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Thermal statistical ensembles of classical extreme black holes

C. Chevalier *, F. Debbasch

Université Pierre et Marie Curie-Paris6, UMR 8112, ERGA-LERMA, 3 rue Galilée 94200 Ivry, France

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ABSTRACT

New statistical ensembles of classical extreme black holes are introduced. Each ensemble is proven to represent a non extreme, finite temperature black hole. This mean or average black hole is surrounded by a mean electromagnetic field and a so-called apparent matter, which is the large scale trace of the averaged upon small scale fluctuations of the gravitational and electromagnetic fields. However, the total mass and the total charge of space–time is not modified by the averaging.

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1. Introduction

Black holes were first discovered as exact solutions of Einstein equations [11,6]; loosely speaking, they represent situations in which the purely classical, non quantum gravitational field becomes so intense that a no-escape region is created in space–time, the boundary between this region and the 'outside' being called the event horizon of the black hole. It was later realized that stationary black holes display remarkable thermodynamical properties [27,28,12]; the event horizon of a stationary black hole has an entropy proportional to its area and the temperature of the black hole coincides, up to proportionality constant, with another, apparently purely geometrical quantity called the surface gravity of the horizon; the temperature of an event horizon is always non-negative and black holes for which this temperature vanishes are called extreme.

These thermodynamical properties have been understood by various authors by considering classical black holes as statistical ensembles of degrees of freedom associated with quantum gravity and/or string theory [13,14,16]. The present article develops an alternative point of view; we start from classical extreme black holes, which have vanishing temperature, introduce new statistical ensembles of these extreme black holes, and show that each ensemble actually represents a classical black hole of finite, non-vanishing temperature. The mean or averaged space–time is further characterized by its mass and charge distributions. It is found that the averaging endows the space–time surrounding the black hole with an apparent charged matter with anisotropic pressure tensor. This matter traces the net non linear effect of the averaged out 'small scale' degrees of freedom on the 'large scale' gravitational field. All results are put into perspective in a final summary and discussion section.

2. Statistical ensembles of extreme Reissner–Nordström black holes

2.1. The extreme Reissner–Nordström metric

The Extreme Reissner–Nordström (ERN) metric is a solution of the Einstein–Maxwell equations [11,27] describing a charged spherically symmetric static black hole of total charge equal to its total mass M.[1] We use the so-called Kerr–Schild

* Corresponding author.
E-mail address: chevalier_claire@yahoo.fr (C. Chevalier).

[1] All computations are presented in Planck units, where $c = G = \hbar = (4\pi\varepsilon_0)^{-1} = 1$.

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coordinates $(t, \boldsymbol{r}) \in \mathbb{R}^4$, in which the line element takes the simple form [6]:

$$ds^2 = dt^2 - d\boldsymbol{r}^2 - (1 - f(r))\left(dt - \frac{\boldsymbol{r}\cdot d\boldsymbol{r}}{r}\right)^2, \tag{1}$$

where

$$f(r) = \left(1 - \frac{M}{r}\right)^2. \tag{2}$$

In (1) and (2), $\boldsymbol{r}\cdot d\boldsymbol{r}$ designates the Euclidean scalar product and r is the Euclidean norm of $\boldsymbol{r}$. The electromagnetic field of the ERN space–time is characterized by the electromagnetic 4-potential A. The covariant components of A can be expressed in the Kerr–Schild coordinate system:

$$A_t(\boldsymbol{r}) = -\frac{M}{r} \qquad \boldsymbol{A}(\boldsymbol{r}) = \frac{M}{r^2}\, h(r)\, \boldsymbol{r} \tag{3}$$

where

$$h(r) = 1 - (f(r))^{-1}. \tag{4}$$

2.2. Definition of the statistical ensemble

Let $\boldsymbol{\omega}$ represent 3 real scalar random variables. The ensemble Ω in which the values of $\boldsymbol{\omega}$ are taken is the Euclidean 3-ball of radius a:

$$\Omega = \{\boldsymbol{\omega} \in \mathbb{R}^3; \boldsymbol{\omega}^2 \leqslant a^2\}, \tag{5}$$

endowed with the uniform probability measure dp_ω; the density p of this measure with respect to the Lebesgue measure $d^3\omega$ is $p(\boldsymbol{\omega}) = V_a^{-1}$ where $V_a = 4\pi a^3/3$.

