第二章 流体力学的基本方程

2.1 连续方程

2.2 动量方程

2.3 能量方程

2.4 牛顿流体的基本方程组

2.5 边界条件

2.1 连续方程

质量守恒 → 系统质量在运动过程中不变

$$\frac{Dm}{Dt}=\frac{D}{Dt}\int_{\tau(t)}\rho d\tau=0$$

应用雷诺输运定理 → 积分形式连续方程

$$\frac{\partial}{\partial t}\int_{\tau}\rho d\tau+\int_{S}\rho\vec{u}\cdot\vec{n}dS=0$$

初始时刻系统与控制体重合

由 $\frac{D}{Dt}\int_{\tau}\rho d\tau=\int_{\tau}\left(\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}\right)d\tau=\int_{\tau}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u})\right]d\tau=0$

连续方程2

微分形式连续方程

假定被积函数连续，且体积是任取的，当积分恒等于零时可知被积函数必须恒等于零

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}=0$$

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_k}{\partial x_k}=0$$

⇐ 相对密度变化率等于负的相对体积变化率

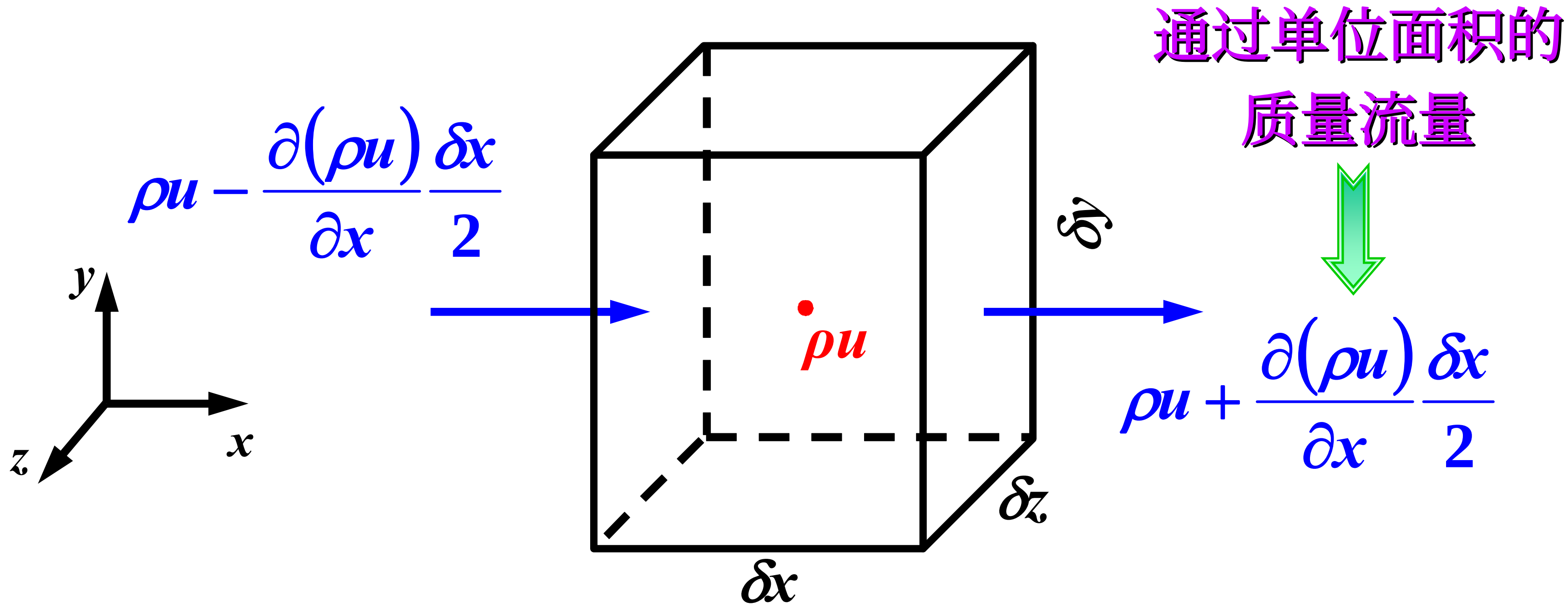
$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{u}\right)=0$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho u_k\right)}{\partial x_k}=0$$

⇐ 单位控制体内流体质量的变化率与净流出控制体的流体质量流量之和为零

连续方程3

对微元控制体应用质量守恒定律



***x* 方向净流出控制体的质量流量** ⇒ $\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$

连续方程4

y、z 方向净流出控制体的质量流量

$$\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z \qquad \frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

控制体内流体质量随时间的变化率

$$\frac{\partial \rho}{\partial t}\delta x\delta y\delta z$$

微分形式连续方程

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0$$

连续方程5

定常流动 → $\partial\rho/\partial t = 0$

→ $\nabla\cdot(\rho\vec{u}) = 0$　　$\dfrac{\partial(\rho u_k)}{\partial x_k} = 0$

定常流动净流出单位控制体的质量流量为零

不可压缩流动 → $D\rho/Dt = 0$

→ $\nabla\cdot\vec{u} = 0$　　$\dfrac{\partial u_k}{\partial x_k} = 0$

不可压缩流动连续方程导出无需定常假设Ma < 0.3可视为不可压缩流动

不可压缩流动净流出单位控制体的体积流量为零

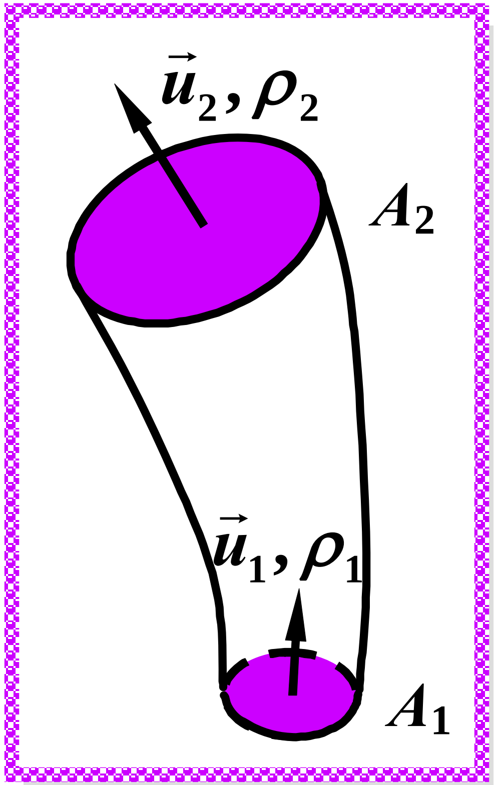
连续方程6

流管内的流动和一维流动

- 引入平均速度和密度，则定常流动

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \implies \rho u A = \text{const}$$

☞ 定常流动时，通过流管任意过流断面的质量流量保持不变



- 不可压缩流动

$$u_1 A_1 = u_2 A_2$$

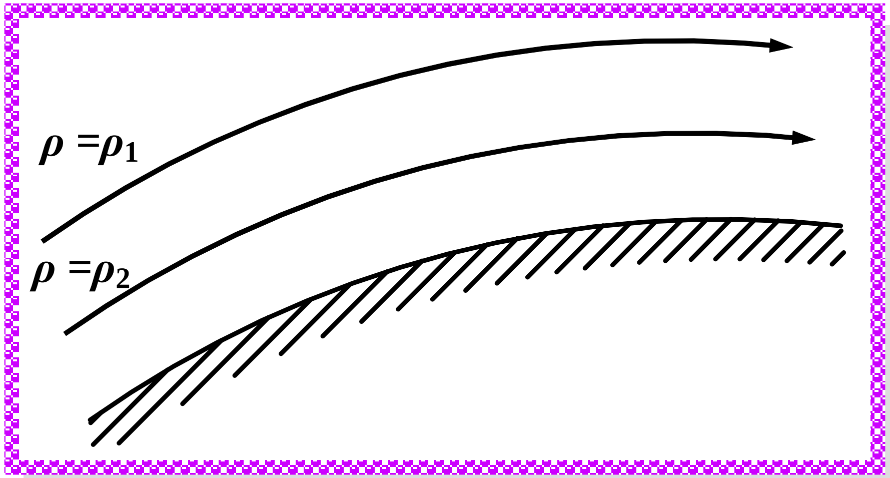
$$uA = \text{const}$$

☞ 不可压缩流动，通过流管任意过流断面的体积流量保持不变

连续方程7

密度的分层流动

流体质点沿$\rho = \rho_1$ 或 $\rho = \rho_2$ 线运动时，密度保持为常数 ρ_1 和 ρ_2



$$D\rho / Dt = 0$$

但

$$\frac{\partial \rho}{\partial x} \neq 0 \text{ , } \frac{\partial \rho}{\partial y} \neq 0$$

