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An equal-order hybridized discontinuous Galerkin method with a small pressure penalty parameter for the Stokes equations

Yanren Hou, Yongbin Han *, Jing Wen

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China

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ABSTRACT

In this paper, an equal-order hybridized discontinuous Galerkin (HDG) method with a small pressure penalty parameter for the Stokes equations is analyzed. When the pressure penalty parameter γ tends to 0 ($\gamma > 0$), the velocity approximation tends to be H(div)-conforming and exactly divergence-free, and the velocity error bound tends to be pressure-robust. However, taking the value of γ too small will cause the over-stabilization of the pressure. Then, we can provide a post-processing procedure to obtain a stable pressure approximation.

1. Introduction

Recently, pressure-robust numerical methods are becoming more and more popular for the numerical simulations of the incompressible flows, which are often realized by constructing divergence-free finite element pairs [1–5] or non-divergence-free finite element pairs with divergence-free reconstruction operators [6–9]. As we see, these pairs of finite elements are inf–sup stable. The application of inf–sup stable pairs of finite elements often requires the use of different order spaces for velocity and pressure. The pairs of equal-order finite elements for velocity and pressure are often non-inf–sup stable finite elements pairs. In the equal-order finite element methods, the pressure stabilization, namely, the pressure–pressure coupling, is needed to achieve the discrete stability. The introduction of the pressure–pressure coupling perturbs the continuity equation, which prevents the method from being pressure-robust. The numerical methods based on equal-order velocities and pressures have been widely studied and applied to incompressible flows, such as the pressure stabilization Petrov–Galerkin (PSPG) method [10], continuous interior penalty (CIP) method [11,12], local projection stabilization (LPS) method [13,14], equal-order discontinuous Galerkin (DG) methods [15,16]. As far as we know, there may be not a standard equal-order pressure-robust finite element method in the literature.

In usual equal-order finite element methods, to alleviate the effect of small viscosity and large pressure on the velocity error, we often need to add some stabilization terms, such as grad-div stabilization and mass flux penalization. However, taking grad-div stabilization parameter too large will lead to an over-stabilizing effect in conforming finite element methods [17,18]. For the equal-order DG method, both grad-div stabilization and mass flux penalization are usually adopted simultaneously by following the idea of [19]. In addition, the DG methods are known to be computationally expensive. The worse thing is that the equal-order DG methods have more degrees of freedom than the mixed-order DG methods.

In view of the expensive computational cost of the discontinuous Galerkin (DG) methods, the hybridized discontinuous Galerkin (HDG) methods have been developed in [20]. The global degrees of freedom of HDG methods is defined on cell boundaries instead of the interior of the cells, the number of which is evidently smaller than that of DG methods, obviating the common criticism of DG methods. The HDG methods have also been proposed for the velocity–pressure, vorticity–velocity–pressure and velocity–pressure-gradient formulations of the Stokes equations [21–24]. The HDG method which we consider in this paper is for the velocity–pressure formulation of the Stokes equations. In 2017, stability and error analysis of both equal-order and mixed-order HDG methods for Stokes equations is presented in [23]. In 2019, a mixed-order embedded-hybridized discontinuous Galerkin (E-HDG) method is presented and analyzed for the Stokes equations in [25]. In [25], it is proved that the mixed-order E-HDG and HDG methods are H(div)-conforming, exactly divergence-free and pressure-robust. For the equal-order HDG method, the number of the global degree of freedom defined on cell boundaries is the same as that of the mixed-order divergence-free HDG method. However, for the equal-order HDG method, the pressure–pressure coupling prevents the equal-order HDG method from being pressure-robust. The equal-order HDG method is not H(div)-conforming, exactly divergence-free and pressure-robust [23].

In this paper, an equal-order HDG method with a small pressure penalty parameter for the Stokes equations is analyzed. The stability and error analysis of the equal-order HDG method with a small pressure penalty parameter are obtained. As γ, the pressure penalty parameter,

* Corresponding author.
E-mail addresses: yrhou@mail.xjtu.edu.cn (Y. Hou), hyb204514@stu.xjtu.edu.cn (Y. Han), wjzuishuai@stu.xjtu.edu.cn (J. Wen).

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tends to zero, the velocity approximation tends to be H(div)-conforming and exactly divergence-free, and the velocity error bound tends to be pressure-robust. As far as we know, it is the first equal-order method, in which by taking the small value of the pressure penalty parameter, the velocity error bound tends to be pressure-robust. However, a sufficiently small pressure penalty parameter will lead to the over-stabilization of the pressure. Furthermore, we provide a post-processing procedure for the over-stabilized pressure. It is natural to propose an equal-order embedded-hybridized discontinuous Galerkin (E-HDG) method for the Stokes equation. For the equal-order E-HDG method, the facet velocity functions are continuous, so it has fewer degrees of freedom than the equal-order HDG method on a given mesh. All the results in this paper hold true verbatim for the equal-order E-HDG method.

This paper is arranged as follows. In Section 2, we present the Stokes equations. In Section 3, we present an equal-order HDG method for the Stokes equations. In Section 4, we give some preliminaries. Energy estimates and error estimates are provided in Section 5. In Section 6, we present the post-processing. In Section 7, the analytical results are supported by some numerical experiments. Finally, conclusions are drawn in Section 8.

2. Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain ($d = 2$) or polyhedral domain ($d = 3$) with Lipschitz boundary $\partial\Omega$. We consider the Stokes equations

$$\begin{cases} -\nu\Delta u + \nabla p = f, & \Omega, \\ \nabla\cdot u = 0, & \Omega, \\ u = 0, & \partial\Omega, \end{cases} \tag{1}$$

where u is the velocity, p the pressure, $\nu > 0$ the viscosity, and f the external body force.

Introduce

$$X = \left[H_0^1(\Omega)\right]^d, \quad Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q dx = 0\right\}.$$

We present the weak formulation of (1): given $f \in \left[L^2(\Omega)\right]^d$, find $(u, p) \in (X, Q)$, such that

$$\begin{aligned} a(u, v) + b(v, p) &= F(v), & \forall v \in X, \\ b(u, q) &= 0, & \forall q \in Q, \end{aligned} \tag{2}$$

with

$$a(u, v) = \nu \int_\Omega \nabla u : \nabla v \, dx, \quad b(u, q) = -\int_\Omega q(\nabla\cdot u)\, dx, \quad F(v) = \int_\Omega f\cdot v\, dx.$$

3. An equal-order hybridized discontinuous Galerkin method

In this section, let us recall the equal-order HDG method for the Stokes equations in [23].

3.1. Notation

Let $\mathcal{T}_h$ be a shape-regular simplicial mesh of Ω. h_K denotes the diameter of each element $K \in \mathcal{T}_h$, and mesh size $h = max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{F}_h$ be the set of all facets and $\mathcal{F}$ be the mesh skeleton. The boundary of a cell K is denoted by ∂K and the outward unit normal vector on ∂K is denoted by n.

Consider the following finite element spaces on Ω and $\mathcal{F}$, respectively:

$$V_h = \left\{v_h \in \left[L^2(\Omega)\right]^d : \ v_h \in \left[P_k(K)\right]^d, \forall K \in \mathcal{T}_h\right\},$$
$$Q_h = \left\{q_h \in L^2(\Omega) : \ q_h \in P_k(K), \forall K \in \mathcal{T}_h\right\},$$

and

$$\bar{V}_h = \left\{\bar{v}_h \in \left[L^2(\mathcal{F})\right]^d : \ \bar{v}_h \in \left[P_k(F)\right]^d, \forall F \in \mathcal{F}_h, \ \bar{v}_h = 0 \text{ on } \partial\Omega\right\},$$
$$\bar{Q}_h = \left\{\bar{q}_h \in L^2(\mathcal{F}) : \ \bar{q}_h \in P_k(F), \forall F \in \mathcal{F}_h\right\},$$

where the space of polynomials of degree $l > 0$, on a domain M, is denoted by $P_l(M)$. Introduce the spaces $V_h^\star = V_h \times \bar{V}_h$, $Q_h^\star = Q_h \times \bar{Q}_h$ and $X_h^\star = V_h^\star \times Q_h^\star$. Function pairs in $V_h^\star$ and $Q_h^\star$ will be denoted by boldface, e.g., $\boldsymbol{v}_h = (v_h, \bar{v}_h) \in V_h^\star$ and $\boldsymbol{q}_h = (q_h, \bar{q}_h) \in Q_h^\star$.

