u}_{1,h}^{i+1}-\boldsymbol{u}_{1,h}^i)\|_{\Omega_f}^2$$

$$
\begin{aligned}
&+\varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{u}_{2,h}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 \\
&+\frac{C_t^2 C_p}{4\varepsilon} \Delta t \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\nabla(\boldsymbol{u}_{1,h}^{i+1} - \boldsymbol{u}_{1,h}^{i})\|_{\Omega_f}^2 \\
&\le \varepsilon \nu \Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + \varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{u}_{2,h}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 \\
&\quad + \left(\frac{1}{\varepsilon} + \frac{C_t^2 C_p (d-1)\alpha g}{\varepsilon \sqrt{\nu g k_{min}}}\right) \nu \Delta t \sum_{i=m_k}^{m_{k+1}} \|\nabla \boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta s a_p(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k}, \varphi_{2,h}^{m_{k+1}}) &\le g \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p} \|\boldsymbol{K}^{\frac{1}{2}} \nabla(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k})\|_{\Omega_p} \\
&\le \varepsilon g \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 \\
&\quad + \frac{1}{4\varepsilon} g \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k})\|_{\Omega_p}^2.
\end{aligned}
$$

We bound the fifth and sixth terms on the right-hand side of (23) by (13),

$$
\begin{aligned}
&g \Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{1,h}^{i+1} - \varphi_{1,h}^{i}) \boldsymbol{u}_{2,h}^{i+1} \cdot \boldsymbol{n}_f \\
&\le \varepsilon \nu \Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + \frac{g C_1 C_2}{4\varepsilon \nu k_{min}} g \Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{K}^{\frac{1}{2}} \nabla(\varphi_{1,h}^{i+1} - \varphi_{1,h}^{i})\|_{\Omega_p}^2 \\
&\le \varepsilon \nu \Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + \frac{g C_1 C_2}{\varepsilon \nu k_{min}} g \Delta t \sum_{i=m_k}^{m_{k+1}} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{1,h}^{i}\|_{\Omega_p}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
g \Delta s \int_\Gamma \varphi_{2,h}^{m_{k+1}} (\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{m_k}) \cdot \boldsymbol{n}_f &\le \varepsilon g \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 \\
&\quad + \frac{g C_1 C_2}{4\varepsilon \nu k_{min}} \nu \Delta s \|\nabla(\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{m_k})\|_{\Omega_f}^2.
\end{aligned}
$$

For the remains of the right-hand side of (23), using (10), Young and Hölder inequalities, we have

$$
\Delta t \sum_{i=m_k}^{m_{k+1}-1} (\boldsymbol{f}_1^{i+1} + \boldsymbol{f}_1^{i}, \boldsymbol{u}_{2,h}^{i+1})_{\Omega_f} + g \Delta s (f_2^{m_{k+1}} + f_2^{m_k}, \varphi_{2,h}^{m_{k+1}})_{\Omega_p}
$$

$$\leq \Delta t C_p \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{f}_1^{i+1} + \boldsymbol{f}_1^i\|_{\Omega_f} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f} + g\Delta s \tilde{C}_p \|f_2^{m_{k+1}} + f_2^{m_k}\|_{\Omega_p} \|\nabla \varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}$$

$$\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + \varepsilon g\Delta s \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + \frac{C_p^2}{\varepsilon\nu}\Delta t \sum_{i=m_k}^{m_{k+1}} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 + \frac{g\tilde{C}_p^2}{4\varepsilon k_{min}}\Delta s \|f_2^{m_{k+1}} + f_2^{m_k}\|_{\Omega_p}^2.$$

Let us combine the above estimates with (23). Then considering $\Delta s = r\Delta t = \sum_{i=m_k}^{m_{k+1}-1} \Delta t$ and setting $\varepsilon = 1/6$, we obtain

$$\|\boldsymbol{u}_{2,h}^{m_{k+1}}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2 + \nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 - \frac{1}{3}\nu\Delta t\|\nabla \boldsymbol{u}_{2,h}^{m_k}\|_{\Omega_f}^2$$
$$+ gS_0(\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 - \|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2 + \|\varphi_{2,h}^{m_{k+1}} - \varphi_{2,h}^{m_k}\|_{\Omega_p}^2) + g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 - \frac{1}{3}g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_k}\|_{\Omega_p}^2$$
$$\leq \frac{3gC_1}{\sqrt{\nu S_0 k_{min}}}\Delta t \sum_{i=m_k}^{m_{k+1}} \|\boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2 + \frac{3gC_1}{2\sqrt{\nu S_0 k_{min}}}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \times \left(gS_0\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + gS_0\|\varphi_{2,h}^{m_k}\|_{\Omega_p}^2\right)$$
$$+ \left(\frac{24rgC_1C_2}{\nu k_{min}} + 6 + \frac{6C_t^2C_p(d-1)\alpha g}{\sqrt{\nu g k_{min}}}\right)\nu\Delta t \sum_{i=m_k}^{m_{k+1}} \|\nabla \boldsymbol{u}_{1,h}^i\|_{\Omega_f}^2 + \frac{3gC_1C_2}{2\nu k_{min}}\nu\Delta s\|\nabla(\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{m_k})\|_{\Omega_f}^2$$
$$+ \left(\frac{24rgC_1C_2}{\nu k_{min}} + \frac{6gC_1C_2}{\nu k_{min}}\right) g\Delta t \sum_{i=m_k}^{m_{k+1}} \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^i\|_{\Omega_p}^2 + \frac{3}{2}g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla(\varphi_{1,h}^{m_{k+1}} - \varphi_{1,h}^{m_k})\|_{\Omega_p}^2$$
$$+ \frac{6C_p^2}{\nu}\Delta t \sum_{i=m_k}^{m_{k+1}} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 + \frac{3g\tilde{C}_p^2}{2k_{min}}\Delta s\|f_2^{m_{k+1}} + f_2^{m_k}\|_{\Omega_p}^2.$$

Then we sum it over $k = 0, \dots, l$ with $0 \le l \le M-1$,

$$
\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{l+1}}\|_{\Omega_f}^2 + \frac{2}{3}\nu\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + gS_0\|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \\
&+\frac{2}{3}g\Delta s\sum_{k=0}^{l}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + \frac{1}{3}g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \\
&\le \frac{3gC_1}{\sqrt{\nu S_0 k_{min}}}\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}}\|\boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + \frac{3gC_1}{\sqrt{\nu S_0 k_{min}}} \\
&\times\left(\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}gS_0\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2 + \Delta t\sum_{i=m_k}^{m_{k+1}-1}gS_0\|\varphi_{2,h}^{m_0}\|_{\Omega_p}^2\right) \\
&+\left(\frac{24rgC_1C_2}{\nu k_{min}} + 6 + \frac{6C_t^2C_p(d-1)\alpha g}{\sqrt{\nu k_{min}}}\right)\nu\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}}\|\nabla\boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2 \\
&+\frac{6gC_1C_2}{\nu k_{min}}\nu\Delta s\sum_{k=0}^{l+1}\|\nabla\boldsymbol{u}_{1,h}^{m_k}\|_{\Omega_f}^2 \\
&+\left(\frac{24rgC_1C_2}{\nu k_{min}} + \frac{6gC_1C_2}{\nu k_{min}}\right)g\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^{i}\|_{\Omega_p}^2 \\
&+6g\Delta s\sum_{k=0}^{l+1}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^{m_k}\|_{\Omega_p}^2 + \frac{6C_p^2}{\nu}\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}}\|\boldsymbol{f}_1^{i}\|_{\Omega_f}^2 \\
&+\frac{3g\tilde{C}_p^2}{k_{min}}\Delta s\sum_{k=0}^{l+1}\|f_2^{m_k}\|_{\Omega_p}^2 + \|\boldsymbol{u}_{2,h}^{m_0}\|_{\Omega_f}^2 + gS_0\|\varphi_{2,h}^{m_0}\|_{\Omega_p}^2 + \frac{1}{3}\nu\Delta t\|\nabla\boldsymbol{u}_{2,h}^{m_0}\|_{\Omega_f}^2 \\
&+\frac{1}{3}g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_0}\|_{\Omega_p}^2. \qquad (24)
\end{aligned}
$$

Taking $\boldsymbol{v}_h = 2\Delta t\boldsymbol{u}_{2,h}^{m+1}$ in (18) and summing over $m = m_{l+1}, m_{l+1}+1, \dots, m_{l+1}+J$ with $0 \le J \le r-1$ yield

$$
\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_{l+1}}\|_{\Omega_f}^2 + \sum_{i=m_{l+1}}^{m_{l+1}+J}\|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + 2\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}a_f(\boldsymbol{u}_{2,h}^{i+1}, \boldsymbol{u}_{2,h}^{i+1}) \\
&= -2g\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}\int_\Gamma\varphi_{2,h}^{m_{l+1}}\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f + 2g\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}\int_\Gamma(\varphi_{1,h}^{m_{l+1}} - \varphi_{1,h}^{i+1})\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f \\
&+\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}a_f(\boldsymbol{u}_{1,h}^{i+1} - \boldsymbol{u}_{1,h}^{i}, \boldsymbol{u}_{2,h}^{i+1}) + g\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}\int_\Gamma(\varphi_{1,h}^{i+1} - \varphi_{1,h}^{i})\boldsymbol{u}_{2,h}^{i+1}\cdot\boldsymbol{n}_f \\
&+\Delta t\sum_{i=m_{l+1}}^{m_{l+1}+J}(\boldsymbol{f}_1^{i+1} + \boldsymbol{f}_1^{i}, \boldsymbol{u}_{2,h}^{i+1})_{\Omega_f}.
\end{aligned}
$$

Then, similar to the previous estimates, we obtain

$$
\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_{l+1}}\|_{\Omega_f}^2 + \sum_{i=m_{l+1}}^{m_{l+1}+J} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + 2\Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} a_f(\boldsymbol{u}_{2,h}^{i+1}, \boldsymbol{u}_{2,h}^{i+1}) \\
&\le 6\varepsilon_1 \nu \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + 2\varepsilon_1 \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \\
&+\varepsilon_1 \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{u}_{2,h}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 \\
&\quad + \frac{gC_1}{8\varepsilon_1 \sqrt{\nu S_0 k_{min}}} \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} \left(\|\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + gS_0 \|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \right) \\
&\quad + \left(\frac{1}{\varepsilon_1} + \frac{C_p C_t^2 (d-1)\alpha g}{\varepsilon_1 \sqrt{\nu g k_{min}}} \right) \nu \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\nabla \boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2 \\
&\quad + \left(\frac{4rgC_1C_2}{\varepsilon_1 \nu k_{min}} + \frac{gC_1C_2}{\varepsilon_1 \nu k_{min}} \right) g\Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{1,h}^{i}\|_{\Omega_p}^2 + \frac{C_p^2}{\varepsilon_1 \nu} \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 .
\end{aligned}
$$

Taking $\varepsilon_1 = 1/6$ in the above equation leads to

$$
\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2 - \|\boldsymbol{u}_{2,h}^{m_{l+1}}\|_{\Omega_f}^2 + \sum_{i=m_{l+1}}^{m_{l+1}+J} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + \nu \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 \\
&\le \frac{3gC_1}{4\sqrt{\nu S_0 k_{min}}} \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J} \left(\|\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 + gS_0 \|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \right) + \frac{g}{3} \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2 \\
&\quad + \left(6 + \frac{6C_p C_t^2 (d-1)\alpha g}{\sqrt{\nu g k_{min}}} \right) \nu \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\nabla \boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2 \\
&\quad + \left(\frac{24rgC_1C_2}{\nu k_{min}} + \frac{6gC_1C_2}{\nu k_{min}} \right) g\Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_{1,h}^{i}\|_{\Omega_p}^2 + \frac{6C_p^2}{\nu} \Delta t \sum_{i=m_{l+1}}^{m_{l+1}+J+1} \|\boldsymbol{f}_1^i\|_{\Omega_f}^2 .
\end{aligned}
\tag{25}
$$

If $l = -1$, we obtain

$$
\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{J+1}\|_{\Omega_f}^2 + \sum_{i=0}^{J} \|\boldsymbol{u}_{2,h}^{i+1} - \boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + \nu \Delta t \sum_{i=0}^{J} \|\nabla \boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2 \\
&\le \frac{3gC_1}{4\sqrt{\nu S_0 k_{min}}} \Delta t \sum_{i=0}^{J+1} \left(\|\boldsymbol{u}_{2,h}^{i}\|_{\Omega_f}^2 + gS_0 \|\varphi^0\|_{\Omega_p}^2 \right) + \|\boldsymbol{u}^0\|_{\Omega_f}^2 + \frac{g}{3} \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi^0\|_{\Omega_p}^2 \\
&\quad + \left(6 + \frac{6C_p C_t^2 (d-1)\alpha g}{\sqrt{\nu g k_{min}}} \right) \nu \Delta t \sum_{i=0}^{J+1} \|\nabla \boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2
\end{aligned}
$$

$$+\left(\frac{24rgC_1C_2}{\nu k_{min}}+\frac{6gC_1C_2}{\nu k_{min}}\right)g\Delta t\sum_{i=0}^{J+1}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^i\|_{\Omega_p}^2+\frac{6C_p^2}{\nu}\Delta t\sum_{i=0}^{J+1}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2.$$

From (22), we have

$$\kappa_1\Delta t:=\frac{3gC_1}{4\sqrt{\nu S_0k_{min}}}\Delta t<\kappa\Delta t<1.$$

Then using the Gronwall inequality (16) and (21) yields

$$\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{J+1}\|_{\Omega_f}^2+\nu\Delta t\sum_{i=0}^{J}\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2\\
&\leq C(r,d,\alpha,g,\nu,k_{min})\exp\left(\Delta t\sum_{i=0}^{J+1}\frac{\kappa_1}{1-\Delta t\kappa_1}\right)\\
&\quad\left(\|\boldsymbol{u}^0\|_{\Omega_f}^2+gS_0\|\varphi^0\|_{\Omega_p}^2+\frac{g}{3}\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi^0\|_{\Omega_p}^2+\nu\Delta t\sum_{i=0}^{J+1}\|\nabla\boldsymbol{u}_{1,h}^i\|_{\Omega_f}^2\right.\\
&\quad\left.+g\Delta t\sum_{i=0}^{J+1}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^i\|_{\Omega_p}^2+\Delta t\sum_{i=0}^{J+1}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2\right)\\
&\leq\widehat{C}\exp\left(\Delta t\sum_{i=0}^{J+1}\frac{\kappa}{1-\Delta t\kappa}\right)\left(\|\underline{\boldsymbol{w}}^0\|_0^2+g\Delta t\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi^0\|_{\Omega_p}^2+\Delta t\sum_{i=0}^{J+1}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2\right.\\
&\qquad\left.+g\Delta t\sum_{i=0}^{J+1}\|f_2^i\|_{\Omega_p}^2\right).
\end{aligned}$$