流体质点运动过程中密度不变，但密度场不均匀

- 由空气温度变化引起大气中的分层流动
- 由于水的含盐量变化引起海洋中的分层流动

连续方程8

不可压缩均质流动

设流体不可压缩且均质

$$D\rho / Dt = 0 \qquad \nabla \rho = 0$$

由物质导数定义 $$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\vec{u} \cdot \nabla)\rho \Rightarrow \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \quad \rho = \text{const}$$

- **绝大多数情况下，单质流体可以看作不可压缩的，流体密度就等于常数**

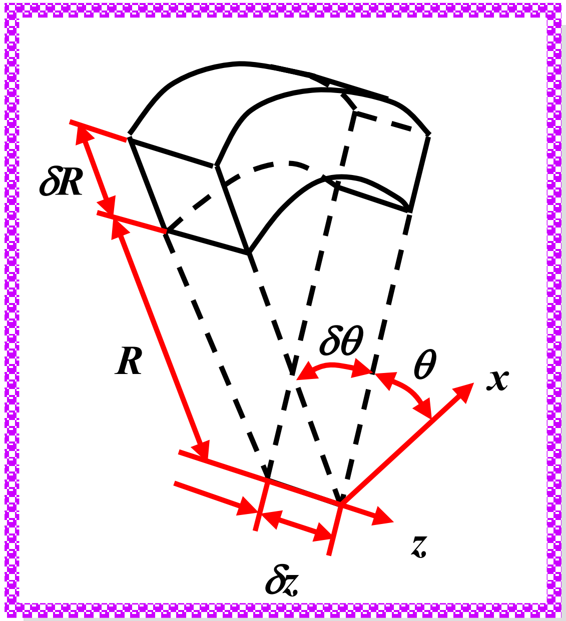
连续方程例题1

例：试对柱坐标形式的微元六面体推导连续方程

质量守恒：

控制体内质量的增加率 + 净流出控制体的质量流率 = 0

δt 时间内控制体质量的变化



$$\left[\rho R\delta\theta\delta R\delta z+\frac{\partial(\rho R\delta\theta\delta R\delta z)}{\partial t}\delta t\right]$$

$$-\rho R\delta\theta\delta R\delta z=\frac{\partial(\rho R\delta\theta\delta R\delta z)}{\partial t}\delta t$$

控制体内质量的变化率 ⇒ $\frac{\partial\rho}{\partial t}R\delta\theta\delta R\delta z$

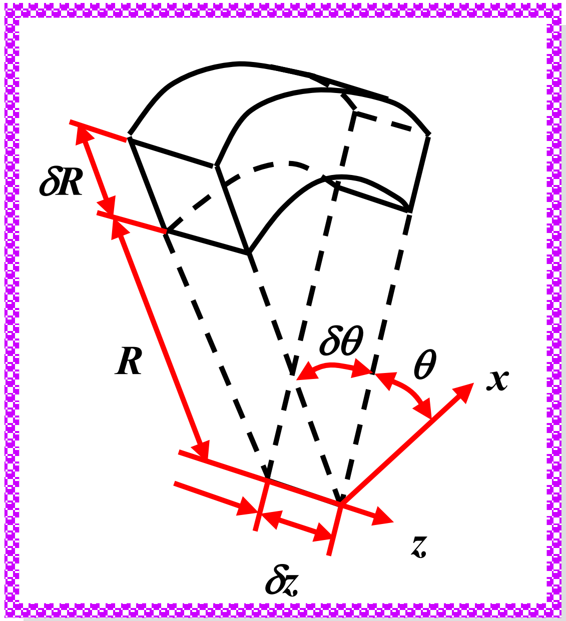
连续方程例题2

δt 时间沿 R 方向净流出控制体的质量

$$\left[\rho u_R R + \frac{\partial(\rho u_R R)}{\partial R}\frac{\delta R}{2}\right]\delta\theta\delta z\delta t$$

$$-\left[\rho u_R R - \frac{\partial(\rho u_R R)}{\partial R}\frac{\delta R}{2}\right]\delta\theta\delta z\delta t$$

$$= \frac{\partial(\rho u_R R)}{\partial R}\delta R\,\delta\theta\delta z\delta t$$



沿 R 方向净流出控制体的质量流率

$$\Rightarrow \quad \frac{\partial(\rho u_R R)}{\partial R}\delta R\,\delta\theta\delta z$$

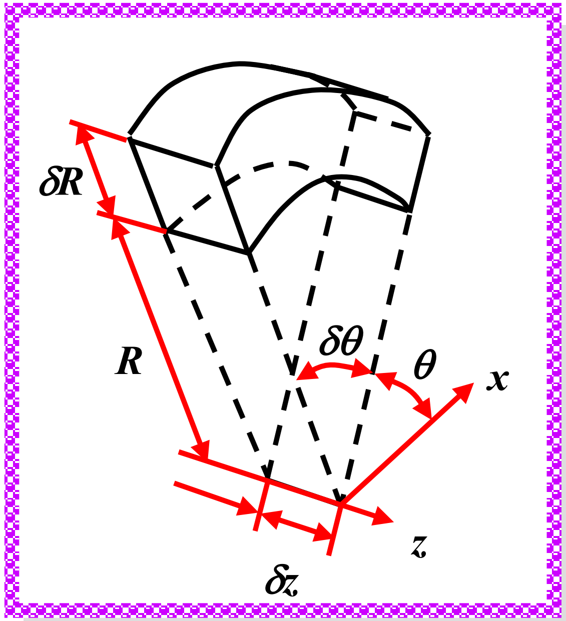
连续方程例题3

δt 时间沿 θ 方向净流出控制体的质量

$$\left[\rho u_\theta + \frac{\partial(\rho u_\theta)}{\partial\theta}\frac{\delta\theta}{2}\right]\delta R\,\delta z\,\delta t$$

$$-\left[\rho u_\theta - \frac{\partial(\rho u_\theta)}{\partial\theta}\frac{\delta\theta}{2}\right]\delta R\,\delta z\,\delta t$$

$$=\frac{\partial(\rho u_\theta)}{\partial\theta}\delta R\,\delta\theta\,\delta z\,\delta t$$



沿 θ 方向净流出控制体的质量流率

$$\Rightarrow \quad \frac{\partial(\rho u_\theta)}{\partial\theta}\delta R\,\delta\theta\,\delta z$$