3.2. Weak formulation

The equal-order HDG formulation of (2) is given by: for $f \in \left[L^2(\Omega)\right]^d$, find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in X_h^\star$ satisfying

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, v_h) &= \sum_{K\in\mathcal{T}_h} \int_K f\cdot v_h dx, & \forall \boldsymbol{v}_h \in V_h^\star, \\ b_h(\boldsymbol{q}_h, u_h) - c_h(\boldsymbol{p}_h, \boldsymbol{q}_h) &= 0, & \forall \boldsymbol{q}_h \in Q_h^\star, \end{aligned} \tag{3}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) = & \sum_{K\in\mathcal{T}_h} \int_K \nu\nabla u : \nabla v dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\eta\nu}{h_K}(u - \bar{u})\cdot(v - \bar{v}) ds \\ & - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left[\nu(u - \bar{u})\cdot\partial_n v + \nu\partial_n u\cdot(v - \bar{v})\right] ds, \end{aligned}$$
$$b_h(\boldsymbol{p}, v) = -\sum_{K\in\mathcal{T}_h} \int_K p\nabla\cdot v dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} v\cdot n\bar{p} ds,$$
$$c_h(\boldsymbol{p}, \boldsymbol{q}) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \gamma h_K (p - \bar{p})(q - \bar{q}) ds.$$

Notice that η is the velocity penalty parameter, and $\gamma = \frac{a_p}{\nu}$ the pressure penalty parameter. In order to ensure the existence and uniqueness of the method, η should be sufficiently large and $\gamma > 0$ (if γ is set to zero, the method is unstable, see [23]).

Remark 1. Here, we present a unified formulation of the equal-order embedded, hybridized, and embedded-hybridized discontinuous Galerkin methods: for $f \in \left[L^2(\Omega)\right]^d$, find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in X_h^\star$ satisfying (3) where $X_h^\star$ is given by

$$X_h^\star = V_h^\star \times Q_h^\star = \begin{cases} (V_h \times \bar{V}_h) \times (Q_h \times \bar{Q}_h), & \text{equal-order HDG method}, \\ (V_h \times (\bar{V}_h \cap C^0(\mathcal{F}))) \times (Q_h \times \bar{Q}_h), & \text{equal-order E-HDG method}, \\ (V_h \times (\bar{V}_h \cap C^0(\mathcal{F}))) \times (Q_h \times (\bar{Q}_h \cap C^0(\mathcal{F}))), & \text{equal-order EDG method}. \end{cases}$$

4. Preliminaries

For scalar-valued functions p and q, define the inner-product $(p, q)_M = \int_M pq dx$ on a domain M with norm $\|p\|_M = \sqrt{(p, p)_M}$. The L^2-norm on Ω is denoted by $\|p\| = \sqrt{(p, p)_{\mathcal{T}_h}}$. The inner product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. For vector-valued functions and tensor-valued functions, they are similarly defined. The standard notation $H^m(\Omega) \overset{\text{def}}{=} W^{m,2}(\Omega)$. The norm of $H^m(\Omega)$ is denoted by $\|\cdot\|_m$. The trace operator $\text{tr} : H^s(\Omega) \to H^{s-1/2}(\mathcal{F})$ ($s \geq 1$) is introduced to restrict functions in $H^s(\Omega)$ to $\mathcal{F}$. Let $\mathcal{F}_h = \mathcal{F}_I \cup \mathcal{F}_B$, where $\mathcal{F}_I$ and $\mathcal{F}_B$ are the subset of interior facets and boundary facets, respectively. Define the jump $[\![\cdot]\!]$ operator across the interior facets $F = \partial K^- \cap \partial K^+ \in \mathcal{F}_I$ by $[\![\phi]\!] = \phi^+ - \phi^-$, where $\phi^\pm$ denote the trace of ϕ from the interior of $K^\pm$. For the boundary facets $F \in \mathcal{F}_B$, let $[\![\phi]\!] = \phi$.

Introduce the broken Sobolev space $H^m(\mathcal{T}_h) = \left\{w \in L^2(\Omega) : w|_K \in H^m(K), \forall K \in \mathcal{T}_h\right\}$ with $m > 0$. Define the broken divergence $\nabla_h\cdot : \left[H^1(\mathcal{T}_h)\right]^d \to L^2(\Omega)$ by

$$\left.(\nabla_h\cdot w)\right|_K := \nabla\cdot(w|_K), \forall K \in \mathcal{T}_h,$$

and similarly define the broken gradient.

Introduce the following extended function spaces

$$V(h) = V_h + \left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d, \quad Q(h) = Q_h + L_0^2(\Omega) \cap H^1(\Omega),$$
$$\bar{V}(h) = \bar{V}_h + \left[H_0^{3/2}(\mathcal{F})\right]^d, \quad \bar{Q}(h) = \bar{Q}_h + H_0^{1/2}(\mathcal{F}).$$

Here, $\left[H_0^{3/2}(\mathcal{F})\right]^d$ and $H_0^{1/2}(\mathcal{F})$ stand for the trace spaces of $\left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d$ and $L_0^2(\Omega) \cap H^1(\Omega)$ on $\mathcal{F}$, respectively. Set $V^\star(h) = V(h) \times \bar{V}(h)$, $Q^\star(h) = Q(h) \times \bar{Q}(h)$ and $X^\star(h) = V^\star(h) \times Q^\star(h)$.

Define the following norms on $V^\star(h)$, $Q^\star(h)$ and $X^\star(h)$, respectively:

$$|||\boldsymbol{v}|||_v^2 = \sum_{K\in\mathcal{T}_h} \|\nabla v\|_K^2 + \sum_{K\in\mathcal{T}_h} \eta h_K^{-1} \|\bar{v} - v\|_{\partial K}^2,$$
$$|||\boldsymbol{v}|||_{v'}^2 = |||\boldsymbol{v}|||_v^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K}{\eta} \| \frac{\partial v}{\partial n} \|_{\partial K}^2,$$
$$|||\boldsymbol{q}|||_p^2 = \|q\|^2 + |\boldsymbol{q}|_p^2,$$
$$|||\boldsymbol{q}|||_{p'}^2 = \|q\|^2 + |\boldsymbol{q}|_p^2 + \sum_{K\in\mathcal{T}_h} h_K \| \bar{q} \|_{\partial K}^2,$$

and

$$|||(\boldsymbol{v},\boldsymbol{q})|||_{v,p}^2 = \nu|||\boldsymbol{v}|||_v^2 + \nu^{-1}|||\boldsymbol{q}|||_p^2,$$
$$|||(\boldsymbol{v},\boldsymbol{q})|||_{v',p'}^2 = \nu|||\boldsymbol{v}|||_{v'}^2 + \nu^{-1}|||\boldsymbol{q}|||_{p'}^2,$$

where $|\boldsymbol{q}|_p^2 = \sum_{K\in\mathcal{T}_h} a_p h_K \|\bar{q}-q\|_{\partial K}^2$, and $|||\cdot|||_{v'}$ and $|||\cdot|||_v$ are equivalent on $V_h^\star$, namely,

$$|||\boldsymbol{v}_h|||_v \le |||\boldsymbol{v}_h|||_{v'} \le c|||\boldsymbol{v}_h|||_v, \tag{4}$$

with $c > 0$ independent of h, see [23, Eq.(28)]. The following discrete Poincaré inequality will be used [26, Eq.(13)]: there is a positive constant C_p independent of h, such that

$$\|v_h\| \le C_p|||\boldsymbol{v}_h|||_v, \quad \forall \boldsymbol{v}_h \in V_h^\star. \tag{5}$$

Define the following semi-norm

$$|v_h|_{\mathrm{nj}}^2 = \sum_{F\in\mathcal{F}_h} \frac{1}{h_F} \int_F \left([\![v_h]\!]\cdot n_F\right)^2 \mathrm{d}s, \quad \forall v_h \in V_h,$$

where h_F denotes the diameter of each facet $F \in \mathcal{F}_h$.

Next, we recall the stability and boundedness of the bilinear forms a_h, b_h and c_h in [23].

Lemma 1 (*[23, Lemmas 4.2 and 4.3] Coercivity and Boundedness of a_h*). *For sufficiently large η, there exist constants $C_a^c > 0$ and $C_a^b > 0$, independent of h and ν, such that for all $\boldsymbol{v}_h \in V_h^\star$ and $\boldsymbol{u} \in V^\star(h)$*

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \nu C_a^c|||\boldsymbol{v}_h|||_v^2 \qquad \text{and} \qquad \left|a_h(\boldsymbol{u},\boldsymbol{v}_h)\right| \le \nu C_a^b|||\boldsymbol{u}|||_{v'}|||\boldsymbol{v}_h|||_v. \tag{6}$$

Lemma 2 (*Boundedness of b_h*). *There exists a constant $C_b^b > 0$, independent of h, such that for all $\boldsymbol{v} \in V^\star(h)$ and $\boldsymbol{q} \in Q^\star(h)$*

$$\left|b_h(\boldsymbol{q},v)\right| \le C_b^b|||\boldsymbol{v}|||_v|||\boldsymbol{q}|||_{p'}. \tag{7}$$

Proof. The proof of this lemma was provided in the proof of Lemma 4.8 in [23]. □

Lemma 3 (*[23, Lemma 4.4] Stability of b_h*). *There exists a constant $\beta_p > 0$, independent of h, such that for all $\boldsymbol{q}_h \in Q_h^\star$*

$$\beta_p\|q_h\| \le \sup_{\boldsymbol{w}_h\in V_h^\star} \frac{b_h(\boldsymbol{q}_h,w_h)}{|||\boldsymbol{w}_h|||_v} + |\boldsymbol{q}_h|_p.$$

Lemma 4 (*[23, Lemma 4.7] Discrete Inf–sup Stability*). *For sufficiently large η, there exists a constant $\sigma > 0$, independent of h and ν, such that for all $(\boldsymbol{v}_h,\boldsymbol{q}_h) \in X_h^\star$*

$$\sigma|||(\boldsymbol{v}_h,\boldsymbol{q}_h)|||_{v,p} \le \sup_{(\boldsymbol{w}_h,\boldsymbol{r}_h)\in X_h^\star} \times \frac{a_h(\boldsymbol{v}_h,\boldsymbol{w}_h) + b_h(\boldsymbol{q}_h,w_h) - b_h(\boldsymbol{r}_h,v_h) + c_h(\boldsymbol{q}_h,\boldsymbol{r}_h)}{|||(\boldsymbol{w}_h,\boldsymbol{r}_h)|||_{v,p}}. \tag{8}$$

Remark 2. We notice that if $\bar{V}_h$ is replaced by a smaller facet velocity space $\bar{V}_h^c$,

$$\bar{V}_h^c = \left\{\bar{v}_h \in \left[L^2(\mathcal{F})\right]^d : \bar{v}_h \in \left[P_1(F)\right]^d, \forall F \in \mathcal{F}_h,\ \bar{v}_h = 0 \text{ on } \partial\Omega\right\} \cap C^0(\mathcal{F}),$$

Lemmas 3 and 4 still hold true. Their proofs are the same as that of Lemma 4.4 and Lemma 4.7 in [23], respectively.

