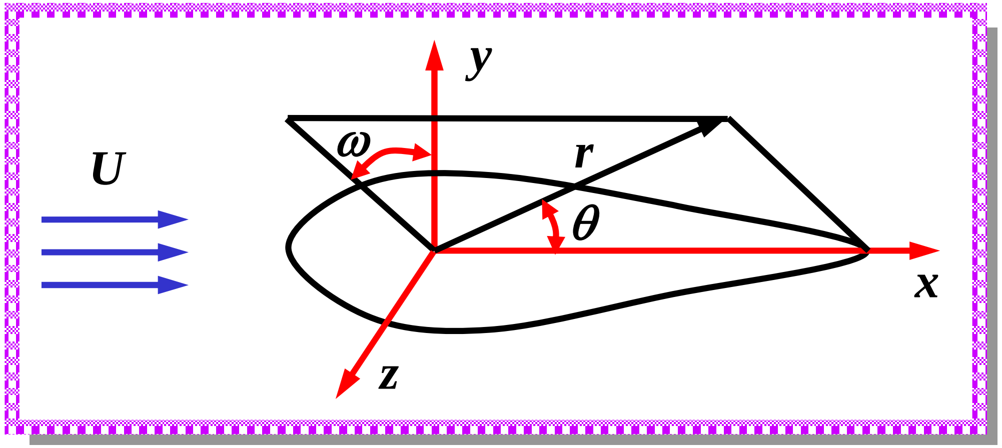
# 第5章 空间轴对称势流

# 轴对称流动

## 形成轴对称流动

- 物体外形必须是轴对称的（旋转体）




- 来流必须沿着对称轴方向

轴对称流动中，任一通过对称轴的平面上的流动图案都是相同的

- 采用球坐标描述空间轴对称流动时

$$u_\omega = 0 \ , \ \frac{\partial}{\partial \omega} = 0$$

三维问题退化为二维问题

# 5.1 速度势函数和斯托克斯流函数

## 势函数

$$u_\omega = 0 \ , \ \frac{\partial}{\partial \omega} = 0$$

- 无旋流动中存在速度势 $\phi$，轴对称流动在球坐标下

$$\vec{u} = \nabla \phi = \frac{\partial \phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{e}_\theta = u_r \vec{e}_r + u_\theta \vec{e}_\theta$$

$$\Rightarrow \quad u_r = \frac{\partial \phi}{\partial r} \ , \ u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

- 不可压缩流动速度势函数满足拉普拉斯方程 $\nabla^2 \phi = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

## 斯托克斯流函数

不可压缩流体轴对称流动球坐标系下的连续方程

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta u_\theta\right)=0$$

$$\Rightarrow\quad \frac{\partial}{\partial r}\left(r^2\sin\theta u_r\right)+\frac{\partial}{\partial\theta}\left(r\sin\theta u_\theta\right)=0$$

令 $\frac{\partial\psi}{\partial\theta}=r^2\sin\theta u_r \quad -\frac{\partial\psi}{\partial r}=r\sin\theta u_\theta$ 则$\psi$自动满足连续方程

$$\Rightarrow\quad u_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\ ,\quad u_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$\psi$为Stokes流函数

# 速度势函数和斯托克斯流函数3

$\psi$等于常数的曲面是流面

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta},\quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

流线上速度 $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta$ 与线元 $d\vec{r} = dr\vec{e}_r + rd\theta\,\vec{e}_\theta$ 平行

由 $\vec{u}\times d\vec{r} = 0$ ⇒

$$\frac{dr}{u_r} = \frac{rd\theta}{u_\theta}$$

流线方程

⇒

$$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}rd\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}dr = \frac{1}{r\sin\theta}d\psi = 0$$

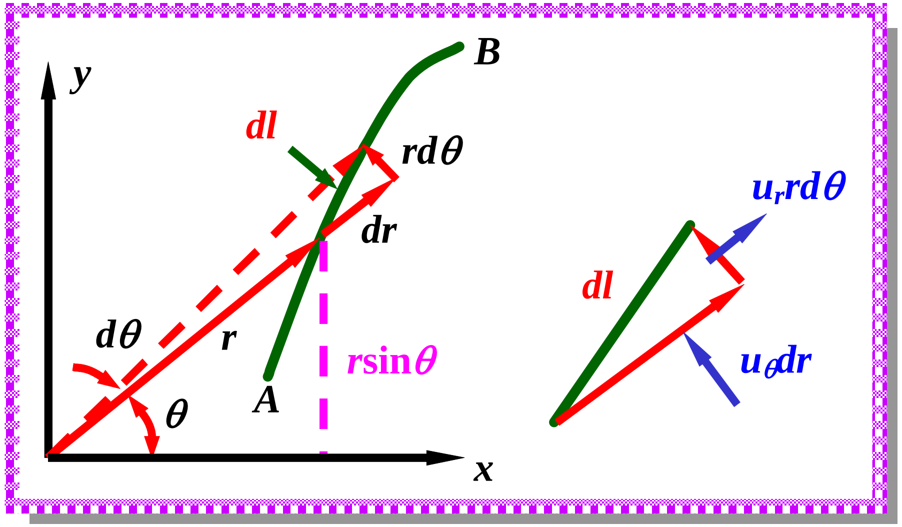
即 $d\psi = 0$ ⇒ $\psi = \text{const}$

子午面上的流线，也是关于对称轴的旋转流面

# 速度势函数和斯托克斯流函数4

过对称轴的平面内任意两点流函数的差乘以2π，等于通过以这两点的任意连线绕对称轴旋转形成的旋转面的流量

$$dQ = \left(u_r r d\theta - u_\theta dr\right) 2\pi r \sin\theta$$




**代入速度与流函数的关系式**

$$dQ = \left( \frac{r}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} d\theta + \frac{1}{r\sin\theta} \frac{\partial \psi}{\partial r} dr \right) 2\pi r \sin\theta$$

$$= 2\pi \left( \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta \right) = 2\pi d\psi$$

$$\Rightarrow \quad Q = 2\pi\left(\psi_B - \psi_A\right)$$

# 速度势函数和斯托克斯流函数5

斯托克斯流函数满足的方程

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\ ,\ u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

设无旋流动

$$\nabla\times\vec{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\omega \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & 0 \\ u_r & ru_\theta & 0 \end{vmatrix} = \frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]\vec{e}_\omega = 0$$

代入速度与流函数的关系式

$$\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0$$

$$r^2\frac{\partial^2\psi}{\partial r^2} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0$$

不是拉氏方程，但仍然是线性方程

# 速度势函数和斯托克斯流函数6

不可压缩轴对称无旋流动

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0$$

$$r^2\frac{\partial^2\psi}{\partial r^2}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)=0$$

- 求解 $\phi$ 的拉氏方程以及流函数方程均可得出不可压缩流体轴对称无旋运动的解，均可采用简单势流叠加获得复杂势流流场
- 有旋流动中势函数不存在，只有应用流函数才能找到一个标量方程来代替矢量形式的运动方程

## 速度势函数拉氏方程

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0$$

分离变量，令 $\phi = R(r)T(\theta)$ ，代入上式

$$\Longrightarrow \quad \frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{R}{r\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)=0$$

两边同乘以 $r^2/RT$

$$\Longrightarrow \quad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)$$

左边只是 $r$ 的函数，右边只是 $\theta$ 的函数，要恒等必为常数

# 势流方程的解2

令 $$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)$$

$$-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)=l(l+1)$$

变量$T(\theta)$的方程可改写为 $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)+Tl(l+1)=0$$

令 $x=\cos\theta$ ⇒ $$\frac{\partial}{\partial\theta}=\frac{\partial}{\partial x}\frac{\partial x}{\partial\theta}=-\sin\theta\frac{\partial}{\partial x}$$

代入 ⇒ $$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dT}{dx}\right]+Tl(l+1)=0$$