Next, considering (24) and (25), we have

$$\begin{aligned}
&\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2+\frac{2}{3}\nu\Delta t\sum_{i=0}^{m_{l+1}+J}\|\nabla\boldsymbol{u}_{2,h}^{i+1}\|_{\Omega_f}^2+gS_0\|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2\\
&\quad+\frac{2}{3}g\Delta s\sum_{k=0}^{l}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2\\
&\leq\frac{3gC_1}{\sqrt{\nu S_0k_{min}}}\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}+J+1}\left(\|\boldsymbol{u}_{2,h}^i\|_{\Omega_f}^2+gS_0\|\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2\right)\\
&\quad+\frac{3gC_1}{\sqrt{\nu S_0k_{min}}}\Delta t\sum_{i=m_k}^{m_{k+1}-1}gS_0\|\varphi_{2,h}^{m_0}\|_{\Omega_p}^2\\
&\quad+\left(6+\frac{6C_t^2C_p(d-1)\alpha g}{\sqrt{\nu gk_{min}}}+\frac{30rgC_1C_2}{\nu k_{min}}\right)\nu\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|\nabla\boldsymbol{u}_{1,h}^i\|_{\Omega_f}^2
\end{aligned}$$

$$+\left(\frac{24rgC_1C_2}{\nu k_{min}}+\frac{6gC_1C_2}{\nu k_{min}}+6r\right)g\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^{i}\|_{\Omega_p}^2$$
$$+\frac{6C_p^2}{\nu}\Delta t\sum_{i=0}^{m_{l+1}+J}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2$$
$$+\frac{3g\tilde{C}_p^2}{2k_{min}}\Delta s\sum_{k=0}^{l+1}\|f_2^{m_k}\|_{\Omega_p}^2+\|\boldsymbol{u}_{2,h}^{m_0}\|_{\Omega_f}^2+gS_0\|\varphi_{2,h}^{m_0}\|_{\Omega_p}^2+\frac{1}{3}\nu\Delta t\|\nabla\boldsymbol{u}_{2,h}^{m_0}\|_{\Omega_f}^2$$
$$+\frac{1}{3}g\Delta s\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_0}\|_{\Omega_p}^2.$$

Take note that $\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}+J+1}\le 2\sum_{i=0}^{m_{l+1}+J+1}$ with $-1\le J\le r-1$. Then considering (21) and (22) and using the Gronwall's inequality (16), we arrive at

$$\|\boldsymbol{u}_{2,h}^{m_{l+1}+J+1}\|_{\Omega_f}^2+\frac{2}{3}\nu\Delta t\sum_{i=0}^{m_{l+1}+J}\|\nabla\boldsymbol{u}_{2,h}^{l+1}\|_{\Omega_f}^2+gS_0\|\varphi_{2,h}^{m_{l+1}}\|_{\Omega_p}^2$$
$$+\frac{2}{3}\Delta s\sum_{k=0}^{l}g\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{2,h}^{m_{k+1}}\|_{\Omega_p}^2$$
$$\le C(r,d,\alpha,g,\nu,k_{min})\exp\left(\Delta t\sum_{i=0}^{m_{l+1}+J+1}\frac{\kappa}{1-\Delta t\kappa}\right)$$
$$\Bigg(rgS_0\|\varphi_{2,h}^{m_0}\|_{\Omega_p}^2+\nu\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|\nabla\boldsymbol{u}_{1,h}^{i}\|_{\Omega_f}^2$$
$$+g\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_{1,h}^{i}\|_{\Omega_p}^2+\Delta t\sum_{i=0}^{m_{l+1}+J}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2$$
$$+g\Delta s\sum_{k=0}^{l+1}\|f_2^{m_k}\|_{\Omega_p}^2+\|\underline{\boldsymbol{w}}_{2,h}^{m_0}\|_0^2+\Delta t\|\underline{\boldsymbol{w}}_{2,h}^{m_0}\|_W^2\Bigg)$$
$$\le\widehat{C}\exp\left(\Delta t\sum_{i=0}^{m_{l+1}+J+1}\frac{\kappa}{1-\Delta t\kappa}\right)\Bigg(\|\underline{\boldsymbol{w}}^0\|_0^2+\Delta t\|\underline{\boldsymbol{w}}^0\|_W^2+\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|\boldsymbol{f}_1^i\|_{\Omega_f}^2$$
$$+g\Delta t\sum_{i=0}^{m_{l+1}+J+1}\|f_2^i\|_{\Omega_p}^2\Bigg).$$

Thus, we complete the proof. □

5 Convergence analysis

In this section, we provide the convergence analysis of Algorithm 1. We denote $(\boldsymbol{u}_{1,h}^m, p_{1,h}^m, \varphi_{1,h}^m)$ and $(\boldsymbol{u}_{2,h}^m, p_{2,h}^m, \varphi_{2,h}^{m_k})$ as the solutions of (17) and Algorithm 1, respectively. Then we decompose the errors into numerical errors and approximation errors as follows: for $i = 1, 2$, $m = 0, 1, \ldots, N$ and $k = 0, 1, \ldots, M$,

$$
\begin{aligned}
&\boldsymbol{u}_{i,h}^m - \boldsymbol{u}^m = \boldsymbol{u}_{i,h}^m - \tilde{\boldsymbol{u}}^m - (\boldsymbol{u}^m - \tilde{\boldsymbol{u}}^m) := \boldsymbol{e}_{i,f}^m - \boldsymbol{\xi}_f^m, && \boldsymbol{u}^m = \boldsymbol{u}(t^m), && \tilde{\boldsymbol{u}}^m = P_h^w \boldsymbol{u}^m,\\
&p_{i,h}^m - p^m = p_{i,h}^m - \tilde{p}^m - (p^m - \tilde{p}^m) := \varepsilon_i^m - \eta^m, && p^m = p(t^m), && \tilde{p}^m = P_h^p p^m,\\
&\varphi_{1,h}^m - \varphi^m = \varphi_{1,h}^m - \tilde{\varphi}^m - (\varphi^m - \tilde{\varphi}^m) := e_{1,p}^m - \xi_p^m, && \varphi^m = \varphi(t^m), && \tilde{\varphi}^m = P_h^w \varphi^m,\\
&\varphi_{2,h}^{m_k} - \varphi^{m_k} = \varphi_{2,h}^{m_k} - \tilde{\varphi}^{m_k} - (\varphi^{m_k} - \tilde{\varphi}^{m_k}) := e_{2,p}^{m_k} - \xi_p^{m_k}, && \varphi^{m_k} = \varphi(t^{m_k}), && \tilde{\varphi}^{m_k} = P_h^w \varphi^{m_k}.
\end{aligned}
$$

For the numerical errors in time derivatives, we use the following notations: for $m = 1, \ldots, N$ and $k = 1, \ldots, M$,

$$
\begin{aligned}
d_t \boldsymbol{e}_{1,f}^m &= \frac{\boldsymbol{e}_{1,f}^m - \boldsymbol{e}_{1,f}^{m-1}}{\Delta t}, \quad d_t e_{1,p}^m = \frac{e_{1,p}^m - e_{1,p}^{m-1}}{\Delta t}, \quad d_t \boldsymbol{e}_{2,f}^m = \frac{\boldsymbol{e}_{2,f}^m - \boldsymbol{e}_{2,f}^{m-1}}{\Delta t},\\
d_s \boldsymbol{e}_{1,f}^{m_k} &= \frac{\boldsymbol{e}_{1,f}^{m_k} - \boldsymbol{e}_{1,f}^{m_{k-1}}}{\Delta s}, \quad d_s e_{1,p}^{m_k} = \frac{e_{1,p}^{m_k} - e_{1,p}^{m_{k-1}}}{\Delta s}, \quad d_s e_{2,p}^{m_k} = \frac{e_{2,p}^{m_k} - e_{2,p}^{m_{k-1}}}{\Delta s}.
\end{aligned}
$$

If we assume the true solutions of Stokes/Darcy equations (6) satisfy $(\boldsymbol{u}, p, \varphi) \in H^{k_1+1}(\Omega_f)^d \times H^{k_1}(\Omega_f) \times H^{k_2+1}(\Omega_p)$, then considering the approximation properties (12) yields

$$
\|\boldsymbol{\xi}_f^m\|_{\Omega_f} + h\|\nabla \boldsymbol{\xi}_f^m\|_{\Omega_f} \le C h^{k_1+1}, \quad \|\xi_p^{m_k}\|_{\Omega_p} + h\|\nabla \xi_p^{m_k}\|_{\Omega_p} \le C h^{k_2+1}, \tag{26}
$$

$$
\|\eta^m\|_{\Omega_f} \le C h^{k_1}. \tag{27}
$$

Here C is a positive constant which is different in different places but independent of mesh size and time step length.

The following lemma shows the convergence results of the coupled backward Euler scheme (17) and we omit the details of its proof which is classical. We refer readers to [26, 33] for its details.

Lemma 4 *Suppose that* $(\boldsymbol{u}_{1,h}^{m+1}, p_{1,h}^{m+1}, \varphi_{1,h}^{m+1})$*,* $m = 0, \ldots, N-1$*, is given by* (17)*. Assume the true solution* $(\boldsymbol{u}, \varphi)$ *satisfies the following regularities*

$$
\begin{aligned}
&\boldsymbol{u}(t) \in L^\infty(0, T; H^{k_1+1}(\Omega_f)^d), && \varphi(t) \in L^\infty(0, T; H^{k_2+1}(\Omega_p)),\\
&\boldsymbol{u}^0 \in H^{k_1+1}(\Omega_f)^d, && \varphi^0 \in H^{k_2+1}(\Omega_p),\\
&\boldsymbol{u}_t \in L^2(0, T; H^{k_1+1}(\Omega_f)^d), && \varphi_t \in L^2(0, T; H^{k_2+1}(\Omega_p)),\\
&\boldsymbol{u}_{tt} \in L^2(0, T; H^{k_1}(\Omega_f)^d), && \varphi_{tt} \in L^2(0, T; H^{k_2}(\Omega_p)),\\
&\boldsymbol{u}_{ttt} \in L^2(0, T; L^2(\Omega_f)^d), && \varphi_{ttt} \in L^2(0, T; L^2(\Omega_p)).
\end{aligned} \tag{28}
$$

Let us denote $\underline{\boldsymbol{e}}_1^{m+1} = (\boldsymbol{e}_{1,f}^{m+1}, e_{1,p}^{m+1})$ *and* $d_t\underline{\boldsymbol{e}}_1^{m+1} = (d_t\boldsymbol{e}_{1,f}^{m+1}, d_t e_{1,f}^{m+1})$*, then we have*

$$\|\underline{\boldsymbol{e}}_1^m\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|\underline{\boldsymbol{e}}_1^{n+1}\|_W^2 \leq C\left(\Delta t^2 + h^{2k_1+2} + h^{2k_2+2}\right), \tag{29}$$

$$\|d_t\underline{\boldsymbol{e}}_1^m\|_0^2 + \Delta t \sum_{n=0}^{m-1} \|d_t\underline{\boldsymbol{e}}_1^{n+1}\|_W^2 \leq C(\Delta t^2 + h^{2k_1} + h^{2k_2}). \tag{30}$$

In addition, we take the average of (6) at time t^{m+1} and t^m: $\forall \underline{\boldsymbol{z}} = (\boldsymbol{v}, \psi) \in \boldsymbol{W}$ and $q \in Q$, we have

$$\begin{cases} \left[\frac{\underline{\boldsymbol{w}}_t^{m+1}+\underline{\boldsymbol{w}}_t^m}{2}, \underline{\boldsymbol{z}}\right] + a\left(\frac{\underline{\boldsymbol{w}}^{m+1}+\underline{\boldsymbol{w}}^m}{2}, \underline{\boldsymbol{z}}\right) + b\left(\underline{\boldsymbol{z}}, \frac{p^{m+1}+p^m}{2}\right) = \left(\frac{\boldsymbol{f}^{m+1}+\boldsymbol{f}^m}{2}, \underline{\boldsymbol{z}}\right), \\ b\left(\frac{\underline{\boldsymbol{w}}^{m+1}+\underline{\boldsymbol{w}}^m}{2}, q\right) = 0. \end{cases}$$

It can be rewritten as follows:

$$\begin{cases} \left(\frac{\boldsymbol{u}_t^{m+1}+\boldsymbol{u}_t^m}{2}, \boldsymbol{v}\right) + a_f\left(\frac{\boldsymbol{u}^{m+1}+\boldsymbol{u}^m}{2}, \boldsymbol{v}\right) + g\int_\Gamma \left(\frac{\varphi^{m+1}+\varphi^m}{2}\right)\boldsymbol{v}\cdot\boldsymbol{n}_f + b\left(\boldsymbol{v}, \frac{p^{m+1}+p^m}{2}\right) = \left(\frac{\boldsymbol{f}_1^{m+1}+\boldsymbol{f}_1^m}{2}, \boldsymbol{v}\right)_{\Omega_f}, \\ b(\boldsymbol{u}^{m+1}+\boldsymbol{u}^m, q) = 0, \\ gS_0\left(\frac{\varphi_t^{m+1}+\varphi_t^m}{2}, \psi\right) + a_p\left(\frac{\varphi^{m+1}+\varphi^m}{2}, \psi\right) - g\int_\Gamma \psi\left(\frac{\boldsymbol{u}^{m+1}+\boldsymbol{u}^m}{2}\right)\cdot\boldsymbol{n}_f = g\left(\frac{f_2^{m+1}+f_2^m}{2}, \psi\right)_{\Omega_p}. \end{cases} \tag{31}$$

From (11) and (31), we obtain, $\forall \underline{\boldsymbol{w}}_h = (\boldsymbol{v}_h, \psi_h) \in \boldsymbol{W}_h$ and $q_h \in Q_h$,

$$\begin{cases} \left(\frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^m}{\Delta t}, \boldsymbol{v}_h\right) + a_f(\tilde{\boldsymbol{u}}^{m+1}, \boldsymbol{v}_h) + g\int_\Gamma \tilde{\varphi}^{m+1}\boldsymbol{v}_h\cdot\boldsymbol{n}_f + b(\boldsymbol{v}_h, \tilde{p}^{m+1}) = \left(\frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^m}{\Delta t} - \frac{\boldsymbol{u}_t^{m+1}+\boldsymbol{u}_t^m}{2}, \boldsymbol{v}_h\right) \\ \quad +a_f\left(\frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^m}{2}, \boldsymbol{v}_h\right) + g\int_\Gamma \left(\frac{\tilde{\varphi}^{m+1}-\tilde{\varphi}^m}{2}\right)\boldsymbol{v}_h\cdot\boldsymbol{n}_f + b\left(\boldsymbol{v}_h, \frac{\tilde{p}^{m+1}-\tilde{p}^m}{2}\right) + \left(\frac{\boldsymbol{f}_1^{m+1}+\boldsymbol{f}_1^m}{2}, \boldsymbol{v}_h\right)_{\Omega_f}, \\ b(\tilde{\boldsymbol{u}}^{m+1}, q_h) = 0, \end{cases} \tag{32}$$

and

$$\begin{aligned} &gS_0\left(\frac{\tilde{\varphi}^{m+1}-\tilde{\varphi}^m}{\Delta t}, \psi_h\right) + a_p(\tilde{\varphi}^{m+1}, \psi_h) - g\int_\Gamma \psi_h\tilde{\boldsymbol{u}}^{m+1}\cdot\boldsymbol{n}_f \\ &= gS_0\left(\frac{\tilde{\varphi}^{m+1}-\tilde{\varphi}^m}{\Delta t} - \frac{\varphi_t^{m+1}+\varphi_t^m}{2}, \psi_h\right) \\ &\quad +a_p\left(\frac{\tilde{\varphi}^{m+1}-\tilde{\varphi}^m}{2}, \psi_h\right) - g\int_\Gamma \psi_h\left(\frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^m}{2}\right)\cdot\boldsymbol{n}_f \\ &\quad +g\left(\frac{f_2^{m+1}+f_2^m}{2}, \psi_h\right)_{\Omega_p}. \end{aligned} \tag{33}$$