Finally, we have the following consistency result.

Lemma 5 (*[23, Lemma 4.1] Consistency*). *Let $(u,p) \in \left(\left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d\right) \times \left(L_0^2(\Omega) \cap H^1(\Omega)\right)$ solves the Stokes problem (1), and $\boldsymbol{u} = (u, \mathrm{tr}(u))$ and $\boldsymbol{p} = (p, \mathrm{tr}(p))$, then*

$$a_h(\boldsymbol{u},\boldsymbol{v}_h) + b_h(\boldsymbol{p},v_h) - b_h(\boldsymbol{q}_h,u) + c_h\left(\boldsymbol{p},\boldsymbol{q}_h\right) = \left(f,v_h\right)_{\mathcal{T}_h}, \ \forall(\boldsymbol{v}_h,\boldsymbol{q}_h) \in X_h^\star. \tag{9}$$

Remark 3. For the equal-order E-HDG and EDG methods, Lemmas 1, 2 and 5 still hold true, because the corresponding spaces in the two methods are subspaces of that in the equal-order HDG method. And, Lemmas 3 and 4 also hold true, by following the proofs of Lemma 4.4 and Lemma 4.7 in [23], respectively.

5. Energy estimates and error estimates

In this section, we present energy estimates and error estimates for the equal-order HDG method with a small pressure penalty parameter.

5.1. Energy estimates

The well-posedness of the equal-order HDG method follows the discrete inf–sup stability (8). The energy estimates of the solution are given in the following theorem. Here, we provide the sharp energy estimates by introducing the Helmholtz–Hodge decomposition, see [27, Theorem 3.168]: for every vector field $g \in [L^2(\Omega)]^d$, there exist a divergence-free vector field $g_0 \in H_0(\mathrm{div},\Omega) = \{v \in H(\mathrm{div},\Omega) : \nabla\cdot v = 0,\ v\cdot n = 0 \text{ on } \partial\Omega\}$ and a scalar function $\phi \in H^1(\Omega)/\mathbb{R}$ with $g = g_0 + \nabla\phi$ and $(g_0,\nabla\phi) = 0$. The decomposition is unique. The function $g_0 = P_H g$ is called the Helmholtz-Hodge projector of g.

Theorem 1. *Let $f \in [L^2(\Omega)]^d$ and assume that $\eta > 0$ is sufficiently large. Then, the solution $(\boldsymbol{u}_h,\boldsymbol{p}_h) \in X_h^\star$ to (3) satisfies the following energy estimates:*

$$\begin{aligned} |||\boldsymbol{u}_h|||_v &\le C_p(\nu C_a^c)^{-1}\|P_H f\| + C\gamma^{1/2}\nu^{-1/2}\|\nabla\psi\|,\\ \|\nabla_h\cdot u_h\| &\le C\gamma^{1/2}\nu^{-1/2}\|P_H f\| + C\gamma\|\nabla\psi\|,\\ |u_h|_{\mathrm{nj}} &\le C\gamma^{1/2}\nu^{-1/2}\|P_H f\| + C\gamma\|\nabla\psi\|,\\ |||\boldsymbol{p}_h|||_p &\le C_p\sigma^{-1}\|f\|, \end{aligned} \tag{10}$$

with $P_H f$ the Helmholtz–Hodge projector of f and $\nabla\psi = f - P_H f$ with $\psi \in H^1(\Omega)/\mathbb{R}$.

Proof. Taking $\boldsymbol{q}_h = (\nabla_h\cdot u_h, 0)$ and $\boldsymbol{q}_h = (0,\bar{q}_h)$ in the second equation of (3), respectively, we have

$$0 = -\sum_K \int_K (\nabla_h\cdot u_h)^2 \mathrm{d}x + \sum_K \int_{\partial K} \gamma h_K(\bar{p}_h - p_h)\nabla_h\cdot u_h \mathrm{d}s, \tag{11}$$

and

$$\begin{aligned} [\![u_h]\!]\cdot n|_F &= \gamma h_{K^+}(\bar{p}_h - p_h^+) + \gamma h_{K^-}(\bar{p}_h - p_h^-), && \forall F \in \mathcal{F}_I,\\ [\![u_h]\!]\cdot n|_F &= \gamma h_K\left(\bar{p}_h - p_h\right), && \forall F \in \mathcal{F}_B. \end{aligned} \tag{12}$$

For (11), we apply Cauchy–Schwarz inequality and the discrete trace inequality to obtain

$$\|\nabla_h\cdot u_h\| \le C\gamma^{1/2}\nu^{-1/2}\left|\boldsymbol{p}_h\right|_p. \tag{13}$$

By applying (12), the triangle inequality and shape regularity of mesh, we have

$$|u_h|_{\rm nj}^2 = \sum_{F\in\mathcal{T}_h}\frac{1}{h_F}\|[\![u_h]\!]\cdot n_F\|_{L^2(F)}^2 \le C\gamma\nu^{-1}|\boldsymbol{p}_h|_p^2. \tag{14}$$

Taking $\boldsymbol{v}_h=\boldsymbol{u}_h, \boldsymbol{q}_h=\boldsymbol{p}_h$ in (3), together with the coercivity of a_h in (6), Cauchy–Schwarz inequality, the discrete Poincaré inequality (5), (13) and (14) leads to

$$\begin{aligned}
&\nu C_a^c\|\!|\boldsymbol{u}_h\|\!|_v^2+\nu^{-1}|\boldsymbol{p}_h|_p^2\\
&\le (P_H f, u_h)_{\mathcal{T}_h} + (f-P_H f, u_h)_{\mathcal{T}_h}\\
&\le C_p\|P_H f\|\|\!|\boldsymbol{u}_h\|\!|_v - \sum_{K\in\mathcal{T}_h}\int_K \psi\nabla_h\cdot u_h\,{\rm d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\psi u_h\cdot n\,{\rm d}s\\
&\le C_p\|P_H f\|\|\!|\boldsymbol{u}_h\|\!|_v + C\|\nabla_h\cdot u_h\|\|\psi\|\\
&\quad + C|u_h|_{\rm nj}(\sum_{K\in\mathcal{T}_h}(\|\psi\|_K^2+h_K^2\|\nabla\psi\|_K^2))^{1/2}\\
&\le C_p\|P_H f\|\|\!|\boldsymbol{u}_h\|\!|_v + C(\|\nabla_h\cdot u_h\|+|u_h|_{\rm nj})\|\nabla\psi\|\\
&\le C_p\|P_H f\|\|\!|\boldsymbol{u}_h\|\!|_v + C\nu^{-1/2}|\boldsymbol{p}_h|_p\gamma^{1/2}\|\nabla\psi\|,
\end{aligned} \tag{15}$$

where we use the well-known inequality $\|w\| \le C\|\nabla w\|, \forall w \in H^1(\Omega)/\mathbb{R}$. We apply Young's inequality to the right-hand side of (15) to obtain

$$\begin{aligned}
&\|\!|\boldsymbol{u}_h\|\!|_v \le C_p(\nu C_a^c)^{-1}\|P_H f\| + C\gamma^{1/2}\nu^{-1/2}\|\nabla\psi\|,\\
&|\boldsymbol{p}_h|_p \leqslant C\|P_H f\| + C\gamma^{1/2}\nu^{1/2}\|\nabla\psi\|.
\end{aligned} \tag{16}$$

Thus, we insert the pressure estimate in (16), into (13) and (14) to obtain

$$\begin{aligned}
&\|\nabla_h\cdot u_h\| \le C\gamma^{1/2}\nu^{-1/2}\|P_H f\| + C\gamma\|\nabla\psi\|,\\
&|u_h|_{\rm nj} \le C\gamma^{1/2}\nu^{-1/2}\|P_H f\| + C\gamma\|\nabla\psi\|.
\end{aligned} \tag{17}$$