勒让德方程的标准形式

其通解为 $$T_l(\theta)=C_lP_l(\cos\theta)+D_lQ_l(\cos\theta)$$

# 势流方程的解3

$$T_l(\theta)=C_l P_l(\cos\theta)+D_l Q_l(\cos\theta)$$

$Q_l(\cos\theta)$ ⇩

**第二类勒让德函数，当 $\cos\theta = \pm 1$ 时对所有的 $l$ 值发散所以应取 $D_l = 0$**

$P_l(\cos\theta)$ ⇩

**第一类勒让德函数，当 $l$ 不为整数时其在 $\cos\theta = \pm 1$ 时发散所以应取 $l = 0, 1, 2, 3, \ldots$**

⇨ $$T_l(\theta)=C_l P_l(\cos\theta)$$

# 势流方程的解4

变量$R(r)$的方程可改写为 $r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0$

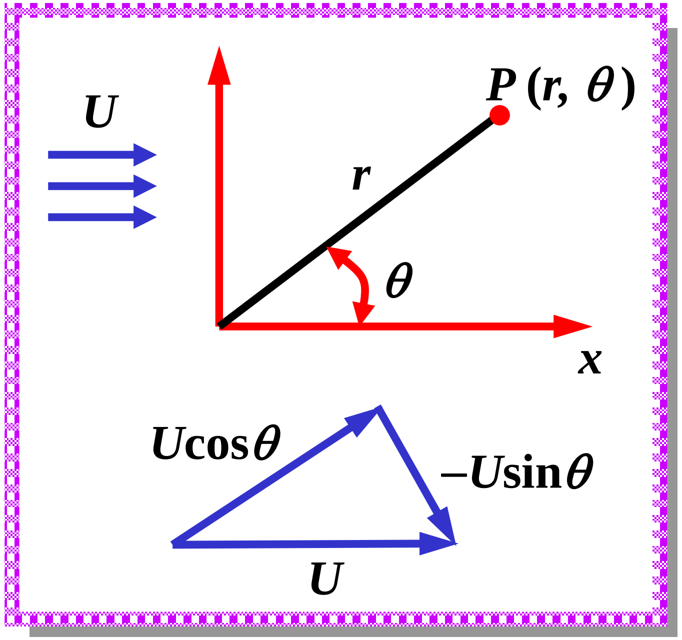
上式为欧拉方程，对于非负整数 $l$，其通解为 $\Rightarrow$ $R_l(r)=A_lr^l+\frac{B_l}{r^{l+1}}$

@势函数的解 $\Rightarrow$ $\phi_l(r,\theta)=\sum_{l=0}^{\infty}\left(A_lr^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

勒让德函数的表达式 $\Rightarrow$ $P_l(x)=\frac{1}{2^l\,l!}\frac{d^l}{dx^l}(x^2-1)^l$

前三项 $P_0(x)=1$，$P_1(x)=x$，$P_2(x)=\frac{1}{2}(3x^2-1)$

均匀流 → 势函数




沿 $x$ 方向均匀流，速度为 $U$，$P$ 点势函数

$$u_r = U\cos\theta = \frac{\partial\phi}{\partial r}$$

$$u_\theta = -U\sin\theta = \frac{\partial\phi}{r\partial\theta}$$

$$\phi = Ur\cos\theta$$

流函数

$$u_r = U\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

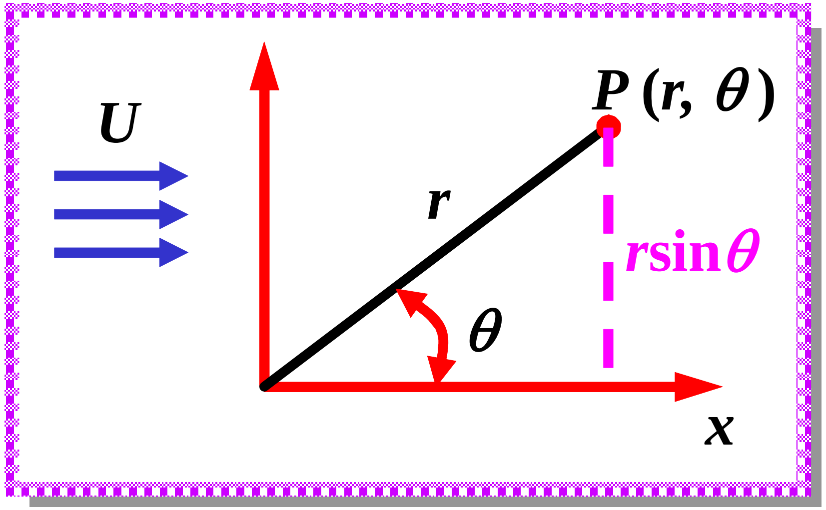
$$u_\theta = -U\sin\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$\psi = \frac{1}{2}Ur^2\sin^2\theta$$

# 均匀流流函数

令原点流函数为 0，$P$ 点为 $\psi$

则均匀流穿过位置矢量围绕对称轴旋转形成的圆锥面的流量




⟹ $2\pi\psi$

圆锥面垂直于流动方向的投影

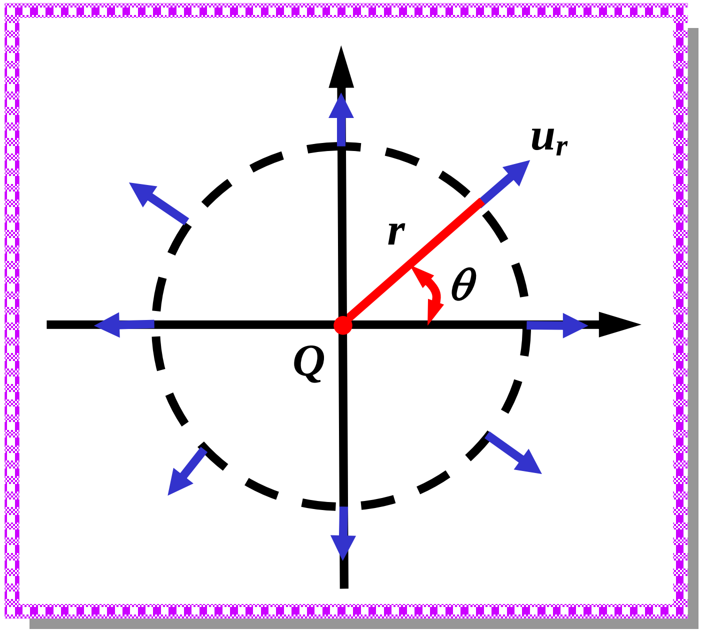
⟹ $\pi(r\sin\theta)^2$

穿过圆锥面的流体将全部通过该投影面积

⟹ $U\pi(r\sin\theta)^2 = 2\pi\psi$ ⟹ $\psi = \dfrac{1}{2}Ur^2\sin^2\theta$

# 点源和点汇1

点源和点汇 → 势函数




设空间点源位于原点，强度为 $Q$，球面上只有径向速度分量，由连续方程

$$Q = 4\pi r^2 u_r \Rightarrow u_r = \frac{Q}{4\pi r^2}$$

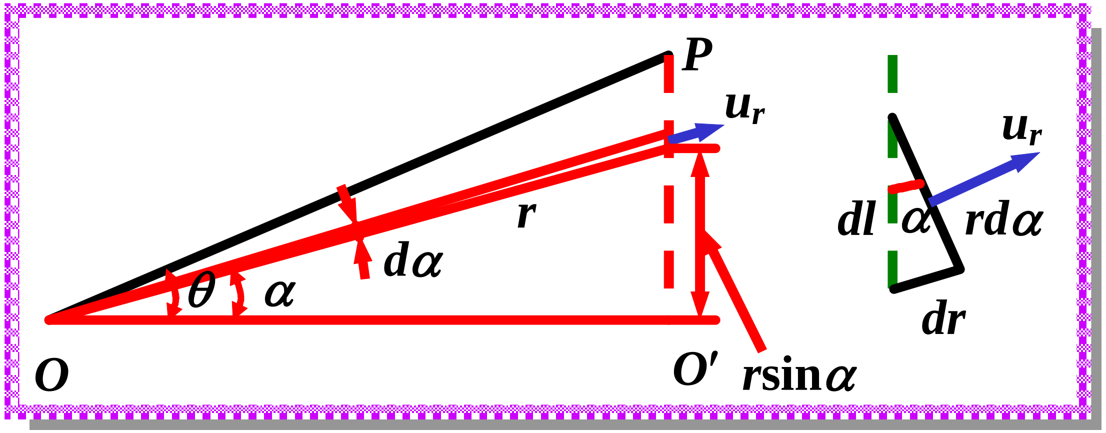
由 $u_r = \dfrac{Q}{4\pi r^2} = \dfrac{\partial\phi}{\partial r}$， $u_\theta = 0 = \dfrac{\partial\phi}{r\partial\theta}$ $\Rightarrow$ $\phi = -\dfrac{Q}{4\pi r}$

