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Robust error analysis of H(div)-conforming DG method for the time-dependent incompressible Navier–Stokes equations ☆

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ABSTRACT

In this paper, we mainly consider the velocity error analysis of H(div)-conforming DG method for the semi-discrete time-dependent Navier–Stokes equations. Firstly, we prove that the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity is optimal and pressure-robust, but the constants in the velocity error bound are dependent on the inverse power of the viscosity, so it is not semi-robust. Secondly, we focus on pressure-robust and semi-robust velocity error analysis at high Reynolds numbers. By introducing the Raviart–Thomas interpolation operator, we prove that when the condition $\nu < Ch$ is satisfied, the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity, which is pressure-robust and semi-robust, is quasi-optimal. Numerical experiments are carried out to verify the analytical results.

1. Introduction

In this paper, we focus on pressure-robust and semi-robust analysis of H(div)-conforming DG method for the time-dependent incompressible Navier–Stokes equations. Pressure-robust means that the error estimate of kinetic energy and dissipated energy is independent of the pressure. Most classical methods relax divergence constraint by discretely enforcing the divergence constraint. Such relaxation of the divergence constraint can lead to a pressure-dependent velocity error, which can contaminate the computed velocity [1]. The error bound of kinetic energy and dissipated energy where the constants are independent of the Reynolds number Re (or ν^{-1}) [2], is called 'Re-semi-robust'. It is very important for the simulation of the high Reynolds number flows. For the sake of brevity, we use the term 'semi-robust' instead of 'Re-semi-robust'.

For pressure-robust and semi-robust analysis, some work has been done recently. Using continuous interior penalty (CIP) finite element method, the $L^\infty(\boldsymbol{L}^2)$ (short for $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$) error of the velocity is quasi-optimal in the case of $\nu < Ch$ [3]. For local projection stabilization (LPS) method, with inf–sup stable finite elements, the $L^\infty(\boldsymbol{L}^2)$ error of the velocity is suboptimal [4]. Using continuous equal-order finite elements for the velocity and the pressure, local projection stabilization methods corresponding to different stabilization terms have been analyzed. For one of the methods, it is proved that the $L^\infty(\boldsymbol{L}^2)$ error of the velocity is quasi-optimal convergent in the case of $\nu < Ch$ [5]. They are semi-robust, rather than pressure-robust [3–5]. For pressure-robust isogeometric finite element method, the $L^\infty(\boldsymbol{L}^2)$ error of the velocity which is optimal, is not semi-robust [6]. For exactly divergence-free and H^1-conforming finite element methods, it is pressure-robust and semi-robust, however, the corresponding convergence rate is suboptimal [7].

In this paper, we consider H(div)-conforming DG method with upwind scheme for the time-dependent incompressible Navier–Stokes equations. H(div)-conforming DG method provides an exactly divergence-free velocity, which of course is

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also pressure-robust, and an opportunity to stabilize the high Reynolds number flows, since an upwind stabilization term is employed [8]. And, the fact that less stability is required allows discretizations to achieve minimization of numerical dissipation. At present, it has successfully simulated some high Reynolds number flows [9–12]. We particularly emphasize numerical analysis of H(div)-conforming DG method for incompressible flows. H(div)-conforming DG methods using the central flux have been analyzed for the incompressible Euler equations. However, numerical experiments suggest that the analysis is not sharp for the upwind flux [13]. H(div)-conforming DG methods for the incompressible Oseen and Navier–Stokes equations is presented in [9,14]. The $L^\infty(\boldsymbol{L}^2)$ error of the velocity is pressure-robust and semi-robust, but it is suboptimal.

The objective of this paper is to conduct detailed numerical analysis of H(div)-conforming DG method for the semi-discrete time-dependent Navier–Stokes equations. Firstly, we prove the optimal convergence rate of the velocity by means of the Stokes projection. The velocity error is pressure-robust, but is not semi-robust. Secondly, we focus on pressure-robust and semi-robust velocity error analysis at high Reynolds number. For the scalar convection–diffusion problem, a specific technique can be applied to the convection terms, allowing additional rate 1/2 when the viscosity is small enough [15]. It has been an open problem whether the similar technique can be used for H(div)-conforming FEM for the Navier–Stokes equations [14]. In this paper, by introducing the Raviart–Thomas interpolation operator, we applied the similar technique to the convection terms and obtained the quasi-optimal convergence rate for the $L^\infty(\boldsymbol{L}^2)$ error bound of the velocity, in which the constants are not explicitly dependent on ν^{-1}. Thus, the velocity error has the same convergence rate as the CIP method [3].

The structure of the article is as follows: In Section 2, the weak form of the Navier–Stokes equations is presented. In Section 3, we introduce H(div)-conforming and inf–sup stable FEM for the time-dependent Navier–Stokes problem. Then, in Section 4, the optimal convergence rate of the velocity error is proved by means of the Stokes projection. In Section 5, by introducing the Raviart–Thomas interpolation operator, we prove that when the condition $\nu < Ch$ is satisfied, the convergence rate of the velocity error which is pressure-robust and semi-robust, is quasi-optimal. Finally, Section 6 presents numerical experiments to verify our theoretical results. We end in Section 7 with some concluding remarks.

2. Navier–Stokes equations

Throughout the paper, for $D \subseteq R^d$, $d \in \{2, 3\}$, we use the Sobolev spaces $W^{m,p}(D)$ for scalar-valued functions with associated norms $\|\cdot\|_{W^{m,p}(D)}$ and seminorms $|\cdot|_{W^{m,p}(D)}$ for $m \geqslant 0$ and $p \geqslant 1$. In the case $m = 0$, $W^{0,p}(D) = L^p(D)$, and when $p = 2$, $W^{m,2}(D) = H^m(D)$. Spaces for vector- and tensor-valued functions are indicated with bold letters. In addition, for the Bochner space $L^p(0, T; \boldsymbol{Y})(1 \le p \le \infty)$, where $\boldsymbol{Y}$ is a Banach space, the abbreviation $L^p(\boldsymbol{Y}) = L^p(0, T; \boldsymbol{Y})$ is frequently used.

We consider the following time-dependent incompressible Navier–Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & (0, T] \times \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & (0, T] \times \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & (0, T] \times \partial\Omega, \\ \boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) & \Omega, \end{cases} \tag{2.1}$$

in a polygonal ($d = 2$) or polyhedral ($d = 3$) domain Ω. Introduce

$$\boldsymbol{X} = \boldsymbol{H}_0^1(\Omega), \quad Q = L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q dx = 0\}.$$

The weak formulation of the unsteady Navier–Stokes equations takes the form: find $(\boldsymbol{u}, p) : (0, T] \to (\boldsymbol{X}, Q)$, satisfying

$$\begin{cases} (\partial_t \boldsymbol{u}, \mathbf{v}) + \nu a(\boldsymbol{u}, \boldsymbol{v}) + c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{X}, \\ b(\boldsymbol{u}, q) = 0, & \forall q \in Q. \end{cases} \tag{2.2}$$

Here, the multilinear forms are given by

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\boldsymbol{x}, \qquad c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x},$$

$$b(\boldsymbol{u}, q) = -\int_\Omega q(\nabla \cdot \boldsymbol{u}) \, d\boldsymbol{x}.$$

We introduce the space of weakly divergence-free space

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X} : b(\boldsymbol{v}, q) = 0, \quad \forall q \in Q\}.$$

3. H(div)-conforming DG finite element method

Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform simplicial mesh of Ω and mesh size $h = max_{T\in\mathcal{T}_h} h_T$, where h_T denotes the diameter of each element $T \in \mathcal{T}_h$. The skeleton $\mathcal{F}_h$ denotes the set of all facets, and h_F denotes the diameter of each facet $F \in \mathcal{F}_h$. $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$, where $\mathcal{F}_h^i$ and $\mathcal{F}_h^\partial$ are the subset of interior facets and boundary facets, respectively. Then, we define the jump $[\![\cdot]\!]_F$ and average $\{\cdot\}_F$ operator across the interior facets $F \in \mathcal{F}_h^i$ by

$$[\![\phi]\!]_F = \phi^+ - \phi^-, \quad \{\phi\}_F = \frac{\phi^+ + \phi^-}{2}.$$

For the boundary facets $F \in \mathcal{F}_h^\partial$, we set

$$[\![\phi]\!]_F = \{\phi\}_F = \phi.$$

Define the broken gradient $\nabla_h : \boldsymbol{H}^1(\mathcal{T}_h) \to \boldsymbol{L}^2(\Omega)$ by $(\nabla_h \boldsymbol{w})|_T = \nabla(\boldsymbol{w}|_T)$, and the broken Sobolev space $\boldsymbol{H}^m(\mathcal{T}_h) = \{\boldsymbol{w} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{w}|_T \in \boldsymbol{H}^m(T), \forall T \in \mathcal{T}_h\}$.