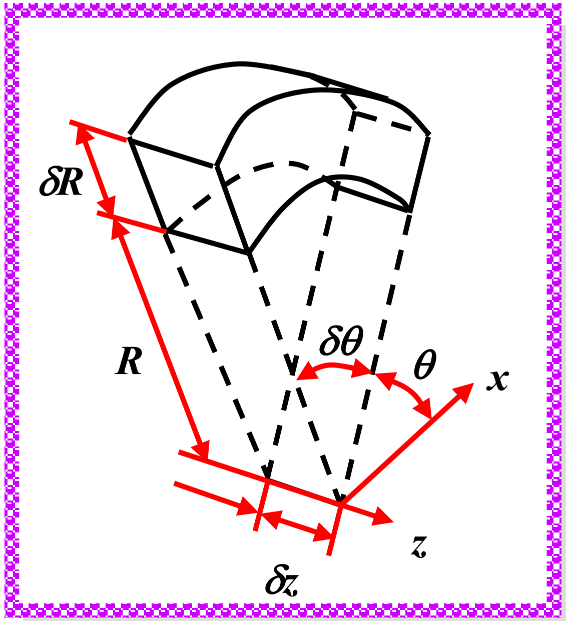
连续方程例题4

δt 时间沿 z 方向净流出控制体的质量

$$\left[\rho u_z + \frac{\partial(\rho u_z)}{\partial z}\frac{\delta z}{2}\right]R\delta\theta\delta R\delta t$$

$$-\left[\rho u_z - \frac{\partial(\rho u_z)}{\partial z}\frac{\delta z}{2}\right]R\delta\theta\delta R\delta t$$

$$=\frac{\partial(\rho u_z)}{\partial z}R\delta\theta\delta R\delta z\delta t$$



沿 z 方向净流出控制体的质量流率

$$\Rightarrow \quad \frac{\partial(\rho u_z)}{\partial z}R\delta\theta\delta R\delta z$$

连续方程例题5

连续方程：

控制体内质量的变化率 + 净流出控制体的质量流量 = 0

$$\frac{\partial \rho}{\partial t} R\delta\theta\delta R\delta z + \frac{\partial(\rho u_R R)}{\partial R}\delta R\delta\theta\delta z$$

$$+\frac{\partial(\rho u_\theta)}{\partial \theta}\delta R\delta\theta\delta z + \frac{\partial(\rho u_z)}{\partial z} R\delta\theta\delta R\delta z = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial(\rho u_R R)}{\partial R} + \frac{1}{R}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z} = 0$$

动量定理 → 系统中流体动量的变化率等于作用在该系统上的质量力和表面力之和

$$\frac{D}{Dt}\int_\tau \rho\vec{u}\,d\tau = \int_\tau \rho\vec{f}\,d\tau + \int_S \vec{p}_n\,ds$$

系统动量变化率　质量力　表面力

积分形式动量方程　　初始时刻系统与控制体重合

$$\int_\tau \rho\vec{f}\,d\tau + \int_S \vec{p}_n\,ds = \frac{\partial}{\partial t}\int_\tau \rho\vec{u}\,d\tau + \int_S \rho\vec{u}\vec{u}\cdot\vec{n}\,dS$$

动量方程2

$$\int_\tau \rho\vec{f}d\tau + \int_S \vec{p}_n ds = \frac{\partial}{\partial t}\int_\tau \rho\vec{u}d\tau + \int_S \rho\vec{u}\vec{u}\cdot\vec{n}dS$$

$\int_\tau \rho\vec{f}d\tau + \int_S \vec{p}_n ds$ → 作用在控制体上的外力之和

$\frac{\partial}{\partial t}\int_\tau \rho\vec{u}d\tau$ → 控制体中流体的动量对时间的变化率，定常该项为零

$\int_S \rho\vec{u}\vec{u}\cdot\vec{n}dS$ → 流出控制体的流体动量的净流率

动量定理3

由 $$\frac{D}{Dt}\int_\tau \rho\vec{u}\,d\tau=\int_\tau \rho\frac{D\vec{u}}{Dt}d\tau$$

$$\sigma_{ni}=n_j\sigma_{ji}$$
$$\vec{p}_n=\vec{n}\cdot\Sigma$$

由高斯公式 $$\int_S \vec{p}_n ds=\int_S(\vec{n}\cdot\Sigma)ds=\int_\tau(\nabla\cdot\Sigma)d\tau$$

$$\Rightarrow\quad \int_\tau \rho\frac{D\vec{u}}{Dt}d\tau=\int_\tau \rho\vec{f}d\tau+\int_\tau(\nabla\cdot\Sigma)d\tau$$

由于体积是任取的，所以积分相等时被积函数必然相等

$$\Rightarrow\quad \rho\frac{D\vec{u}}{Dt}=\rho\vec{f}+\nabla\cdot\Sigma$$

微分形式动量方程

动量方程4

$$\rho\frac{\partial u_j}{\partial t}+\rho u_i\frac{\partial u_j}{\partial x_i}=\rho f_j+\frac{\partial\sigma_{ij}}{\partial x_j}$$

作用在单位体积流体上的质量力

$$\rho\frac{\partial\vec{u}}{\partial t}+\rho(\vec{u}\cdot\nabla)\vec{u}=\rho\vec{f}+\nabla\cdot\Sigma$$

作用在单位体积流体上的表面力

单位体积流体的动量随时间的变化率

$\rho\frac{\partial\vec{u}}{\partial t}$ ⇒ 密度（单位体积流体的质量）与当地加速度乘积，由速度的非定常性引起当地加速度

$\rho(\vec{u}\cdot\nabla)\vec{u}$ ⇒ 密度与对流加速度项乘积，由流体质点运动及速度分布的不均匀性引起。对流加速度项是非线性的

动量方程5

由高斯公式

$$\int_S \rho\vec{u}\vec{u}\cdot\vec{n}dS=\int_\tau \nabla\cdot(\rho\vec{u}\vec{u})d\tau$$

$$\int_S \vec{p}_n ds=\int_S(\vec{n}\cdot\Sigma)ds=\int_\tau(\nabla\cdot\Sigma)d\tau$$

⟹

$$\int_\tau \frac{\partial(\rho\vec{u})}{\partial t}d\tau+\int_\tau\nabla\cdot(\rho\vec{u}\vec{u})d\tau=\int_\tau\rho\vec{f}d\tau+\int_\tau(\nabla\cdot\Sigma)d\tau$$

守恒形式的微分动量方程 ⟹

$$\frac{\partial(\rho\vec{u})}{\partial t}+\nabla\cdot(\rho\vec{u}\vec{u})=\rho\vec{f}+\nabla\cdot\Sigma$$

$$\frac{\partial}{\partial t}(\rho u_j)+\frac{\partial}{\partial x_i}(\rho u_i u_j)=\rho f_j+\frac{\partial\sigma_{ij}}{\partial x_i}$$

纳维—斯托克斯方程1

$$\rho\frac{Du_j}{Dt}=\rho f_j+\frac{\partial\sigma_{ij}}{\partial x_j}$$

由 $\sigma_{ij}=-p\delta_{ij}+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$

$$\Rightarrow\quad \frac{\partial\sigma_{ij}}{\partial x_i}=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]$$

$$\Rightarrow\quad \rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\rho f_j$$

$$\rho\frac{D\vec{u}}{Dt}=-\nabla p+\nabla(\lambda\nabla\cdot\vec{u})+\nabla\cdot(2\mu S)+\rho\vec{f}$$

纳维—斯托克斯方程2

不可压缩流动

$$\rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\rho f_j$$

$$\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]=\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_i}{\partial x_j}\right)+\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right)$$

$$=\mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right)+\mu\frac{\partial^2 u_j}{\partial x_i^2}=\mu\frac{\partial^2 u_j}{\partial x_i^2}$$