☞ 式中负号对应点源，正号对应点汇

☞ $r \to 0$，$u_r \to \infty$，原点为奇点；$r \to \infty$，$u_r \to 0$

## 流函数

设点源稍偏向于原点右侧




点源释放的流体通过圆锥面向外流出的流量 ⟹ $-2\pi\psi$

点源释放的流体通过圆锥面投影面流出的流量

⟹ $$\int_0^{\theta} u_r r d\alpha 2\pi r\sin\alpha$$

点源释放的流体通过圆锥面及投影面流出的总流量

⟹ $$Q = -2\pi\psi + \int_0^{\theta} u_r r d\alpha 2\pi r\sin\alpha = -2\pi\psi + 2\pi\int_0^{\theta} u_r r^2 \sin\alpha d\alpha$$

# 点源和点汇3

代入径向速度 $u_r = \dfrac{Q}{4\pi r^2}$

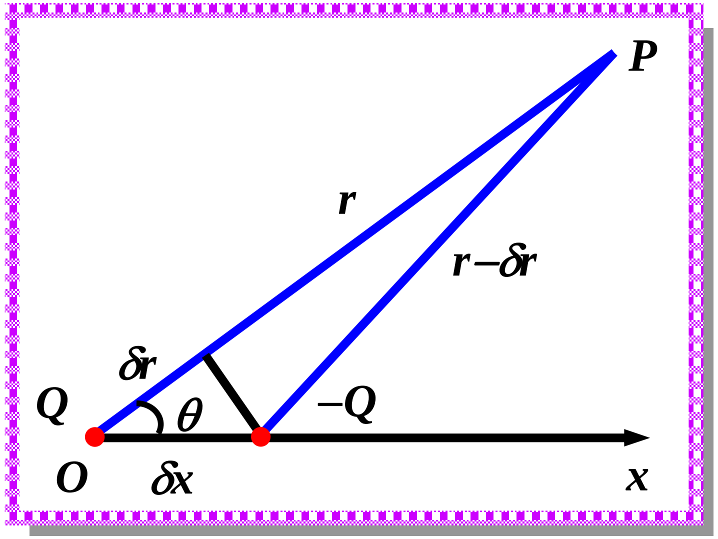
$$\Rightarrow \quad Q = -2\pi\psi + \frac{Q}{2}(1-\cos\theta)$$

$$\Rightarrow \quad \psi = -\frac{Q}{4\pi}(1+\cos\theta)$$

☞ 如果让点源位于原点左边，则得到的流函数与上式仅相差一个常数，并不影响得到的流场分布

# 偶极子1

偶极子 → 势函数




**一对相等强度的点源和点汇在 $P$ 点的势函数**

$$\phi = -\frac{Q}{4\pi r} + \frac{Q}{4\pi(r-\delta r)}$$

$$= -\frac{Q}{4\pi r}\left(1 - \frac{1}{1-\delta r/r}\right)$$

**当$\delta r / r \ll 1$时**

$$\phi = -\frac{Q}{4\pi r}\left\{1 - \left[1 + \frac{\delta r}{r} + O\left(\frac{\delta r^2}{r^2}\right)\right]\right\} = \frac{Q}{4\pi r}\left[\frac{\delta r}{r} + O\left(\frac{\delta r^2}{r^2}\right)\right]$$

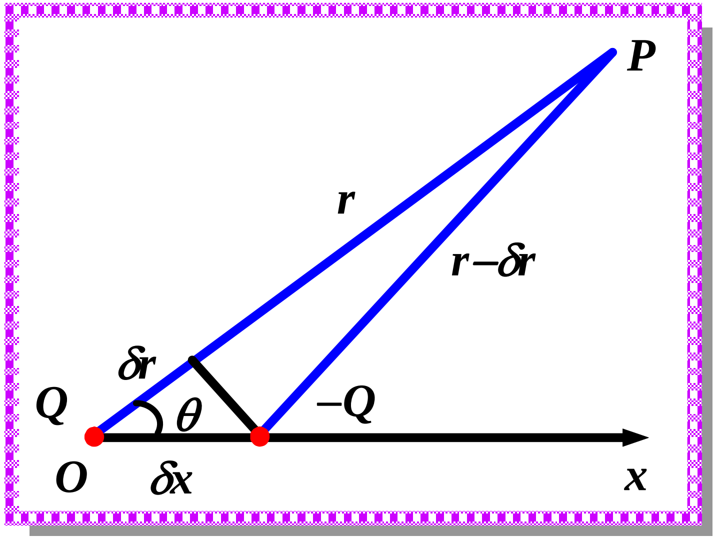
**当$\delta r / r$很小时，**

$$\delta r \approx \delta x \cos\theta$$

⟹ $$\phi = \frac{Q\delta x}{4\pi r^2}\cos\theta$$

**设** $$\lim_{\substack{\delta x \to 0 \\ Q \to \infty}} Q\delta x = \mu$$

⟹ $$\phi = \frac{\mu}{4\pi r^2}\cos\theta$$




☞ **点汇在右，点源在左，偶极子方向指向 $x$ 轴负向**

# 偶极子3

流函数

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$\phi = \frac{\mu}{4\pi r^2}\cos\theta$$

**由势函数可得速度分量为**

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{\mu}{2\pi r^3}\cos\theta, \quad u_\theta = \frac{\partial\phi}{r\partial\theta} = -\frac{\mu}{4\pi r^3}\sin\theta$$

$$\Longrightarrow \frac{\partial\psi}{\partial\theta} = -\frac{\mu}{2\pi r}\cos\theta\sin\theta \Longrightarrow \psi = -\frac{\mu}{4\pi r}\sin^2\theta + f(r)$$

由

$$\frac{\partial\psi}{\partial r} = \frac{\mu}{4\pi r^2}\sin^2\theta + f'(r) = -r\sin\theta u_\theta = \frac{\mu}{4\pi r^2}\sin^2\theta$$

$$\Longrightarrow f(r) = C \Longrightarrow$$

$$\psi = -\frac{\mu}{4\pi r}\sin^2\theta$$

# 偶极子4

**均匀流，点源（汇）和偶极子流动的势函数是势流方程的基本解**

$$\phi_l(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

**均匀流**

$$P_0(x)=1\ ,\ P_1(x)=x\ ,\ P_2(x)=\frac{1}{2}\left(3x^2-1\right)$$

$$A_l=\begin{cases}0 & l\neq 1\\ U & l=1\end{cases}\qquad B_l=0\quad \text{对所有 } l \text{ 值}$$

$$\Rightarrow\quad \phi=UrP_1(\cos\theta)=Ur\cos\theta$$

# 偶极子5

$$\phi_l(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$P_0(x)=1\ ,\ P_1(x)=x\ ,\ P_2(x)=\frac{1}{2}(3x^2-1)$$

## 点源（汇）

$A_l=0$ 对所有 $l$ 值

$$B_l=\begin{cases}0 & l\neq 0\\ -\dfrac{Q}{4\pi} & l=0\end{cases} \Rightarrow \phi=-\frac{Q}{4\pi r}P_0(\cos\theta)=-\frac{Q}{4\pi r}$$