3.1. Velocity space and pressure space

We introduce H(div)-conforming velocity space and pressure space,

$$\boldsymbol{X}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_T \in \boldsymbol{V}_k(T),\ \forall T \in \mathcal{T}_h \text{ and } \boldsymbol{v}_h \cdot \boldsymbol{n} = 0, \text{ on } \partial\Omega\right\},$$
$$Q_h = \left\{q \in L_0^2(\Omega) : q|_T \in P_l(T),\ \forall T \in \mathcal{T}_h\right\}.$$

For simplicial mesh, we choose the local space $\boldsymbol{V}_k(T)$ as $RT_k(T)$ or $BDM_k(T)$, $k \geq 1$, then the corresponding pressure space $P_l(T)$ is $P_k(T)$ and $P_{k-1}(T)$, respectively. In these cases, the global spaces $\boldsymbol{X}_h$ and Q_h form the inf–sup stable FE pair [9], that is, there exists $\beta_h > 0$, independent of the mesh size h, such that

$$\inf_{q_h\in Q_h\setminus\{0\}} \sup_{\boldsymbol{v}_h\in \boldsymbol{X}_h\setminus\{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}_h, q_h)}{|||\boldsymbol{v}_h|||_e \|q_h\|_{L^2}} \geqslant \beta_h, \tag{3.1}$$

where $|||\cdot|||_e$ is defined in (3.6). The global spaces $\boldsymbol{X}_h$ and Q_h satisfy the following relationship

$$\nabla\cdot\boldsymbol{X}_h \subseteq Q_h. \tag{3.2}$$

3.2. Finite element method

The space-semidiscrete weak formulation of (2.2) reads as follows: find $(\boldsymbol{u}_h, p_h) : (0, T) \to (\boldsymbol{X}_h, Q_h)$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{u}_h, q_h) + c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h), \tag{3.3}$$

for any $(\boldsymbol{v}_h, q_h) \in (\boldsymbol{X}_h, Q_h)$.

For the discretization of the dissipative term, we use the standard symmetric interior penalty form

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \nabla_h \boldsymbol{u}_h : \nabla_h \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h} \int_F \Big[(\{\nabla_h \boldsymbol{u}_h\}\boldsymbol{n}_F \cdot [\![\boldsymbol{v}_h]\!]) + ([\![\boldsymbol{u}_h]\!] \cdot \{\nabla_h \boldsymbol{v}_h\}\boldsymbol{n}_F)\Big] \,\mathrm{d}\boldsymbol{s}$$
$$+ \sum_{F\in\mathcal{F}_h} \int_F (\frac{\sigma}{h_F} [\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{v}_h]\!]) \,\mathrm{d}\boldsymbol{s}.$$

Because of the normal continuity of H(div)-conforming finite elements, the bilinear form b_h of velocity and pressure is the same as the continuous Galerkin method

$$b_h(\boldsymbol{u}_h, q_h) = -\int_\Omega q_h(\nabla_\mathrm{h} \cdot \boldsymbol{u}_h) \,\mathrm{d}\boldsymbol{x}. \tag{3.4}$$

For the inertia term, we choose the following convection term with standard upwind mechanism [8,15]. Because of $\nabla \cdot \boldsymbol{u}_h = 0$ and $[\![\boldsymbol{u}_h]\!]_F \cdot \boldsymbol{n}_F = 0$, $\forall F \in \mathcal{F}_h$, it can be abbreviated, as follows:

$$c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega (\boldsymbol{u}_h \cdot \nabla_h)\boldsymbol{u}_h \cdot \boldsymbol{v}_h \,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h^i} \int_F (\boldsymbol{u}_h \cdot \boldsymbol{n}_F)[\![\boldsymbol{u}_h]\!]\{\boldsymbol{v}_h\} \,\mathrm{d}\boldsymbol{s}$$
$$+ \sum_{F\in\mathcal{F}_h^i} \int_F \frac{1}{2} |(\boldsymbol{u}_h \cdot \boldsymbol{n}_F)| [\![\boldsymbol{u}_h]\!][\![\boldsymbol{v}_h]\!] \,\mathrm{d}\boldsymbol{s}. \tag{3.5}$$

The energy norm corresponding to the bilinear form a_h is given by

$$|||\boldsymbol{v}_h|||_e^2 = \|\nabla_h \boldsymbol{v}_h\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h} \frac{\sigma}{h_F}\|[\![\boldsymbol{v}_h]\!]\|_{L^2(F)}^2, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{3.6}$$

In addition, we introduce a larger space

$$\boldsymbol{X}(h) = \boldsymbol{X}_h \oplus \left[\boldsymbol{X} \cap \boldsymbol{H}^{\frac{3}{2}+\varepsilon}(\mathcal{T}_h)\right],$$

and define a stronger norm on $\boldsymbol{X}(h)$

$$|||\boldsymbol{w}|||_{e,\sharp}^2 = |||\boldsymbol{w}|||_e^2 + \sum_{T\in\mathcal{T}_h} h_T \|\nabla_h \boldsymbol{w}\cdot \boldsymbol{n}_T\|_{L^2(\partial T)}^2, \quad \forall \boldsymbol{w} \in \boldsymbol{X}(h).$$

Let us summarize the most important properties of a_h [9,15].

Lemma 3.1 (*Discrete Coercivity of a_h*)**.** *Assume that $\sigma > 0$ is sufficiently large. Then, there exists a constant $C_\sigma > 0$, independent of h, such that*

$$C_\sigma |||\boldsymbol{v}_h|||_e^2 \le a_h(\boldsymbol{v}_h, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{3.7}$$

Lemma 3.2 (*Boundedness of a_h*)**.** *There exists a constant $C > 0$, independent of h, such that*

$$a_h(\boldsymbol{w}, \boldsymbol{v}_h) \le C|||\boldsymbol{w}|||_{e,\sharp}|||\boldsymbol{v}_h|||_e, \quad \forall(\boldsymbol{w}, \boldsymbol{v}_h) \in \boldsymbol{X}(h)\times \boldsymbol{X}_h. \tag{3.8}$$

The property in (3.2) ensures that the velocity approximation will be exactly divergence-free. We introduce the exactly divergence-free space

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h \in \boldsymbol{X}_h : b(\boldsymbol{v}_h, q_h) = 0,\ \forall q_h \in Q_h\}.$$

Moreover, we introduce the jump seminorm

$$|\boldsymbol{v}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2 = \sum_{F\in\mathcal{F}_h^i} \int_F \frac{1}{2}|(\boldsymbol{u}_h\cdot \boldsymbol{n}_F)||[\![\boldsymbol{v}_h]\!]|^2\,\mathrm{d}\boldsymbol{s}.$$

In addition, the well-posedness of (3.3) and the velocity energy estimate are given in [14, Corollary 3.5].

3.3. Important inequalities

First, we introduce the space $P_d^n(\mathcal{T}_h)$

$$P_d^n(\mathcal{T}_h) = \{\boldsymbol{v}_h \in L^2(\Omega) : \boldsymbol{v}_h|_T \in (P_n(T))^d, \forall T \in \mathcal{T}_h\},$$

where $n \ge 0$ is an integer.

Lemma 3.3 (*[15, Remark 1.47] Trace Inequality*)**.** *There exists a constant $C_{\mathrm{tr}} > 0$, independent of h, such that*

$$\|\boldsymbol{v}_h\|_{L^2(\partial T)} \leqslant C_{\mathrm{tr}} h_T^{-\frac{1}{2}} \|\boldsymbol{v}_h\|_{L^2(T)}, \quad \forall T \in \mathcal{T}_h, \forall \boldsymbol{v}_h \in P_d^n(\mathcal{T}_h). \tag{3.9}$$

Lemma 3.4 (*[16, Lemma 1.138] Inverse Inequality)*)**.** *Let $0 \leqslant m \leqslant \ell$ and $1 \leqslant p, q \leqslant \infty$. The space $P_d^n(\mathcal{T}_h)$ satisfies the local inverse inequality*

$$\|\boldsymbol{v}_h\|_{W^{\ell,p}(T)} \leqslant C_{\mathrm{inv}} h_T^{m-\ell+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|\boldsymbol{v}_h\|_{W^{m,q}(T)}, \quad \forall T \in \mathcal{T}_h, \forall \boldsymbol{v}_h \in P_d^n(\mathcal{T}_h). \tag{3.10}$$

Lemma 3.5. *There exists a constant $C > 0$, independent of h, such that*

$$|||\boldsymbol{v}_h|||_e \le Ch^{-1}\|\boldsymbol{v}_h\|_{L^2(\Omega)}, \quad \forall \boldsymbol{v}_h \in P_d^n(\mathcal{T}_h).$$

Proof. By Lemmas 3.3 and 3.4, and quasi-uniformity of the mesh, this inequality can be proved. □

Lemma 3.6 (*[16, Proposition 1.135]*)**.** *Let $\rho_n \boldsymbol{w}$ denotes the L^2-projection of $\boldsymbol{w}$ onto $P_d^n(\mathcal{T}_h)$. There exists a constant $C > 0$, independent of h, such that, for $0 \leqslant l \leqslant n+1$ and $1 \leqslant p \leqslant \infty$,*

$$\|\boldsymbol{w} - \rho_n \boldsymbol{w}\|_{L^p(\Omega)} \le Ch^l |\boldsymbol{w}|_{W^{l,p}(\Omega)}, \quad \forall \boldsymbol{w} \in \boldsymbol{W}^{l,p}(\Omega).$$

4. Pressure-robust analysis for the velocity

In this section, we derive the convergence rate of the discrete velocity by means of the Stokes projection.