不可压缩流动连续方程

$$\rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_i^2}+\rho f_j$$

$$\rho\frac{D\vec{u}}{Dt}=-\nabla p+\mu\nabla\cdot\nabla\vec{u}+\rho\vec{f}$$

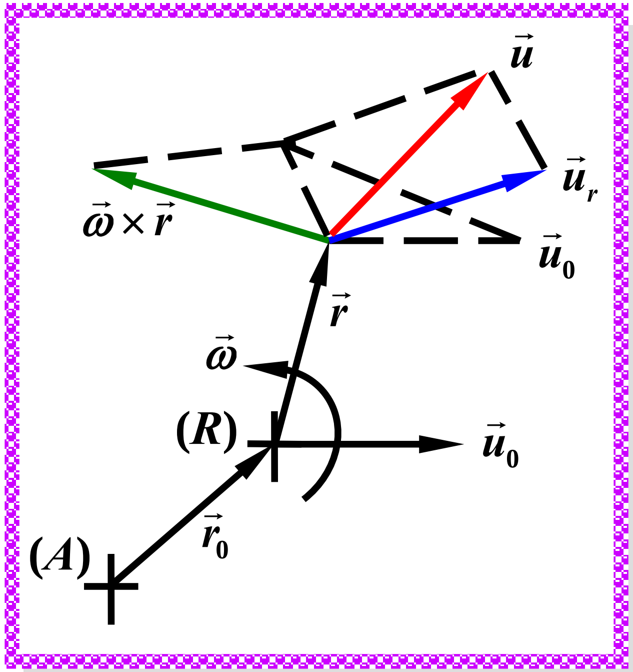
非惯性系中的动量方程1

绝对速度

$$\vec{u} = \vec{u}_r + \vec{u}_e = \vec{u}_r + \vec{u}_0 + \vec{\omega} \times \vec{r}$$

$\vec{u}_0$ ⇨ 非惯性系平移速度

$\vec{u}_r$ ⇨ 流体质点在运动坐标系中的速度，相对速度



绝对加速度

向心加速度

$$\vec{a} = \left(\frac{d\vec{u}_r}{dt}\right)_r + \frac{d\vec{u}_0}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{u}_r$$

相对加速度　平移加速度　哥氏加速度

非惯性系中的动量方程2

非惯性系动量方程

$$\rho\left(\frac{D\vec{u}_r}{Dt}\right)_r = \rho\vec{f} + \nabla\cdot\Sigma - \rho\left(\frac{d\vec{u}_0}{dt} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{\omega}\times\vec{u}_r\right)$$

$-\rho\frac{d\vec{u}_0}{dt}$ ⟹ **运动坐标系平移加速度引起的力**

运动坐标系静止或作匀速直线运动时该项为零

$-\rho\frac{d\vec{\omega}}{dt}\times\vec{r}$ ⟹ **运动坐标系角加速度引起的力**

旋转角速度为常矢量时，该项为零

$-\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$ ⟹ **离心力** 由运动坐标系旋转引起的力

$-2\rho\vec{\omega}\times\vec{u}_r$ ⟹ **哥氏力** 流体质点相对于运动坐标系静止时为零

动量方程例题1

例：从 N-S 方程出发，作出适当的假定，推导以下各方程。设不可压缩流体

$$(a)\quad \frac{\partial u}{\partial t}+v_0(t)\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$(b)\quad \frac{dv_0(t)}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+g_y$$

$$(c)\quad \frac{\partial\vec{\Omega}}{\partial t}+(\vec{u}\cdot\nabla)\vec{\Omega}=\nu\nabla^2\vec{\Omega}$$

(a) 设 $v=v_0(t)$ **根据不可压缩流动连续方程**

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \;\Longrightarrow\; \frac{\partial u}{\partial x}=0\Rightarrow u=u(y,t)$$

动量方程例题2

$$\rho\frac{Du_j}{Dt}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_i^2}+\rho f_j$$

$$u=u(y,t)\ ,\ v=v_0(t)$$

x 方向 N-S 方程

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+g_x$$

u 与 x 无关，重力沿 y 方向

$$\frac{\partial u}{\partial t}+v_0(t)\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$$

(b) y 方向 N-S 方程

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+g_y$$

动量方程例题3

$$\frac{dv_0(t)}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y$$

$$\rho\frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_i^2} + \rho f_j$$

$$u = u(y,t)\ ,\ v = v_0(t)$$

(c) 令下式对 x 求导

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) + \nu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) + \nu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad ①$$

动量方程例题4

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+g_x$$ **对 y 求导**

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}$$

$$=-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad ②$$

用 ① 式减 ② 式，且 $\Omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

动量方程例题5

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+u\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

⇒ $$\frac{\partial \Omega_z}{\partial t}+u\frac{\partial \Omega_z}{\partial x}+v\frac{\partial \Omega_z}{\partial y}=\nu\left(\frac{\partial^2 \Omega_z}{\partial x^2}+\frac{\partial^2 \Omega_z}{\partial y^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$=\frac{\partial^2}{\partial x^2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]+\frac{\partial^2}{\partial y^2}\left[\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right]=\frac{\partial^2 \Omega_z}{\partial x^2}+\frac{\partial^2 \Omega_z}{\partial y^2}$$

令 $\vec{\Omega}=\Omega_z\vec{k}$

⇒ $$\frac{\partial \vec{\Omega}}{\partial t}+(\vec{u}\cdot\nabla)\vec{\Omega}=\nu\nabla^2\vec{\Omega}$$

2.3 能量方程

热力学第一定理

设流体力学系统偏离平衡态不远：系统总能量的变化率（包括内能和动能）等于外力对系统的作功功率与通过导热向系统的传热功率之和

系统总能量 → $\int_{\tau}\rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau$ 内能 + 动能

质量力作功功率 → $\int_{\tau}\vec{u}\cdot\vec{f}\rho d\tau$ 单位质量力

表面力作功功率 → $\int_{A}\vec{u}\cdot\vec{p}_{n}dA$ 应力矢量

能量方程2

传热功率 → $\int_A -\vec{n}\cdot\vec{q}\,dA$ 热流密度

积分形式能量方程

$$\frac{D}{Dt}\int_\tau \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau=\int_\tau \vec{u}\cdot\vec{f}\rho\,d\tau+\int_A \vec{u}\cdot\vec{p}_n\,dA-\int_A \vec{n}\cdot\vec{q}\,dA$$

根据雷诺输运定理

$$\frac{\partial}{\partial t}\int_\tau \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau+\int_S \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\vec{u}\cdot\vec{n}\,dS$$
$$=\int_\tau \vec{u}\cdot\vec{f}\rho\,d\tau+\int_A \vec{u}\cdot\vec{p}_n\,dA-\int_A \vec{n}\cdot\vec{q}\,dA$$

能量方程3

微分形式能量方程

$$\frac{D}{Dt}\int_\tau \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau=\int_\tau \vec{u}\cdot\vec{f}\rho d\tau+\int_A \vec{u}\cdot\vec{p}_n dA-\int_A \vec{n}\cdot\vec{q}dA$$

$$\int_A \vec{n}\cdot\vec{a}dA=\int_\tau \nabla\cdot\vec{a}d\tau$$

由

$$\frac{D}{Dt}\int_\tau \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau=\int_\tau \rho\frac{D}{Dt}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau$$

$$\int_A \vec{u}\cdot\vec{p}_n dA=\int_A \vec{u}\cdot(\vec{n}\cdot\Sigma)dA=\int_A \vec{n}\cdot(\Sigma\cdot\vec{u})dA=\int_\tau \nabla\cdot(\Sigma\cdot\vec{u})d\tau$$

$$\int_A \vec{n}\cdot q dA=\int_\tau \nabla\cdot q d\tau$$

⇓

$$\int_\tau \rho\frac{D}{Dt}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau=\int_\tau \vec{u}\cdot\vec{f}\rho d\tau+\int_\tau \nabla\cdot(\Sigma\cdot\vec{u})d\tau-\int_\tau \nabla\cdot q d\tau$$

能量方程4

$$\int_\tau \rho \frac{D}{Dt}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau=\int_\tau \vec{u}\cdot\vec{f}\rho d\tau+\int_\tau \nabla\cdot(\Sigma\cdot\vec{u})d\tau-\int_\tau \nabla\cdot q d\tau$$

$$\Rightarrow \int_\tau\left[\rho\frac{D}{Dt}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)-\vec{u}\cdot\vec{f}\rho-\nabla\cdot(\Sigma\cdot\vec{u})+\nabla\cdot q\right]d\tau=0$$