## 偶极子

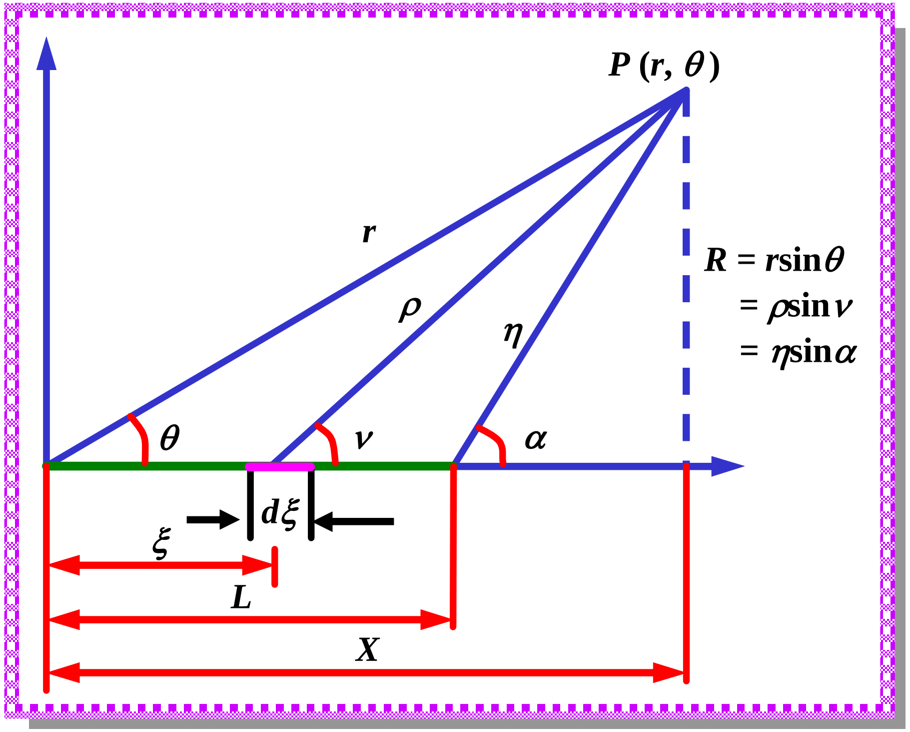
$A_l=0$ 对所有 $l$ 值

$$B_l=\begin{cases}0 & l\neq 1\\ \dfrac{\mu}{4\pi} & l=1\end{cases} \Rightarrow \phi=\frac{\mu}{4\pi r^2}P_1(\cos\theta)=\frac{\mu}{4\pi r^2}\cos\theta$$

# 线源1

☞设源（汇）在区间 $0 \le x \le L$ 均匀连续分布，线源单位长度的源强为 $q$

**流函数**




如图取线元 $d\zeta$，线元的源强为 $qd\zeta$，则 $P$ 点流函数

$$\psi = -\int_0^L \frac{qd\xi}{4\pi}\left(1+\cos\nu\right)$$

由 $X-\xi = R\,\mathrm{ctg}\,\nu$ —— X 保持不变 ⟶ $-d\xi = -R\csc^2\nu d\nu$

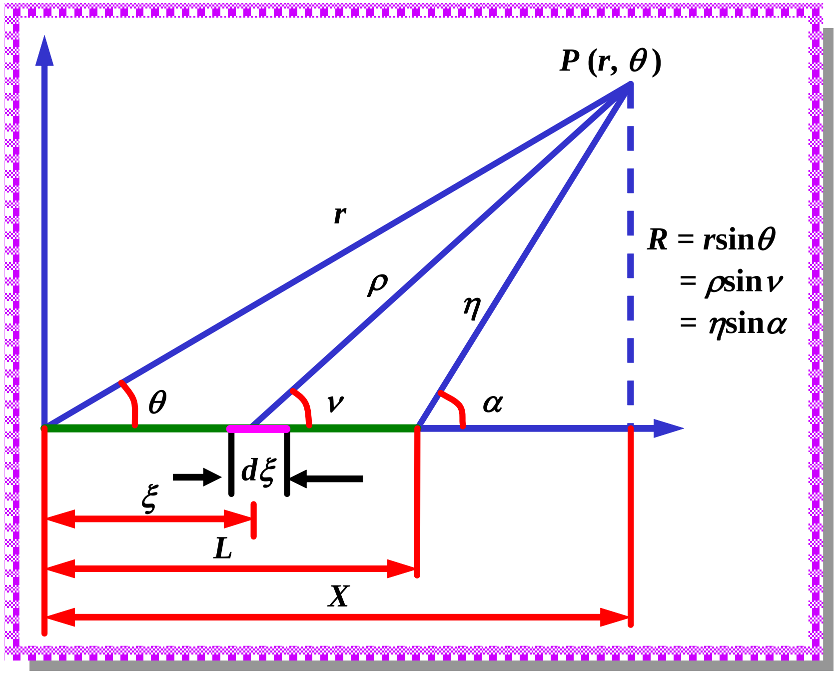
# 线源2

$$\psi = -\int_0^L \frac{qR}{4\pi}\csc^2\nu(1+\cos\nu)d\nu = \frac{qR}{4\pi}\left(\mathrm{ctg}\,\nu + \frac{1}{\sin\nu}\right)\Bigg|_\theta^\alpha$$

$$\psi = \frac{qR}{4\pi}\left(\mathrm{ctg}\,\alpha - \mathrm{ctg}\,\theta + \frac{1}{\sin\alpha} - \frac{1}{\sin\theta}\right)$$

$$= \frac{qR}{4\pi}\left(\frac{X-L}{R} - \frac{X}{R} + \frac{\eta}{R} - \frac{r}{R}\right)$$

$$\psi = -\frac{q}{4\pi}(L + r - \eta)$$



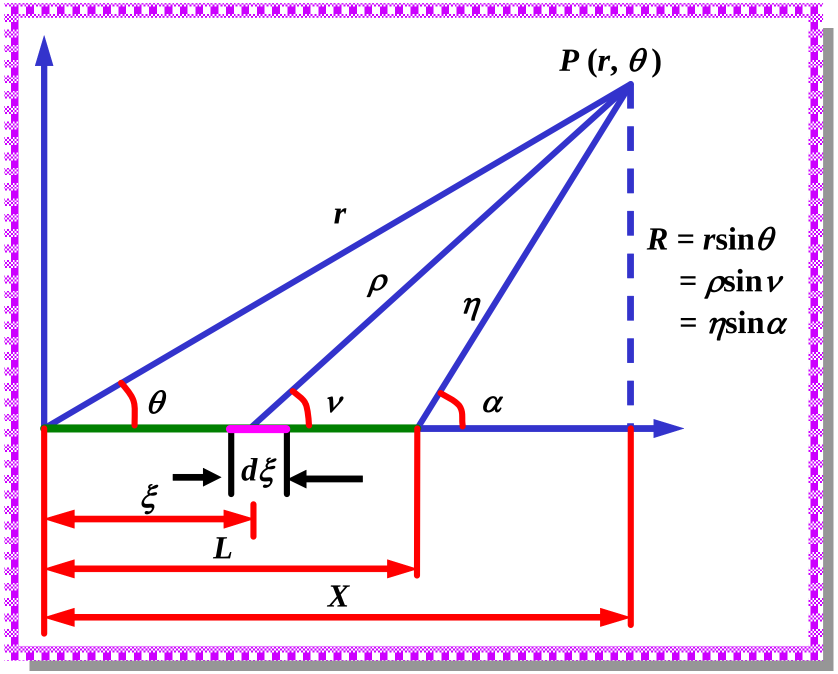
## 势函数

$$-d\xi = -R\csc^2\nu d\nu$$

$$\phi = -\int_0^L \frac{q d\xi}{4\pi\rho} = -\frac{q}{4\pi}\int_\theta^\alpha \frac{R\csc^2\nu}{R/\sin\nu} d\nu = -\frac{q}{4\pi}\int_\theta^\alpha \frac{1}{\sin\nu} d\nu$$

$$\Rightarrow \quad \phi = -\frac{q}{4\pi}\ln \mathrm{tg}\frac{\nu}{2}\Big|_\theta^\alpha$$

$$\Rightarrow \quad \phi = -\frac{q}{4\pi}\ln\frac{\mathrm{tg}(\alpha/2)}{\mathrm{tg}(\theta/2)}$$



**流函数**　　均匀流和一个位于原点的点源叠加

$$\psi(r,\theta)=\frac{1}{2}Ur^2\sin^2\theta-\frac{Q}{4\pi}(1+\cos\theta)$$

⇒

$$r=\sqrt{\frac{2\psi}{U\sin^2\theta}+\frac{Q}{2\pi U}\frac{(1+\cos\theta)}{\sin^2\theta}}$$