Lemma 4.1 (*[14, Theorem 5.3]*). *Let $\rho_{st}\boldsymbol{w} \in \boldsymbol{V}_h$ be the Stokes projection of $\boldsymbol{w}$ with $\nabla \cdot \boldsymbol{w} = 0$*

$$a_h(\rho_{st}\boldsymbol{w}, \boldsymbol{v}_h) = a_h(\boldsymbol{w}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.1}$$

and 'elliptic regularity' as stated in [14, Assumption D], be fulfilled. Then, provided $\boldsymbol{w} \in \boldsymbol{H}^r(\Omega)$ with $r > 3/2$ and with $r_{\boldsymbol{w}} = \min\{r, k+1\}$,

$$\|\boldsymbol{w} - \boldsymbol{\rho}_{st}\boldsymbol{w}\|_{L^2} + h|||\boldsymbol{w} - \boldsymbol{\rho}_{st}\boldsymbol{w}|||_{e,\sharp} \le Ch \inf_{\forall \boldsymbol{w}_h \in \boldsymbol{V}_h} |||\boldsymbol{w} - \boldsymbol{w}_h|||_{e,\sharp} \le Ch^{r_{\boldsymbol{w}}}|\boldsymbol{w}|_{H^{r_{\boldsymbol{w}}}(\Omega)}. \tag{4.2}$$

Lemma 4.2. *Assume $\boldsymbol{w} \in L^2(0,T;\boldsymbol{W}^{1,\infty}(\Omega)) \cap L^2(0,T;\boldsymbol{H}^{d/2+1}(\Omega))$ and $\nabla \cdot \boldsymbol{w} = 0$. In the setting of Lemma 4.1, with $k \ge \frac{d}{2}$, there exists a constant $C > 0$, independent of h, such that*

$$\|\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} \leqslant Ch\|\nabla \boldsymbol{w}\|^*_{L^\infty}, \tag{4.3}$$

where

$$\|\nabla \boldsymbol{w}\|^*_{L^\infty} = \|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)} + |\boldsymbol{w}|_{H^{d/2+1}(\Omega)}.$$

In addition, $\rho_{st}\boldsymbol{w} \in L^2\big(0,T;\boldsymbol{L}^\infty(\Omega)\big)$.

Proof. Using the triangle inequality, we have

$$\|\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} \le \|\boldsymbol{w} - \rho_k\boldsymbol{w}\|_{L^\infty(\Omega)} + \|\rho_k\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)}. \tag{4.4}$$

Due to Lemma 3.6, we can get

$$\|\boldsymbol{w} - \rho_k\boldsymbol{w}\|_{L^\infty(\Omega)} \leqslant Ch\|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}, \tag{4.5}$$

where $\rho_k\boldsymbol{w}$ denotes the L^2-projection of $\boldsymbol{w}$ onto $P_d^k(\mathcal{T}_h)$, with $k \ge \frac{d}{2}$. For the second term on the right-hand side of (4.4), we use the inverse inequality (3.10) and quasi-uniformity of the mesh to obtain

$$\|\rho_k\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} \le Ch^{-\frac{d}{2}}\|\rho_k\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^2(\Omega)}. \tag{4.6}$$

Using again the triangle inequality, (4.2) and $\|\boldsymbol{w} - \rho_k\boldsymbol{w}\|_{L^2(\Omega)} \le Ch^{d/2+1}|\boldsymbol{w}|_{H^{d/2+1}(\Omega)}$, we have

$$\|\rho_k\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^2(\Omega)} \le \|\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^2(\Omega)} + \|\boldsymbol{w} - \rho_k\boldsymbol{w}\|_{L^2(\Omega)} \le Ch^{d/2+1}|\boldsymbol{w}|_{H^{d/2+1}(\Omega)}. \tag{4.7}$$

Using Eqs. (4.4)–(4.7), we obtain

$$\|\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} \leqslant Ch(\|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)} + |\boldsymbol{w}|_{H^{d/2+1}(\Omega)}). \quad \square$$

Remark 1. As for the maximum norm estimates of H(div) Stokes projection, it makes the following assumptions in [14, Assumption E]

$$\|\boldsymbol{w} - \rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} + h\|\nabla_h\rho_{st}\boldsymbol{w}\|_{L^\infty(\Omega)} \leqslant Ch\|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}. \tag{4.8}$$

At present, there is no strict mathematical proof of (4.8). Here we evade the question by improving the regularity of $\boldsymbol{w}$. Due to the elliptic regularity, thus we only appropriately improve the regularity of $\boldsymbol{w}$ from $H^2(\Omega)$ to $H^{\frac{5}{2}}(\Omega)$ for the three-dimensional problem. In [5], the regularity of the exact solution $\boldsymbol{u}$ of Navier–Stokes equations belonging to $H^3(\Omega)$ is used. In fact, we make a more gentle improvement of the regularity.

4.1. Optimal error bound for the velocity

We split the velocity error as

$$\boldsymbol{u} - \boldsymbol{u}_h = (\boldsymbol{u} - \rho_{st}\boldsymbol{u}) - (\boldsymbol{u}_h - \rho_{st}\boldsymbol{u}) = \boldsymbol{\eta} - \boldsymbol{e}_h, \tag{4.9}$$

Now, we firstly make an error analysis for the convection term, which allows for the optimal error estimate for the velocity. The proof of the following Lemma is similar to that of Lemma 5.5 in [14], but there are some important differences.

Lemma 4.3. *In the setting of Lemma 4.2, there holds the following estimation:*

$$\begin{aligned} c_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{e}_h) - c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{e}_h) \le{}& \frac{C_\sigma \nu}{2}|||\boldsymbol{e}_h|||_e^2 + C\|\nabla \boldsymbol{u}\|^*_{L^\infty}\|\boldsymbol{e}_h\|_{L^2}^2 + \|\nabla \boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|_{L^2}^2 \\ &+ \frac{C}{\nu}(\|\boldsymbol{u}\|_{L^\infty} + \|\nabla \boldsymbol{u}\|^*_{L^\infty})^2(\|\boldsymbol{\eta}\|_{L^2}^2 + h^2\|\nabla_h\boldsymbol{\eta}\|_{L^2}^2). \end{aligned}$$

Proof. Firstly, by using (3.5) and $[\![\boldsymbol{u}]\!]_F = 0$ for $\forall F \in \mathcal{F}_h^i$, we have

$$\begin{aligned}
\boldsymbol{I} &= c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{e}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{e}_h) \\
&= \int_\Omega \Big[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h - (\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{u}_h\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\{\boldsymbol{e}_h\}\,\mathrm{d}\boldsymbol{s} \\
&\quad + \sum_{F\in\mathcal{F}_h^i}\int_F \frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{u}-\boldsymbol{u}_h]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s} = \boldsymbol{I}_1+\boldsymbol{I}_2+\boldsymbol{I}_3.
\end{aligned} \tag{4.10}$$

By using $\boldsymbol{u}-\boldsymbol{u}_h = \boldsymbol{\eta}-\boldsymbol{e}_h$, we make error splitting for $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$, respectively, as shown below.

$$\begin{aligned}
\boldsymbol{I}_1 &= \int_\Omega\Big[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h - (\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h + (\boldsymbol{u}_h\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h - (\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{u}_h\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega\Big[((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} + (\boldsymbol{u}_h\cdot\nabla_h)(\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega\Big[((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} + \int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{\eta}\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} \\
&= \boldsymbol{I}_{1,1}+\boldsymbol{I}_{1,2}+\boldsymbol{I}_{1,3},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\boldsymbol{I}_2 &= -\sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{\eta}]\!]\{\boldsymbol{e}_h\}\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{e}_h\}\,\mathrm{d}\boldsymbol{s} \\
&= \boldsymbol{I}_{2,1}+\boldsymbol{I}_{2,2},
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
\boldsymbol{I}_3 &= \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s} - \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{e}_h]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s} \\
&= \boldsymbol{I}_{3,1} - |\boldsymbol{e}_h|^2_{\boldsymbol{u}_h,\mathrm{upw}}.
\end{aligned} \tag{4.13}$$