体积任取，积分恒等于零，被积函数必然为零

$$\Rightarrow \rho\frac{D}{Dt}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)=\vec{u}\cdot\vec{f}\rho+\nabla\cdot(\Sigma\cdot\vec{u})-\nabla\cdot q$$

$$\rho\frac{D}{Dt}\left(e+\frac{1}{2}u_iu_i\right)=\rho u_i f_i+\frac{\partial}{\partial x_i}\left(\sigma_{ij}u_j\right)-\frac{\partial q_i}{\partial x_i}$$

能量方程5

守恒形式的微分能量方程

$$\frac{\partial}{\partial t}\int_\tau \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)d\tau+\int_S \rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\vec{u}\cdot\vec{n}dS$$
$$=\int_\tau \vec{u}\cdot\vec{f}\rho d\tau+\int_A \vec{u}\cdot\vec{p}_n dA-\int_A \vec{n}\cdot\vec{q}dA$$

利用高斯公式

$$\Rightarrow \int_\tau \frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\right]d\tau+\int_\tau \nabla\cdot\left[\rho\vec{u}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\right]d\tau$$
$$=\int_\tau \vec{u}\cdot\vec{f}\rho d\tau+\int_\tau \nabla\cdot(\Sigma\cdot\vec{u})d\tau-\int_\tau \nabla\cdot q d\tau$$

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\right]+\nabla\cdot\left[\rho\vec{u}\left(e+\frac{1}{2}\vec{u}\cdot\vec{u}\right)\right]=\vec{u}\cdot\vec{f}\rho+\nabla\cdot(\Sigma\cdot\vec{u})-\nabla\cdot q$$

能量方程6

表面力做功项分析

$$\frac{\partial}{\partial x_i}\left(\sigma_{ij}u_j\right)=\frac{\partial \sigma_{ij}}{\partial x_i}u_j+\sigma_{ij}\frac{\partial u_j}{\partial x_i}$$

$$\frac{\partial \sigma_{ij}}{\partial x_i}u_j$$

→ 单位体积流体受到的表面力与速度相乘表示表面力在流体微团运动过程中的作功功率，使流体动能增加

$$\sigma_{ij}\frac{\partial u_j}{\partial x_i}=\sigma_{ij}\left(s_{ji}+a_{ji}\right)=\sigma_{ij}s_{ji}+\sigma_{ij}a_{ji}=\sigma_{ij}s_{ij}$$

→ 应力张量与应变率张量相乘，表示在流体变形过程中表面力的作功功率，称变形功，它将使流体内能增加

能量方程7

变形功

$$\sigma_{ij}=-p\delta_{ij}+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ij}s_{ij}=\left(-p\delta_{ij}+\tau_{ij}\right)s_{ij}=-ps_{ii}+\tau_{ij}s_{ij}$$

$$-ps_{ii}=-p\frac{\partial u_i}{\partial x_i}$$

→ 压缩功功率，表示流体体积变化时外部压强对单位体积流体作功的功率，这种转变是可逆的

$$\tau_{ij}s_{ij}$$

→ 表示流体变形时粘性应力对单位体积流体的作功功率，这部分机械能向内能的转变是不可逆的，总大于零

机械能方程

动量方程 $$\rho\frac{Du_j}{Dt}=\rho f_j+\frac{\partial\sigma_{ij}}{\partial x_j}$$ 两端同乘以 u_j

$$\Rightarrow\quad \rho u_j\frac{Du_j}{Dt}=\rho u_j f_j+u_j\frac{\partial\sigma_{ij}}{\partial x_j}$$

可看作在 j 方向的受力平衡式和速度点乘，表示力的机械功功率，所以上式是机械能守恒方程

$$\Rightarrow\quad \rho\frac{D}{Dt}\left(\frac{1}{2}u_j u_j\right)=\rho u_j f_j+u_j\frac{\partial\sigma_{ij}}{\partial x_j}$$

能量方程9

内能方程

$$\rho\frac{D}{Dt}\left(e+\frac{1}{2}u_j u_j\right)=\rho u_j f_j+\frac{\partial}{\partial x_i}\left(\sigma_{ij}u_j\right)-\frac{\partial q_j}{\partial x_j}$$

$$\rho\frac{D}{Dt}\left(e+\frac{1}{2}u_j u_j\right)=\rho u_j f_j+\frac{\partial \sigma_{ij}}{\partial x_i}u_j+\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}u_j u_j\right)=\rho u_j f_j+u_j\frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\Rightarrow\quad \rho\frac{De}{Dt}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

能量方程10

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

$\rho\frac{\partial e}{\partial t}$ → 内能的当地变化率

$\rho u_k\frac{\partial e}{\partial x_k}$ → 内能的对流变化率，是由于流体质点从一个区域运动到另一个区域引起的

$\sigma_{ij}\frac{\partial u_j}{\partial x_i}$ → 由于表面力的作用引起的机械能向内能的转换功率

$-\frac{\partial q_j}{\partial x_j}$ → 由于导热从外界向系统内部的传热功率

能量方程11

引入傅里叶定律 $\vec{q} = -k\nabla T$

$$\sigma_{ij}\frac{\partial u_j}{\partial x_i} = \sigma_{ij}s_{ij} = -p\frac{\partial u_k}{\partial x_k} + \tau_{ij}s_{ij}$$

令 $\Phi = \tau_{ij}s_{ij}$ 耗散函数，粘性应力的作功功率

$$\rho\frac{De}{Dt} = -p\frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \Phi$$

$$\rho\frac{De}{Dt} = -p\nabla\cdot\vec{u} + \nabla\cdot\left(k\nabla T\right) + \Phi$$

- 总能量方程是机械能方程和内能方程的和

能量方程的其它形式

$$\rho\frac{De}{Dt}=-p\nabla\cdot\vec{u}+\nabla\cdot(k\nabla T)+\Phi$$

由连续方程 $\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}=0 \Rightarrow \nabla\cdot\vec{u}=-\frac{1}{\rho}\frac{D\rho}{Dt}$

$$\Rightarrow p\nabla\cdot\vec{u}=-\frac{p}{\rho}\frac{D\rho}{Dt}=p\rho\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

代入内能方程

$$\Rightarrow \rho\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=\nabla\cdot(k\nabla T)+\Phi$$

由热力学关系式

$$h=e+\frac{p}{\rho} \Rightarrow Tds=de+pd\left(\frac{1}{\rho}\right)=dh-\frac{1}{\rho}dp$$

能量方程13

$$T\frac{Ds}{Dt}=\frac{De}{Dt}+p\frac{D(1/\rho)}{Dt}=\frac{Dh}{Dt}-\frac{1}{\rho}\frac{Dp}{Dt}$$

$$\rho\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=\nabla\cdot(k\nabla T)+\Phi$$

以熵表示的能量方程

$$\rho T\frac{Ds}{Dt}=\nabla\cdot(k\nabla T)+\Phi$$

$$\rho T\frac{Ds}{Dt}=\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)+\Phi$$

以焓表示的能量方程

$$\rho\frac{Dh}{Dt}=\frac{Dp}{Dt}+\nabla\cdot(k\nabla T)+\Phi$$

$$\rho\frac{Dh}{Dt}=\frac{Dp}{Dt}+\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)+\Phi$$

2.8 牛顿流体的基本方程组

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t}+\dfrac{\partial}{\partial x_k}(\rho u_k)=0 \\
\rho\dfrac{\partial u_j}{\partial t}+\rho u_k\dfrac{\partial u_j}{\partial x_k}=-\dfrac{\partial p}{\partial x_j}+\dfrac{\partial}{\partial x_j}\left(\lambda\dfrac{\partial u_k}{\partial x_k}\right)+\dfrac{\partial}{\partial x_i}\left[\mu\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\right)\right]+\rho f_j \\
\rho\dfrac{\partial e}{\partial t}+\rho u_k\dfrac{\partial e}{\partial x_k}=-p\dfrac{\partial u_k}{\partial x_k}+\dfrac{\partial}{\partial x_j}\left(k\dfrac{\partial T}{\partial x_j}\right)+\lambda\dfrac{\partial u_k}{\partial x_k}+\mu\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\right)\dfrac{\partial u_j}{\partial x_i} \\
p=p(\rho,T) \\
e=e(\rho,T)
\end{cases}
$$