由

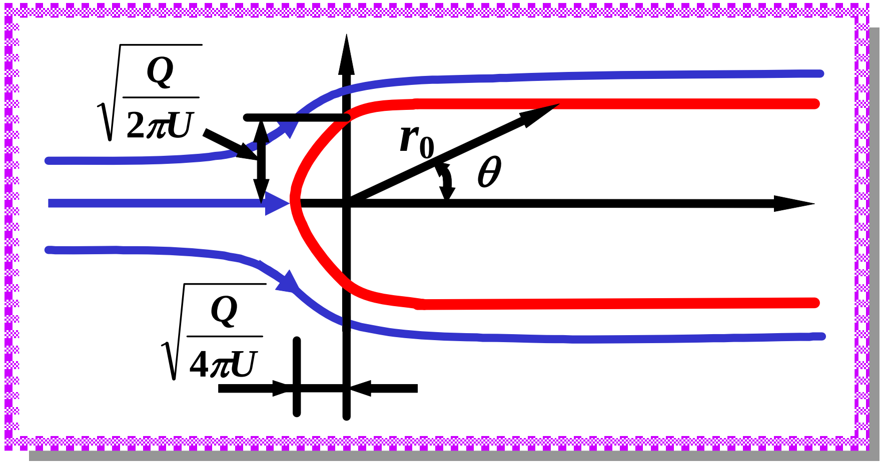
$$1+\cos\theta=2\cos^2\frac{\theta}{2}\ ,\quad \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

⇒

$$r=\sqrt{\frac{2\psi}{U\sin^2\theta}+\frac{Q}{4\pi U\sin^2\dfrac{\theta}{2}}}$$

# 半无穷体绕流2

令 $\psi = 0$ $r_0$ 为物面坐标




$$\Rightarrow \quad r_0 = \sqrt{\frac{Q}{4\pi U}} \frac{1}{\sin(\theta/2)}$$

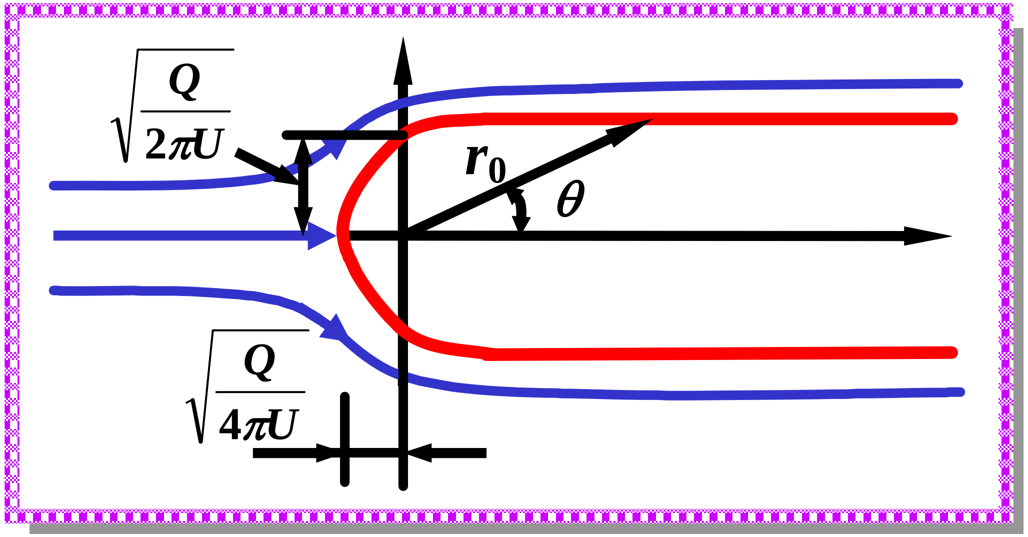
$\theta = 0 \Rightarrow$ $r_0 \to \infty$

$\theta = \frac{\pi}{2} \Rightarrow$ $r_0 = \sqrt{\frac{Q}{2\pi U}}$

$\theta = \pi \Rightarrow$ $r_0 = \sqrt{\frac{Q}{4\pi U}}$

令 $R_0 = r_0 \sin\theta = \sqrt{\frac{Q}{4\pi U}} \frac{\sin\theta}{\sin(\theta/2)} = \sqrt{\frac{Q}{\pi U}} \cos\frac{\theta}{2} \xRightarrow{\theta = 0}$ $R_0 = \sqrt{\frac{Q}{\pi U}}$

# 半无穷体绕流3




**半无穷体把流动分为两部分，均匀来流在体外，点源流动在体内，互不相混**

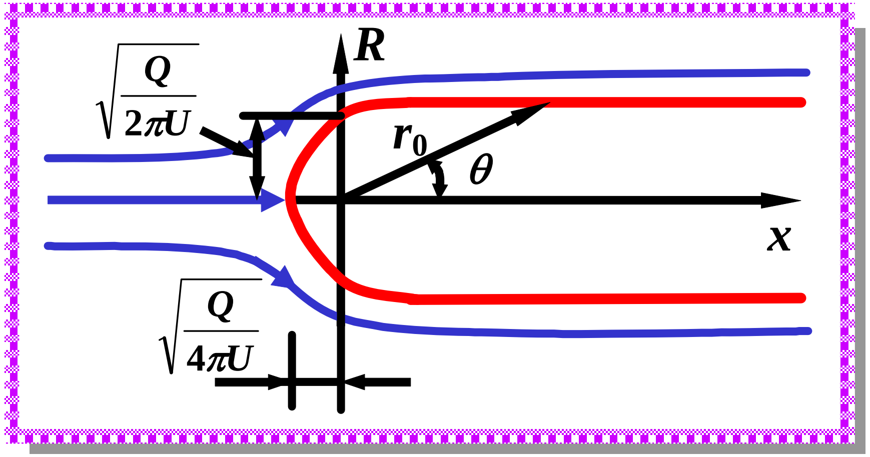
**势函数** ⟹

$$\phi = Ur\cos\theta - \frac{Q}{4\pi r}$$

# 半无穷体绕流例题1

**例：求半无穷体绕流物面上的速度和压强分布**

## (1) 速度分布




**均匀流与点源的叠加**

$$\phi = Ur\cos\theta - \frac{Q}{4\pi r}$$

**物面速度分布**

$$u_{r0} = \left.\frac{\partial\phi}{\partial r}\right|_{r=r_0} = U\cos\theta + \frac{Q}{4\pi r_0^2} \qquad u_{\theta 0} = \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=r_0} = -U\sin\theta$$

$\theta = \pi \quad r_0 = \sqrt{\frac{Q}{4\pi U}} \Rightarrow u_{r0} = u_{\theta 0} = 0$ **柱体头部为驻点**

# 半无穷体绕流例题2

## (2) 物面压强分布

$$u_{r0}=U\cos\theta+\frac{Q}{4\pi r_0^2}\ ,\ u_{\theta 0}=-U\sin\theta$$

由物面方程 $r_0=\sqrt{\frac{Q}{4\pi U}\frac{1}{\sin(\theta/2)}}$ ⟹ $\frac{Q}{4\pi r_0^2}=U\sin^2\frac{\theta}{2}$

对物面一点及无穷远点列伯努利方程

⟹ $$p_\infty+\frac{1}{2}\rho U^2=p+\frac{1}{2}\rho\left(u_{r0}^2+u_{\theta 0}^2\right)$$

$$u_{r0}^2+u_{\theta 0}^2=U^2\left[\sin^2\theta+\cos^2\theta+\sin^4\left(\frac{\theta}{2}\right)+2\cos\theta\sin^2\left(\frac{\theta}{2}\right)\right]$$

$$=U^2\left\{1+\sin^4\left(\frac{\theta}{2}\right)+2\left[1-2\sin^2\left(\frac{\theta}{2}\right)\right]\sin^2\left(\frac{\theta}{2}\right)\right\}$$

# 半无穷体绕流例题3

$$u_{r0}^2+u_{\theta 0}^2=U^2\left[1+2\sin^2\left(\frac{\theta}{2}\right)-3\sin^4\left(\frac{\theta}{2}\right)\right]$$