Notice that

$$\boldsymbol{I}_{1,3}+\boldsymbol{I}_{2,2} = 0, \tag{4.14}$$

and

$$\begin{aligned}
\boldsymbol{I}_{1,2}+\boldsymbol{I}_{2,1} &= \int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{\eta}\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} - \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{\eta}]\!]\{\boldsymbol{e}_h\}\,\mathrm{d}\boldsymbol{s} \\
&= -\int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{x} + \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{\eta}\}\,\mathrm{d}\boldsymbol{s}.
\end{aligned} \tag{4.15}$$

Now using the above identities (4.10)–(4.15), we can get

$$\begin{aligned}
\boldsymbol{I} &= \int_\Omega\Big[((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{x} + \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{\eta}\}\,\mathrm{d}\boldsymbol{s} \\
&\quad + \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s} - \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{e}_h]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s} \\
&= \boldsymbol{I}_{vol}+\boldsymbol{I}_{fac}.
\end{aligned} \tag{4.16}$$

For the volume term $\boldsymbol{I}_{vol}$, applying $\boldsymbol{u}_h = \boldsymbol{e}_h + \rho_{st}\boldsymbol{u}$, Hölder's inequality, Young's inequality, (4.3) and inverse inequality (3.10), we can obtain

$$\begin{aligned}
\boldsymbol{I}_{vol} &= \int_\Omega\Big[((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\Big]\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\boldsymbol{u}_h\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{\eta}\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\boldsymbol{e}_h\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{e}_h\,\mathrm{d}\boldsymbol{x} - \int_\Omega\Big[(\boldsymbol{e}_h\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{\eta} + (\rho_{st}\boldsymbol{u}\cdot\nabla_h)\boldsymbol{e}_h\cdot\boldsymbol{\eta}\Big]\,\mathrm{d}\boldsymbol{x} \\
&\leqslant \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2} + 2\|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{e}_h\|^2_{L^2} + C\frac{1}{\nu}\|\rho_{st}\boldsymbol{u}\|^2_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2} + \frac{C_\sigma\nu}{2}\|\nabla_h\boldsymbol{e}_h\|^2_{L^2} \\
&\quad + C\|\nabla\boldsymbol{u}\|^*_{L^\infty}\|\boldsymbol{e}_h\|^2_{L^2} \\
&\leqslant \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2} + C\frac{1}{\nu}\|\rho_{st}\boldsymbol{u}\|^2_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2} + \frac{C_\sigma\nu}{2}\|\nabla_h\boldsymbol{e}_h\|^2_{L^2} + C\|\nabla\boldsymbol{u}\|^*_{L^\infty}\|\boldsymbol{e}_h\|^2_{L^2}.
\end{aligned} \tag{4.17}$$

By using the discrete trace inequality and quasi-uniformity of the mesh, we observe that

$$\begin{aligned}\sum_{F\in\mathcal{F}_h^i}\int_F |\{\boldsymbol{e}_h\}|^2\,\mathrm{d}\boldsymbol{s} &\leqslant 2\sum_{F\in\mathcal{F}_h^i}\left[\|\boldsymbol{e}_h{}^+\|_{L^2(F)}^2+\|\boldsymbol{e}_h{}^-\|_{L^2(F)}^2\right]\\ &\leqslant 2\sum_{T\in\mathcal{T}_h}\|\boldsymbol{e}_h\|_{L^2(\partial T)}^2\leqslant Ch^{-1}\|\boldsymbol{e}_h\|_{L^2(\Omega)}^2.\end{aligned}\tag{4.18}$$

In addition, given $\boldsymbol{\eta}\in\boldsymbol{H}^1(\mathcal{T}_h)$ and applying a continuous trace inequality and quasi-uniformity of the mesh, we have

$$\begin{aligned}\sum_{F\in\mathcal{F}_h^i}\int_F |\{\boldsymbol{\eta}\}|^2\,\mathrm{d}\boldsymbol{s} &\leqslant 2\sum_{F\in\mathcal{F}_h^i}\left[\|\boldsymbol{\eta}^+\|_{L^2(F)}^2+\|\boldsymbol{\eta}^-\|_{L^2(F)}^2\right]\\ &\leqslant 2\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\eta}\|_{L^2(\partial T)}^2\leqslant C\sum_{T\in\mathcal{T}_h}(h_T^{-1}\|\boldsymbol{\eta}\|_{L^2(T)}^2+h_T\|\nabla_h\boldsymbol{\eta}\|_{L^2(T)}^2)\\ &\leqslant C(h^{-1}\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2+h\|\nabla_h\boldsymbol{\eta}\|_{L^2(\Omega)}^2).\end{aligned}\tag{4.19}$$

The same estimate can be obtained when the average $\{\}$ is replaced by the jump $[\![]\!]$, then we have

$$\begin{aligned}&\sum_{F\in\mathcal{F}_h^i}\int_F |[\![\boldsymbol{e}_h]\!]|^2\,\mathrm{d}\boldsymbol{s}\leqslant Ch^{-1}\|\boldsymbol{e}_h\|_{L^2(\Omega)}^2,\\ &\sum_{F\in\mathcal{F}_h^i}\int_F |[\![\boldsymbol{\eta}]\!]|^2\,\mathrm{d}\boldsymbol{s}\leqslant C(h^{-1}\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2+h\|\nabla_h\boldsymbol{\eta}\|_{L^2(\Omega)}^2).\end{aligned}\tag{4.20}$$

For the face term $\boldsymbol{I}_{fac}$, applying $\boldsymbol{u}_h=\boldsymbol{e}_h+\rho_{st}\boldsymbol{u}$, Hölder's inequality, Young's inequality, (4.3), (3.9), (4.19) and (4.20), we can obtain

$$\begin{aligned}\boldsymbol{I}_{fac} &= \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{\eta}\}\,\mathrm{d}\boldsymbol{s}+\sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s}-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2\\ &\leq \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{e}_h\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{\eta}\}\,\mathrm{d}\boldsymbol{s}+\sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{e}_h\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s}\\ &\quad+\sum_{F\in\mathcal{F}_h^i}\int_F(\rho_{st}\boldsymbol{u}\cdot\boldsymbol{n}_F)[\![\boldsymbol{e}_h]\!]\{\boldsymbol{\eta}\}\,\mathrm{d}\boldsymbol{s}+\sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\rho_{st}\boldsymbol{u}\cdot\boldsymbol{n}_F)|[\![\boldsymbol{\eta}]\!][\![\boldsymbol{e}_h]\!]\,\mathrm{d}\boldsymbol{s}-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2\\ &\leqslant C\|\eta\|_{L^\infty}\Big(\sum_{F\in\mathcal{F}_h^i}\int_F|[\![\boldsymbol{e}_h]\!]|^2\,\mathrm{d}\boldsymbol{s}\Big)^{\frac{1}{2}}\Big(\sum_{F\in\mathcal{F}_h^i}\int_F|\boldsymbol{e}_h\cdot\boldsymbol{n}_F|^2\,\mathrm{d}\boldsymbol{s}\Big)^{\frac{1}{2}}+\frac{1}{2}\sum_{F\in\mathcal{F}_h^i}\frac{C_\sigma\nu\sigma}{h_F}\int_F|[\![\boldsymbol{e}_h]\!]|^2\,\mathrm{d}\boldsymbol{s}\\ &\quad+C\sum_{F\in\mathcal{F}_h^i}\frac{h_F}{\nu}\int_F|(\rho_{st}\boldsymbol{u}\cdot\boldsymbol{n}_F)|^2\,|[\![\boldsymbol{\eta}]\!]|^2\,\mathrm{d}\boldsymbol{s}+C\sum_{F\in\mathcal{F}_h^i}\frac{h_F}{\nu}\int_F|(\rho_{st}\boldsymbol{u}\cdot\boldsymbol{n}_F)|^2\,|\{\boldsymbol{\eta}\}|^2\,\mathrm{d}\boldsymbol{s}\\ &\quad-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2\\ &\leq C\|\nabla\boldsymbol{u}\|_{L^\infty}^*\|\boldsymbol{e}_h\|_{L^2}^2+\frac{1}{2}\sum_{F\in\mathcal{F}_h}\frac{C_\sigma\nu\sigma}{h_F}\|[\![\boldsymbol{e}_h]\!]\|_{L^2(F)}^2+C\frac{1}{\nu}\|\rho_{st}\boldsymbol{u}\|_{L^\infty}^2(\|\boldsymbol{\eta}\|_{L^2}^2+h^2\|\nabla_h\boldsymbol{\eta}\|_{L^2}^2)\\ &\quad-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2.\end{aligned}\tag{4.21}$$