- 7个标量方程，7个未知量，方程组封闭
- μ、λ、k 为压强和温度的函数，由实验确定
- 完全气体 ⟹ $p=\rho RT$ $e=C_V T$

牛顿流体的基本方程组2

不可压缩流体，μ 为常数

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}\left(\rho u_k\right)=0$$

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}+\rho f_j$$

- 密度 ρ 为常数时，上述方程共 4 个标量方程
- 未知量 u_j、p 也是 4 个，方程组封闭
- 流体力学问题和热力学问题可分开求解

牛顿流体的基本方程组3

理想不可压缩流体

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho u_k)=0$$

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\rho f_j$$

静止流体

$$0=-\frac{\partial p}{\partial x_j}+\rho f_j$$

2.9 边界条件

初始条件 → 流体运动应该满足的初始状态，即 $t = t_0$ 时

$$\vec{u}(\vec{r},t) = \vec{u}_0(\vec{r}) \ , \ p(\vec{r},t) = p_0(\vec{r})$$
$$\rho(\vec{r},t) = \rho_0(\vec{r}) \ , \ T(\vec{r},t) = T_0(\vec{r})$$

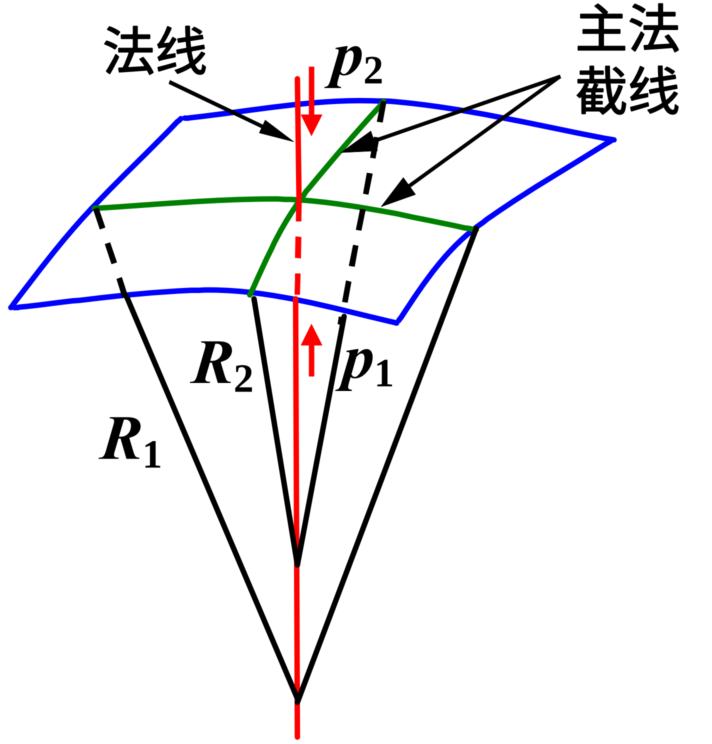
边界条件 → 边界上方程组的解应满足的条件

- 本节主要研究两种介质界面上的边界条件
- 假设分界面两边的物质互不渗透，原来的边界在以后时刻永远是两介质的界面

边界条件1

曲面上的表面张力

$$p_1 - p_2 = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



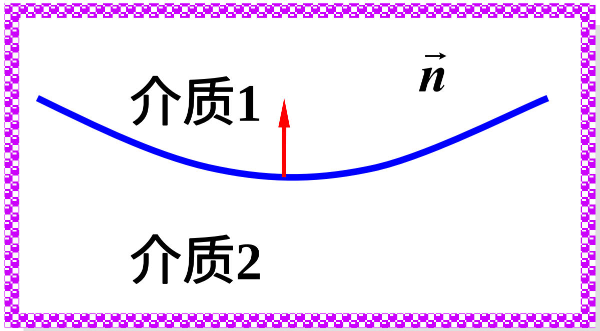
$p_1 - p_2$ ⇒ 曲面两侧压强差

α ⇒ 表面张力系数

R_1, R_2 ⇒ 曲面在考虑点的两个主曲率半径

- 表面张力的合力指向凹面一侧，与压力差平衡

液液分界面边界条件



作用在界面两侧的表面力和表面张力相平衡

$$\Sigma^{(1)}\cdot\vec{n}-\Sigma^{(2)}\cdot\vec{n}+\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\vec{n}=0$$

$\vec{n}$ 指向介质 1，当曲率中心在 $\vec{n}$ 指向的一侧时，R_1、R_2 取正值

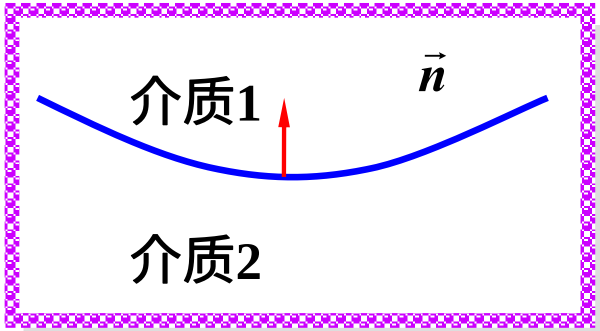
将上式向法向和切向两个方向分解

$$\sigma_{nn}^{(1)}-\sigma_{nn}^{(2)}+\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=0$$

$$\sigma_{n\tau}^{(1)}-\sigma_{n\tau}^{(2)}=0$$

边界条件3

- 界面曲率不为零时，表面张力会导致法向应力突跃；而界面两侧的切向应力总是连续的



- 界面两侧介质运动速度相等

$$\vec{u}^{(1)} = \vec{u}^{(2)}$$

$$\vec{u}_n^{(1)} = \vec{u}_n^{(2)} \qquad \vec{u}_\tau^{(1)} = \vec{u}_\tau^{(2)}$$

无穿透条件　粘附条件或无滑移条件

- 界面两侧温度和热流密度相等

$$T^{(1)} = T^{(2)} \qquad \left(k\frac{\partial T}{\partial n}\right)^{(1)} = \left(k\frac{\partial T}{\partial n}\right)^{(2)}$$

固壁边界条件

在固体边界上通常给定的条件是固壁的运动

$$\vec{u} = \vec{U}$$

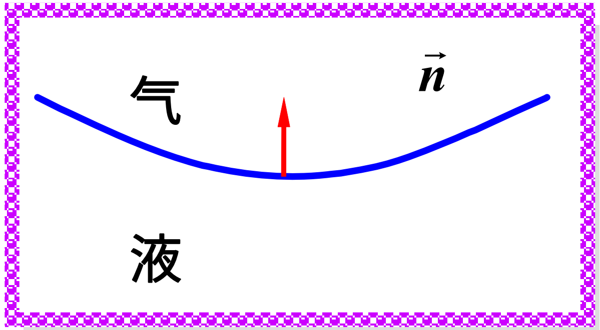
$$T^{(1)} = T^{(2)}$$

$$k\left(\frac{\partial T}{\partial n}\right)^{(1)} = k\left(\frac{\partial T}{\partial n}\right)^{(2)}$$