Subtracting (32) from (18), we obtain the error equations

$$\begin{cases} \left(\frac{\boldsymbol{e}_{2,f}^{m+1}-\boldsymbol{e}_{2,f}^{m}}{\Delta t}, \boldsymbol{v}_h\right) + a_f(\boldsymbol{e}_{2,f}^{m+1}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \varepsilon_2^{m+1}) \\ = \left(\frac{\boldsymbol{u}_t^{m+1}+\boldsymbol{u}_t^{m}}{2} - \frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^{m}}{\Delta t}, \boldsymbol{v}_h\right) - g\int_\Gamma(\varphi_{2,h}^{m_k} - \tilde{\varphi}^{m+1})\boldsymbol{v}_h\cdot\boldsymbol{n}_f + g\int_\Gamma(\varphi_{1,h}^{m_k} - \varphi_{1,h}^{m+1})\boldsymbol{v}_h\cdot\boldsymbol{n}_f \\ \quad +a_f\left(\frac{\boldsymbol{e}_{1,f}^{m+1}-\boldsymbol{e}_{1,f}^{m}}{2}, \boldsymbol{v}_h\right) + g\int_\Gamma\left(\frac{e_{1,p}^{m+1}-e_{1,p}^{m}}{2}\right)\boldsymbol{v}_h\cdot\boldsymbol{n}_f + b\left(\boldsymbol{v}_h, \frac{\varepsilon_1^{m+1}-\varepsilon_1^{m}}{2}\right), \\ b(\boldsymbol{e}_{2,f}^{m+1}, q_h) = 0. \end{cases} \tag{34}$$

Considering the large time step length $\Delta s = r\Delta t$ and subtracting (33) from (19), we have

$$\begin{aligned} & gS_0\left(\frac{e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k}}{\Delta s}, \psi_h\right) + a_p(e_{2,p}^{m_{k+1}}, \psi_h) \\ & = gS_0\left(\frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\tilde{\varphi}^{m_{k+1}} - \tilde{\varphi}^{m_k}}{\Delta s}, \psi_h\right) \\ & \quad +g\int_\Gamma \psi_h\left(\frac{1}{r}\sum_{i=m_k}^{m_{k+1}-1}\boldsymbol{u}_{2,h}^i - \tilde{\boldsymbol{u}}^{m_{k+1}}\right)\cdot\boldsymbol{n}_f \\ & \quad +\frac{g}{r}\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma\psi_h(\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^i)\cdot\boldsymbol{n}_f + a_p\left(\frac{e_{1,p}^{m_{k+1}} - e_{1,p}^{m_k}}{2}, \psi_h\right) \\ & \quad -g\int_\Gamma\psi_h\left(\frac{\boldsymbol{e}_{1,f}^{m_{k+1}} - \boldsymbol{e}_{1,f}^{m_k}}{2}\right)\cdot\boldsymbol{n}_f. \end{aligned} \tag{35}$$

Theorem 2 *Let the assumptions (28) be satisfied. Also assume that*

$$\kappa_2\Delta t := \frac{18rgC_1}{h}\max\left\{\frac{\tilde{C}_pC_I}{k_{min}}, \frac{C_p\tilde{C}_I}{\nu S_0}\right\}\Delta t < 1, \tag{36}$$

then we have the following error estimate at the large time steps: for $0 \le l \le M-1$,

$$\begin{aligned} & \|\boldsymbol{e}_{2,f}^{m_{l+1}}\|_{\Omega_f}^2 + \nu\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla\boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 + gS_0\|e_{2,p}^{m_{l+1}}\|_{\Omega_p}^2 \\ & \quad +g\Delta s\sum_{k=0}^{l}\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \\ & \le C(r, d, \alpha, g, \nu, k_{min}, S_0, T)(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}). \end{aligned} \tag{37}$$

Here $C(r, d, \alpha, g, \nu, k_{min}, S_0, T)$ denotes a generic positive constant depending on data $(r, d, \alpha, g, \nu, k_{min}, S_0, T)$.

Proof Taking $\boldsymbol{v}_h = 2\Delta t \boldsymbol{e}_{2,f}^{m+1}$ in (34), using the divergence-free property and summing it over $m = m_k, m_k + 1, \ldots, m_{k+1} - 1$, we get

$$
\begin{aligned}
&\|\boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 - \|\boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2 \\
&+2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_f(\boldsymbol{e}_{2,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) \\
&= 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, \boldsymbol{e}_{2,f}^{i+1}\right) \\
&-2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{2,h}^{m_k} - \tilde{\varphi}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&+2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{1,h}^{m_k} - \varphi_{1,h}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&+\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) \\
&+g\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma d_t e_{1,p}^{i+1} \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f .
\end{aligned}
$$

Taking $\psi_h = 2\Delta s e_{2,p}^{m_{k+1}}$ in (35), we have

$$
\begin{aligned}
&gS_0(\|e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 - \|e_{2,p}^{m_k}\|_{\Omega_p}^2 + \|e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k}\|_{\Omega_p}^2) \\
&\quad +2\Delta s a_p(e_{2,p}^{m_{k+1}}, e_{2,p}^{m_{k+1}}) \\
&= 2gS_0\Delta s \left(\frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\tilde{\varphi}^{m_{k+1}} - \tilde{\varphi}^{m_k}}{\Delta s}, e_{2,p}^{m_{k+1}}\right) \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma e_{2,p}^{m_{k+1}} (\boldsymbol{u}_{2,h}^{i} - \tilde{\boldsymbol{u}}^{m_{k+1}}) \cdot \boldsymbol{n}_f \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma e_{2,p}^{m_{k+1}} (\boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{i}) \cdot \boldsymbol{n}_f + \Delta s^2 a_p(d_s e_{1,p}^{m_{k+1}}, e_{2,p}^{m_{k+1}}) \\
&\quad -g\Delta s^2 \int_\Gamma e_{2,p}^{m_{k+1}} d_s \boldsymbol{e}_{1,f}^{m_{k+1}} \cdot \boldsymbol{n}_f .
\end{aligned}
$$

Then combining the above two equations yields

$$
\begin{aligned}
&\|\boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2-\|\boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2+\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{e}_{2,f}^{i+1}-\boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2+2\Delta t\sum_{i=m_k}^{m_{k+1}-1}a_f(\boldsymbol{e}_{2,f}^{i+1},\boldsymbol{e}_{2,f}^{i+1})\\
&+gS_0(\|e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2-\|e_{2,p}^{m_k}\|_{\Omega_p}^2+\|e_{2,p}^{m_{k+1}}-e_{2,p}^{m_k}\|_{\Omega_p}^2)+2\Delta s a_p(e_{2,p}^{m_{k+1}},e_{2,p}^{m_{k+1}})\\
&=2\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^{i}}{2}-\frac{\tilde{\boldsymbol{u}}^{i+1}-\tilde{\boldsymbol{u}}^{i}}{\Delta t},\boldsymbol{e}_{2,f}^{i+1}\right)\\
&+2gS_0\Delta s\left(\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\tilde{\varphi}^{m_{k+1}}-\tilde{\varphi}^{m_k}}{\Delta s},e_{2,p}^{m_{k+1}}\right)\\
&\quad-2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\varphi_{2,h}^{m_k}-\tilde{\varphi}^{i+1})\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f\\
&\quad+2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(\varphi_{1,h}^{m_k}-\varphi_{1,h}^{i+1})\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f\\
&\quad+2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma e_{2,p}^{m_{k+1}}(\boldsymbol{u}_{2,h}^{i}-\tilde{\boldsymbol{u}}^{m_{k+1}})\cdot\boldsymbol{n}_f\\
&\quad+2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma e_{2,p}^{m_{k+1}}(\boldsymbol{u}_{1,h}^{m_{k+1}}-\boldsymbol{u}_{1,h}^{i})\cdot\boldsymbol{n}_f\\
&\quad+\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}a_f(d_t\boldsymbol{e}_{1,f}^{i+1},\boldsymbol{e}_{2,f}^{i+1})+\Delta s^2a_p(d_se_{1,p}^{m_{k+1}},e_{2,p}^{m_{k+1}})\\
&\quad+g\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma d_te_{1,p}^{i+1}\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f-g\Delta s^2\int_\Gamma e_{2,p}^{m_{k+1}}d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\cdot\boldsymbol{n}_f.
\end{aligned}\tag{38}
$$

Using (10), Young and Hölder inequalities, we obtain

$$
\begin{aligned}
I&=2\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^{i}}{2}-\frac{\tilde{\boldsymbol{u}}^{i+1}-\tilde{\boldsymbol{u}}^{i}}{\Delta t},\boldsymbol{e}_{2,f}^{i+1}\right)\\
&+2gS_0\Delta s\left(\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\tilde{\varphi}^{m_{k+1}}-\tilde{\varphi}^{m_k}}{\Delta s},e_{2,p}^{m_{k+1}}\right)\\
&\le 2C_p\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^{i}}{2}-\frac{\boldsymbol{u}^{i+1}-\boldsymbol{u}^{i}}{\Delta t}+\frac{\boldsymbol{\xi}_f^{i+1}-\boldsymbol{\xi}_f^{i}}{\Delta t}\right\|_{\Omega_f}\|\nabla\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}
\end{aligned}
$$

$$
\begin{aligned}
&+2\tilde{C}_p g S_0 \Delta s \left\| \frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}}-\varphi^{m_k}}{\Delta s} + \frac{\xi_p^{m_{k+1}}-\xi_p^{m_k}}{\Delta s} \right\|_{\Omega_p} \|\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p} \\
&\leq \varepsilon \nu \Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \frac{2C_p^2}{\varepsilon \nu}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left\| \frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^{i}}{2} - \frac{\boldsymbol{u}^{i+1}-\boldsymbol{u}^{i}}{\Delta t} \right\|_{\Omega_f}^2 \\
&+\frac{2C_p^2}{\varepsilon \nu}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left\| \frac{\boldsymbol{\xi}_f^{i+1}-\boldsymbol{\xi}_f^{i}}{\Delta t} \right\|_{\Omega_f}^2 \\
&+\varepsilon g \Delta s \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 + \frac{2\tilde{C}_p^2 g S_0^2}{\varepsilon k_{min}}\Delta s \left\| \frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}}-\varphi^{m_k}}{\Delta s} \right\|_{\Omega_p}^2 \\
&+\frac{2\tilde{C}_p^2 g S_0^2}{\varepsilon k_{min}}\Delta s \left\| \frac{\xi_p^{m_{k+1}}-\xi_p^{m_k}}{\Delta s} \right\|_{\Omega_p}^2 . \qquad (39)
\end{aligned}
$$

For the next four interface terms, we denote

$$
\begin{aligned}
II &= -2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (\varphi_{2,h}^{m_k} - \tilde{\varphi}^{i+1} - \varphi_{1,h}^{m_k} + \varphi_{1,h}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma e_{2,p}^{m_{k+1}} (\boldsymbol{u}_{2,h}^{i} - \tilde{\boldsymbol{u}}^{m_{k+1}} + \boldsymbol{u}_{1,h}^{m_{k+1}} - \boldsymbol{u}_{1,h}^{i}) \cdot \boldsymbol{n}_f \\
&= -2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma ((\varphi_{2,h}^{m_k} - \tilde{\varphi}^{m_k}) - (\varphi_{1,h}^{m_k} - \tilde{\varphi}^{m_k}) + (\varphi_{1,h}^{i+1} - \tilde{\varphi}^{i+1})) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma e_{2,p}^{m_{k+1}} ((\boldsymbol{u}_{2,h}^{i} - \tilde{\boldsymbol{u}}^{i}) - (\boldsymbol{u}_{1,h}^{i} - \tilde{\boldsymbol{u}}^{i}) + (\boldsymbol{u}_{1,h}^{m_{k+1}} - \tilde{\boldsymbol{u}}^{m_{k+1}})) \cdot \boldsymbol{n}_f \\
&= -2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (e_{2,p}^{m_k} - e_{1,p}^{m_k} + e_{1,p}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma e_{2,p}^{m_{k+1}} (\boldsymbol{e}_{2,f}^{i} - \boldsymbol{e}_{1,f}^{i} + \boldsymbol{e}_{1,f}^{m_{k+1}}) \cdot \boldsymbol{n}_f \\
&= 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_{k+1}}; \boldsymbol{e}_{2,f}^{i}, e_{2,p}^{m_k}) \\
&\quad +2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_{k+1}}; \boldsymbol{e}_{1,f}^{m_{k+1}} - \boldsymbol{e}_{1,f}^{i}, e_{1,p}^{i+1} - e_{1,p}^{m_k}).
\end{aligned}
$$

Then using (8) and (15), we get

$$
\begin{aligned}
&2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_{k+1}}; \boldsymbol{e}_{2,f}^{i}, e_{2,p}^{m_k}) \\
&= 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_{k+1}}; \boldsymbol{e}_{2,f}^{i} - \boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_k} - e_{2,p}^{m_{k+1}}) \\
&\leq \varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \right) \\
&\quad + \frac{gC_1}{\varepsilon h} \max\left\{ \frac{\tilde{C}_p C_I}{k_{min}}, \frac{C_p \tilde{C}_I}{\nu S_0} \right\} \Delta t \sum_{i=m_k}^{m_{k+1}-1} \\
&\quad \times \left(\|\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2 + gS_0 \|e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k}\|_{\Omega_p}^2 \right).
\end{aligned}
$$