By the discrete inf–sup stability (8) and the discrete Poincaré inequality (5),

$$\sigma\nu^{-\frac12}\|\!|\boldsymbol{p}_h\|\!|_p \le \sigma\|\!|(\boldsymbol{u}_h,\boldsymbol{p}_h)\|\!|_{v,p} \le \sup_{(\boldsymbol{v}_h,\boldsymbol{q}_h)\in X_h^\star}\frac{(f,v_h)_{\mathcal{T}_h}}{\|\!|(\boldsymbol{v}_h,\boldsymbol{q}_h)\|\!|_{v,p}} \le C_p\nu^{-\frac12}\|f\|.$$

Finally, by collecting the above estimates, we can finish the proof. □

5.2. *Error estimates*

Now, we give the error estimates for the velocity and the pressure. Introduce the following approximation and discretization errors for the velocity and the pressure, respectively, as follows:

$$\xi_u = u-\mathcal{I}_{\rm BDM}u,\quad e_u = u_h-\mathcal{I}_{\rm BDM}u,\quad \bar\xi_u = {\rm tr}(u)-\bar{\mathcal{I}}_V u,\quad \bar e_u = \bar u_h-\bar{\mathcal{I}}_V u, \tag{18}$$

where $\mathcal{I}_{\rm BDM} : [H^1(\Omega)]^d \to V_h$ is the usual BDM interpolation operator of order k in [28, Lemma 7] and $\bar{\mathcal{I}}_V$ is L^2 -projection operator onto $\bar V_h$,

$$\xi_p = p-\mathcal{I}_Q p,\quad e_p = p_h-\mathcal{I}_Q p,\quad \bar\xi_p = {\rm tr}(p)-\bar{\mathcal{I}}_Q p,\quad \bar e_p = \bar p_h-\bar{\mathcal{I}}_Q p, \tag{19}$$

with $\mathcal{I}_Q$ and $\bar{\mathcal{I}}_Q$ the L^2 -projection operators onto Q_h and $\bar Q_h$, respectively, and

$$\xi_p' = p-\mathcal{I}_{SZ}p,\quad e_p' = p_h-\mathcal{I}_{SZ}p,\quad \bar\xi_p' = {\rm tr}(p)-\bar{\mathcal{I}}_{SZ}p,\quad \bar e_p' = \bar p_h-\bar{\mathcal{I}}_{SZ}p, \tag{20}$$

with $\mathcal{I}_{SZ}$ the continuous Scott–Zhang interpolant [29], and $\bar{\mathcal{I}}_{SZ}p = \mathcal{I}_{SZ}p|_F \in \bar Q_h$. Set $\boldsymbol{\xi}_u=(\xi_u,\bar\xi_u)$, $\boldsymbol{e}_u=(e_u,\bar e_u)$, $\boldsymbol{\xi}_p=(\xi_p,\bar\xi_p)$, $\boldsymbol{e}_p=(e_p,\bar e_p)$, $\boldsymbol{\xi}_p'=(\xi_p',\bar\xi_p')$ and $\boldsymbol{e}_p'=(e_p',\bar e_p')$.

Theorem 2. *Let $(u,p) \in [H^{k+1}(\Omega)]^d \times H^{k+1}(\Omega)$ and $(\boldsymbol{u}_h,\boldsymbol{p}_h)$ be the solutions to (2) and (3), respectively. Set $\boldsymbol{u}=(u,{\rm tr}(u))$ and $\boldsymbol{p}=(p,{\rm tr}(p))$. Then, the following error estimates hold true:*

$$\begin{aligned}
&\|\!|\boldsymbol{u}-\boldsymbol{u}_h\|\!|_{v'} \le C(h^k\|u\|_{k+1}+\gamma^{\frac12}\nu^{-\frac12}h^{k+1}\|p\|_{k+1}),\\
&\|\!|\boldsymbol{p}-\boldsymbol{p}_h\|\!|_p \le C\left[\nu\sigma^{-1}h^k\|u\|_{k+1}+(\sigma^{-1}+1)h^{k+1}\|p\|_{k+1}\right],
\end{aligned} \tag{21}$$

with $C>0$ independent of h, σ, γ and ν.

Remark 4. Based on the proof of Section 5 in [23], the pressure term in the velocity estimate can be scaled by $\nu^{-\frac12}$, whereas we obtain an improved velocity estimate scaled by $\gamma^{\frac12}\nu^{-\frac12}$ in (21) due to a small pressure penalty parameter.

Proof. By subtracting (3) from (9), we have the following error equation

$$\begin{aligned}
&a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{p}-\boldsymbol{p}_h,v_h)-b_h(\boldsymbol{q}_h,u-u_h)+c_h(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{q}_h)=0,\\
&\forall(\boldsymbol{v}_h,\boldsymbol{q}_h)\in X_h^\star.
\end{aligned} \tag{22}$$

We split the above error equation as

$$\begin{aligned}
&a_h(\boldsymbol{e}_u,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p,v_h)-b_h(\boldsymbol{q}_h,e_u)+c_h(\boldsymbol{e}_p,\boldsymbol{q}_h)\\
&=a_h(\boldsymbol{\xi}_u,\boldsymbol{v}_h)+b_h(\boldsymbol{\xi}_p,v_h)-b_h(\boldsymbol{q}_h,\xi_u)+c_h(\boldsymbol{\xi}_p,\boldsymbol{q}_h).
\end{aligned} \tag{23}$$

Taking $\boldsymbol{v}_h=\boldsymbol{e}_u, \boldsymbol{q}_h=\boldsymbol{e}_p$ in (23) and using the coercivity of a_h in (6) lead to

$$\nu C_a^c\|\!|\boldsymbol{e}_u\|\!|_v^2+\nu^{-1}|\boldsymbol{e}_p|_p^2 \le a_h(\boldsymbol{\xi}_u,\boldsymbol{e}_u)+b_h(\boldsymbol{\xi}_p,e_u)-b_h(\boldsymbol{e}_p,\xi_u)+c_h(\boldsymbol{\xi}_p,\boldsymbol{e}_p). \tag{24}$$

Noting that ξ_u is H(div)-conforming and exactly divergence-free, $\xi_p = p-\mathcal{I}_Q p$ and $\bar\xi_p={\rm tr}(p)-\bar{\mathcal{I}}_Q p$, we have

$$\begin{aligned}
&b_h(\boldsymbol{e}_p,\xi_u) = -\sum_{K\in\mathcal{T}_h}\int_K e_p\nabla_h\cdot\xi_u{\rm d}x+\sum_{K\in\mathcal{T}_h}\int_{\partial K}\xi_u\cdot n\bar e_p{\rm d}s=0,\\
&b_h(\boldsymbol{\xi}_p,e_u) = -\sum_{K\in\mathcal{T}_h}\int_K \xi_p\nabla_h\cdot e_u{\rm d}x+\sum_{K\in\mathcal{T}_h}\int_{\partial K}e_u\cdot n\bar\xi_p{\rm d}s=0.
\end{aligned} \tag{25}$$

Thanks to the boundedness of a_h in (6), Cauchy–Schwarz inequality and Young's inequality,

$$\begin{aligned}
&a_h(\boldsymbol{\xi}_u,\boldsymbol{e}_u) \le \nu C_a^b\|\!|\boldsymbol{\xi}_u\|\!|_{v'}\|\!|\boldsymbol{e}_u\|\!|_v \le C\nu\|\!|\boldsymbol{\xi}_u\|\!|_{v'}^2+\frac12 C_a^c\nu\|\!|\boldsymbol{e}_u\|\!|_v^2,\\
&c_h(\boldsymbol{\xi}_p,\boldsymbol{e}_p) \le \nu^{-1}|\boldsymbol{e}_p|_p|\boldsymbol{\xi}_p|_p \le \frac12\nu^{-1}|\boldsymbol{e}_p|_p^2+\frac12\nu^{-1}|\boldsymbol{\xi}_p|_p^2.
\end{aligned} \tag{26}$$

Then the combination of (24)–(26) admits

$$\frac12 C_a^c\nu\|\!|\boldsymbol{e}_u\|\!|_v^2+\frac12\nu^{-1}|\boldsymbol{e}_p|_p^2 \le \nu C\|\!|\boldsymbol{\xi}_u\|\!|_{v'}^2+\frac12\nu^{-1}|\boldsymbol{\xi}_p|_p^2. \tag{27}$$

By the interpolation estimates of the BDM interpolation operator and the L^2 -projection operators, we have

$$\|\!|\boldsymbol{\xi}_u\|\!|_{v'}^2 \le Ch^{2k}\|u\|_{k+1}^2,\quad |\boldsymbol{\xi}_p|_p^2 \le Ca_p h^{2k+2}\|p\|_{k+1}^2. \tag{28}$$

For the first inequality of (28), the detailed proof is provided in [25, Lemma 9]. Combining (27) and (28) yields

$$\|\!|\boldsymbol{e}_u\|\!|_v^2 \le Ch^{2k}\|u\|_{k+1}^2+C\gamma\nu^{-1}h^{2k+2}\|p\|_{k+1}^2. \tag{29}$$

Then,

$$\|\!|\boldsymbol{u}-\boldsymbol{u}_h\|\!|_{v'} \le Ch^k\|u\|_{k+1}+C\gamma^{\frac12}\nu^{-\frac12}h^{k+1}\|p\|_{k+1}, \tag{30}$$

which follows from the triangle inequality, (29), the equivalence of $\|\!|\cdot\|\!|_{v'}$ and $\|\!|\cdot\|\!|_v$ on $V_h^\star$ and the first inequality of (28).