By using (4.17), (4.21), (3.6) and $\|\rho_{st}\boldsymbol{u}\|_{L^\infty}\leq\|\boldsymbol{u}\|_{L^\infty}+\|\nabla\boldsymbol{u}\|_{L^\infty}^*$ from (4.3), we can get

$$\begin{aligned}\boldsymbol{I} &\leq \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|_{L^2}^2+\frac{\nu C_\sigma}{2}\|\nabla_h\boldsymbol{e}_h\|_{L^2}^2+\frac{1}{2}\sum_{F\in\mathcal{F}_h}\frac{C_\sigma\nu\sigma}{h_F}\|[\![\boldsymbol{e}_h]\!]\|_{L^2(F)}^2+C\|\nabla\boldsymbol{u}\|_{L^\infty}^*\|\boldsymbol{e}_h\|_{L^2}^2\\ &\quad+C\frac{1}{\nu}\|\rho_{st}\boldsymbol{u}\|_{L^\infty}^2(\|\boldsymbol{\eta}\|_{L^2}^2+h^2\|\nabla_h\boldsymbol{\eta}\|_{L^2}^2)-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2\\ &\leq \frac{C_\sigma\nu}{2}|||\boldsymbol{e}_h|||_e^2+C\|\nabla\boldsymbol{u}\|_{L^\infty}^*\|\boldsymbol{e}_h\|_{L^2}^2+C\frac{1}{\nu}\|\rho_{st}\boldsymbol{u}\|_{L^\infty}^2(\|\boldsymbol{\eta}\|_{L^2}^2+h^2\|\nabla_h\boldsymbol{\eta}\|_{L^2}^2)+\|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|_{L^2}^2\\ &\quad-|\boldsymbol{e}_h|_{\boldsymbol{u}_h,\mathrm{upw}}^2\\ &\leq \frac{C_\sigma\nu}{2}|||\boldsymbol{e}_h|||_e^2+C\|\nabla\boldsymbol{u}\|_{L^\infty}^*\|\boldsymbol{e}_h\|_{L^2}^2+C\frac{1}{\nu}(\|\boldsymbol{u}\|_{L^\infty}+\|\nabla\boldsymbol{u}\|_{L^\infty}^*)^2(\|\boldsymbol{\eta}\|_{L^2}^2+h^2\|\nabla_h\boldsymbol{\eta}\|_{L^2}^2)\\ &\quad+\|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|_{L^2}^2.\quad\square\end{aligned}$$

Next, by combining with Lemma 4.3, we give the velocity discretization error estimate in the following theorem.

Theorem 4.4. *If* $\boldsymbol{u} \in L^2(0,T;\boldsymbol{H}^{d/2+1}(\Omega)) \cap L^2\big(0,T;\boldsymbol{W}^{1,\infty}(\Omega)\big) \cap L^\infty\big(0,T;\boldsymbol{H}^r(\Omega)\big)$, $\partial_t\boldsymbol{u} \in L^2\big(0,T;\boldsymbol{H}^r(\Omega)\big)\,(r \geq 2)$ *and* $\boldsymbol{u}_h(0) = \rho_{st}\boldsymbol{u}_0$, *we have the following error estimate:*

$$\|\boldsymbol{e}_h\|^2_{L^\infty\left(\boldsymbol{L}^2\right)} + \int_0^T \nu |||\boldsymbol{e}_h(\tau)|||_e^2 \,\mathrm{d}\tau$$
$$\leqslant Ce^{A(\boldsymbol{u})}\int_0^T \|\partial_t\boldsymbol{\eta}\|^2_{L^2} + \frac{1}{\nu}(\|\boldsymbol{u}\|_{L^\infty} + \|\nabla\boldsymbol{u}\|^*_{L^\infty})^2(\|\boldsymbol{\eta}\|^2_{L^2} + h^2\|\nabla_h\boldsymbol{\eta}\|^2_{L^2}) + \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2}\,\mathrm{d}\tau,$$

with $A(\boldsymbol{u}) = \int_0^T C(1 + \|\nabla\boldsymbol{u}\|^*_{L^\infty})\,\mathrm{d}\tau$.

Proof. Firstly, we present the Galerkin orthogonality for $\forall \boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$(\partial_t(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{v}_h) + \nu a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) = 0. \tag{4.22}$$

Then we take $\boldsymbol{v}_h = \boldsymbol{e}_h \in \boldsymbol{V}_h$ in (4.22) and use the error splitting (4.9) to obtain

$$(\partial_t\boldsymbol{e}_h,\boldsymbol{e}_h) + \nu a_h(\boldsymbol{e}_h,\boldsymbol{e}_h) = (\partial_t\boldsymbol{\eta},\boldsymbol{e}_h) + \nu a_h(\boldsymbol{\eta},\boldsymbol{e}_h) + c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{e}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{e}_h).$$

By using $(\partial_t\boldsymbol{e}_h,\boldsymbol{e}_h) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_h\|^2_{L^2}$ and $a_h(\boldsymbol{\eta},\boldsymbol{e}_h) = 0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_h\|^2_{L^2} + \nu a_h(\boldsymbol{e}_h,\boldsymbol{e}_h) = (\partial_t\boldsymbol{\eta},\boldsymbol{e}_h) + c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{e}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{e}_h). \tag{4.23}$$

We apply discrete coercivity of a_h on the left-hand side of (4.23). On the right-hand side, applying Cauchy–Schwarz inequality, and Lemma 4.3, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_h\|^2_{L^2} + \frac{1}{2}\nu C_\sigma |||\boldsymbol{e}_h|||_e^2 \\
&\leq \frac{1}{2}\|\partial_t\boldsymbol{\eta}\|^2_{L^2} + \frac{1}{2}\|\boldsymbol{e}_h\|^2_{L^2} + C\|\nabla\boldsymbol{u}\|^*_{L^\infty}\|\boldsymbol{e}_h\|^2_{L^2} + C\frac{1}{\nu}(\|\boldsymbol{u}\|_{L^\infty} + \|\nabla\boldsymbol{u}\|^*_{L^\infty})^2(\|\boldsymbol{\eta}\|^2_{L^2} + h^2\|\nabla_h\boldsymbol{\eta}\|^2_{L^2}) \\
&\quad + \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2}.
\end{aligned} \tag{4.24}$$

After rearranging (4.24), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}_h\|^2_{L^2} + \nu|||\boldsymbol{e}_h|||_e^2 \\
&\leq C\left\{(1 + \|\nabla\boldsymbol{u}\|^*_{L^\infty})\|\boldsymbol{e}_h\|^2_{L^2} + \|\partial_t\boldsymbol{\eta}\|^2_{L^2} + \frac{1}{\nu}(\|\boldsymbol{u}\|_{L^\infty} + \|\nabla\boldsymbol{u}\|^*_{L^\infty})^2(\|\boldsymbol{\eta}\|^2_{L^2} + h^2\|\nabla_h\boldsymbol{\eta}\|^2_{L^2})\right. \\
&\quad \left. + \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2}\right\}.
\end{aligned}$$

Applying Gronwall's Lemma, we have

$$\|\boldsymbol{e}_h\|^2_{L^\infty\left(\boldsymbol{L}^2\right)} + \int_0^T \nu |||\boldsymbol{e}_h(\tau)|||_e^2 \,\mathrm{d}\tau$$
$$\leqslant Ce^{A(\boldsymbol{u})}\int_0^T \|\partial_t\boldsymbol{\eta}\|^2_{L^2} + \frac{1}{\nu}(\|\boldsymbol{u}\|_{L^\infty} + \|\nabla\boldsymbol{u}\|^*_{L^\infty})^2(\|\boldsymbol{\eta}\|^2_{L^2} + h^2\|\nabla_h\boldsymbol{\eta}\|^2_{L^2}) + \|\nabla\boldsymbol{u}\|_{L^\infty}\|\boldsymbol{\eta}\|^2_{L^2}\,\mathrm{d}\tau,$$

with $A(\boldsymbol{u}) = \int_0^T C(1 + \|\nabla\boldsymbol{u}\|^*_{L^\infty})\,\mathrm{d}\tau$. □

The following corollary states that the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity is optimal and pressure-robust, but the constants in the velocity error bound are dependent on the inverse power of the viscosity.