Using (13), we get

$$
\begin{aligned}
&2\Delta t \sum_{i=m_k}^{m_{k+1}-1} a_\Gamma(\boldsymbol{e}_{2,f}^{i+1}, e_{2,p}^{m_{k+1}}; \boldsymbol{e}_{1,f}^{m_{k+1}} - \boldsymbol{e}_{1,f}^{i}, e_{1,p}^{i+1} - e_{1,p}^{m_k}) \\
&\leq \varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \right) \\
&\quad + \frac{gC_1C_2}{\varepsilon \nu k_{min}} \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla (\boldsymbol{e}_{1,f}^{m_{k+1}} - \boldsymbol{e}_{1,f}^{i})\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla (e_{1,p}^{i+1} - e_{1,p}^{m_k})\|_{\Omega_p}^2 \right) \\
&\leq \varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \right) \\
&\quad + \frac{rgC_1C_2}{\varepsilon \nu k_{min}} \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla (\boldsymbol{e}_{1,f}^{i+1} - \boldsymbol{e}_{1,f}^{i})\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla (e_{1,p}^{i+1} - e_{1,p}^{i})\|_{\Omega_p}^2 \right) \\
&\leq \varepsilon \Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \right) \\
&\quad + \frac{rgC_1C_2}{\varepsilon \nu k_{min}} \Delta t^3 \sum_{i=m_k}^{m_{k+1}-1} \left(\nu \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 \right).
\end{aligned}
$$

The following two bilinear terms in (38) are bounded by (10), Young and Hölder inequalities,

$$
III = \Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) + \Delta s^2 a_p(d_s e_{1,p}^{m_{k+1}}, e_{2,p}^{m_{k+1}})
$$

$$
\begin{aligned}
&\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \varepsilon\Delta t \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{e}_{2,f}^{i+1}\cdot\tau_j\|_{L^2(\Gamma)}^2 \\
&\quad + \varepsilon g\Delta s \|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 + \left(\frac{1}{4\varepsilon} + \frac{C_t^2 C_p (d-1)\alpha g}{4\varepsilon\sqrt{\nu g k_{min}}}\right)\nu\Delta t^3 \sum_{i=m_k}^{m_{k+1}-1} \\
&\quad \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 \\
&\quad + \frac{g}{4\varepsilon}\Delta s^3 \|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2.
\end{aligned}
$$

The remaining terms on the right-hand side of (38) are bounded by (13)

$$
\begin{aligned}
IV &= g\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma d_t e_{1,p}^{i+1} \boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f - g\Delta s^2 \int_\Gamma e_{2,p}^{m_{k+1}} d_s \boldsymbol{e}_{1,f}^{m_{k+1}}\cdot\boldsymbol{n}_f \\
&\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \varepsilon g\Delta s \|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \\
&\quad + \frac{gC_1C_2}{4\varepsilon\nu k_{min}} g\Delta t^3 \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 + \frac{gC_1C_2}{4\varepsilon\nu k_{min}} \nu\Delta s^3 \|\nabla d_s \boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2.
\end{aligned}
$$

Combining the above estimates with (38) and taking $\varepsilon = 1/5$, we arrive at

$$
\begin{aligned}
&\|\boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 - \|\boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2 + \nu\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 \\
&+ gS_0(\|e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 - \|e_{2,p}^{m_k}\|_{\Omega_p}^2 + \|e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k}\|_{\Omega_p}^2) + g\Delta s \|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \\
\leq\ & \frac{5gC_1}{h}\max\left\{\frac{\tilde{C}_p C_I}{k_{min}}, \frac{C_p\tilde{C}_I}{\nu S_0}\right\}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\|\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2 + gS_0\|e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k}\|_{\Omega_p}^2\right) \\
&+ \frac{10C_p^2}{\nu}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left\|\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}}{\Delta t}\right\|_{\Omega_f}^2 + \frac{10C_p^2}{\nu}\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left\|\frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^{i}}{\Delta t}\right\|_{\Omega_f}^2 \\
&+ \frac{10\tilde{C}_p^2 gS_0^2}{k_{min}}\Delta s \left\|\frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}} - \varphi^{m_k}}{\Delta s}\right\|_{\Omega_p}^2 + \frac{10\tilde{C}_p^2 gS_0^2}{k_{min}}\Delta s \left\|\frac{\xi_p^{m_{k+1}} - \xi_p^{m_k}}{\Delta s}\right\|_{\Omega_p}^2 \\
&+ \left(\frac{5rgC_1C_2}{\nu k_{min}} + \frac{5}{4} + \frac{5C_t^2C_p(d-1)\alpha g}{4\sqrt{\nu g k_{min}}}\right)\nu\Delta t^3 \sum_{i=m_k}^{m_{k+1}-1} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 \\
&+ \frac{5gC_1C_2}{4k_{min}}\Delta s^3 \|\nabla d_s \boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2 \\
&+ \left(\frac{5rgC_1C_2}{\nu k_{min}} + \frac{5gC_1C_2}{4\nu k_{min}}\right) g\Delta t^3 \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 + \frac{5g}{4}\Delta s^3 \|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2.
\end{aligned}
$$

Sum the above equation over $k = 0, \ldots, l$ with $0 \le l \le M - 1$. Using (36), we obtain

$$
\begin{aligned}
&\|\boldsymbol{e}_{2,f}^{m_{l+1}}\|_{\Omega_f}^2 + \nu \Delta t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \|\nabla \boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 + g S_0 \|e_{2,p}^{m_{l+1}}\|_{\Omega_p}^2 + g \Delta s \sum_{k=0}^{l} \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \\
&\le C(r, d, \alpha, g, \nu, k_{min}, S_0) \left(\Delta t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} \right\|_{\Omega_f}^2 \right. \\
&\quad + \Delta t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \left\| \frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} \right\|_{\Omega_f}^2 \\
&\quad + g S_0 \Delta s \sum_{k=0}^{l} \left\| \frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}} - \varphi^{m_k}}{\Delta s} \right\|_{\Omega_p}^2 + g S_0 \Delta s \sum_{k=0}^{l} \left\| \frac{\xi_p^{m_{k+1}} - \xi_p^{m_k}}{\Delta s} \right\|_{\Omega_p}^2 \\
&\quad + \nu \Delta t^3 \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 + \nu \Delta s^3 \sum_{k=0}^{l} \|\nabla d_s \boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2 \\
&\quad \left. + g \Delta t^3 \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 + g \Delta s^3 \sum_{k=0}^{l} \|\boldsymbol{K}^{\frac{1}{2}} \nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2 \right). \qquad (40)
\end{aligned}
$$

Next, we bound these terms on the right-hand side of (40). First, from the Taylor expansion, we have

$$
\begin{aligned}
\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} &= \frac{1}{8} \Delta t^2 \boldsymbol{u}_{ttt}(\theta_1^i) - \frac{1}{24} \Delta t^2 \boldsymbol{u}_{ttt}(\theta_2^i), \\
\frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}} - \varphi^{m_k}}{\Delta s} &= \frac{1}{8} \Delta s^2 \varphi_{ttt}(\vartheta_1^{m_k}) - \frac{1}{24} \Delta s^2 \varphi_{ttt}(\vartheta_2^{m_k}), \qquad (41)
\end{aligned}
$$

where $\theta_1^i,\ \theta_2^i \in (t^i, t^{i+1})$ and $\vartheta_1^{m_k},\ \vartheta_2^{m_k} \in (t^{m_k}, t^{m_{k+1}})$. Under the assumption (28), we obtain

$$
\begin{aligned}
&\Delta t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} \right\|_{\Omega_f}^2 \\
&+ g S_0 \Delta s \sum_{k=0}^{l} \left\| \frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\varphi^{m_{k+1}} - \varphi^{m_k}}{\Delta s} \right\|_{\Omega_p}^2 \\
&\le C \Delta t^5 \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{2} \|\boldsymbol{u}_{ttt}(\theta_j^i)\|_{\Omega_f}^2 + C g S_0 \Delta s^5 \sum_{k=0}^{l} \sum_{j=1}^{2} \|\varphi_{ttt}(\vartheta_j^{m_k})\|_{\Omega_p}^2 \\
&\le C(r, g, S_0) (\Delta t^4 \|\boldsymbol{u}_{ttt}\|_{L^2(0,T;L^2(\Omega_f)^d)}^2 + \Delta s^4 \|\varphi_{ttt}\|_{L^2(0,T;L^2(\Omega_p))}^2). \qquad (42)
\end{aligned}
$$

Considering (12) and (28), we have

$$\Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{\xi}_f^{i+1}-\boldsymbol{\xi}_f^{i}}{\Delta t}\right\|_{\Omega_f}^2 \le C\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\int_{t^i}^{t^{i+1}}\|(\boldsymbol{\xi}_f)_t\|_{\Omega_f}^2 dt \le Ch^{2k_1+2}\|\boldsymbol{u}_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f)^d)}, \quad (43)$$

and

$$gS_0\Delta s\sum_{k=0}^{l}\|\frac{\xi_p^{m_{k+1}}-\xi_p^{m_k}}{\Delta s}\|_{\Omega_p}^2 = \frac{gS_0}{r^2}\Delta s\sum_{k=0}^{l}\left\|\sum_{i=m_k}^{m_{k+1}-1}\left(\frac{\xi_p^{i+1}-\xi_p^{i}}{\Delta t}\right)\right\|_{\Omega_p}^2 \le \frac{gS_0}{r}\Delta s\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\xi_p^{i+1}-\xi_p^{i}}{\Delta t}\right\|_{\Omega_p}^2 \le gS_0\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\int_{t^i}^{t^{i+1}}\|(\xi_p)_t\|_{\Omega_p}^2 dt \le Ch^{2k_2+2}\|\varphi_t\|^2_{L^2(0,T;H^{k_2+1}(\Omega_p))}. \quad (44)$$

From (30), we have

$$\Delta s\sum_{k=0}^{l}\|d_s\underline{\boldsymbol{e}}_1^{m_{k+1}}\|_W^2 = \Delta s\sum_{k=0}^{l}\left\|\frac{\underline{\boldsymbol{e}}_1^{m_{k+1}}-\underline{\boldsymbol{e}}_1^{m_k}}{\Delta s}\right\|_W^2 = \frac{\Delta s}{r^2}\sum_{k=0}^{l}\left\|\sum_{i=m_k}^{m_{k+1}-1}d_t\underline{\boldsymbol{e}}_1^{i+1}\right\|_W^2 \le \Delta t\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|d_t\underline{\boldsymbol{e}}_1^{i+1}\|_W^2 \le C(\Delta t^2+h^{2k_1}+h^{2k_2}). \quad (45)$$

Hence

$$\nu\Delta t^3\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2+\nu\Delta s^3\sum_{k=0}^{l}\|\nabla d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2 + g\Delta t^3\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\boldsymbol{K}^{\frac12}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2+g\Delta s^3\sum_{k=0}^{l}\|\boldsymbol{K}^{\frac12}\nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2 \le C(r)\Delta t^2(\Delta t^2+h^{2k_1}+h^{2k_2}) \le C(r)(\Delta t^4+h^{2k_1+2}+h^{2k_2+2}).$$

Combining the above estimates with (40) leads to the final result (37). □

The next theorem gives the error estimate of Stokes equations at the small time step.

Theorem 3 *Under the assumptions of Theorem 2, we have the following error estimate for fluid velocity on the small time step: for* $k = 0, 1, \ldots, M-1$ *and* $J = 0, 1, \ldots, r-1$,

$$\|\boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 + \nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 \leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T)(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}). \tag{46}$$

Proof Taking $\boldsymbol{v}_h = 2\Delta t \boldsymbol{e}_{2,f}^{m+1}$ in (34) and summing it over $m = m_k, m_k+1, \ldots, m_k+J$ with $0 \leq J \leq r-1$, we obtain

$$\begin{aligned}
&\|\boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 - \|\boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 + \sum_{i=m_k}^{m_k+J} \|\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2 + 2\Delta t \sum_{i=m_k}^{m_k+J} a_f(\boldsymbol{e}_{2,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) \\
&= 2\Delta t \sum_{i=m_k}^{m_k+J} \left(\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, \boldsymbol{e}_{2,f}^{i+1} \right) \\
&\quad -2g\Delta t \sum_{i=m_k}^{m_k+J} \int_\Gamma (e_{2,p}^{m_k} - e_{1,p}^{m_k} + e_{1,p}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&\quad + \Delta t^2 \sum_{i=m_k}^{m_k+J} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) + g\Delta t^2 \sum_{i=m_k}^{m_k+J} \int_\Gamma d_t e_{1,p}^{i+1} \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f.
\end{aligned} \tag{47}$$

Similar to the estimate of (39), we have

$$\begin{aligned}
&2\Delta t \sum_{i=m_k}^{m_k+J} \left(\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, \boldsymbol{e}_{2,f}^{i+1} \right) \\
&\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \frac{2C_p^2}{\varepsilon\nu} \Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}}{\Delta t} \right\|_{\Omega_f}^2 \\
&\quad + \frac{2C_p^2}{\varepsilon\nu} \Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^{i}}{\Delta t} \right\|_{\Omega_f}^2.
\end{aligned}$$

Using (13), Young and Hölder inequalities, we have

$$\begin{aligned}
&-2g\Delta t \sum_{i=m_k}^{m_k+J} \int_\Gamma (e_{2,p}^{m_k} - e_{1,p}^{m_k} + e_{1,p}^{i+1}) \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \frac{2gC_1C_2}{\varepsilon\nu k_{min}} \left(\Delta t \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2 + \Delta t \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla (e_{1,p}^{i+1} - e_{1,p}^{m_k})\|_{\Omega_p}^2 \right) \\
&\leq \varepsilon\nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \frac{2gC_1C_2}{\varepsilon\nu k_{min}} \left(\Delta t \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2 + r\Delta t^3 \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 \right).
\end{aligned}$$

The remaining terms are bounded by the Young and Hölder inequalities and (13),

$$\Delta t^2 \sum_{i=m_k}^{m_k+J} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, \boldsymbol{e}_{2,f}^{i+1}) + g\Delta t^2 \sum_{i=m_k}^{m_k+J} \int_\Gamma d_t e_{1,p}^{i+1} \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f$$
$$\leq 2\varepsilon\nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \frac{\nu}{4\varepsilon}\Delta t^3 \sum_{i=m_k}^{m_k+J} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2$$
$$+\varepsilon\Delta t \sum_{i=m_k}^{m_k+J} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{e}_{2,f}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2$$
$$+\frac{C_t^2 C_p}{4\varepsilon}\Delta t^3 \sum_{i=m_k}^{m_k+J} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_p}^2$$
$$+\frac{gC_1C_2}{4\varepsilon\nu k_{min}}\Delta t^3 \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2.$$