Next, we split the error Eq. (22) with (18) and (20) to obtain

$$\begin{aligned}
&a_h(\boldsymbol{e}_u,\boldsymbol{v}_h)+b_h(\boldsymbol{e}_p',v_h)-b_h(\boldsymbol{q}_h,e_u)+c_h(\boldsymbol{e}_p',\boldsymbol{q}_h)\\
&=a_h(\boldsymbol{\xi}_u,\boldsymbol{v}_h)+b_h(\boldsymbol{\xi}_p',v_h)-b_h(\boldsymbol{q}_h,\xi_u)+c_h(\boldsymbol{\xi}_p',\boldsymbol{q}_h),
\end{aligned} \tag{31}$$

By the discrete inf–sup stability (8), (31), and noting that $c_h(\boldsymbol{\xi}_p',\boldsymbol{q}_h)=0$ and $b_h(\boldsymbol{q}_h,\xi_u)=0$,

$$\begin{aligned}
\sigma\nu^{-\frac12}\|\!|\boldsymbol{e}_p'\|\!|_p \le \sigma\|\!|(\boldsymbol{e}_u,\boldsymbol{e}_p')\|\!|_{v,p} &\le \sup_{(\boldsymbol{v}_h,\boldsymbol{q}_h)\in X_h^\star}\frac{a_h(\boldsymbol{\xi}_u,\boldsymbol{v}_h)+b_h(\boldsymbol{\xi}_p',v_h)}{\|\!|(\boldsymbol{v}_h,\boldsymbol{q}_h)\|\!|_{v,p}}\\
&\le C_a^b\nu^{1/2}\|\!|\boldsymbol{\xi}_u\|\!|_{v'}+C_b^b\nu^{-1/2}\|\!|\boldsymbol{\xi}_p'\|\!|_{p'}.
\end{aligned} \tag{32}$$

By the triangle inequality and (32), we get

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_p \le \nu\sigma^{-1}C_a^b\|\xi_u\|_{v'} + C_b^b\sigma^{-1}\|\xi_p'\|_{p'} + \|\xi_p'\|_p.$$

Finally, we have

$$\|\boldsymbol{p}-\boldsymbol{p}_h\|_p \le C\left[\nu\sigma^{-1}h^k\|u\|_{k+1} + (\sigma^{-1}+1)h^{k+1}\|p\|_{k+1}\right],$$

which follows from the first inequality of (28) and the interpolation estimate of the Scott–Zhang interpolation operator. □

From Theorem 2, we can conclude that when the pressure penalty parameter γ is equal to ν, the velocity error bound is independent of the negative power of viscosity. When γ tends to 0, the velocity error bound tends to be pressure-robust. In addition, we also notice that the convergence rate of the pressure is $k+1$ when $\nu \le Ch$, so the equal-order method is beneficial in terms of accuracy and convergence rate for the pressure. For the velocity error in the L^2-norm, we refer readers to Appendix.

Remark 5. Here, we will comment on whether the current analysis still works for the equal-order E-HDG and EDG methods. For the equal-order E-HDG method, all the results of the section hold verbatim by following the above proofs. For the equal-order EDG method, from the proof of Theorem 1, we can notice that $\|\nabla_h\cdot u_h\|$ tends to 0 as $\gamma\to 0$. However, $|u_h|_{\mathrm{nj}}$ does not tend to 0 as $\gamma\to 0$, because (12) is not true. And, the velocity error bound does not tend to be pressure-robust as $\gamma\to 0$, because the second equation of (25) is not true.

6. Post-processing

In this section, we present a Stokes-based post-processing. Notice that from Theorem 2, the dependence of σ^{-1} in the pressure error bound indicates that when we take the value of γ too small, it may lead to the risk of over-stabilization of the pressure, because $\gamma\to 0$, then $\sigma\to 0$ (see Lemma 4.7 in [23]). Then, the post-processing can be used to avoid the over-stabilization of the pressure.

Consider the following finite element spaces on the $\mathcal{F}$:

$$\bar{V}_h^c = \left\{\bar{v}_h \in \left[L^2(\mathcal{F})\right]^d : \ \bar{v}_h \in \left[P_1(F)\right]^d, \forall F\in\mathcal{F}_h,\ \bar{v}_h = 0 \text{ on } \partial\Omega\right\} \cap C^0(\mathcal{F}),$$

$$\bar{Q}_h^c = \left\{\bar{q}_h \in L^2(\mathcal{F}) : \ \bar{q}_h \in P_k(F), \forall F\in\mathcal{F}_h\right\} \cap C^0(\mathcal{F}).$$

Notice that $\bar{V}_h^c$ is piecewise continuous P_1 polynomials on $\mathcal{F}$, and $\bar{Q}_h^c$ piecewise continuous P_k polynomials on $\mathcal{F}$. Set the following spaces $V_h^{\star c} = V_h\times\bar{V}_h^c$, $Q_h^{\star c} = Q_h\times\bar{Q}_h^c$ and $X_h^{\star c} = V_h^{\star c}\times Q_h^{\star c}$.

The post-processing is given by: for $\boldsymbol{p}_h$, the pressure solution of the equal-order HDG or E-HDG method with a sufficiently small pressure penalty parameter, find $(\boldsymbol{u}_h^\star, \boldsymbol{p}_h^\star)\in X_h^{\star c}$ satisfying

$$\begin{aligned} &a_h(\boldsymbol{u}_h^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^\star, v_h) = b_h(\boldsymbol{p}_h, v_h), && \forall \boldsymbol{v}_h\in V_h^{\star c},\\ &b_h(\boldsymbol{q}_h, u_h^\star) - c_h(\boldsymbol{p}_h^\star, \boldsymbol{q}_h) = 0, && \forall \boldsymbol{q}_h \in Q_h^{\star c}, \end{aligned} \tag{33}$$

with a mild pressure penalty parameter γ_o in the term c_h, say $\gamma_o = 1$ with $a_p = \nu$. Notice that we devise the post-processing to get a better pressure approximation $\boldsymbol{p}_h^\star$, and the velocity $\boldsymbol{u}_h^\star$ is only an auxiliary variable.

Remark 6. In the implementation, $b_h(\boldsymbol{p}_h, v_h)$ is replaced by $\left(f, v_h\right)_{\mathcal{T}_h} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$, because $b_h(\boldsymbol{p}_h, v_h) = \left(f, v_h\right)_{\mathcal{T}_h} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)$, for $\forall \boldsymbol{v}_h \in V_h^{\star c}$.

Next, consider the following Stokes problem

$$\begin{aligned} \nu\nabla^2 u^\star + \nabla p^\star &= \nabla p, && \text{in } \Omega,\\ \nabla\cdot u^\star &= 0, && \text{in } \Omega,\\ u^\star &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{34}$$

It is obvious that $u^\star = 0$, $p^\star = p$.

We give a corresponding consistent form

$$\begin{aligned} &a_h(\boldsymbol{u}^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{p}^\star, v_h) = b_h(\boldsymbol{p}, v_h), && \forall \boldsymbol{v}_h\in V_h^{\star c},\\ &b_h(\boldsymbol{q}_h, u^\star) - c_h(\boldsymbol{p}^\star, \boldsymbol{q}_h) = 0, && \forall \boldsymbol{q}_h \in Q_h^{\star c}. \end{aligned} \tag{35}$$

Introduce

$$\begin{aligned} &\xi_u^\star = u^\star - \mathcal{I}_{\mathrm{BDM}}u^\star,\ e_u^\star = u_h^\star - \mathcal{I}_{\mathrm{BDM}}u^\star,\ \bar{\xi}_u^\star = \mathrm{tr}(u^\star) - \bar{\mathcal{I}}_V^c u^\star,\\ &\bar{e}_u^\star = \bar{u}_h^\star - \bar{\mathcal{I}}_V^c u^\star,\\ &\xi_p^\star = p^\star - \mathcal{I}_Q p^\star,\ e_p^\star = p_h^\star - \mathcal{I}_Q p^\star,\ \bar{\xi}_p^\star = \mathrm{tr}(p^\star) - \bar{\mathcal{I}}_Q^c p^\star,\\ &\bar{e}_p^\star = \bar{p}_h^\star - \bar{\mathcal{I}}_Q^c p^\star, \end{aligned}$$

where $\bar{\mathcal{I}}_V^c$ and $\bar{\mathcal{I}}_Q^c$ are the L^2 -projection operators onto $\bar{V}_h^c$ and $\bar{Q}_h^c$, respectively. Set $\boldsymbol{\xi}_u^\star = (\xi_u^\star, \bar{\xi}_u^\star) = (0,0)$, $\boldsymbol{e}_u^\star = (e_u^\star, \bar{e}_u^\star)$, $\boldsymbol{\xi}_p^\star = (\xi_p^\star, \bar{\xi}_p^\star)$ and $\boldsymbol{e}_p^\star = (e_p^\star, \bar{e}_p^\star)$.