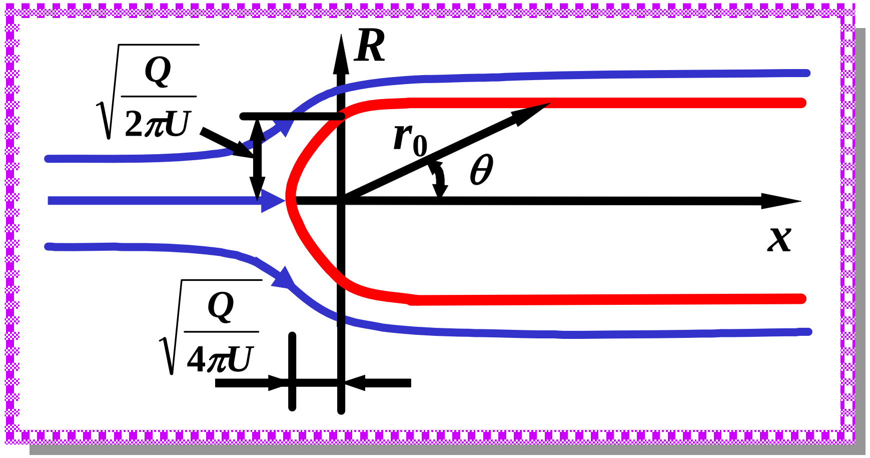
**定义无量纲压强**

$$\bar{p}=\frac{p-p_\infty}{\frac{1}{2}\rho U^2}=1-\frac{u_{r0}^2+u_{\theta 0}^2}{U^2}=3\sin^4\left(\frac{\theta}{2}\right)-2\sin^2\left(\frac{\theta}{2}\right)$$

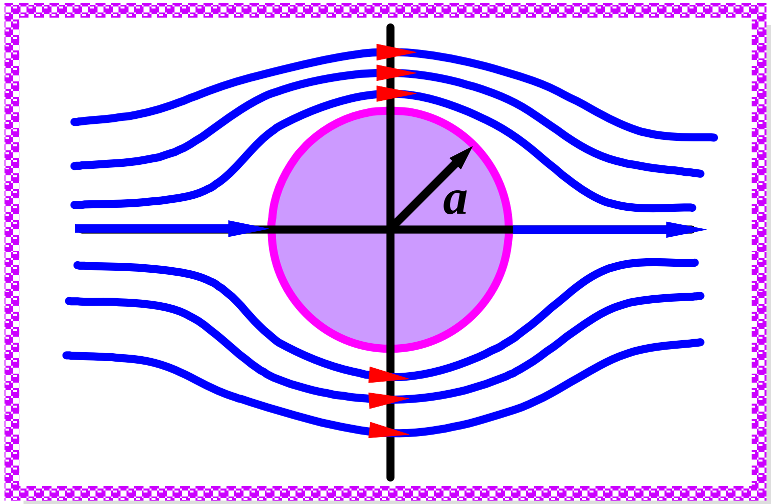
$\theta=\pi$ ⟹ $\bar{p}=1$ **驻点**

$x/\sqrt{Q/\pi U}>3$ ⟹ $\bar{p}\approx 0$

**压强恢复到来流静压**



## 流函数




### 均匀流和位于原点的偶极子

$$\psi = \frac{1}{2}Ur^2\sin^2\theta - \frac{\mu}{4\pi r}\sin^2\theta$$

令 $\psi = 0$ $r_0$ 为物面坐标 ⇒ $r_0 = \left(\frac{\mu}{2\pi U}\right)^{1/3} = a = \text{const}$

$\psi = 0$ 的流面是一个球面 $\mu = 2\pi U a^3$

⇒ $$\psi = \frac{1}{2}U\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta$$

# 圆球绕流2

## 势函数

$\mu = 2\pi U a^3$

$$\phi = Ur\cos\theta + \frac{\mu}{4\pi r^2}\cos\theta$$

⇒

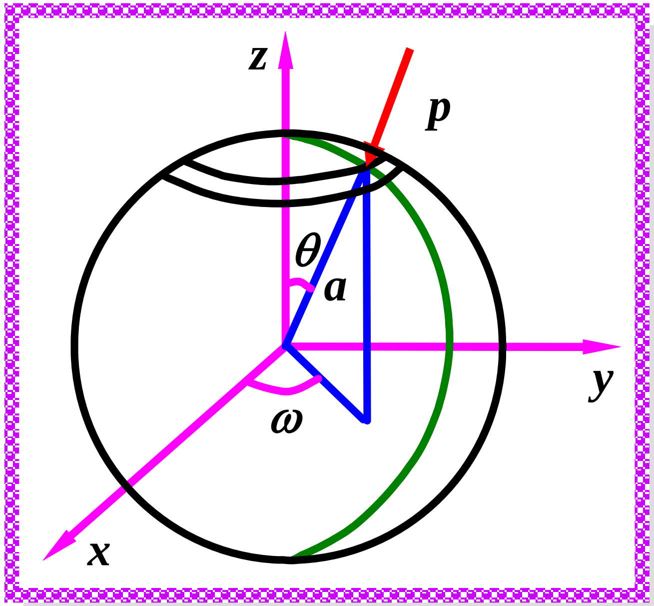
$$\phi = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta$$

# 圆球绕流例题1

例：利用流体绕流圆球的势函数 $\phi = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta$

求圆球表面的压强分布，并计算流体作用在圆球上的力

**速度场** $$\phi = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta$$




$$\Rightarrow \quad u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$u_\theta = \frac{\partial\phi}{r\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$$

# 圆球绕流例题2

$$u_r = U\left(1-\frac{a^3}{r^3}\right)\cos\theta$$

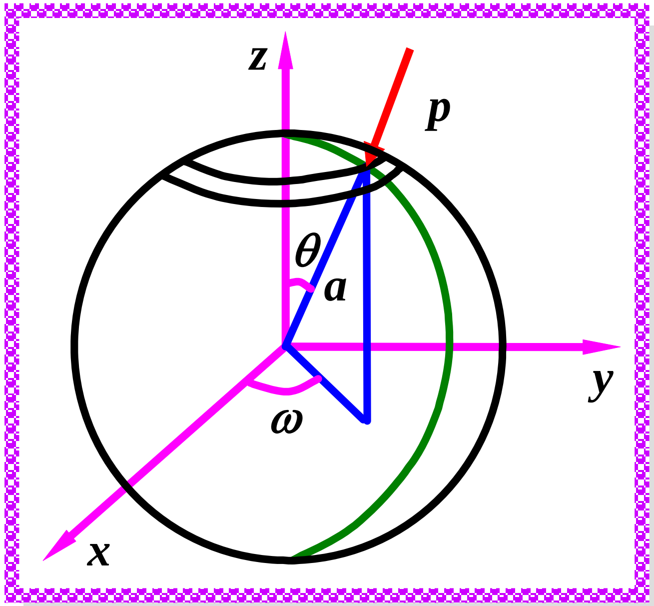
$$u_\theta = -U\left(1+\frac{a^3}{2r^3}\right)\sin\theta$$

**圆球表面** ⇒ $u_r = 0\ ,\ u_\theta = -\frac{3}{2}U\sin\theta$

**伯努利方程**

$$p_\infty + \frac{1}{2}\rho U^2 = p + \frac{1}{2}\rho\left(u_{r0}^2 + u_{\theta 0}^2\right) \Rightarrow p = p_\infty + \frac{1}{2}\rho U^2\left(1-\frac{9}{4}\sin^2\theta\right)$$

**流动方向流体作用于圆球的力**




$$\Rightarrow F_z = \int_A - p\cos\theta dA$$

$$= \int_A - p\cos\theta 2\pi a\sin\theta a d\theta$$

$$= 0$$

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# 兰金卵体绕流例题1

例：兰金卵球体绕流可通过均匀流和一对等强度的点源和点汇叠加得到。设均匀流速度为 $U$；点源和点汇的强度均为$Q$，分别位于原点两侧，距原点距离为 $l$。求物面方程，并求特征尺寸 $h$ 和 $L$ 的计算式