固壁静止时

$$\vec{u} = 0$$

液气分界面边界条件



- 气体密度和粘度都很低，其运动一般不会对液体产生显著影响
- 通常只关心液体内部的流动，气相运动未知
- 气体稀薄时界面上切向速度和温度可能发生间断，但仍要求法向速度相等，U 为界面速度

$$\vec{u}\cdot\vec{n}=\vec{U}\cdot\vec{n}$$

边界条件6

- 忽略气相粘性，界面上切应力为零

$$\vec{\sigma}_{n\tau} = 0$$

- 设 p_a 为大气压强，p 为液气边界面上的液体侧压强自由面曲率中心在气相一侧，法应力条件可写为

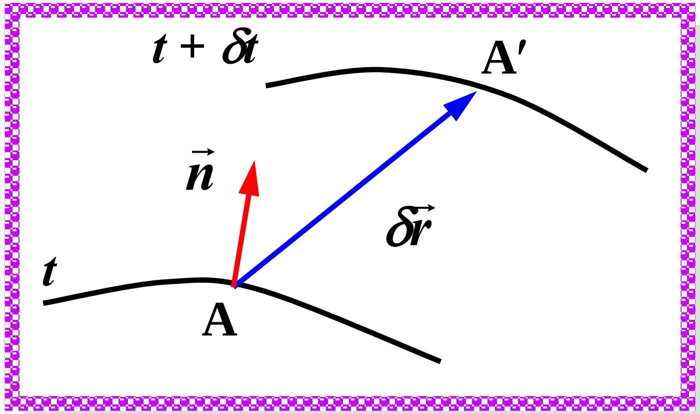
$$p_a - p = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

- 不考虑表面张力影响 ⟹ $p = p_a$

界面法向速度1

定义界面边界条件时，需要知道界面法向速度，设界面方程为

$$F(\vec{r},t)=0$$



设界面上一点 A 在 t 时刻的位置矢量为 $\vec{r}$ ，该点的法向单位矢量为 $\vec{n}$ ，经过 $t+\delta t$ 时间后，A 点运动到 $\vec{r}+\delta\vec{r}$ ，则

$$\delta F = F(\vec{r}+\delta\vec{r},t+\delta t)-F(\vec{r},t)$$

$$=F(\vec{r},t)+\frac{\partial F}{\partial t}\delta t+\frac{\partial F}{\partial x}\delta x+\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial z}\delta z-F(\vec{r},t)$$

$$=\frac{\partial F}{\partial t}\delta t+\delta\vec{r}\cdot\nabla F=0$$

界面法向速度2

界面在 t 时刻的法向速度为

$$U_n = \lim_{\delta t \to 0} \frac{\delta\vec{r} \cdot \vec{n}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\vec{r} \cdot \nabla F}{\delta t \left|\nabla F\right|} = \frac{-\partial F / \partial t}{\left|\nabla F\right|}$$

$$U_n = \frac{-\partial F / \partial t}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

设 t 时刻在A点的流体质点的速度为 $\vec{u}$ ，则其法向速度为

$$u_n = \vec{u} \cdot \vec{n}$$

流体质点的法向速度等于界面本身在该点的法向速度

界面法向速度3

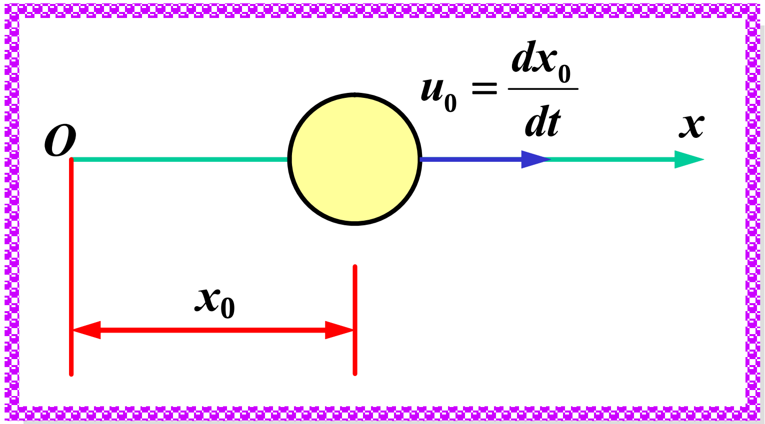
$$\Rightarrow \quad u_n = U_n \quad \Rightarrow \quad \vec{u}\cdot\frac{\nabla F}{|\nabla F|} = \frac{-\partial F/\partial t}{|\nabla F|}$$

$$\Rightarrow \quad \frac{\partial F}{\partial t} + \vec{u}\cdot\nabla F = 0 \qquad \frac{DF}{Dt} = 0$$

- 界面上流体质点的位置矢量始终满足方程 $F(\vec{r},t)=0$
  流体质点始终保持在界面上，或者说界面始终由同一些流体质点所组成
- 上式既适用于固壁和气液界面，也适用于液液界面和流体中的其他物质间断面

边界条件例题1-1

例：小球在理想流体中作缓慢直线运动，试给出小球表面流体速度 u, v, w 所必须满足的边界条件。小球的半径为 a。



解1：取固定坐标系如图，球面方程为

$$[x - x_0(t)]^2 + y^2 + z^2 = a^2$$

令 $F = [x - x_0(t)]^2 + y^2 + z^2 - a^2 = 0$

物面边界条件为 $\dfrac{\partial F}{\partial t} + u\dfrac{\partial F}{\partial x} + v\dfrac{\partial F}{\partial y} + w\dfrac{\partial F}{\partial z} = 0$

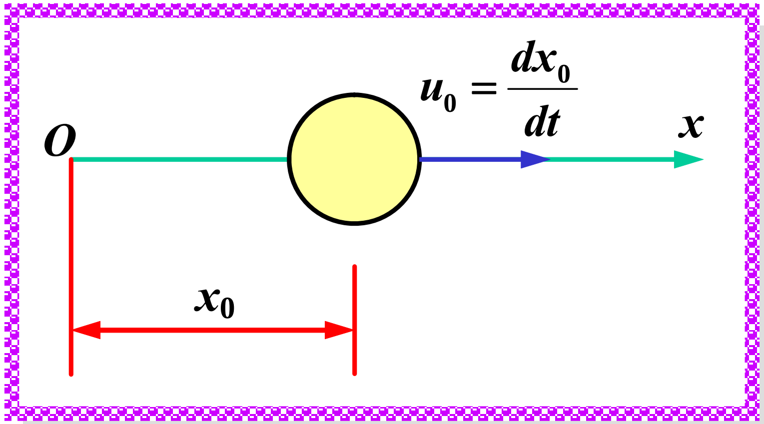
边界条件例题1-2

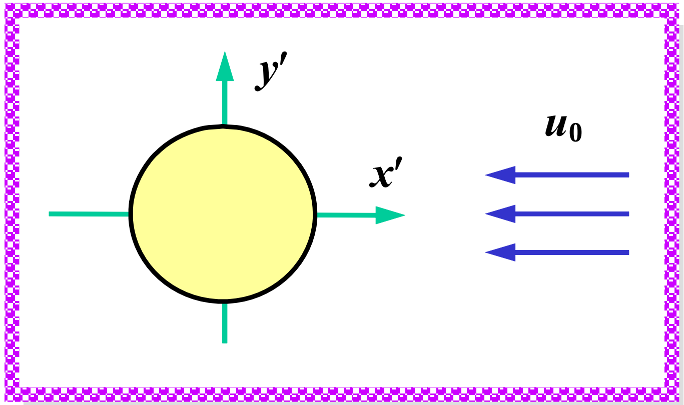
$$-2u_0(x-x_0)+2u(x-x_0)+2vy+2wz=0$$

$$(u-u_0)(x-x_0)+vy+wz=0$$

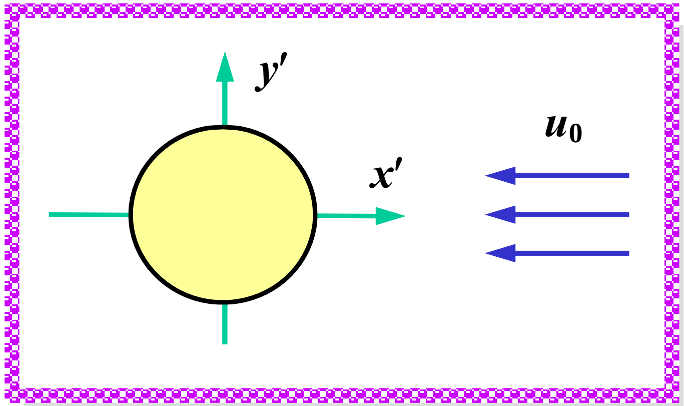
式中， $u_0=\dfrac{dx_0}{dt}$ 　u,v,w 为物面上流体速度