Theorem 3. *There is a unique solution* $\left(\boldsymbol{u}_h^\star, \boldsymbol{p}_h^\star\right)\in X_h^{\star c}$ *to* (33)*, the pressure solution of which satisfies the following stability estimate and error estimate:*

$$\|\boldsymbol{p}_h^\star\|_p \le C_p{\sigma_o}^{-1}\|f\| + \frac{{\sigma_o}^{-1}cC_a^bC_p}{C_a^c}\|P_H f\| + C{\sigma_o}^{-1}\gamma^{1/2}\nu^{1/2}\|\nabla\psi\|,$$

$$\|\boldsymbol{p}-\boldsymbol{p}_h^\star\|_p \le C\left[\nu h^k\|u\|_{k+1} + (\nu^{1/2}\gamma^{\frac{1}{2}}+1)h^{k+1}\|p\|_{k+1}\right],$$

with $C>0$ a constant independent of h, γ and ν, and σ_o in (36)*, $P_H f$ the Helmholtz-Hodge projector of f and $\nabla\psi = f - P_H f$ with $\psi\in H^1(\Omega)/\mathbb{R}$.*

Remark 7. Comparing the stability and error estimates of the pressure in Theorems 1 and 2, the estimates of the post-processed pressure in Theorem 3 have no dependence on the value of σ^{-1}.

Proof. By Remark 2, there is a constant $\sigma_o > 0$ independent of h and ν such that for $\forall(\boldsymbol{v}_h, \boldsymbol{q}_h)\in X_h^{\star c}$

$$\sigma_o\|(\boldsymbol{v}_h, \boldsymbol{q}_h)\|_{v,p} \le \sup_{(\boldsymbol{w}_h, \boldsymbol{r}_h)\in X_h^{\star c}} \times \frac{a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) + b_h(\boldsymbol{q}_h, w_h) - b_h(\boldsymbol{r}_h, v_h) + c_h(\boldsymbol{q}_h, \boldsymbol{r}_h)}{\|(\boldsymbol{w}_h, \boldsymbol{r}_h)\|_{v,p}}. \tag{36}$$

The well-posedness of (33) follows using this discrete inf–sup stability.

By (33) and $b_h(\boldsymbol{p}_h, v_h) = (f, v_h)_{\mathcal{T}_h} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in V_h^{\star c}$, we have

$$a_h(\boldsymbol{u}_h^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^\star, v_h) - b_h(\boldsymbol{q}_h, u_h^\star) + c_h(\boldsymbol{p}_h^\star, \boldsymbol{q}_h) = \left(f, v_h\right)_{\mathcal{T}_h} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h). \tag{37}$$

Using the discrete inf–sup stability (36), Cauchy–Schwarz inequality, the discrete Poincaré inequality (5), the boundedness of a_h in (6) and (4) yields

$$\sigma_o\|(\boldsymbol{u}_h^\star, \boldsymbol{p}_h^\star)\|_{v,p} \le \sup_{(\boldsymbol{v}_h, \boldsymbol{q}_h)\in X_h^{\star c}} \frac{\left(f, v_h\right)_{\mathcal{T}_h} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)}{\|(\boldsymbol{v}_h, \boldsymbol{q}_h)\|_{v,p}} \le C_p\nu^{-\frac{1}{2}}\|f\| + cC_a^b\nu^{\frac{1}{2}}\|\boldsymbol{u}_h\|_v. \tag{38}$$

Then, the stability estimate follows from the energy estimate (16) of the velocity.

By (35) and $b_h(\boldsymbol{p}, v_h) = (f, v_h)_{\mathcal{T}_h} - a_h(\boldsymbol{u}, \boldsymbol{v}_h), \forall \boldsymbol{v}_h \in V_h^{\star c}$, we have

$$a_h(\boldsymbol{u}^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{p}^\star, v_h) - b_h(\boldsymbol{q}_h, u^\star) + c_h(\boldsymbol{p}^\star, \boldsymbol{q}_h) = (f, v_h)_{\mathcal{T}_h} - a_h(\boldsymbol{u}, \boldsymbol{v}_h). \tag{39}$$

Subtracting (37) from (39) leads to

$$\begin{aligned} &a_h(\boldsymbol{e}_u^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{e}_p^\star, v_h) - b_h(\boldsymbol{q}_h, e_u^\star) + c_h(\boldsymbol{e}_p^\star, \boldsymbol{q}_h)\\ &= a_h(\boldsymbol{\xi}_u^\star, \boldsymbol{v}_h) + b_h(\boldsymbol{\xi}_p^\star, v_h) - b_h(\boldsymbol{q}_h, \xi_u^\star) + c_h(\boldsymbol{\xi}_p^\star, \boldsymbol{q}_h) + a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h). \end{aligned} \tag{40}$$

We notice $a_h(\boldsymbol{\xi}_u^\star, \boldsymbol{v}_h) = b_h(\boldsymbol{q}_h, \xi_u^\star) = 0$ with $\boldsymbol{\xi}_u^\star = (0,0)$. Then we use the discrete inf–sup stability (36) and (40), yielding

$$\sigma_o\|(\boldsymbol{e}_u^\star, \boldsymbol{e}_p^\star)\|_{v,p} \le \sup_{(\boldsymbol{v}_h, \boldsymbol{q}_h)\in X_h^{\star c}} \frac{a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{\xi}_p^\star, v_h) + c_h(\boldsymbol{\xi}_p^\star, \boldsymbol{q}_h)}{\|(\boldsymbol{v}_h, \boldsymbol{q}_h)\|_{v,p}}. \tag{41}$$

We use the boundedness of a_h in (6), the boundedness of b_h (7) and Cauchy–Schwarz inequality to obtain

$$\begin{aligned} a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) &\le \nu C_a^b |||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'} |||\boldsymbol{v}_h|||_v, \\ b_h(\xi_p^\star, v_h) &\le C_b^b |||\boldsymbol{v}_h|||_v |||\xi_p^\star|||_{p'}, \\ c_h(\xi_p^\star, \boldsymbol{q}_h) &\le \nu^{-1}|\boldsymbol{q}_h|_p|\xi_p^\star|_p. \end{aligned} \tag{42}$$

By inserting (42) into (41),

$$|||(\boldsymbol{e}_u^\star,\boldsymbol{e}_p^\star)|||_{v,p} \le C\nu^{1/2}|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'} + C\nu^{-1/2}|||\xi_p^\star|||_{p'}. \tag{43}$$

Using the triangle inequality and (43), we have

$$\begin{aligned} |||(\boldsymbol{u}^\star-\boldsymbol{u}_h^\star, \boldsymbol{p}^\star-\boldsymbol{p}_h^\star)|||_{v,p} &\le |||(\xi_u^\star,\xi_p^\star)|||_{v,p} + |||(\boldsymbol{e}_u^\star,\boldsymbol{e}_p^\star)|||_{v,p} \\ &\le C\nu^{1/2}|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'} + C\nu^{-1/2}|||\xi_p^\star|||_{p'}. \end{aligned} \tag{44}$$

Then, the pressure error estimate follows from (44), the interpolation estimates of the L^2 -projection operators and the velocity error bound (21). □

Remark 8. When we take the value of γ too small, we will comment on the post-processing in solving the Navier–Stokes equations. For the steady Navier–Stokes problem, it only need a post-processing for the pressure solution after the Picard iterations. For the time-dependent Navier–Stokes problem, when we only need the pressure solution at the final time, in fact, we also only need to do a post-processing for the pressure solution at the final time. When we need to compute the discrete error of the pressure in the $L^2(0,t;L^2(\Omega))$ norm, we have to do a post-processing for every time step, which is indeed very expensive. In the numerical simulation of incompressible flows, the velocity solution is often the most important variable which we care about. Thus when we do not care about the pressure solution, we do not need to do a post-processing for the pressure.

7. Numerical examples

In this section, some numerical experiments are presented to support our analytical results. Numerical examples have been run on the high order finite element library NGSolve [30]. In all numerical examples, the velocity penalty parameter η is chosen to be $10k^2$, and the regular triangulations with diagonals (from bottom right to top left), with the same number of subdivisions on each coordinate direction, are used.

7.1. No-flow problem

In this numerical example, we test the equal-order HDG and E-HDG methods with small pressure penalty parameters. We consider the No-flow problem where the exact solution for the Stokes equations is $u = (0,0)$, $p = \mathrm{R}\left(y^3 - y^2/2 + y - 7/12\right)$ in the unit square $\Omega = (0,1)^2$ [9]. The right-hand side $f = \left(0, \mathrm{R}\left(1 - y + 3y^2\right)\right)$ and $\mathrm{R} = 100$.

Set $\nu = 10^{-4}$ and the polynomial order $k = 2$. We use the mesh with $N = 10$ subdivisions in each coordinate direction. It can be observed in Table 1 that the velocity errors for the equal-order HDG and E-HDG methods tend to be pressure-robust, and $\|\nabla_h \cdot u_h\|$ and $|u_h|_{\mathrm{nj}}$ tend to 0 as $\gamma \to 0$. For the equal-order EDG method, it does not tend to be pressure-robust, and $|u_h|_{\mathrm{nj}}$ does not tend to 0 as $\gamma \to 0$.