Combining the above estimates with (47) and taking $\varepsilon = 1/4$, we obtain

$$\|\boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 - \|\boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 + \nu\Delta t \sum_{i=m_k}^{m_k+J} \|\nabla \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2$$
$$+\Delta t \sum_{i=m_k}^{m_k+J} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{e}_{2,f}^{i+1} \cdot \tau_j\|_{L^2(\Gamma)}^2$$
$$\leq \frac{4C_p^2}{\nu}\Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} \right\|_{\Omega_f}^2 + \frac{4C_p^2}{\nu}\Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} \right\|_{\Omega_f}^2$$
$$+\frac{8gC_1C_2}{\nu k_{min}}\Delta t \sum_{i=m_k}^{m_k+J} g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2$$
$$+\left(1 + \frac{C_t^2 C_p (d-1)\alpha g}{\sqrt{\nu g k_{min}}}\right)\nu\Delta t^3 \sum_{i=m_k}^{m_k+J} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2$$
$$+\left(\frac{8rgC_1C_2}{\nu k_{min}} + \frac{gC_1C_2}{\nu k_{min}}\right) g\Delta t^3 \sum_{i=m_k}^{m_k+J} \|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2.$$

From (41), (12), (30) and (46), we have

$$\Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} \right\|_{\Omega_f}^2 \leq C\Delta t^5 \sum_{i=m_k}^{m_k+J} \sum_{j=1}^{2} \|\boldsymbol{u}_{ttt}(\theta_j^i)\|_{\Omega_f}^2$$
$$\leq C\Delta t^4 \|\boldsymbol{u}_{ttt}\|_{L^2(0,T;L^2(\Omega_f)^d)}^2,$$

$$\Delta t \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} \right\|_{\Omega_f}^2 \leq C \sum_{i=m_k}^{m_k+J} \int_{t^i}^{t^{i+1}} \|(\boldsymbol{\xi}_f)_t\|_{\Omega_f}^2 dt$$
$$\leq C h^{2k_1+2} \|\boldsymbol{u}_t\|^2_{L^2(0,T;H^{k_1+1}(\Omega_f)^d)},$$
$$\Delta t \sum_{i=m_k}^{m_k+J} g \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2 \leq r \Delta t g \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2$$
$$\leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T)(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}),$$
$$\Delta t^3 \left(\nu \sum_{i=m_k}^{m_k+J} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 + g \sum_{i=m_k}^{m_k+J} \|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2 \right)$$
$$\leq C \Delta t^2 (\Delta t^2 + h^{2k_1} + h^{2k_2}).$$

Hence, we arrive at the result (46). Thus, we complete the proof. □

Corollary 1 *Under the assumptions of Theorem 2, we have the following error bounds: for* $0 \leq m \leq N-1$ *and* $0 \leq l \leq M-1$,

$$\begin{aligned} \|\boldsymbol{u}_{2,h}^{m+1} - \boldsymbol{u}^{m+1}\|_{\Omega_f}^2 &\leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T)(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}), \\ g S_0 \|\varphi_{2,p}^{m_{l+1}} - \varphi^{m_{l+1}}\|_{\Omega_p}^2 &\leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T)(\Delta t^4 + h^{2k_1+2} + h^{2k_2+2}). \end{aligned} \quad (48)$$

Proof Using the triangle inequalities and combining the approximation properties (26) and the results of Theorem 2 and 3, we obtain the result (48). □

Theorem 4 *Under the assumptions of Theorem 2, we have, for* $0 \leq l \leq M-1$,

$$\nu \|\nabla \boldsymbol{e}_{2,f}^{m_{l+1}}\|_{\Omega_f}^2 + g \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{l+1}}\|_{\Omega_p}^2 \leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T) \left(\Delta t^3 + h^{2k_1+1} + h^{2k_2+1} + \frac{h^{2k_1+2} + h^{2k_2+2}}{\Delta t} \right). \quad (49)$$

In addition, for all $0 \leq k \leq M-1$ *and* $-1 \leq J \leq r-1$, *we have the following error estimate at the smaller time steps for the Stokes equations:*

$$\nu \|\nabla \boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 \leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T) \times \left(\Delta t^3 + h^{2k_1+1} + h^{2k_2+1} + \frac{h^{2k_1+2} + h^{2k_2+2}}{\Delta t} \right). \quad (50)$$

Proof Taking $\boldsymbol{v}_h = 2(\boldsymbol{e}_{2,f}^{m+1} - \boldsymbol{e}_{2,f}^m) = 2\Delta t d_t \boldsymbol{e}_{2,f}^{m+1}$ in (34) and summing it over $m = m_k, m_k+1, \ldots, m_{k+1}-1$, we get

$$2\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \nu \|\nabla \boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 - \nu \|\nabla \boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2$$

$$
\begin{aligned}
&+\nu\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} \|\nabla d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 \\
&+\sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\operatorname{tr}(\boldsymbol{K})}} \left(\|\boldsymbol{e}_{2,f}^{m_{k+1}} \cdot \tau_j\|_{L^2(\Gamma)}^2 - \|\boldsymbol{e}_{2,f}^{m_k} \cdot \tau_j\|_{L^2(\Gamma)}^2\right) \\
&+ \sum_{i=m_k}^{m_{k+1}-1} \sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\operatorname{tr}(\boldsymbol{K})}} \|(\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}) \cdot \tau_j\|_{L^2(\Gamma)}^2 \\
&= 2\Delta t \sum_{i=m_k}^{m_{k+1}-1} \left(\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, d_t \boldsymbol{e}_{2,f}^{i+1}\right) \\
&-2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma (e_{2,p}^{m_k} - e_{1,p}^{m_k} + e_{1,p}^{i+1}) d_t \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f \\
&+\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, d_t \boldsymbol{e}_{2,f}^{i+1}) + g\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma d_t e_{1,p}^{i+1} d_t \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f. \qquad (51)
\end{aligned}
$$

Taking $\psi_h = 2\Delta s d_s e_{2,p}^{m_{k+1}} = 2(e_{2,p}^{m_{k+1}} - e_{2,p}^{m_k})$ in (35) leads to

$$
\begin{aligned}
&2gS_0\Delta s\|d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 + g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 - g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_k}\|_{\Omega_p}^2 \\
&\quad +g\Delta s^2\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2 \\
&= 2gS_0\Delta s\left(\frac{\varphi_t^{m_{k+1}} + \varphi_t^{m_k}}{2} - \frac{\tilde{\varphi}^{m_{k+1}} - \tilde{\varphi}^{m_k}}{\Delta s}, d_s e_{2,p}^{m_{k+1}}\right) \\
&\quad +2g\Delta t \sum_{i=m_k}^{m_{k+1}-1} \int_\Gamma d_s e_{2,p}^{m_{k+1}} (\boldsymbol{e}_{2,f}^{i} - \boldsymbol{e}_{1,f}^{i} + \boldsymbol{e}_{1,f}^{m_{k+1}}) \cdot \boldsymbol{n}_f \\
&\quad +\Delta s^2 a_p(d_s e_{1,p}^{m_{k+1}}, d_s e_{2,p}^{m_{k+1}}) - g\Delta s^2 \int_\Gamma d_s e_{2,p}^{m_{k+1}} d_s \boldsymbol{e}_{1,f}^{m_{k+1}} \cdot \boldsymbol{n}_f.
\end{aligned}
$$

Combining the above two equations together, we have

$$
\begin{aligned}
&2\Delta t \sum_{i=m_k}^{m_{k+1}-1} \|d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \nu\|\nabla \boldsymbol{e}_{2,f}^{m_{k+1}}\|_{\Omega_f}^2 - \nu\|\nabla \boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 \\
&+\nu\Delta t^2 \sum_{i=m_k}^{m_{k+1}-1} \|\nabla d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\operatorname{tr}(\boldsymbol{K})}} \\
&\left(\|\boldsymbol{e}_{2,f}^{m_{k+1}} \cdot \tau_j\|_{L^2(\Gamma)}^2 - \|\boldsymbol{e}_{2,f}^{m_k} \cdot \tau_j\|_{L^2(\Gamma)}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=m_k}^{m_{k+1}-1}\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|(\boldsymbol{e}_{2,f}^{m_{i+1}}-\boldsymbol{e}_{2,f}^{m_i})\cdot\tau_j\|_{L^2(\Gamma)}^2\\
&+2gS_0\Delta s\|d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2-g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_k}\|_{\Omega_p}^2\\
&+g\Delta s^2\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2\\
=\ &2\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^i}{2}-\frac{\tilde{\boldsymbol{u}}^{i+1}-\tilde{\boldsymbol{u}}^i}{\Delta t},d_t\boldsymbol{e}_{2,f}^{i+1}\right)\\
&+2gS_0\Delta s\left(\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\tilde{\varphi}^{m_{k+1}}-\tilde{\varphi}^{m_k}}{\Delta s},d_s e_{2,p}^{m_{k+1}}\right)\\
&-2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma(e_{2,p}^{m_k}-e_{1,p}^{m_k}+e_{1,p}^{i+1})d_t\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f\\
&+2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma d_s e_{2,p}^{m_{k+1}}(\boldsymbol{e}_{2,f}^i-\boldsymbol{e}_{1,f}^i+\boldsymbol{e}_{1,f}^{m_{k+1}})\cdot\boldsymbol{n}_f\\
&+\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}a_f(d_t\boldsymbol{e}_{1,f}^{i+1},d_t\boldsymbol{e}_{2,f}^{i+1})+r\Delta s^2 a_p(d_s e_{1,p}^{m_{k+1}},d_s e_{2,p}^{m_{k+1}})\\
&+g\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma d_t e_{1,p}^{i+1}d_t\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f-g\Delta s^2\int_\Gamma d_s e_{2,p}^{m_{k+1}}d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\cdot\boldsymbol{n}_f. \qquad (52)
\end{aligned}
$$

Using (10) and the Young and Hölder inequalities, we obtain

$$
\begin{aligned}
&2\Delta t\sum_{i=m_k}^{m_{k+1}-1}\left(\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^i}{2}-\frac{\tilde{\boldsymbol{u}}^{i+1}-\tilde{\boldsymbol{u}}^i}{\Delta t},d_t\boldsymbol{e}_{2,f}^{i+1}\right)\\
&\quad+2gS_0\Delta s\left(\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\tilde{\varphi}^{m_{k+1}}-\tilde{\varphi}^{m_k}}{\Delta s},d_s e_{2,p}^{m_{k+1}}\right)\\
\le\ &\varepsilon\nu\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2+\frac{2C_p^2}{\varepsilon\nu}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^i}{2}-\frac{\boldsymbol{u}^{i+1}-\boldsymbol{u}^i}{\Delta t}\right\|_{\Omega_f}^2\\
&\quad+\frac{2C_p^2}{\varepsilon\nu}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{\xi}_f^{i+1}-\boldsymbol{\xi}_f^i}{\Delta t}\right\|_{\Omega_f}^2\\
&\quad+\varepsilon g\Delta s^2\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2+\frac{2\tilde{C}_p^2 gS_0^2}{\varepsilon k_{min}}\left\|\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\varphi^{m_{k+1}}-\varphi^{m_k}}{\Delta s}\right\|_{\Omega_p}^2\\
&\quad+\frac{2\tilde{C}_p^2 gS_0^2}{\varepsilon k_{min}}\left\|\frac{\xi_p^{m_{k+1}}-\xi_p^{m_k}}{\Delta s}\right\|_{\Omega_p}^2.
\end{aligned}
$$

We bound the next two interface terms by (13),

$$
\begin{aligned}
&-2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma (e_{2,p}^{m_k}-e_{1,p}^{m_k}+e_{1,p}^{i+1})d_t\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f\\
&+2g\Delta t\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma d_s e_{2,p}^{m_{k+1}}(\boldsymbol{e}_{2,f}^{i}-\boldsymbol{e}_{1,f}^{i}+\boldsymbol{e}_{1,f}^{m_{k+1}})\cdot\boldsymbol{n}_f\\
&=2\Delta t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(d_t\boldsymbol{e}_{2,f}^{i+1},d_s e_{2,p}^{m_{k+1}};\boldsymbol{e}_{2,f}^{i},e_{2,p}^{m_k})\\
&+2\Delta t\sum_{i=m_k}^{m_{k+1}-1}a_\Gamma(d_t\boldsymbol{e}_{2,f}^{i+1},d_s e_{2,p}^{m_{k+1}};\boldsymbol{e}_{1,f}^{m_{k+1}}-\boldsymbol{e}_{1,f}^{i},e_{1,p}^{i+1}-e_{1,p}^{m_k})\\
&\le \varepsilon\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla d_t\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2\right)\\
&+\frac{gC_1C_2}{2\varepsilon\nu k_{min}}\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla e_{2,p}^{m_k}\|_{\Omega_p}^2\right)\\
&+\frac{gC_1C_2}{2\varepsilon\nu k_{min}}\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla(\boldsymbol{e}_{1,f}^{m_{k+1}}-\boldsymbol{e}_{1,f}^{i})\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla(e_{1,p}^{i+1}-e_{1,p}^{m_k})\|_{\Omega_p}^2\right)\\
&\le \varepsilon\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla d_t\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2\right)\\
&+\frac{gC_1C_2}{2\varepsilon\nu k_{min}}\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{e}_{2,f}^{i}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla e_{2,p}^{m_k}\|_{\Omega_p}^2\right)\\
&+\frac{rgC_1C_2}{2\varepsilon\nu k_{min}}\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac12}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2\right).
\end{aligned}
$$

For the last four terms, using the Young and Hölder inequalities and (13), we obtain

$$
\begin{aligned}
&\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}a_f(d_t\boldsymbol{e}_{1,f}^{i+1},d_t\boldsymbol{e}_{2,f}^{i+1})+\Delta s^2 a_p(d_s e_{1,p}^{m_{k+1}},d_s e_{2,p}^{m_{k+1}})\\
&\le \varepsilon\nu\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2+\varepsilon\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\boldsymbol{e}_{2,f}^{i+1}\cdot\tau_j\|_{L^2(\Gamma)}^2\\
&+\varepsilon g\Delta s^2\|\boldsymbol{K}^{\frac12}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2
\end{aligned}
$$

$$+\frac{\nu}{4\varepsilon}\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2+\frac{C_t^2C_p(d-1)\alpha g}{4\varepsilon\sqrt{\nu g k_{min}}}\nu\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2$$
$$+\frac{g}{4\varepsilon}\Delta s^2\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2,$$

and

$$g\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\int_\Gamma d_t e_{1,p}^{i+1}d_t\boldsymbol{e}_{2,f}^{i+1}\cdot\boldsymbol{n}_f-g\Delta s^2\int_\Gamma d_s e_{2,p}^{m_{k+1}}d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\cdot\boldsymbol{n}_f$$
$$\leq\varepsilon\nu\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2+\varepsilon g\Delta s^2\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{2,p}^{m_{k+1}}\|_{\Omega_p}^2$$
$$+\frac{gC_1C_2}{4\varepsilon\nu k_{min}}\Delta t^2\sum_{i=m_k}^{m_{k+1}-1}g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2+\frac{gC_1C_2}{4\varepsilon\nu k_{min}}\Delta s^2\nu\|\nabla d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2.$$