Corollary 4.5. *Under the assumptions of the previous theorem, with* $r_{\boldsymbol{u}} = \min\{r, k+1\}$ *and a constant* C *independent of* h *and* ν^{-1}, *we obtain the following estimate:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^\infty\left(\boldsymbol{L}^2\right)} \leqslant h^{2r_{\boldsymbol{u}}}\Big(B(\boldsymbol{u})C(\boldsymbol{u}) + D_1(\boldsymbol{u})\Big),$$

and

$$\int_0^T \nu|||(\boldsymbol{u}-\boldsymbol{u}_h)(\tau)|||_e^2\,\mathrm{d}\tau \leqslant h^{2r_{\boldsymbol{u}}-2}\Big(h^2B(\boldsymbol{u})C(\boldsymbol{u}) + D_2(\boldsymbol{u})\Big),$$

where

$$B(\boldsymbol{u}) = Ce^{T+\|\nabla \boldsymbol{u}\|^*_{L^1(\boldsymbol{L}^\infty)}},$$
$$C(\boldsymbol{u}) = \|\partial_t \boldsymbol{u}\|^2_{L^2(\boldsymbol{H}^{r_{\boldsymbol{u}}})} + \left(\frac{1}{\nu}(\|\boldsymbol{u}\|^2_{L^2(\boldsymbol{L}^\infty)} + \|\nabla \boldsymbol{u}\|^{*2}_{L^2(\boldsymbol{L}^\infty)}) + \|\nabla \boldsymbol{u}\|_{L^1(\boldsymbol{L}^\infty)}\right) \|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})},$$
$$D_1(\boldsymbol{u}) = C\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})},$$
$$D_2(\boldsymbol{u}) = C\nu T\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}.$$

Proof. By means of triangular inequality and Lemma 4.1, we can conclude the proof. □

5. Pressure-robust and semi-robust analysis for the velocity

In this section, we consider pressure-robust and semi-robust analysis for the velocity error at high Reynolds numbers. To the end, we introduce the Raviart–Thomas interpolation operator [17, Example 2.5.3]

$$\begin{cases} \int_{\partial T} (\boldsymbol{w} - \rho_{rt}\boldsymbol{w}) \cdot \boldsymbol{n} p_k ds = 0, \forall p_k \in P_k(\partial T), \\ \int_T (\boldsymbol{w} - \rho_{rt}\boldsymbol{w}) \cdot \boldsymbol{p}_{k-1} d\boldsymbol{x} = 0, \forall \boldsymbol{p}_{k-1} \in (P_{k-1}(T))^d. \end{cases} \tag{5.1}$$

Lemma 5.1 (*[17, Proposition 2.5.2]*). *Let ρ_{rt} be the interpolation operator: $\boldsymbol{H}^1(\Omega) \to RT_k$, and π_k be the L^2-projection on $\nabla \cdot RT_k$. Then we have, for all $\boldsymbol{q} \in \boldsymbol{H}^1(\Omega)$,*

$$\nabla \cdot (\rho_{rt}\boldsymbol{q}) = \pi_k \nabla \cdot \boldsymbol{q}.$$

The Raviart–Thomas interpolation operator satisfies the following approximation properties [13, p. 1737] that for $\forall T \in \mathcal{T}_h, \forall \boldsymbol{w} \in \boldsymbol{H}^m(T)$ $(m \geq 1)$,

$$\|\boldsymbol{w} - \rho_{rt}\boldsymbol{w}\|_{L^2(T)} + h_T\|\nabla \boldsymbol{w} - \nabla_h \rho_{rt}\boldsymbol{w}\|_{L^2(T)} \leqslant Ch_T^m |\boldsymbol{w}|_{H^m(T)}, \tag{5.2}$$

and for $\forall T \in \mathcal{T}_h, \forall \boldsymbol{w} \in \boldsymbol{W}^{1,\infty}(T)$,

$$\|\boldsymbol{w} - \rho_{rt}\boldsymbol{w}\|_{L^\infty(T)} + h_T\|\nabla \boldsymbol{w} - \nabla_h \rho_{rt}\boldsymbol{w}\|_{L^\infty(T)} \leqslant Ch_T\|\nabla \boldsymbol{w}\|_{L^\infty(T)}. \tag{5.3}$$

Remark 2. Thanks to (5.3), we do not need to make the additional regularity assumption of the solution $\boldsymbol{u}$ of the Navier–Stokes equation, except $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$. However, in the previous section, due to Lemma 4.2, the regularity assumption $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega) \cap \boldsymbol{H}^{d/2+1}(\Omega)$ in space is made, instead of only $\boldsymbol{W}^{1,\infty}(\Omega)$.

Lemma 5.2 (*[17, Corollary 2.3.1]*). *Let T be an simplicial (triangular or tetrahedral) element. Then we have*

$$BDM_k^0(T) = RT_k^0(T) \subset (P_k(T))^d,$$

where

$$RT_k^0(T) = \{\boldsymbol{q} \in RT_k(T) | \nabla \cdot \boldsymbol{q} = 0\},$$
$$BDM_k^0(T) = \{\boldsymbol{q} \in BDM_k(T) | \nabla \cdot \boldsymbol{q} = 0\}.$$

5.1. Quasi-optimal error bound for the velocity

We use the above Raviart–Thomas interpolation operator to make the error splitting

$$\boldsymbol{u} - \boldsymbol{u}_h = (\boldsymbol{u} - \rho_{rt}\boldsymbol{u}) - (\boldsymbol{u}_h - \rho_{rt}\boldsymbol{u}) = \bar{\boldsymbol{\eta}} - \bar{\boldsymbol{e}}_h. \tag{5.4}$$

Whether we choose BDM_k or RT_k $(k \geq 1)$ elements to be the velocity space, we introduce the Raviart–Thomas interpolation operator, respectively.

In order to make a sharp analysis for the convective term, we give an important lemma.

Lemma 5.3. *Assume $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$. There exists a $C > 0$, independent of h, such that*

$$\int_\Omega \left[(\boldsymbol{u}\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\right] d\boldsymbol{x} \leq C\|\nabla \boldsymbol{u}\|_{L^\infty}(\|\bar{\boldsymbol{\eta}}\|^2_{L^2} + \|\bar{\boldsymbol{e}}_h\|^2_{L^2}).$$

Proof. Let $\langle \boldsymbol{u}\rangle_T$ denotes the mean value of $\boldsymbol{u}$ on each cell $T \in \mathcal{T}_h$

$$\langle \boldsymbol{u}\rangle_T = \frac{\int_T \boldsymbol{u}\, \mathrm{d}\boldsymbol{x}}{|T|}.$$

On the one hand,

$$\|\boldsymbol{u} - \langle \boldsymbol{u}\rangle_T\|_{L^\infty(T)} \leqslant Ch_T\|\nabla \boldsymbol{u}\|_{L^\infty(T)}, \tag{5.5}$$

since $\boldsymbol{u}$ is Lipschitz continuous [15,18].

On the other hand, $\bar{\boldsymbol{e}}_h = (\boldsymbol{u}_h - \rho_{rt}\boldsymbol{u})$, where $\rho_{rt}\boldsymbol{u} \in RT_k^0(T)$ from Lemma 5.1 and $\boldsymbol{u}_h \in RT_k^0(T)$ or $\boldsymbol{u}_h \in BDM_k^0(T)$. Due to Lemma 5.2, we have $\bar{\boldsymbol{e}}_h \in (P_k(T))^d$, so $(\langle \boldsymbol{u}\rangle_T\cdot\nabla_h)\bar{\boldsymbol{e}}_h \in (P_{k-1}(T))^d$. Using (5.1), we have

$$\int_T (\langle \boldsymbol{u}\rangle_T\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x} = 0, \quad \forall T \in \mathcal{T}_h. \tag{5.6}$$

Using Eqs. (5.5) and (5.6), Holder's inequality, inverse inequality and Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_\Omega (\boldsymbol{u}\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x} &= \sum_{T\in\mathcal{T}_h}\int_T ((\boldsymbol{u} - \langle \boldsymbol{u}\rangle_T)\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x}\\ &\le C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2\\ &\le C\|\nabla \boldsymbol{u}\|_{L^\infty}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + \|\bar{\boldsymbol{e}}_h\|_{L^2}^2). \quad \square\end{aligned}$$

Next, we give an error estimate for the convection terms, which allows for the semi-robust estimate for the velocity.

Lemma 5.4. *Let $\boldsymbol{u}$ be the solution of (2.2), $\boldsymbol{u}_h$ be the solution of (3.3), and assume that $\boldsymbol{u} \in \boldsymbol{W}^{1,\infty}(\Omega)$. Then, we obtain*

$$c_h(\boldsymbol{u}, \boldsymbol{u}, \bar{\boldsymbol{e}}_h) - c_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \bar{\boldsymbol{e}}_h) \le C(1 + h^{-1})\|\boldsymbol{u}\|_{W^{1,\infty}}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h^2\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) + C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2.$$

Proof. Now, similar to (4.10)–(4.15) in the proof of Lemma 4.3, we have

$$\begin{aligned}\boldsymbol{I} = &\int_\Omega \Big[((\boldsymbol{u} - \boldsymbol{u}_h)\cdot\nabla)\boldsymbol{u}\cdot\bar{\boldsymbol{e}}_h\Big]\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{u}_h\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x} + \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\bar{\boldsymbol{e}}_h]\!]\{\bar{\boldsymbol{\eta}}\}\, \mathrm{d}\boldsymbol{s}\\ &+ \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)|[\![\bar{\boldsymbol{\eta}}]\!][\![\bar{\boldsymbol{e}}_h]\!]\, \mathrm{d}\boldsymbol{s} - \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)||[\![\bar{\boldsymbol{e}}_h]\!]|^2\, \mathrm{d}\boldsymbol{s}\\ =&\boldsymbol{I}_{vol} + \boldsymbol{I}_{fac}.\end{aligned}$$