7.2. The numerical performance of post-processing

In this subsection, we consider the numerical performances of the post-processing for the equal-order HDG method. We take the analytical solution (u,p) of the Stokes equations on the two-dimensional square domain $\Omega = (0,1)^2$, as follows:

$$u = curl(\chi), \quad p = x^5 + y^5 - 1/3, \tag{45}$$

Table 1
The numerical performances of equal-order HDG, E-HDG, and EDG methods with small pressure penalty parameter.

γ	$\|u-u_h\|$	$\|\nabla_h(u-u_h)\|$	$\|p-p_h\|$	$\|\nabla_h\cdot u_h\|$	$\|u_h\|_{\mathrm{nj}}$
Equal-order HDG					
1.00e+00	3.18e−05	2.66e−03	2.60e−04	2.85e−03	6.88e−04
1.00e−01	3.19e−06	2.66e−04	2.60e−04	2.86e−04	6.91e−05
1.00e−02	3.19e−07	2.66e−05	2.60e−04	2.86e−05	6.91e−06
1.00e−03	3.19e−08	2.66e−06	2.60e−04	2.86e−06	6.91e−07
1.00e−04	3.19e−09	2.66e−07	2.60e−04	2.86e−07	6.91e−08
Equal-order E-HDG					
1.00e+00	3.48e−05	2.65e−03	2.60e−04	2.85e−03	6.88e−04
1.00e−01	3.48e−06	2.65e−04	2.60e−04	2.86e−04	6.91e−05
1.00e−02	3.49e−07	2.65e−05	2.60e−04	2.86e−05	6.91e−06
1.00e−03	3.49e−08	2.65e−06	2.60e−04	2.86e−06	6.91e−07
1.00e−04	3.49e−09	2.65e−07	2.60e−04	2.86e−07	6.91e−08
Equal-order EDG					
1.00e+00	3.90e−03	2.83e−01	3.28e−04	4.62e−03	1.81e−01
1.00e−01	3.86e−03	2.81e−01	3.28e−04	4.62e−04	1.81e−01
1.00e−02	3.85e−03	2.81e−01	3.28e−04	4.62e−05	1.81e−01
1.00e−03	3.85e−03	2.81e−01	3.28e−04	4.62e−06	1.81e−01
1.00e−04	3.85e−03	2.81e−01	3.28e−04	4.62e−07	1.81e−01

Table 2
The numerical performance of the equal-order HDG method with the post-processing, with different parameters γ, and '×' represents no post-processing.

γ	$\|u-u_h\|$	$\|\nabla_h(u-u_h)\|$	$\|p-p_h\|$	$\|p-p_h^\star\|$	$\|\nabla_h\cdot u_h\|$	$\|u_h\|_{\mathrm{nj}}$
1.00e+02	1.02e−06	6.61e−05	2.89e−08	×	5.91e−05	2.61e−06
1.00e+01	3.33e−07	2.09e−05	2.48e−08	×	8.02e−06	6.16e−07
1.00e+00	3.21e−07	1.98e−05	2.45e−08	×	8.46e−07	7.09e−08
1.00e−01	3.22e−07	1.99e−05	2.45e−08	×	8.51e−08	7.20e−09
1.00e−02	3.23e−07	1.99e−05	2.45e−08	×	8.52e−09	7.21e−10
1.00e−03	3.23e−07	1.99e−05	2.45e−08	×	8.52e−10	7.21e−11
1.00e−04	3.23e−07	1.99e−05	2.45e−08	×	8.52e−11	7.21e−12
1.00e−05	3.23e−07	1.99e−05	2.45e−08	×	8.52e−12	7.21e−13
1.00e−06	3.23e−07	1.99e−05	2.45e−08	×	8.69e−13	7.60e−14
1.00e−07	3.23e−07	1.99e−05	2.50e−08	×	1.79e−13	2.26e−14
1.00e−08	3.23e−07	1.99e−05	5.88e−08	3.08e−08	1.61e−13	1.94e−14
1.00e−09	3.23e−07	1.99e−05	5.34e−07	3.08e−08	1.54e−13	1.86e−14
1.00e−10	3.23e−07	1.99e−05	5.34e−06	3.08e−08	9.07e−14	2.43e−14
1.00e−11	3.23e−07	1.99e−05	5.34e−05	3.08e−08	1.66e−13	2.59e−14
1.00e−12	3.23e−07	1.99e−05	5.34e−04	3.08e−08	1.56e−13	2.21e−14
1.00e−13	3.23e−07	1.99e−05	5.34e−03	3.08e−08	1.39e−13	2.08e−14
1.00e−14	3.23e−07	1.99e−05	5.34e−02	3.08e−08	3.74e−13	2.37e−14

where $\chi = x^2(x-1)^2y^2(y-1)^2$, see [25, Subsection 4.2]. Here, we take $\nu = 10^{-4}$. The right-hand side f and the Dirichlet boundary condition are derived from the exact solution.

Use the mesh with $N = 10$ subdivisions in each coordinate direction and the polynomial order $k = 4$. Here, for smooth solutions, it is preferable to use higher-order finite elements with the post-processing. From Table 2, we can observe that for the equal-order HDG method, a sufficiently small pressure penalty parameter γ cause the over-stabilization of the pressure (when γ is less than 10^{-8}). After the post-processing, we get a stable and accurate pressure approximation. Then, by taking $\gamma = 10^{-11}$, we test the convergence rates of the equal-order HDG method with the post-processing, with different viscosity ν. The meshes with $N = 5, 10, 20$ and 40 subdivisions in each coordinate direction are used. It can be observed in Table 3 that the errors in the velocity for the equal-order HDG method with the post-processing are indeed independent of viscosity, as expected from Theorem 2. Optimal rates of convergence are observed for $\nu = 1$ and $\nu = 10^{-5}$, respectively. We also notice that the convergence rate of the pressure is 5 for small viscosity $\nu = 10^{-5}$, as expected from the pressure estimate in Theorem 3.

In addition, the equal-order E-HDG method with the post-processing also have similar numerical performance(for brevity not shown here).

Table 3
The equal-order HDG method with the post-processing, $\gamma = 10^{-11}$, polynomial order $k = 4$.

N	$\|u-u_h\|$	Rate	$\|\nabla_h(u-u_h)\|$	Rate	$\|p-p_h^\star\|$	Rate
$\nu = 1$						
5	3.39e−06		1.32e−04		1.55e−04	
10	1.05e−07	5.0	8.06e−06	4.0	8.93e−06	4.1
20	3.20e−09	5.0	4.91e−07	4.0	5.02e−07	4.2
40	9.83e−11	5.0	3.01e−08	4.0	2.89e−08	4.1
$\nu = 10^{-5}$						
5	3.39e−06		1.32e−04		3.25e−07	
10	1.05e−07	5.0	8.06e−06	4.0	9.81e−09	5.1
20	3.20e−09	5.0	4.91e−07	4.0	3.03e−10	5.0
40	1.08e−10	4.9	3.01e−08	4.0	9.45e−12	5.0

8. Conclusions

In this paper, we analyze an equal-order HDG method with a small pressure penalty parameter for the Stokes equations. When the pressure penalty parameter γ tends to 0, the velocity approximation tends to be H(div)-conforming and exactly divergence-free, and the velocity error tends to be pressure-robust. To avoid that a sufficiently small pressure penalty parameter cause the over-stabilization of the pressure, we provide a post-processing procedure.

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Appendix. Velocity error in the L^2-norm

Firstly, we consider the following Stokes problem:

$$\begin{aligned} -\nu\nabla^2\chi + \nabla\lambda &= g, && \text{in } \Omega,\\ \nabla\cdot\chi &= 0, && \text{in } \Omega,\\ \chi &= 0, && \text{on } \partial\Omega. \end{aligned} \tag{A.1}$$

The solution (χ, λ) to (A.1) has the following regularity assumption [31, Assumption A7]:

$$\sqrt{\nu}\,\|\chi\|_2 + \|\lambda\|_1 \le C\,\|g\|\,. \tag{A.2}$$

Introducing the following approximation errors

$$\xi_\chi = \chi - \mathcal{I}_{\mathrm{BDM}}\chi, \quad \bar{\xi}_\chi = \mathrm{tr}(\chi) - \bar{\mathcal{I}}_V\chi,$$
$$\xi_\lambda = \lambda - \mathcal{I}_{SZ}\lambda, \quad \bar{\xi}_\lambda = \mathrm{tr}(\lambda) - \bar{\mathcal{I}}_{SZ}\lambda,$$

where $\mathcal{I}_{SZ}\lambda$ is the continuous Scott–Zhang interpolant [29] and $\bar{\mathcal{I}}_{SZ}\lambda = \mathcal{I}_{SZ}\lambda|_F \in \bar{Q}_h$. Set $\boldsymbol{\xi}_\chi = (\xi_\chi, \bar{\xi}_\chi)$ and $\boldsymbol{\xi}_\lambda = (\xi_\lambda, \bar{\xi}_\lambda)$.