Combine the above bounds with (52) and take $\varepsilon=1/4$. Then considering $\boldsymbol{e}_{2,f}^0=\boldsymbol{0}$, $e_{2,p}^0=0$ and summing it over $k=0,1,\ldots,l$ with $0\leq l\leq M-1$, we get

$$\nu\|\nabla\boldsymbol{e}_{2,f}^{m_{l+1}}\|_{\Omega_f}^2+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|\boldsymbol{e}_{2,f}^{m_{l+1}}\cdot\tau_j\|_{L^2(\Gamma)}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_{l+1}}\|_{\Omega_p}^2$$
$$\leq\frac{8C_p^2}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{u}_t^{i+1}+\boldsymbol{u}_t^i}{2}-\frac{\boldsymbol{u}^{i+1}-\boldsymbol{u}^i}{\Delta t}\right\|_{\Omega_f}^2+\frac{8C_p^2}{\nu}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\left\|\frac{\boldsymbol{\xi}_f^{i+1}-\boldsymbol{\xi}_f^i}{\Delta t}\right\|_{\Omega_f}^2$$
$$+\frac{8\tilde{C}_p^2gS_0^2}{k_{min}}\sum_{k=0}^{l}\left\|\frac{\varphi_t^{m_{k+1}}+\varphi_t^{m_k}}{2}-\frac{\varphi^{m_{k+1}}-\varphi^{m_k}}{\Delta s}\right\|_{\Omega_p}^2+\frac{8\tilde{C}_p^2gS_0^2}{k_{min}}\sum_{k=0}^{l}\left\|\frac{\xi_p^{m_{k+1}}-\xi_p^{m_k}}{\Delta s}\right\|_{\Omega_p}^2$$
$$+\frac{2gC_1C_2}{\nu k_{min}}\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\left(\nu\|\nabla\boldsymbol{e}_{2,f}^i\|_{\Omega_f}^2+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_k}\|_{\Omega_p}^2\right)$$
$$+\left(\frac{2rgC_1C_2}{\nu k_{min}}+1+\frac{C_t^2C_p(d-1)\alpha g}{\sqrt{\nu g k_{min}}}\right)$$
$$\nu\Delta t^2\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2+\left(\frac{2rgC_1C_2}{\nu k_{min}}+\frac{gC_1C_2}{\nu k_{min}}\right)\Delta t^2\sum_{k=0}^{l}\sum_{i=m_k}^{m_{k+1}-1}$$
$$g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2$$
$$+g\Delta s^2\sum_{k=0}^{l}\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_s e_{1,p}^{m_{k+1}}\|_{\Omega_p}^2+\frac{gC_1C_2}{\nu k_{min}}\Delta s^2\sum_{k=0}^{l}\nu\|\nabla d_s\boldsymbol{e}_{1,f}^{m_{k+1}}\|_{\Omega_f}^2. \tag{53}$$

From (30), we have

$$\Delta s \sum_{k=0}^{l} \|d_s \underline{\boldsymbol{e}}_1^{m_{k+1}}\|_W^2 = \Delta s \sum_{k=0}^{l} \left\| \frac{\underline{\boldsymbol{e}}_1^{m_{k+1}} - \underline{\boldsymbol{e}}_1^{m_k}}{\Delta s} \right\|_W^2 = \frac{\Delta s}{r^2} \sum_{k=0}^{l} \left\| \sum_{i=m_k}^{m_{k+1}-1} d_t \underline{\boldsymbol{e}}_1^{i+1} \right\|_W^2$$
$$\leq \Delta t \sum_{k=0}^{l} \sum_{i=m_k}^{m_{k+1}-1} \|d_t \underline{\boldsymbol{e}}_1^{i+1}\|_W^2.$$

Under the condition (36), it is easy to get $C\Delta t < h$. Considering (42), (43), (44), (37), (30) and (45), we can get

$$\nu \|\nabla \boldsymbol{e}_{2,f}^{m_{l+1}}\|_{\Omega_f}^2 + g\|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_{l+1}}\|_{\Omega_p}^2 \leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T)$$
$$\times \left(\Delta t^3 + h^{2k_1+1} + h^{2k_2+1} + \frac{h^{2k_1+2} + h^{2k_2+2}}{\Delta t} \right). \quad (54)$$

Now, we prove the estimate (50). Taking $\boldsymbol{v}_h = 2(\boldsymbol{e}_{2,f}^{m+1} - \boldsymbol{e}_{2,f}^{m}) = 2\Delta t d_t \boldsymbol{e}_{2,f}^{m+1}$ in (34) and summing it over $m = m_k, m_k + 1, \ldots, m_k + J$ with $0 \leq k \leq M - 1$ and $-1 \leq J \leq r - 1$, we get

$$2\Delta t \sum_{i=m_k}^{m_k+J} \|d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \nu \|\nabla \boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 - \nu \|\nabla \boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2$$
$$+\nu \Delta t^2 \sum_{i=m_k}^{m_k+J} \|\nabla d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2$$
$$+\sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \left(\|\boldsymbol{e}_{2,f}^{m_k+J+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 - \|\boldsymbol{e}_{2,f}^{m_k} \cdot \tau_j\|_{L^2(\Gamma)}^2 \right)$$
$$+ \sum_{i=m_k}^{m_k+J} \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|(\boldsymbol{e}_{2,f}^{i+1} - \boldsymbol{e}_{2,f}^{i}) \cdot \tau_j\|_{L^2(\Gamma)}^2$$
$$= 2\Delta t \sum_{i=m_k}^{m_k+J} \left(\frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^{i}}{2} - \frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, d_t \boldsymbol{e}_{2,f}^{i+1} \right)$$
$$-2g\Delta t \sum_{i=m_k}^{m_k+J} \int_\Gamma (e_{2,p}^{m_k} - e_{1,p}^{m_k} + e_{1,p}^{i+1}) d_t \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f$$
$$+\Delta t^2 \sum_{i=m_k}^{m_k+J} a_f(d_t \boldsymbol{e}_{1,f}^{i+1}, d_t \boldsymbol{e}_{2,f}^{i+1}) + g\Delta t^2 \sum_{i=m_k}^{m_k+J} \int_\Gamma d_t e_{1,p}^{i+1} d_t \boldsymbol{e}_{2,f}^{i+1} \cdot \boldsymbol{n}_f.$$

Then similar to the estimation of (51) in the above part, we finally arrive at

$$
\begin{aligned}
&\Delta t \sum_{i=m_k}^{m_k+J} \|d_t \boldsymbol{e}_{2,f}^{i+1}\|_{\Omega_f}^2 + \nu \|\nabla \boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 + \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{e}_{2,f}^{m_k+J+1} \cdot \tau_j\|_{L^2(\Gamma)}^2 \\
&\leq \nu \|\nabla \boldsymbol{e}_{2,f}^{m_k}\|_{\Omega_f}^2 + \sum_{j=1}^{d-1} \alpha \sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}} \|\boldsymbol{e}_{2,f}^{m_k} \cdot \tau_j\|_{L^2(\Gamma)}^2 + \frac{2gC_1C_2}{\nu k_{min}} \sum_{i=m_k}^{m_k+J} g \|\boldsymbol{K}^{\frac{1}{2}} \nabla e_{2,p}^{m_k}\|_{\Omega_p}^2 \\
&\quad + \frac{8C_P^2}{\nu} \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{u}_t^{i+1} + \boldsymbol{u}_t^i}{2} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} \right\|_{\Omega_f}^2 + \frac{8C_P^2}{\nu} \sum_{i=m_k}^{m_k+J} \left\| \frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} \right\|_{\Omega_f}^2 \\
&\quad + \left(1 + \frac{C_t^2 C_P (d-1)\alpha g}{\sqrt{\nu g k_{min}}}\right) \nu \Delta t^2 \sum_{i=m_k}^{m_k+J} \|\nabla d_t \boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}^2 \\
&\quad + \left(\frac{2rgC_1C_2}{\nu k_{min}} + \frac{gC_1C_2}{\nu k_{min}}\right) \Delta t^2 \sum_{i=m_k}^{m_k+J} g \|\boldsymbol{K}^{\frac{1}{2}} \nabla d_t e_{1,p}^{i+1}\|_{\Omega_p}^2.
\end{aligned}
$$

Then considering (54), (42), (43), (44), (37), (30) and (45), we can obtain

$$
\begin{aligned}
\nu \|\nabla \boldsymbol{e}_{2,f}^{m_k+J+1}\|_{\Omega_f}^2 \leq\ & C(r, d, \alpha, g, \nu, k_{min}, S_0) \\
& \times \left(\Delta t^3 + h^{2k_1+1} + h^{2k_2+1} + \frac{h^{2k_1+2} + h^{2k_2+2}}{\Delta t}\right).
\end{aligned}
$$

Thus we complete the proof. □

Corollary 2 *Under the assumptions of Theorem 2, we have, for* $0 \leq m \leq N-1$ *and* $0 \leq l \leq M-1$*,*

$$
\begin{aligned}
& \nu \|\nabla(\boldsymbol{u}_{2,f}^{m+1} - \boldsymbol{u}^{m+1})\|_{\Omega_f}^2 + g \|\boldsymbol{K}^{\frac{1}{2}} \nabla(\varphi_{2,p}^{m_{l+1}} - \varphi^{m_{l+1}})\|_{\Omega_p}^2 \\
& \leq C(r, d, \alpha, g, \nu, k_{min}, S_0, T) \left(\Delta t^3 + h^{2k_1} + h^{2k_2} + \frac{h^{2k_1+2} + h^{2k_2+2}}{\Delta t}\right).
\end{aligned}
$$

Proof Using the triangle inequality and considering the approximation properties (26) and (49), we can obtain this result. □

Now, we are going to provide the estimate of pressure. Before we do that, let us present a lemma of the estimates for errors in derivatives of the first-order coupled scheme (17).

Lemma 5 *Under assumptions (28), we have, for* $1 \leq m \leq N-1$*,*

$$
\|d_t \varepsilon_{1,f}^{m+1}\|_{\Omega_f} \lesssim 1. \tag{55}
$$

Proof Considering (17) and (11), we obtain the error equation, for $0 \leq i \leq N-1$,

$$\begin{cases} \left[\frac{\underline{\boldsymbol{e}}_1^{i+1}-\underline{\boldsymbol{e}}_1^i}{\Delta t}, \underline{z}_h\right] + a(\underline{\boldsymbol{e}}_1^{i+1}, \underline{z}_h) + b(\underline{z}_h, \varepsilon_1^{i+1}) = \left[\underline{\boldsymbol{w}}_t^{i+1} - \frac{\underline{\boldsymbol{w}}^{i+1}-\underline{\boldsymbol{w}}^i}{\Delta t}, \underline{z}_h\right] + \left[\frac{\underline{\boldsymbol{\xi}}^{i+1}-\underline{\boldsymbol{\xi}}^i}{\Delta t}, \underline{z}_h\right], \\ b(\underline{\boldsymbol{e}}_1^{i+1}, q_h) = 0. \end{cases} \quad (56)$$

Subtracting (56) on two adjacent time levels, we have

$$\left[\frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}, \underline{z}_h\right] + a(d_t\underline{\boldsymbol{e}}_1^{i+1}, \underline{z}_h) + b(\underline{z}_h, d_t\varepsilon_1^{i+1})$$
$$= \frac{1}{\Delta t}\left[\underline{\boldsymbol{w}}_t^{i+1} - \frac{\underline{\boldsymbol{w}}^{i+1} - \underline{\boldsymbol{w}}^i}{\Delta t} - \underline{\boldsymbol{w}}_t^i + \frac{\underline{\boldsymbol{w}}^i - \underline{\boldsymbol{w}}^{i-1}}{\Delta t}, \underline{z}_h\right] + \frac{1}{\Delta t}\left[\frac{\underline{\boldsymbol{\xi}}^{i+1} - \underline{\boldsymbol{\xi}}^i}{\Delta t} - \frac{\underline{\boldsymbol{\xi}}^i - \underline{\boldsymbol{\xi}}^{i-1}}{\Delta t}, \underline{z}_h\right]. \quad (57)$$

Taking $\underline{z}_h = (\boldsymbol{v}_h, 0)$ in (57), we have

$$\left(\frac{d_t\boldsymbol{e}_{1,f}^{i+1} - d_t\boldsymbol{e}_{1,f}^i}{\Delta t}, \boldsymbol{v}_h\right) + a_f(d_t\boldsymbol{e}_{1,f}^{i+1}, \boldsymbol{v}_h) + g\int_\Gamma d_t e_{1,p}^{i+1}\boldsymbol{v}_h \cdot \boldsymbol{n}_f + b(\boldsymbol{v}_h, d_t\varepsilon_1^{i+1})$$
$$= \frac{1}{\Delta t}\left(\boldsymbol{u}_t^{i+1} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} - \boldsymbol{u}_t^i + \frac{\boldsymbol{u}^i - \boldsymbol{u}^{i-1}}{\Delta t}, \boldsymbol{v}_h\right) + \frac{1}{\Delta t}\left(\frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} - \frac{\boldsymbol{\xi}_f^i - \boldsymbol{\xi}_f^{i-1}}{\Delta t}, \boldsymbol{v}_h\right). \quad (58)$$

Therefore

$$b(\boldsymbol{v}_h, d_t\varepsilon_1^{i+1}) \leq \|\nabla\boldsymbol{v}_h\|_{\Omega_f}\left(C_p\|\frac{d_t\boldsymbol{e}_{1,f}^{i+1} - d_t\boldsymbol{e}_{1,f}^i}{\Delta t}\|_{\Omega_f} + \nu\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}\right.$$
$$+C_t C_p^{\frac{1}{2}}\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\boldsymbol{e}_{1,f}^{i+1}\cdot\tau_j\|_{L^2(\Gamma)}$$
$$+C_1^{\frac{1}{2}}C_2^{\frac{1}{2}}g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p} + \frac{C_p}{\Delta t}\left(\|\boldsymbol{u}_t^{i+1} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} - \boldsymbol{u}_t^i + \frac{\boldsymbol{u}^i - \boldsymbol{u}^{i-1}}{\Delta t}\|_{\Omega_f}\right.$$
$$\left.\left.+\ \|\frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} - \frac{\boldsymbol{\xi}_f^i - \boldsymbol{\xi}_f^{i-1}}{\Delta t}\|_{\Omega_f}\right)\right).$$

Then the discrete inf-sup condition (9) implies

$$\|d_t\varepsilon_1^{i+1}\|_{\Omega_f} \leq C(d, \alpha, \nu, g, k_{min})\left(\|\frac{d_t\boldsymbol{e}_{1,f}^{i+1} - d_t\boldsymbol{e}_{1,f}^i}{\Delta t}\|_{\Omega_f} + \nu\|\nabla d_t\boldsymbol{e}_{1,f}^{i+1}\|_{\Omega_f}\right.$$
$$+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p} + \frac{1}{\Delta t}\|\boldsymbol{u}_t^{i+1} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} - \boldsymbol{u}_t^i + \frac{\boldsymbol{u}^i - \boldsymbol{u}^{i-1}}{\Delta t}\|_{\Omega_f}$$
$$\left.+\frac{1}{\Delta t}\|\frac{\boldsymbol{\xi}_f^{i+1} - \boldsymbol{\xi}_f^i}{\Delta t} - \frac{\boldsymbol{\xi}_f^i - \boldsymbol{\xi}_f^{i-1}}{\Delta t}\|_{\Omega_f}\right). \quad (59)$$