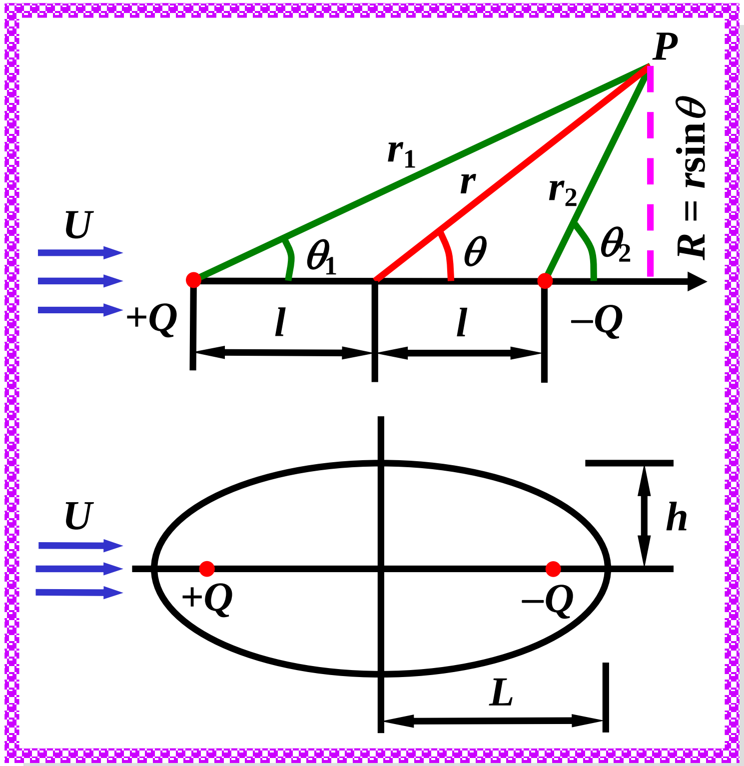
**(1) 流函数**

$$\psi=\frac{1}{2}Ur^2\sin^2\theta-\frac{Q}{4\pi}(\cos\theta_1-\cos\theta_2)$$

设$\psi=0$面上，$r=r_0$，$R_0=r_0\sin\theta$

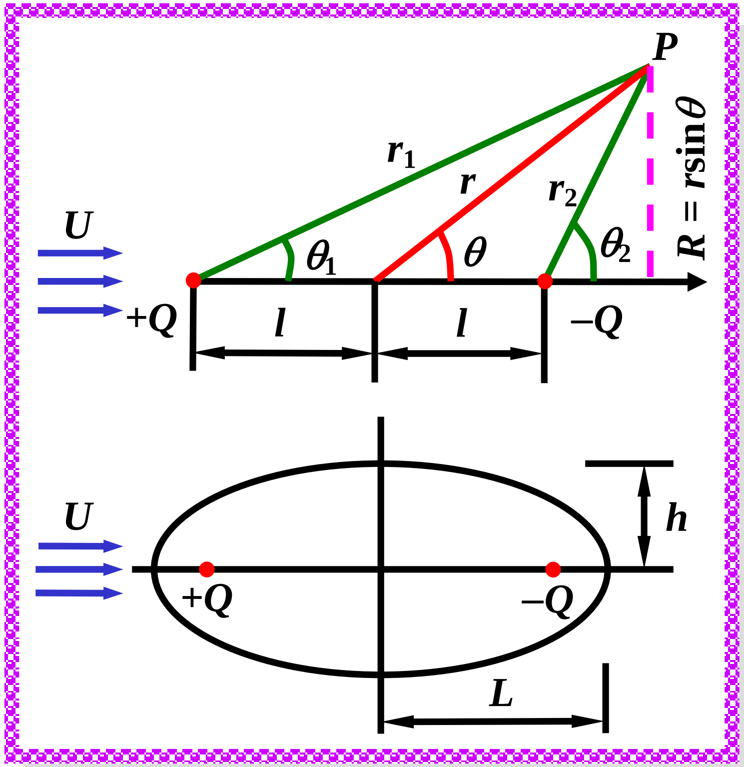
$$0=\frac{1}{2}UR_0^2-\frac{Q}{4\pi}(\cos\theta_1-\cos\theta_2)$$

$$\Rightarrow\quad R_0^2=\frac{Q}{2\pi U}(\cos\theta_1-\cos\theta_2)$$



# 兰金卵体绕流例题2

$$R_0^2 = \frac{Q}{2\pi U}\left(\cos\theta_1 - \cos\theta_2\right)$$




$\theta_1 = \theta_2 = 0$ 及 $\theta_1 = \theta_2 = \pi$ 时

⇒ $R_0 = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ⇒ $R_0$ 取最大值

(2) 求对称卵球体的 $L$ 和 $h$

卵球体后驻点速度为零，后驻点的速度由均匀流、点源点汇在该点的速度叠加获得

$$\Rightarrow \quad U + \frac{Q}{4\pi(L+l)^2} - \frac{Q}{4\pi(L-l)^2} = 0$$

# 兰金卵体绕流例题3

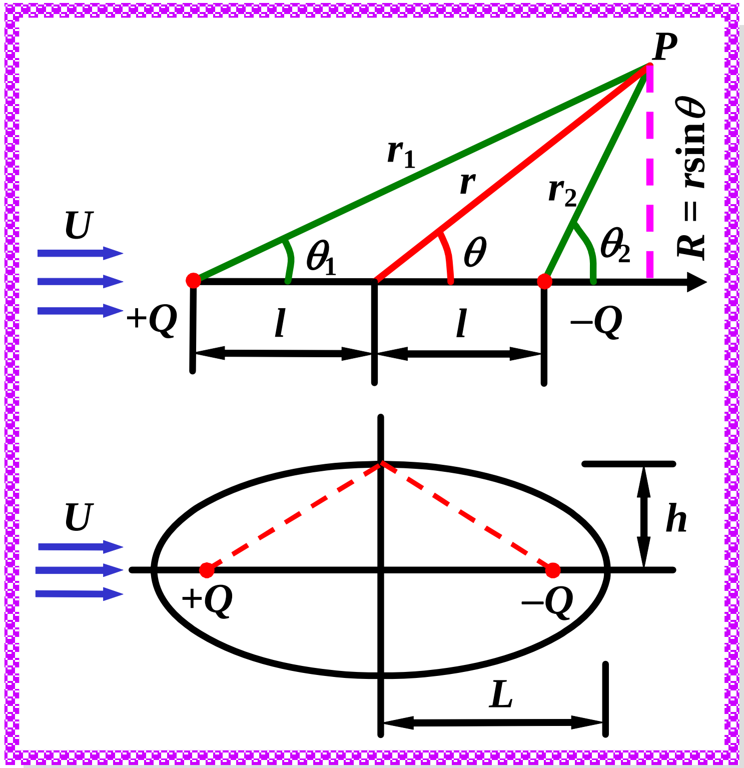
$$\left(L^2 - l^2\right)^2 - \frac{Ql}{\pi U}L = 0$$

**求解上式可得到 $L$**

**卵球体体表面** $\theta = \frac{\pi}{2}$

$$R_0 = h \Longrightarrow h^2 = \frac{Q}{2\pi U}\left(\cos\theta_1 - \cos\theta_2\right)$$

$$\cos\theta_1 = \frac{l}{\sqrt{l^2 + h^2}}, \quad \cos\theta_2 = -\frac{l}{\sqrt{l^2 + h^2}}$$