Theorem 4. *Under the settings of Theorem 2, we have*

$$\|u-u_h\| \le C(h^{k+1}\|u\|_{k+1} + \gamma^{\frac{1}{2}}\nu^{-\frac{1}{2}}h^{k+2}\|p\|_{k+1}), \tag{A.3}$$

with $C > 0$ independent of h, γ and ν.

Proof. Firstly, taking $g = u - u_h$ in (A.1), it is easy to obtain

$$\begin{aligned} &a_h(\boldsymbol{u}-\boldsymbol{u}_h, (\chi, \mathrm{tr}(\chi))) + b_h(\boldsymbol{p}-\boldsymbol{p}_h, \chi) + b_h((\lambda, \mathrm{tr}(\lambda)), u-u_h)\\ &\quad - c_h\left(\boldsymbol{p}-\boldsymbol{p}_h, (\lambda, \mathrm{tr}(\lambda))\right)\\ &= \sum_{K\in\mathcal{T}_h}\int_K (u-u_h)\cdot\left(-\nu\nabla^2\chi + \nabla\lambda\right)\mathrm{d}x = \|u-u_h\|^2. \end{aligned} \tag{A.4}$$

In addition, we have the following error equation

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}-\boldsymbol{p}_h, v_h) + b_h(\boldsymbol{q}_h, u-u_h) - c_h\left(\boldsymbol{p}-\boldsymbol{p}_h, \boldsymbol{q}_h\right) = 0. \tag{A.5}$$

We subtract (A.5) from (A.4) and take $\boldsymbol{v}_h = (\mathcal{I}_{\mathrm{BDM}}\chi, \bar{\mathcal{I}}_V\chi)$ and $\boldsymbol{q}_h = (\mathcal{I}_{SZ}\lambda, \bar{\mathcal{I}}_{SZ}\lambda)$ to obtain

$$\begin{aligned} \|u-u_h\|^2 &= a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{\xi}_\chi) + b_h(\boldsymbol{p}-\boldsymbol{p}_h, \xi_\chi) + b_h(\boldsymbol{\xi}_\lambda, u-u_h)\\ &\quad - c_h(\boldsymbol{p}-\boldsymbol{p}_h, \boldsymbol{\xi}_\lambda)\\ &= T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{A.6}$$

Now, we estimate each term in (A.6). By the boundedness of a_h, the first inequality of (28) and the regularity condition (A.2),

$$\begin{aligned} T_1 &\le C\nu|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}|||\boldsymbol{\xi}_\chi|||_{v'} \le C\nu h|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}\|\chi\|_2\\ &\le C\sqrt{\nu}h|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}\,\|u-u_h\|\,, \end{aligned}$$

where for the first inequality, we use the boundedness of a_h on the extended space $V^\star(h)$ in [25, Lemma 10]. Noticing that $T_2 = 0$. For T_3, we use the boundedness of b_h (7), the continuous Scott–Zhang interpolant estimate, and the regularity condition (A.2) to obtain

$$T_3 \le C_b^b|||\boldsymbol{u}-\boldsymbol{u}_h|||_v|||\boldsymbol{\xi}_\lambda|||_{p'} \le Ch|||\boldsymbol{u}-\boldsymbol{u}_h|||_v\|\lambda\|_1 \le Ch|||\boldsymbol{u}-\boldsymbol{u}_h|||_v\|u-u_h\|.$$

For T_4, noting $\bar{\mathcal{I}}_{SZ}\lambda = \mathcal{I}_{SZ}\lambda|_F$, we have $T_4 = 0$. We collect the error bounds of the right-hand side of (A.6), and divide both sides of (A.6) by $\|u-u_h\|$. Then, we have

$$\|u-u_h\| \le Ch|||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}.$$

This completes the proof. □

References

[1] B. Cockburn, G. Kanschat, D. Schötzau, A note on discontinuous Galerkin divergence-free solutions of the Navier–Stokes equations, J. Sci. Comput. 31 (1–2) (2007) 61–73.
[2] J. Gopalakrishnan, P.L. Lederer, J. Schöberl, A mass conserving mixed stress formulation for the Stokes equations, IMA J. Numer. Anal. 40 (3) (2020) 1838–1874.
[3] Johnny Guzmán, L.R. Scott, The Scott-Vogelius finite elements revisited, Math. Comp. 88 (2019) 515–529.
[4] J.A. Evans, T.J.R. Hughes, Isogeometric divergence-conforming B-splines for the steady Navier–Stokes equations, Math. Models Methods Appl. Sci. 23 (08) (2013) 1421–1478.
[5] J. Guzmán, M. Neilan, Inf-sup stable finite elements on barycentric refinements producing divergence–free approximations in arbitrary dimensions, SIAM J. Numer. Anal. 56 (5) (2018) 2826–2844.
[6] P.L. Lederer, A. Linke, C. Merdon, et al., Divergence-free reconstruction operators for pressure-robust Stokes discretizations with continuous pressure finite elements, SIAM J. Numer. Anal. 55 (3) (2017) 1291–1314.
[7] A. Linke, C. Merdon, Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 311 (2016) 304–326.
[8] P.L. Lederer, S. Rhebergen, A pressure-robust embedded discontinuous Galerkin method for the Stokes problem by reconstruction operators, 2020, arXiv preprint arXiv:2002.04951.
[9] V. John, A. Linke, C. Merdon, et al., On the divergence constraint in mixed finite element methods for incompressible flows, SIAM Rev. 59 (3) (2017) 492–544.
[10] T.J.R. Hughes, L.P. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1) (1986) 85–99.
[11] E. Burman, P. Hansbo, Edge stabilization for the generalized Stokes problem: A continuous interior penalty method, Comput. Methods Appl. Mech. Engrg. 195 (19–22) (2006) 2393–2410.
[12] Burman E. Fernández, A. Miguel, P. Hansbo, Continuous interior penalty finite element method for Oseen's equations, SIAM J. Numer. Anal. 44 (3) (2006) 1248–1274.
[13] J. De Frutos, García-Archilla Bosco, V. John, et al., Error analysis of non inf-sup stable discretizations of the time-dependent Navier–Stokes equations with local projection stabilization, IMA J. Numer. Anal. 39 (4) (2019) 1747–1786.
[14] S. Ganesan, G. Matthies, L. Tobiska, Local projection stabilization of equal order interpolation applied to the Stokes problem, Math. Comp. 77 (264) (2008) 2039–2060.
[15] B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab, Local discontinuous Galerkin methods for the Stokes system, SIAM J. Numer. Anal. 40 (1) (2002) 319–343.
[16] B. Cockburn, G. Kanschat, D. Schötzau, An equal-order DG method for the incompressible Navier–Stokes equations, J. Sci. Comput. 40 (1–3) (2009) 188–210.

[17] E.W. Jenkins, V. John, A. Linke, L.G. Rebholz, On the parameter choice in grad-div stabilization for the Stokes equations, Adv. Comput. Math. 40 (2) (2014) 491–516.

[18] M. Olshanskii, G. Lube, T. Heister, et al., Grad-div stabilization and subgrid pressure models for the incompressible Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 198 (49–52) (2009) 3975–3988.

[19] Mine Akbas, A. Linke, L.G. Rebholz, et al., The analogue of grad-div stabilization in DG methods for incompressible flows: Limiting behavior and extension to tensor-product meshes, Comput. Methods Appl. Mech. Engrg. 341 (2017) 917–938.

[20] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2) (2009) 1319–1365.

[21] B. Cockburn, J. Gopalakrishnan, The derivation of hybridizable discontinuous Galerkin methods for Stokes flow, SIAM J. Numer. Anal. 47 (2) (2009) 1092–1125.

[22] B. Cockburn, J. Gopalakrishnan, N. Nguyen, et al., Analysis of HDG methods for Stokes flow, Math. Comp. 80 (274) (2011) 723–760.

[23] S. Rhebergen, G. Wells, Analysis of a hybridized/interface stabilized finite element method for the Stokes equations, SIAM J. Numer. Anal. 55 (4) (2017) 1982–2003.

[24] B. Cockburn, F.J. Sayas, Divergence-conforming HDG methods for Stokes flows, Math. Comp. 83 (288) (2014) 1571–1598.

[25] S. Rhebergen, G.N. Wells, An embedded–hybridized discontinuous Galerkin finite element method for the Stokes equations, Comput. Methods Appl. Mech. Engrg. 358 (11) (2020) 112619.

[26] K.L.A. Kirk, S. Rhebergen, Analysis of a pressure-robust hybridized discontinuous Galerkin method for the stationary Navier–Stokes equations, J. Sci. Comput. 81 (2) (2019) 881–897.

[27] V. John, Finite Element Methods for Incompressible Flow Problems, Springer International Publishing, Cham, 2016.

[28] P. Hansbo, M.G. Larson, Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method, Comput. Methods Appl. Mech. Engrg. 191 (17–18) (2002) 1895–1908.

[29] L.R. Scott, S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp. 54 (190) (1990) 483.

[30] J. Schöberl, C++ 11 Implementation of Finite Elements in NGSolve, Institute for analysis and scientific computing, Vienna University of Technology, 2014.

[31] P.W. Schroeder, G. Lube, Divergence-free H(div)-FEM for time-dependent incompressible flows with applications to high Reynolds number vortex dynamics, J. Sci. Comput. 75 (2) (2018) 830–858.