Taking $\underline{z}_h = 2(d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i)$ in (57) leads to

$$\begin{aligned}
&2\Delta t\|\frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\|_0^2 + \|d_t\underline{\boldsymbol{e}}_1^{i+1}\|_W^2 - \|d_t\underline{\boldsymbol{e}}_1^i\|_W^2 + \|d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i\|_W^2 \\
&+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\left(\|d_t\underline{\boldsymbol{e}}_1^{i+1}\cdot\tau_j\|_{L^2(\Gamma)}^2 - \|d_t\underline{\boldsymbol{e}}_1^i\cdot\tau_j\|_{L^2(\Gamma)}^2\right) \\
&+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|(d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i\cdot\tau_j)\cdot\tau_j\|_{L^2(\Gamma)}^2 \\
&= 2\left[\underline{\boldsymbol{w}}_t^{i+1} - \frac{\underline{\boldsymbol{w}}^{i+1} - \underline{\boldsymbol{w}}^i}{\Delta t} - \underline{\boldsymbol{w}}_t^i + \frac{\underline{\boldsymbol{w}}^i - \underline{\boldsymbol{w}}^{i-1}}{\Delta t}, \frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\right] \\
&+2\left[\frac{\underline{\boldsymbol{\xi}}^{i+1} - \underline{\boldsymbol{\xi}}^i}{\Delta t} - \frac{\underline{\boldsymbol{\xi}}^i - \underline{\boldsymbol{\xi}}^{i-1}}{\Delta t}, \frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\right].
\end{aligned} \tag{60}$$

Since

$$\underline{\boldsymbol{w}}_t^{i+1} - \frac{\underline{\boldsymbol{w}}^{i+1} - \underline{\boldsymbol{w}}^i}{\Delta t} - \underline{\boldsymbol{w}}_t^i + \frac{\underline{\boldsymbol{w}}^i - \underline{\boldsymbol{w}}^{i-1}}{\Delta t} = \Delta t\int_{t^i}^{t^{i+1}} \underline{\boldsymbol{w}}_{ttt}(\tau)d\tau, \quad \frac{\underline{\boldsymbol{\xi}}^{i+1} - \underline{\boldsymbol{\xi}}^i}{\Delta t} - \frac{\underline{\boldsymbol{\xi}}^i - \underline{\boldsymbol{\xi}}^{i-1}}{\Delta t} = \int_{t^i}^{t^{i+1}} \underline{\boldsymbol{\xi}}_{tt}(\tau)d\tau,$$

then using the Young and Hölder inequalities yields

$$\begin{aligned}
&2\left[\underline{\boldsymbol{w}}_t^{i+1} - \frac{\underline{\boldsymbol{w}}^{i+1} - \underline{\boldsymbol{w}}^i}{\Delta t} - \underline{\boldsymbol{w}}_t^i + \frac{\underline{\boldsymbol{w}}^i - \underline{\boldsymbol{w}}^{i-1}}{\Delta t}, \frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\right] \\
&\le \varepsilon\Delta t\|\frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\|_0^2 + \frac{1}{\varepsilon}\Delta t^2\int_{t^i}^{t^{i+1}} \|\underline{\boldsymbol{w}}_{ttt}(\tau)\|_0^2 d\tau, \\
&2\left[\frac{\underline{\boldsymbol{\xi}}^{i+1} - \underline{\boldsymbol{\xi}}^i}{\Delta t} - \frac{\underline{\boldsymbol{\xi}}^i - \underline{\boldsymbol{\xi}}^{i-1}}{\Delta t}, \frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\right] \\
&\le \varepsilon\Delta t\|\frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\|_0^2 + \frac{1}{\varepsilon}\int_{t^i}^{t^{i+1}} \|\underline{\boldsymbol{\xi}}_{tt}(\tau)\|_0^2 d\tau.
\end{aligned}$$

Combine the above bounds with (60) and set $\varepsilon = 1/2$. Since we do not define $d_t\underline{\boldsymbol{e}}_1^0$. Next, we only sum the equation from $i = 1$ to $i = m$ with $1 \le m \le N-1$. Hence, considering (28) and (12) leads to

$$\begin{aligned}
&\Delta t\sum_{i=1}^{m}\left\|\frac{d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i}{\Delta t}\right\|_0^2 + \|d_t\underline{\boldsymbol{e}}_1^{m+1}\|_W^2 + \sum_{i=1}^{m}\|d_t\underline{\boldsymbol{e}}_1^{i+1} - d_t\underline{\boldsymbol{e}}_1^i\|_W^2 \\
&+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^{m+1}\cdot\tau_j\|_{L^2(\Gamma)}^2 \le \|d_t\underline{\boldsymbol{e}}_1^1\|_W^2
\end{aligned}$$

$$+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^1\cdot\tau_j\|^2_{L^2(\Gamma)}+2\Delta t^2\sum_{i=1}^{m}\int_{t^i}^{t^{i+1}}\|\underline{\boldsymbol{w}}_{ttt}(\tau)\|_0^2 d\tau$$
$$+2\sum_{i=1}^{m}\int_{t^i}^{t^{i+1}}\|\underline{\boldsymbol{\xi}}_{tt}(\tau)\|_0^2 d\tau\le\|d_t\underline{\boldsymbol{e}}_1^1\|_W^2+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^1\cdot\tau_j\|^2_{L^2(\Gamma)}$$
$$+C(\Delta t^2+h^{2k_1}+h^{2k_2}). \tag{61}$$

Now, we will estimate $d_t\underline{\boldsymbol{e}}_1^1$. Note that $\underline{\boldsymbol{e}}_1^0=\mathbf{0}$. Then considering (56) at time t^1 and taking $\underline{\boldsymbol{z}}_h=2d_t\underline{\boldsymbol{e}}_1^1$, we obtain

$$2\|d_t\underline{\boldsymbol{e}}_1^1\|_0^2+2\Delta t\|d_t\underline{\boldsymbol{e}}_1^1\|_W^2+2\Delta t\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^1\cdot\tau_j\|^2_{L^2(\Gamma)}$$
$$\le 2\left[\underline{\boldsymbol{w}}_t^2-\frac{\underline{\boldsymbol{w}}^2-\underline{\boldsymbol{w}}^1}{\Delta t},d_t\underline{\boldsymbol{e}}_1^1\right]+2\left[\frac{\underline{\boldsymbol{\xi}}^2-\underline{\boldsymbol{\xi}}^1}{\Delta t},d_t\underline{\boldsymbol{e}}_1^1\right]$$
$$\le 2\varepsilon_1\|d_t\underline{\boldsymbol{e}}_1^1\|_0^2+\frac{1}{\varepsilon_1}\left\|\underline{\boldsymbol{w}}_t^2-\frac{\underline{\boldsymbol{w}}^2-\underline{\boldsymbol{w}}^1}{\Delta t}\right\|_0^2$$
$$+\frac{1}{\varepsilon_1}\left\|\frac{\underline{\boldsymbol{\xi}}^2-\underline{\boldsymbol{\xi}}^1}{\Delta t}\right\|_0^2$$

Setting $\varepsilon_1=1/2$ and considering (28), we obtain

$$\|d_t\underline{\boldsymbol{e}}_1^1\|_0^2+\Delta t\|d_t\underline{\boldsymbol{e}}_1^1\|_W^2+\Delta t\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^1\cdot\tau_j\|^2_{L^2(\Gamma)}\le C(\Delta t^2+h^{2k_1+2}+h^{2k_2+2}).$$

Hence,

$$\|d_t\underline{\boldsymbol{e}}_1^1\|_W^2+\sum_{j=1}^{d-1}\alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|d_t\underline{\boldsymbol{e}}_1^1\cdot\tau_j\|^2_{L^2(\Gamma)}\le C\left(\Delta t+\frac{h^{2k_1+2}+h^{2k_2+2}}{\Delta t}\right).$$

Combining it with (61) yields

$$\Delta t\sum_{i=1}^{m}\left\|\frac{d_t\underline{\boldsymbol{e}}_1^{i+1}-d_t\underline{\boldsymbol{e}}_1^{i}}{\Delta t}\right\|_0^2+\|d_t\underline{\boldsymbol{e}}_1^{m+1}\|_W^2+\sum_{i=1}^{m}\|d_t\underline{\boldsymbol{e}}_1^{i+1}-d_t\underline{\boldsymbol{e}}_1^{i}\|_W^2$$
$$\le C\left(\Delta t+h^{2k_1}+h^{2k_2}+\frac{h^{2k_1+2}+h^{2k_2+2}}{\Delta t}\right). \tag{62}$$

It is easy to obtain that for all $1\le m\le N-1$,

$$\left\|\frac{d_t\underline{\boldsymbol{e}}_1^{m+1}-d_t\underline{\boldsymbol{e}}_1^{m}}{\Delta t}\right\|_0^2\lesssim 1. \tag{63}$$

Combining (62) and (63) with (59), and considering (28) and (12), we can obtain the final result (55). □

Now, we are ready to provide the error estimate of pressure in the decoupled scheme Algorithm 1.

Theorem 5 *Let the assumptions of Theorem 2 be satisfied. Assume also that the true solution p satisfies the following regularity*

$$p \in L^{\infty}(0, T; H^{k_1}(\Omega_f)). \tag{64}$$

Then we have, for $1 \le m \le N-1$,

$$\|p_{2,h}^{m+1} - p^{m+1}\|_{\Omega_f} \le C(r, d, \alpha, g, \nu, k_{min}, S_0, T)\left(\Delta t + h^{k_1} + h^{k_2} + \frac{h^{k_1+1} + h^{k_2+1}}{\Delta t}\right). \tag{65}$$

Proof From (34), for $t^m \in (t^{m_k}, t^{m_{k+1}}]$ with $0 \le k \le M-1$, we have,

$$\begin{aligned} b(\boldsymbol{v}_h, \varepsilon_2^{m+1}) = & -\left(\frac{\boldsymbol{e}_{2,f}^{m+1} - \boldsymbol{e}_{2,f}^{m}}{\Delta t}, \boldsymbol{v}_h\right) - a_f(\boldsymbol{e}_{2,f}^{m+1}, \boldsymbol{v}_h) \\ & + \left(\frac{\boldsymbol{u}_t^{m+1} + \boldsymbol{u}_t^{m}}{2} - \frac{\tilde{\boldsymbol{u}}^{m+1} - \tilde{\boldsymbol{u}}^{m}}{\Delta t}, \boldsymbol{v}_h\right) - g\int_{\Gamma}(\varphi_{2,h}^{m_k} - \tilde{\varphi}^{m+1})\boldsymbol{v}_h \cdot \boldsymbol{n}_f \\ & + g\int_{\Gamma}(\varphi_{1,h}^{m_k} - \varphi_{1,h}^{m+1})\boldsymbol{v}_h \cdot \boldsymbol{n}_f + a_f\left(\frac{\boldsymbol{e}_{1,f}^{m+1} - \boldsymbol{e}_{1,f}^{m}}{2}, \boldsymbol{v}_h\right) \\ & + g\int_{\Gamma}\left(\frac{e_{1,p}^{m+1} - e_{1,p}^{m}}{2}\right)\boldsymbol{v}_h \cdot \boldsymbol{n}_f + b\left(\boldsymbol{v}_h, \frac{\varepsilon_1^{m+1} - \varepsilon_1^{m}}{2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} b(\boldsymbol{v}_h, \varepsilon_2^{m+1}) \le \|\nabla \boldsymbol{v}_h\|_{\Omega_f} \Bigg(& C_p\|d_t\boldsymbol{e}_{2,f}^{m+1}\|_{\Omega_f} + \nu\|\nabla \boldsymbol{e}_{2,f}^{m+1}\|_{\Omega_f} \\ & + C_t C_p^{\frac{1}{2}} \sum_{j=1}^{d-1} \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\boldsymbol{K})}}\|\boldsymbol{e}_{2,f}^{m+1} \cdot \tau_j\|_{L^2(\Gamma)} \\ & + C_p\left\|\frac{\boldsymbol{u}_t^{m+1} + \boldsymbol{u}_t^{m}}{2} - \frac{\tilde{\boldsymbol{u}}^{m+1} - \tilde{\boldsymbol{u}}^{m}}{\Delta t}\right\|_{\Omega_f} + C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_k}\|_{\Omega_p} \\ & + C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} g\|\boldsymbol{K}^{\frac{1}{2}}\nabla(e_{1,p}^{m+1} - e_{1,p}^{m_k})\|_{\Omega_p} \\ & + \frac{\Delta t}{2}\nu\|\nabla d_t\boldsymbol{e}_{1,f}^{m+1}\|_{\Omega_f} + \frac{\Delta t}{2}C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{m+1}\|_{\Omega_p} + \frac{\Delta t}{2}\|d_t\varepsilon_1^{m+1}\|_{\Omega_f}\Bigg). \end{aligned}$$

Using the discrete inf-sup condition (9), we arrive at

$$\|\varepsilon_2^{m+1}\|_{\Omega_f} \leq C(d,\alpha,g,\nu,k_{min}) \Bigg(\|d_t \boldsymbol{e}_{2,f}^{m+1}\|_{\Omega_f} + \nu\|\nabla \boldsymbol{e}_{2,f}^{m+1}\|_{\Omega_f}$$
$$+ \left\| \frac{\boldsymbol{u}_t^{m+1}+\boldsymbol{u}_t^m}{2} - \frac{\tilde{\boldsymbol{u}}^{m+1}-\tilde{\boldsymbol{u}}^m}{\Delta t} \right\|_{\Omega_f}$$
$$+g\|\boldsymbol{K}^{\frac{1}{2}}\nabla e_{2,p}^{m_k}\|_{\Omega_p} + g\Delta t\sum_{i=m_k}^{m}\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{i+1}\|_{\Omega_p} + \Delta t\nu\|\nabla d_t \boldsymbol{e}_{1,f}^{m+1}\|_{\Omega_f}$$
$$+\Delta t g\|\boldsymbol{K}^{\frac{1}{2}}\nabla d_t e_{1,p}^{m+1}\|_{\Omega_p} + \Delta t\|d_t\varepsilon_1^{m+1}\|_{\Omega_f}\Bigg).$$

From (37) and (46), we have

$$\|d_t \boldsymbol{e}_{2,f}^{m+1}\|_{\Omega_f} = \left\| \frac{\boldsymbol{e}_{2,f}^{m+1} - \boldsymbol{e}_{2,f}^m}{\Delta t} \right\|_{\Omega_f} \leq C\left(\Delta t + \frac{h^{k_1+1}+h^{k_2+1}}{\Delta t}\right).$$

By applying (49), (50), (41), (28), (12) and (55), we obtain

$$\|\varepsilon_2^{m+1}\|_{\Omega_f} \leq C\left(\Delta t + h^{k_1} + h^{k_2} + \frac{h^{k_1+1}+h^{k_2+1}}{\Delta t}\right).$$

Combining it with (64) and (27), and using the triangle inequality lead to the final result (65). □

6 Numerical experiments

In this part, we provide some numerical experiments to verify the convergence rates and the effectiveness of our scheme. All the numerical results are obtained by using the software package FreeFem++. The finite element spaces are constructed by the Mini elements for the Stokes equations and the linear Lagrangian elements for the Darcy equations.