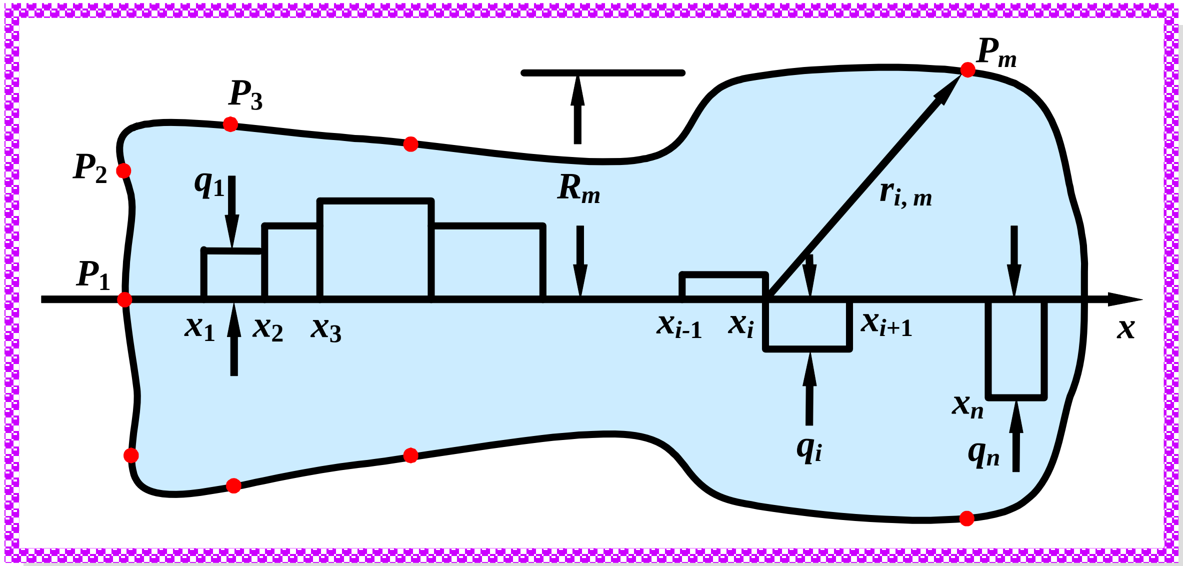



$$\Longrightarrow \quad h^2 = \frac{Q}{2\pi U}\frac{2l}{\sqrt{l^2 + h^2}} \quad \Longrightarrow \quad h^2\sqrt{l^2 + h^2} - \frac{Ql}{\pi U} = 0$$

**求解上式可得到 $h$**

## 正问题

给定旋转体物面形状，求叠加后可以得到旋转体无攻角绕流流场的奇点分布，该流场满足相应的方程和边界条件

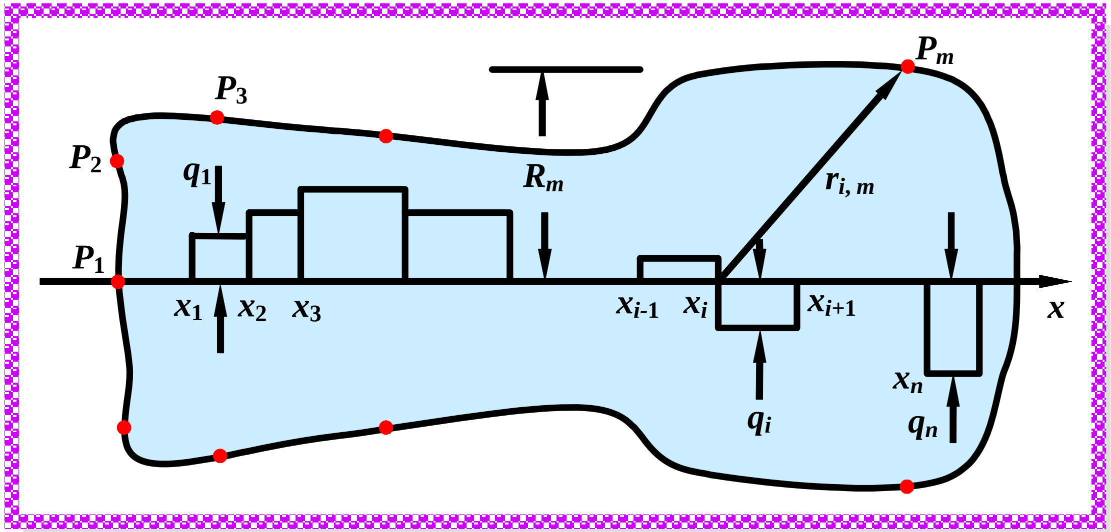



**旋转体方程**

$$r = r(\theta)$$

# 旋转体无攻角绕流2

均匀流与在 $x$ 轴上区间 $a \le x \le b$ 连续分布的线源（汇）相叠加




$$\psi = \frac{1}{2}Ur^2\sin^2\theta - \int_0^L \frac{q d\xi}{4\pi}(1+\cos\nu)$$

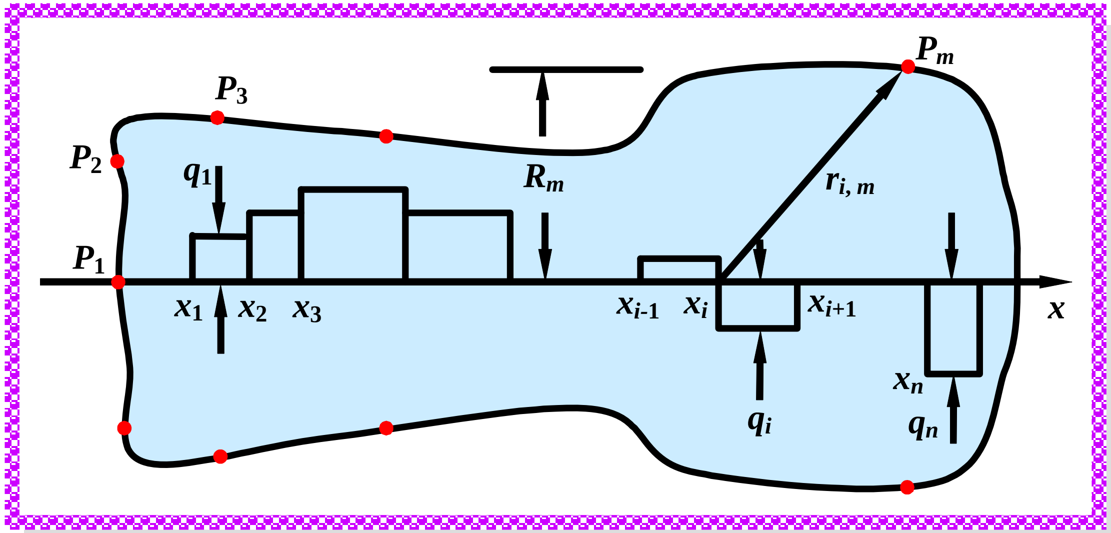
求恰当的线源（汇）分布使得物面上流函数为零

$$\Rightarrow \quad \frac{1}{2}UR - \int_a^b \frac{q d\xi}{4\pi}(1+\cos\nu) = 0$$

很难采用解析方法求解上式，需借助于数值方法

离散线源（汇）为 $n$ 小段，而在每一小段上 $q_i$ 取为常数，对于

$$x_i \le x \le x_{i+1} \quad i = 1,2,\dots n$$




$$-\int_{x_i}^{x_{i+1}} \frac{q d\xi}{4\pi}(1+\cos\nu) = -\frac{q_i}{4\pi}\left[(x_{i+1}-x_i)-(r_{i+1,m}-r_{i,m})\right]$$

$$\Rightarrow \quad \frac{1}{2}UR_m - \sum_{i=1}^{N}\frac{q_i}{4\pi}\left[(x_{i+1}-x_i)-(r_{i+1,m}-r_{i,m})\right] = 0$$

在物面上选 $m$ >= $2N$ 个点，对每一点写上述方程，如此可以获得 $m$ 个方程，联立求解这个方程可得出 $x_i$ 和 $q_i$

# 旋转体无攻角绕流4

$$\frac{1}{2}UR_m - \sum_{i=1}^{N}\frac{q_i}{4\pi}\left[\left(x_{i+1} - x_i\right) - \left(r_{i+1,m} - r_{i,m}\right)\right] = 0$$

也可以先选定 $x_i$，这样就只需在物面上选 $m >= N$ 个点，得到关于 $q_i$ 的 $n$ 个方程，求解这个方程可得到 $q_i$

若物面线封闭，则流出旋转体物面的净流量应为零，物体内部的源汇之和应等于零

$$\sum_{i=1}^{N} q_i\left(x_{i+1} - x_i\right) = 0$$

⇒

$$\frac{1}{2}UR_m - \sum_{i=1}^{N}\frac{q_i}{4\pi}\left(r_{i+1,m} - r_{i,m}\right) = 0$$

# 作业

**作业：P.168～169**

- 5.1
- 5.6
- 5.7
- 5.9