For the volume term $\boldsymbol{I}_{vol}$, inserting $\boldsymbol{u}_h = \bar{\boldsymbol{e}}_h + \boldsymbol{u} - \bar{\boldsymbol{\eta}}$, and applying Hölder's inequality, inverse inequality and Lemma 5.3, we have

$$\begin{aligned}\boldsymbol{I}_{vol} &= \int_\Omega\Big[(\bar{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{u}\cdot\bar{\boldsymbol{e}}_h - (\bar{\boldsymbol{e}}_h\cdot\nabla)\boldsymbol{u}\cdot\bar{\boldsymbol{e}}_h\Big]\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{u}_h\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x}\\ &= \int_\Omega\Big[(\bar{\boldsymbol{\eta}}\cdot\nabla)\boldsymbol{u}\cdot\bar{\boldsymbol{e}}_h - (\bar{\boldsymbol{e}}_h\cdot\nabla)\boldsymbol{u}\cdot\bar{\boldsymbol{e}}_h\Big]\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\bar{\boldsymbol{e}}_h\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{u}\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega (\bar{\boldsymbol{\eta}}\cdot\nabla_h)\bar{\boldsymbol{e}}_h\cdot\bar{\boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{x}\\ &\leqslant C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + C\|\nabla \boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2.\end{aligned} \tag{5.7}$$

For the faces terms,

$$\begin{aligned}\boldsymbol{I}_{fac} &= \sum_{F\in\mathcal{F}_h^i}\int_F(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)[\![\bar{\boldsymbol{e}}_h]\!]\{\bar{\boldsymbol{\eta}}\}\, \mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|\boldsymbol{u}_h\cdot\boldsymbol{n}_F|[\![\bar{\boldsymbol{\eta}}]\!][\![\bar{\boldsymbol{e}}_h]\!]\, \mathrm{d}\boldsymbol{s} - \sum_{F\in\mathcal{F}_h^i}\int_F\frac{1}{2}|\boldsymbol{u}_h\cdot\boldsymbol{n}_F||[\![\bar{\boldsymbol{e}}_h]\!]|^2\, \mathrm{d}\boldsymbol{s}\\ &\leqslant \sum_{F\in\mathcal{F}_h^i}\int_F|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)||\{\bar{\boldsymbol{\eta}}\}|^2\, \mathrm{d}\boldsymbol{s} + \frac{1}{4}\sum_{F\in\mathcal{F}_h^i}\int_F|(\boldsymbol{u}_h\cdot\boldsymbol{n}_F)||[\![\bar{\boldsymbol{\eta}}]\!]|^2\, \mathrm{d}\boldsymbol{s}\\ &= \boldsymbol{I}_{fac1} + \boldsymbol{I}_{fac2},\end{aligned}$$

where we apply Young's inequality to the first two terms on the right-hand side of the equal sign to cancel out the third term.

For the facet term $\boldsymbol{I}_{fac1}$, applying $\boldsymbol{u}_h = \bar{\boldsymbol{e}}_h + \rho_{rt}\boldsymbol{u}$, Hölder's inequality, Young's inequality, $\|\boldsymbol{\rho}_{rt}\boldsymbol{u}\|_{L^\infty} \le C\|\boldsymbol{u}\|_{W^{1,\infty}}$ from (5.3) and trace inequality, we have

$$\begin{aligned}
|\boldsymbol{I}_{fac1}| &= \sum_{F\in\mathcal{F}_h^i}\int_F |(\bar{\boldsymbol{e}}_h\cdot\boldsymbol{n}_F)||\{\bar{\boldsymbol{\eta}}\}|^2\,\mathrm{d}\boldsymbol{s} + \sum_{F\in\mathcal{F}_h^i}\int_F |(\rho_{rt}\boldsymbol{u}\cdot\boldsymbol{n}_F)||\{\bar{\boldsymbol{\eta}}\}|^2\,\mathrm{d}\boldsymbol{s} \\
&\le \|\bar{\boldsymbol{\eta}}\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\bar{\boldsymbol{e}}_h\|_{L^2(F)}^2 + \|\bar{\boldsymbol{\eta}}\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\{\bar{\boldsymbol{\eta}}\}\|_{L^2(F)}^2 + \|\boldsymbol{\rho}_{rt}\boldsymbol{u}\|_{L^\infty}\sum_{F\in\mathcal{F}_h^i}\|\{\bar{\boldsymbol{\eta}}\}\|_{L^2(F)}^2 \\
&\le C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + Ch\|\nabla\boldsymbol{u}\|_{L^\infty}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\quad + C\|\boldsymbol{u}\|_{W^{1,\infty}}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2).
\end{aligned} \tag{5.8}$$

Similarly, for the facet term $\boldsymbol{I}_{fac2}$, it can be inferred that

$$\begin{aligned}
|\boldsymbol{I}_{fac2}| &\le C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + Ch\|\nabla\boldsymbol{u}\|_{L^\infty}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\quad + C\|\boldsymbol{u}\|_{W^{1,\infty}}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2).
\end{aligned} \tag{5.9}$$

Therefore, by (5.8) and (5.9), we have

$$\begin{aligned}
|\boldsymbol{I}_{fac}| &\le C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + Ch\|\nabla\boldsymbol{u}\|_{L^\infty}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\quad + C\|\boldsymbol{u}\|_{W^{1,\infty}}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2).
\end{aligned} \tag{5.10}$$

From (5.7) and (5.10), we get

$$\begin{aligned}
\boldsymbol{I} &\le C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + Ch\|\nabla\boldsymbol{u}\|_{L^\infty}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\quad + C\|\boldsymbol{u}\|_{W^{1,\infty}}(h^{-1}\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\le C(1+h^{-1})\|\boldsymbol{u}\|_{W^{1,\infty}}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h^2\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) + C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2. \quad \square
\end{aligned}$$

Theorem 5.5. *Assuming $\boldsymbol{u} \in L^1\big(0,T;\boldsymbol{W}^{1,\infty}(\Omega)\big) \cap L^\infty\big(0,T;\boldsymbol{H}^r(\Omega)\big)$, $\partial_t\boldsymbol{u} \in L^2\big(0,T;\boldsymbol{H}^r(\Omega)\big)\,(r \ge 2)$ and $\boldsymbol{u}_h(0) = \rho_{rt}\boldsymbol{u}_0$, we have the following error estimate:*

$$\begin{aligned}
&\|\bar{\boldsymbol{e}}_h\|_{L^\infty(\boldsymbol{L}^2)}^2 + \int_0^T \nu|||\bar{\boldsymbol{e}}_h(\tau)|||_e^2\,\mathrm{d}\tau \\
&\leqslant Ce^{H(\boldsymbol{u})}\int_0^T \|\partial_t\bar{\boldsymbol{\eta}}\|_{L^2}^2\,\mathrm{d}\tau + \nu|||\bar{\boldsymbol{\eta}}(\tau)|||_{e,\sharp}^2 + (1+h^{-1})\|\boldsymbol{u}\|_{W^{1,\infty}}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h^2\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2)\,\mathrm{d}\tau,
\end{aligned}$$

with $H(\boldsymbol{u}) = \int_0^T C(1+\|\nabla\boldsymbol{u}\|_{L^\infty})\,\mathrm{d}\tau$.

Proof. Firstly, we present the Galerkin orthogonality for $\forall\boldsymbol{v}_h \in \boldsymbol{V}_h$

$$(\partial_t(\boldsymbol{u}-\boldsymbol{u}_h),\boldsymbol{v}_h) + \nu a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + c_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h) = 0. \tag{5.11}$$

Taking $\boldsymbol{v}_h = \bar{\boldsymbol{e}}_h \in \boldsymbol{V}_h$ in (5.11) and using (5.4), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + \nu a_h(\bar{\boldsymbol{e}}_h,\bar{\boldsymbol{e}}_h) = (\partial_t\bar{\boldsymbol{\eta}},\bar{\boldsymbol{e}}_h) + \nu a_h(\bar{\boldsymbol{\eta}},\bar{\boldsymbol{e}}_h) + c_h(\boldsymbol{u},\boldsymbol{u},\bar{\boldsymbol{e}}_h) - c_h(\boldsymbol{u}_h,\boldsymbol{u}_h,\bar{\boldsymbol{e}}_h). \tag{5.12}$$