Let the computational domain Ω be composed of $\Omega_f = [0,1]\times[1,2]$ and $\Omega_p = [0,1]\times[0,1]$ with the interface $\Gamma = (0,1)\times[1]$. The exact solutions of the unsteady Stokes/Darcy equations are given by

$$\begin{aligned}
\boldsymbol{u}(x,y,t) &= \left(x^2(y-1)^2+y)\cos(t), \left(-\frac{2}{3}x(y-1)^3\right)\cos(t) + (2-\pi\sin(\pi x))\cos(t)\right),\\
p(x,y,t) &= (2-\pi\sin(\pi x))\sin\left(\frac{\pi}{2}y\right)\cos(t),\\
\varphi(x,y,t) &= (2-\pi\sin(\pi x))\,(1-y-\cos(\pi y))\cos(t).
\end{aligned} \tag{66}$$

The initial conditions and the forcing terms follow the above exact solutions. In addition, we set all physical parameters $\alpha, g, \nu, k_{min}, S_0$ equal to 1.

To examine the orders of convergence with respect to the time step Δt or the mesh size h, we give the following measure of the convergence. If we assume that

$$u_h^{\Delta t}(x,t^m) \approx u(x,t^m) + C_1(x,t^m)\Delta t^{\gamma} + C_2(x,t^m)h^{\mu},$$

the measures testing the convergence order of the time step Δt and the mesh size h are defined as follows:

$$\rho_{u,\Delta t,i} = \frac{\| u_h^{\Delta t}(x,t^m) - u_h^{\frac{\Delta t}{2}}(x,t^m) \|_i}{\| u_h^{\frac{\Delta t}{2}}(x,t^m) - u_h^{\frac{\Delta t}{4}}(x,t^m) \|_i} \approx \frac{4^{\gamma}-2^{\gamma}}{2^{\gamma}-1},$$
$$\rho_{u,h,i} = \frac{\| u_h^{\Delta t}(x,t^m) - u_{\frac{h}{2}}^{\Delta t}(x,t^m) \|_i}{\| u_{\frac{h}{2}}^{\Delta t}(x,t^m) - u_{\frac{h}{4}}^{\Delta t}(x,t^m) \|_i} \approx \frac{4^{\mu}-2^{\mu}}{2^{\mu}-1}.$$

Here u can be $\boldsymbol{u}$, p, φ and i can be 0, 1. While $\rho_{u,\Delta t,0}$ and $\rho_{u,h,0}$ approach 4.0 or 2.0, the convergence order will be 2.0 or 1.0.

In Table 1, we provide the convergence order with the fixed mesh size $h = 1/10$ and the varying time step length $\Delta t = 1/40,\ 1/80,\ 1/160,\ 1/320,\ 1/640$ with the large time size $\Delta s = 5\Delta t$. The values of $\rho_{\boldsymbol{u}_{2,h}^m,\Delta t,0}$ and $\rho_{\varphi_{2,h}^m,\Delta t,0}$ approximate 4.0 meaning that the orders of convergence in time is $O(\Delta t^2)$ for the L^2-norm of $\boldsymbol{u}_{2,h}^m$ and $\varphi_{2,h}^m$. These are consist with our theoretical analysis results of (48). However, since the values of $\rho_{\boldsymbol{u}_{2,h}^m,\Delta t,1}$, $\rho_{p_{2,h}^m,\Delta t,0}$ and $\rho_{\varphi_{2,h}^m,\Delta s,1}$ approach 4.0, this means that the error estimates $O(\Delta t^2)$ for H^1-norm of $\boldsymbol{u}_{2,h}^m$ and $\varphi_{2,h}^m$ and L^2-norm of $p_{2,h}^m$, which is better than the results of our analysis. So in the further work, it is necessary to find a finer analysis method to improve the convergence order. In Table 2, we study the convergence order of the mesh size with the fixed time step $\Delta t = 1/20$ and

Table 1 The convergence orders with respect to the time step Δt at time $t^m = 1$, with the fixed mesh size $h = 1/10$ and $\Delta s = 5\Delta t$

Δt	$\|\boldsymbol{u}_{2,h}^{m,\Delta t} - \boldsymbol{u}_{2,h}^{m,\frac{\Delta t}{2}}\|_{\Omega_f}$	$\rho_{\boldsymbol{u}_{2,h}^m,\Delta t,0}$	$\nu\|\nabla(\boldsymbol{u}_{2,h}^{m,\Delta t} - \boldsymbol{u}_{2,h}^{m,\frac{\Delta t}{2}})\|_{\Omega_f}$	$\rho_{\boldsymbol{u}_{2,h}^m,\Delta t,1}$	$\|p_{2,h}^{m,\Delta t} - p_{2,h}^{m,\frac{\Delta t}{2}}\|_{\Omega_f}$	$\rho_{p_{2,h}^m,\Delta t,0}$
1/40	2.19484e−6	3.92634	1.42000e−5	3.91248	8.44944e−5	3.73121
1/80	5.59004e−7	3.98131	3.62940e−6	3.98603	2.26453e−5	3.90132
1/160	1.40407e−7	4.00229	9.10532e−7	4.01140	5.80452e−6	3.96321
1/320	3.50817e−8	4.00601	2.26986e−7	4.01353	1.46460e−6	3.98733
1/640	8.75726e−9		5.65552e−8		3.67313e−7	
Δt	$\|\varphi_{2,h}^{m,\Delta s} - \varphi_{2,h}^{m,\frac{\Delta s}{2}}\|_{\Omega_p}$	$\rho_{\varphi_{2,h}^m,\Delta s,0}$	$g\|\boldsymbol{K}^{\frac{1}{2}}\nabla(\varphi_{2,h}^{m,\Delta s} - \varphi_{2,h}^{m,\frac{\Delta s}{2}})\|_{\Omega_p}$	$\rho_{\varphi_{2,h}^m,\Delta s,1}$		
1/40	2.27759e−5	3.82193	9.88402e−5	3.83789		
1/80	5.95927e−6	3.92282	2.57538e−5	3.98261		
1/160	1.51913e−6	3.96386	6.55543e−6	3.96633		
1/320	3.83245e−7	3.98253	1.65277e−6	3.98366		
1/640	9.62315e−8		4.14887e−7			

Table 2 The convergence orders with respect to the mesh size h at time $t^m = 1$, with the fixed time step length $\Delta t = 1/20$ and $\Delta s = 5\Delta t$

h	$\Vert\boldsymbol{u}_{2,h}^{m,h} - \boldsymbol{u}_{2,h}^{m,\frac{h}{2}}\Vert_{\Omega_f}$	$\rho_{\boldsymbol{u}_{2,h}^m,h,0}$	$\nu\Vert\nabla(\boldsymbol{u}_{2,h}^{m,h} - \boldsymbol{u}_{2,h}^{m,\frac{h}{2}})\Vert_{\Omega_f}$	$\rho_{\boldsymbol{u}_{2,h}^m,h,1}$	$\Vert p_{2,h}^{m,\Delta t} - p_{2,h}^{m,\frac{\Delta t}{2}}\Vert_{\Omega_f}$	$\rho_{p_{2,h}^m,\Delta t,0}$
1/8	1.30214e−2	4.03917	2.67049e−1	2.11343	5.21996e−3	7.85237
1/16	3.22378e−3	3.80942	1.26358e−1	1.56795	6.64762e−4	5.00050
1/32	8.46265e−4	4.12117	8.05883e−2	2.25087	1.32939e−4	4.58208
1/64	2.05346e−4	4.06704	3.58031e−2	2.18565	2.90128e−4	1.78770
1/128	5.04903e−5		1.63810e−2		1.62291e−4	
h	$\Vert\varphi_{2,h}^{m,h} - \varphi_{2,h}^{m,\frac{h}{2}}\Vert_{\Omega_p}$	$\rho_{\varphi_{2,h}^m,h,0}$	$g\Vert\boldsymbol{K}^{\frac{1}{2}}\nabla(\varphi_{2,h}^{m,h} - \varphi_{2,h}^{m,\frac{h}{2}})\Vert_{\Omega_p}$	$\rho_{\varphi_{2,h}^m,h,1}$		
1/8	1.31950e−2	4.02975	3.37379e−1	2.43021		
1/16	3.27440e−3	3.83958	1.38827e−1	1.54550		
1/32	8.52802e−4	4.12035	8.98262e−2	2.38848		
1/64	2.06973e−4	3.86570	3.76081e−2	1.78770		
1/128	5.35409e−5		2.13003e−2			

$\Delta s = 5\Delta t$ and varying spacing $h = 1/8, 1/16, 1/32, 1/64, 1/128$. The results show that our scheme owns the optimal convergence order in space. These numerical results coincide with our theoretical analysis results.

For the size of ratio between two time step lengths, we will show its influence through the errors between the exact solutions and the numerical solutions of Algorithm 1. In Fig. 2, we provide the numerical errors for the decoupled second-order scheme, Algorithm 1, with different ratios r and mesh size h, and the fixed small time step length $\Delta t = 1/1200$. Furthermore, since the ratio $r = \Delta s/\Delta t$ is an integer, there is a finite number of ratio we can pick. These figures show that the smaller of mesh size h, the higher accuracy and the more stable. And in general, the bigger the ratio r, the higher accuracy. Next, we would like to indicate which value of r to be selected with the smallest errors for the different time step Δt. In Fig. 3, we fix the mesh size $h = 1/20$ and pick the different small time step

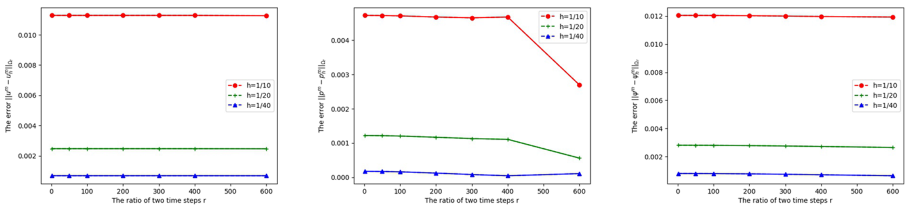

Fig. 2 The numerical errors for velocity (left), pressure (middle) and the hydraulic head (right) of Algorithm 1 at time $t^m = 1$. We fix the small time step length $\Delta t = 1/1200$, with varying mesh size h and the ratio of two time steps $r = \Delta s/\Delta t$

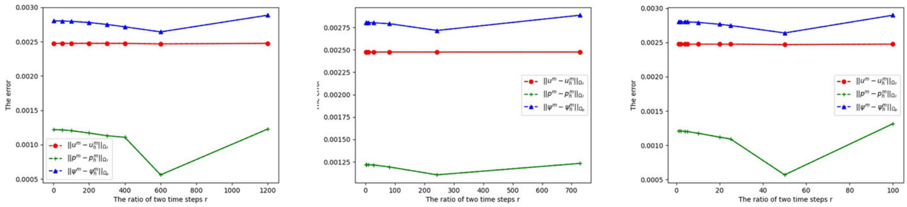

Fig. 3 The numerical errors of Algorithm 1 at time $t^m = 1$. We choose the fixed mesh size $h = 1/20$ and the small time step length $\Delta t = 1/1200$ (left), $\Delta t = 1/729$ (middle) and $\Delta t = 1/100$ (right) with the varying ratio $r = \Delta s/\Delta t$

length Δt. For the left one, we choose $\Delta t = 1200$ and pick the integer ration $r = 1, 50, 100, 200, 300, 400, 600, 1200$. For the next figure in the middle, we choose $\Delta t = 729$ and pick $r = 1, 3, 9, 27, 82, 243, 729$. For the last one, we choose $\Delta t = 1/100$ and pick $r = 1, 2, 4, 5, 10, 20, 25, 50, 100$. Figure 3 strongly indicates that if we want to obtain the smallest numerical errors, we need to pick the ratio r to be the biggest positive integer we can get: $r = 600$ for the left figure, $r = 243$ for the middle one and $r = 50$ for the right one. That means we can choose the biggest Δs that satisfies $\Delta t < \Delta s < T$ to minimize the errors. An explanation of this situation is that the fewer intermediate values we calculate, the smaller accumulative error of the algorithm.

Next, in order to show the improvement of our decoupled scheme (Algorithm 1), we will compare it with the coupled scheme (Algorithm 2). Table 3 gives the comparisons of the errors and the central processing unit (CPU) times for these two schemes. We provide the errors between the numerical solutions and exact solutions at time $t^m = 1$, with fixed times step size $\Delta t = 1/200$ but varying mesh size $h = 1/16,\ 1/64$. It is worth mentioning that since the coupled Stokes equation (18) in Algorithm 1 needs more time to be solved than the Darcy equation (19), the total computational cost of Algorithm 1 is dominated by the cost of (18). So, whatever the ratio is, as long as it is on the same grid, the CPU times for Algorithm 1 is almost the same. So for the decoupled scheme, we pick the largest ratio $r = 100$ leading to $\Delta s = 1/2$. The numerical results indicate that both the coupled and decoupled schemes own almost the same accuracy. However, the decoupled scheme with different subdomain time steps requires less CPU time than the coupled scheme.

Table 3 The comparisons of numerical errors and CPU times at time $t^m = 1$, with the fixed time step length $\Delta t = 1/200$ and $\Delta s = 1/2$

Algorithm	$h = 16$			$h = 64$		
	$\Vert\boldsymbol{u}^m_{2,h} - \boldsymbol{u}^m\Vert_{\Omega_f}$	$\Vert\varphi^m_{2,h} - \varphi^m\Vert_{\Omega_p}$	CPU	$\Vert\boldsymbol{u}^m_{2,h} - \boldsymbol{u}^m\Vert_{\Omega_f}$	$\Vert\varphi^m_{2,h} - \varphi^m\Vert_{\Omega_p}$	CPU
Coupled scheme	4.398248e−3	4.778007e−3	30.419	2.766481e−4	3.030442e−4	457.483
Decoupled scheme	4.388659e−3	4.619500e−3	20.492	2.756915e−4	2.439594e−4	337.470

7 Conclusion

In this paper, we present a second-order decoupled scheme with different subdomain time steps for the non-stationary Stokes/Darcy model, which is based on the second-order spectral deferred correction method in time and the finite element method in space. We prove that this scheme is stable and owns second-order convergence in time. Last, numerical experiments are conducted to demonstrate the accuracy and effectiveness of our decoupled scheme.

Funding This work is supported by the National Natural Science Foundation of China (Grant No. 11971378 & 11571274).

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