解2：取运动坐标系固连在小球上





边界条件例题1-3



球面方程为

$$x'^2 + y'^2 + z'^2 = a^2$$

令 $F = x'^2 + y'^2 + z'^2 - a^2 = 0$

物面流体速度为 $u - u_0$, v , w

运动坐标系与固定坐标系之间的关系为

⟹ $x' = x - x_0$, $y' = y$, $z' = z$

相对于运动坐标系的速度矢量

⟹ $\vec{u}_r = (u - u_0)\vec{i}' + v\vec{j}' + w\vec{k}'$

边界条件例题1-4

法向单位矢量

$$\vec{n}' = \frac{\nabla F'}{|\nabla F'|} = \frac{2x'\vec{i} + 2y'\vec{j} + 2z'\vec{k}}{|\nabla F'|}$$

边界条件 $\vec{u}_r \cdot \vec{n}' = 0$

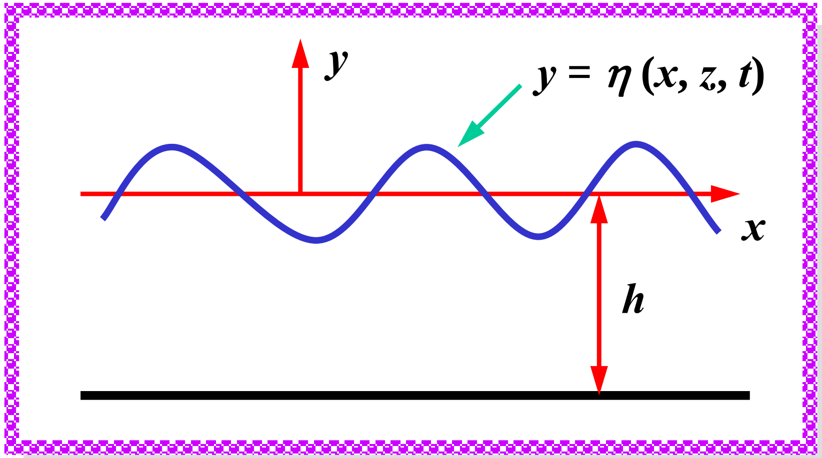
$$(u - u_0)x' + vy' + wz' = 0$$

$$\underline{\underline{(u - u_0)(x - x_0) + vy + wz = 0}}$$

边界条件例题2

例：试写出自由表面波动时的运动学边界条件

设自由面方程为 $y=\eta(x,z,t)$



令 $F=y-\eta(x,z,t)=0$

自由面的运动学边界条件

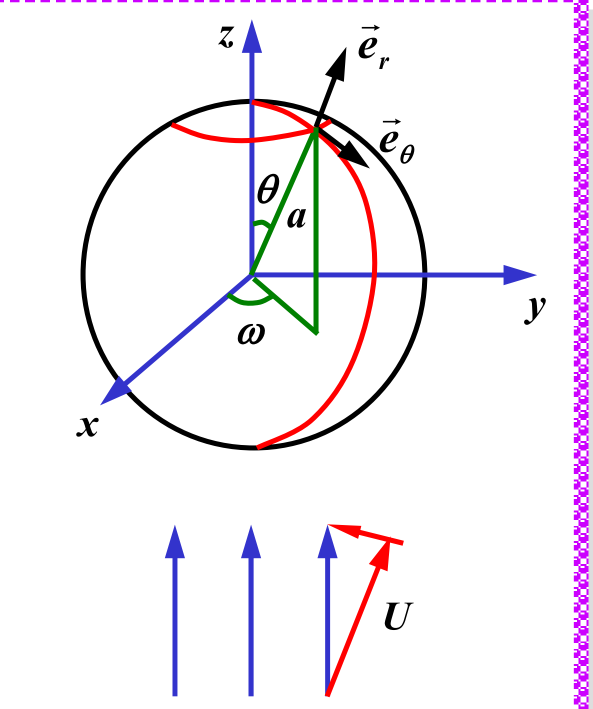
$$\frac{DF}{Dt}=\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

$$\Rightarrow\quad -\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+v+w\frac{\partial \eta}{\partial z}=0 \quad\Rightarrow\quad v=\frac{\partial \eta}{\partial t}-u\frac{\partial \eta}{\partial x}-w\frac{\partial \eta}{\partial z}$$

$$\Rightarrow\quad v=\frac{\partial \eta}{\partial t}-(\vec{u}\cdot\nabla)\eta$$

边界条件例题3-1

例：分别写出均匀来流绕流固体圆球（粘性流体），圆球状液滴和圆球状汽泡时的边界条件。圆球半径为 a，取来流方向沿对称轴。设绕流雷诺数很小 $Re < 1$



设来流沿 z 方向，由流动的对称性

$$u_\omega = 0 \ , \ \frac{\partial}{\partial \omega} = 0$$

此种流动为轴对称流动

(1) 固体圆球，只需研究球外流场

$$r = a \Rightarrow u_\theta = 0 \ , \ u_r = 0$$

$$r \to \infty \Rightarrow u_r = U\cos\theta \ , \ u_\theta = -U\sin\theta \ , \ p = p_\infty$$

边界条件3-2

(2) 液滴，需研究内外两个流场

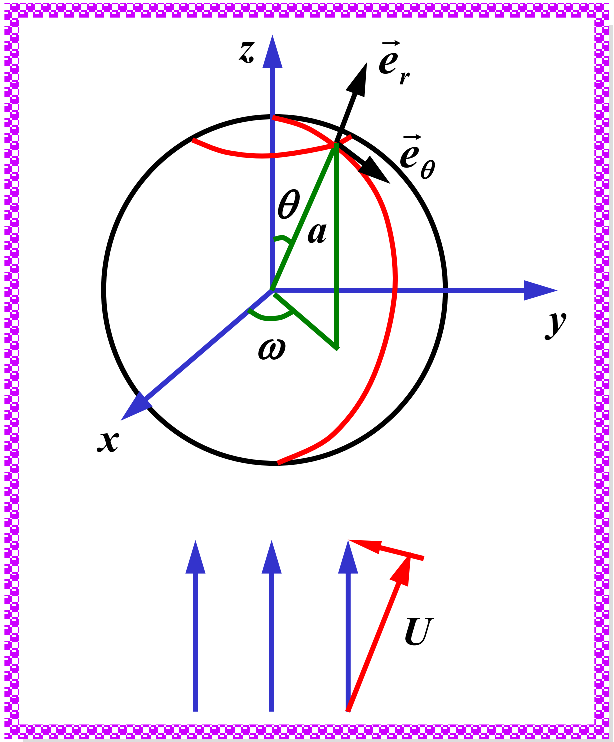
$r=0 \Rightarrow u_r, u_\theta, p$ 为有限值

$r=a \Rightarrow u_r^{(1)}=u_r^{(2)}$, $u_\theta^{(1)}=u_\theta^{(2)}$

$$\sigma_{rr}^{(1)}-\sigma_{rr}^{(2)}=\frac{2\alpha}{a} \ , \ \sigma_{r\theta}^{(1)}=\sigma_{r\theta}^{(2)}$$

$r\to\infty \Rightarrow u_r=U\cos\theta$, $u_\theta=-U\sin\theta$

$p=p_\infty$



(3) 气泡，只研究气泡外液体流场

$r=a \Rightarrow u_r=0$, $\sigma_{r\theta}=0$

$r\to\infty \Rightarrow u_r=U\cos\theta$, $u_\theta=-U\sin\theta$, $p=p_\infty$

作业

作业：P.66～69

- 2.2
- 2.3
- 2.5
- 2.11
- 2.13
- 2.14
- 2.15
- 2.17
- 2.20