We apply discrete coercivity of a_h on the left-hand side of (5.12). On the right-hand side of (5.12), applying Cauchy–Schwarz inequality, boundedness of a_h, Young's inequality and Lemma 5.4, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + \frac{1}{2}\nu C_\sigma|||\bar{\boldsymbol{e}}_h|||_e^2 \\
&\le \frac{1}{2}\|\partial_t\bar{\boldsymbol{\eta}}\|_{L^2}^2 + \frac{1}{2}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2 + \nu C|||\bar{\boldsymbol{\eta}}|||_{e,\sharp}^2 + C(1+h^{-1})\|\boldsymbol{u}\|_{W^{1,\infty}}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h^2\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2) \\
&\quad + C\|\nabla\boldsymbol{u}\|_{L^\infty}\|\bar{\boldsymbol{e}}_h\|_{L^2}^2.
\end{aligned} \tag{5.13}$$

After rearranging (5.13) and applying Gronwall's Lemma, we can obtain

$$\begin{aligned}
&\|\bar{\boldsymbol{e}}_h\|_{L^\infty(\boldsymbol{L}^2)}^2 + \int_0^T \nu|||\bar{\boldsymbol{e}}_h(\tau)|||_e^2\,\mathrm{d}\tau \\
&\leqslant Ce^{H(\boldsymbol{u})}\int_0^T \|\partial_t\bar{\boldsymbol{\eta}}\|_{L^2}^2 + \nu|||\bar{\boldsymbol{\eta}}(\tau)|||_{e,\sharp}^2 + (1+h^{-1})\|\boldsymbol{u}\|_{W^{1,\infty}}(\|\bar{\boldsymbol{\eta}}\|_{L^2}^2 + h^2\|\nabla_h\bar{\boldsymbol{\eta}}\|_{L^2}^2)\,\mathrm{d}\tau,
\end{aligned}$$

with $H(\boldsymbol{u}) = \int_0^T C(1+\|\nabla\boldsymbol{u}\|_{L^\infty})\,\mathrm{d}\tau$. $\square$

The following corollaries state that the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity is pressure-robust and semi-robust. The quasi-optimal error bound for the velocity is presented in Corollary 5.7.

Corollary 5.6. *Under the assumptions of the previous theorem, with $r_{\boldsymbol{u}} = \min\{r, k+1\}$ and a constant C independent of h and ν, we have the following estimate:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^\infty\left(\boldsymbol{L}^2\right)} + \int_0^T \nu |||(\boldsymbol{u}-\boldsymbol{u}_h)(\tau)|||_e^2 \,\mathrm{d}\tau \leqslant h^{2r_{\boldsymbol{u}}-2}\Big(E(\boldsymbol{u})F_1(\boldsymbol{u}) + G_1(\boldsymbol{u})\Big),$$

where

$$\begin{aligned}
&E(\boldsymbol{u}) = Ce^{T+\|\nabla \boldsymbol{u}\|_{L^1(\boldsymbol{L}^\infty)}},\\
&F_1(\boldsymbol{u}) = h^2\,\|\partial_t \boldsymbol{u}\|^2_{L^2(\boldsymbol{H}^{r_{\boldsymbol{u}}})} + \Big(\nu T + (h^2+h)\|\boldsymbol{u}\|_{L^1(\boldsymbol{W}^{1,\infty})}\Big)\,\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})},\\
&G_1(\boldsymbol{u}) = C(h^2+\nu T)\,\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}.
\end{aligned}$$

Proof. By means of triangular inequality and (5.2), we can conclude the proof. □

Corollary 5.7. *Under the assumptions of the previous corollary, assuming that $\nu < Ch$, we have the following estimate:*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^\infty\left(\boldsymbol{L}^2\right)} + \int_0^T \nu |||(\boldsymbol{u}-\boldsymbol{u}_h)(\tau)|||_e^2 \,\mathrm{d}\tau \leqslant h^{2r_{\boldsymbol{u}}-1}\Big(E(\boldsymbol{u})F_2(\boldsymbol{u}) + G_2(\boldsymbol{u})\Big),$$

where

$$\begin{aligned}
&E(\boldsymbol{u}) = Ce^{T+\|\nabla \boldsymbol{u}\|_{L^1(\boldsymbol{L}^\infty)}},\\
&F_2(\boldsymbol{u}) = h\,\|\partial_t \boldsymbol{u}\|^2_{L^2(\boldsymbol{H}^{r_{\boldsymbol{u}}})} + \Big(T + (h+1)\|\boldsymbol{u}\|_{L^1(\boldsymbol{W}^{1,\infty})}\Big)\,\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})},\\
&G_2(\boldsymbol{u}) = C(h+T)\,\|\boldsymbol{u}\|^2_{L^\infty(\boldsymbol{H}^{r_{\boldsymbol{u}}})}.
\end{aligned}$$

Finally, we notice that Corollary 5.7 yields convergence with order $2r_{\boldsymbol{u}}-1$ in the case of $\nu < Ch$, which is one order better than $2r_{\boldsymbol{u}}-2$ in [14, Corollary 5.9].

6. Numerical experiments

In this section, we carry out numerical experiments to verify our analytical results. Simulations were performed at a problem defined in the domain $\Omega = (0,1)^2$ with the exact solution [19]

$$\begin{aligned}
\boldsymbol{u}(x,y,t) &= \frac{6+4\cos(4t)}{10}\begin{bmatrix} 8\sin^2(\pi x)(2y(1-y)(1-2y)) \\ -8\pi\sin(2\pi x)(y(1-y))^2 \end{bmatrix},\\
p(x,y,t) &= \frac{6+4\cos(4t)}{10}\sin(\pi x)\cos(\pi y).
\end{aligned} \tag{6.1}$$

Both the Dirichlet boundary condition and the initial condition are derived from the exact solution.

In our implementation, we use BDM_k/P_{k-1} pair for the velocity and pressure spaces. For the symmetric internal penalty term a_h, the penalty parameter is equal to $10k^2$. We choose $k = 1$, and the quasi-uniform unstructured triangular meshes are used. As for temporal discretization, an implicit/explicit (IMEX) BDF2 scheme is applied, in which $c_h\left(2\boldsymbol{u}_h^{n-1} - \boldsymbol{u}_h^{n-2}, \boldsymbol{u}_h^n, \boldsymbol{v}_h\right)$ is used in the convection term, except in the first time step using $c_h\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n, \boldsymbol{v}_h\right)$. We set the time interval small enough to ensure that the spatial error dominates over the temporal error. We choose the small time step $\triangle t = 1E-3$, and the final time $T = 2$. All numerical experiments are implemented in this NGSolve software [20].

From Table 6.1, we fix the mesh size to observe the variation of the velocity error with respect to ν. We can observe that as the viscosity decreases, the velocity error becomes larger and larger. For large values of ν, the velocity L^2 errors have an explicit dependence on $\nu^{-\frac{1}{2}}$. When the viscosity is small enough, the velocity errors hold unchanged, that is to say, they are independent of the viscosity. This is consistent with our theoretical results, see Corollaries 4.5 and 5.7. It can be also seen that the velocity error has the optimal convergence rate for large values of ν, as we predicted in Corollary 4.5. For small values of ν, the velocity error tends to the optimal convergence rate with mesh refinements, which is better than the quasi-optimal convergence rate as we predicted in Corollary 5.7.

7. Conclusions and outlook

Detailed error analysis of the semi-discrete time-dependent Navier–Stokes equations is presented. Firstly, we prove the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity is optimal and pressure-robust, but not semi-robust. Secondly, we focus on pressure-robust and semi-robust velocity error analysis at high Reynolds number. We prove that the $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ error of the velocity is quasi-optimal ($\nu < Ch$), pressure-robust and semi-robust.

Table 6.1
BDM_1 $/P_0$ pair of finite element spaces, $T = 2$, velocity errors in the L^2-norm.

h	$\nu = 1E-0$		$\nu = 1E-2$		$\nu = 1E-4$	
	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate
1/6	4.99E−02		7.77E−02		1.91E−01	
1/12	1.16E−02	2.10	1.59E−02	2.29	5.98E−02	1.69
1/24	2.86E−03	2.02	3.69E−03	2.11	1.63E−02	1.87
1/48	7.10E−04	2.01	8.99E−04	2.03	4.08E−03	2.00
h	$\nu = 1E-6$		$\nu = 1E-8$		$\nu = 1E-10$	
	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2}$	Rate
1/6	1.96E−01		1.96E−01		1.96E−01	
1/12	6.29E−02	1.63	6.29E−02	1.63	6.29E−02	1.63
1/24	1.84E−02	1.77	1.84E−02	1.77	1.84E−02	1.77
1/48	4.95E−03	1.89	4.95E−03	1.89	4.95E−03	1.89

H(div)-conforming DG method has many advantages in solving the Navier–Stokes equations, such as exactly divergence-free, local momentum conservation, pressure-robust and semi-robust (quasi-optimal). Compared with other finite element methods, the obvious disadvantage of the H(div)-conforming DG method is that it needs to solve more degrees of freedom. However, it can be hybridized, and solving efficiency can be greatly improved by using static condensation method [21]. So, it is a very promising approach for high Reynolds number flows.

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