We now fix the base manifold $\mathbb{R}^4$ and consider, for each $\boldsymbol{\omega} \in \Omega$ and $r > a$, the metric $g(\boldsymbol{\omega})$ defined by the line-element:

$$ds^2_\omega = dt^2 - d\boldsymbol{r}^2 - (1 - f(R))\left(dt - \frac{\boldsymbol{R}\cdot d\boldsymbol{r}}{R}\right)^2, \tag{6}$$

with

$$\boldsymbol{R}(\boldsymbol{r}, \boldsymbol{\omega}) = \boldsymbol{r} - i\boldsymbol{\omega}, \tag{7}$$

and

$$R(\boldsymbol{r}, \boldsymbol{\omega}) = \left(r^2 - \omega^2 - 2\, i\, \boldsymbol{r}.\boldsymbol{\omega}\right)^{1/2}, \tag{8}$$

where the principal value of $(\cdot)^{1/2}$ is retained in (8).

The components of the associated four potential $A(\boldsymbol{\omega})$ read:

$$A_t(\boldsymbol{r}, \boldsymbol{\omega}) = -\frac{M}{R} \qquad \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{\omega}) = \frac{M}{R^2}\, h(R)\, \boldsymbol{R}. \tag{9}$$

The ensemble of metrics $\{g(\boldsymbol{\omega}), \boldsymbol{\omega} \in \Omega\}$ and the probability measure dp_ω define together [8,7] a mean metric $\bar{g} = \int_\Omega dp_\omega g(\boldsymbol{\omega})$ on $\mathcal{B} = \left\{(t, \boldsymbol{r}) \in \mathbb{R}^4, r > a\right\}$; a mean four potential A is defined similarly by $\bar{A}_\mu = \int_\Omega dp_\omega A_\mu(\boldsymbol{\omega})$. From now on, averages over Ω with probability measure dp_ω will be denoted by angular brackets; one thus has, for example, $\bar{g} = \langle g(\boldsymbol{\omega})\rangle$.

2.3. Motivation

Let us comment rapidly on this choice of ensembles. The basic idea underlying the present work is that at least some thermodynamical properties of classical finite temperature real black holes can be understood by viewing these black holes as statistical ensembles of other space–times with vanishing temperature. Extreme black holes are natural candidates for these other, vanishing temperature space–times.

The retained choice of *complex* extreme black holes allows for the following interpretation. Space–time degrees of freedom are observed as real quantities but are in fact complex (see the Section 7.2 for a discussion of and references on complex space–times). Modeling a real black hole space–time thus involves a trace over the imaginary part of the space–time degrees of freedom and this trace (or statistical averaging) can confer a non-vanishing temperature to the real space–time.

3. Determination of the mean metric and the mean electromagnetic 4-potential

3.1. Mean metric

It is convenient to consider the dimensionless quantities $s = r/M$ and $x = a/M$. The exact expression of $\bar{g}$ in Kerr–Schild coordinates reads, for every $a < r$ ($x < s$):

$$\left\langle ds^2 \right\rangle = b_1 dt^2 + b_2 d\boldsymbol{r}^2 + b_3 \left(\frac{\boldsymbol{r} \cdot d\boldsymbol{r}}{r} \right)^2 + b_4 \frac{\boldsymbol{r} \cdot d\boldsymbol{r}}{r} dt, \tag{10}$$

with

$$b_1(s,x) = 1 - \frac{2}{s} + \frac{3}{2x^2} \left(-1 + \left(\frac{s}{x} + \frac{x}{s} \right) \arctan\left(\frac{x}{s} \right) \right) \tag{11}$$

$$b_2(s,x) = -1 - \frac{3}{16s^2} + \frac{2x^2}{5s^3} - \frac{9}{16x^2} - \frac{3s}{16x^3} \left(\frac{x^2}{s^2} + 1 \right) \left(\frac{x^2}{s^2} - 3 \right) \arctan\left(\frac{x}{s} \right) \tag{12}$$

$$b_3(s,x) = -\frac{2}{s} + \frac{9}{16s^2} - \frac{6x^2}{5s^3} + \frac{3}{16x^2} - \frac{3s}{16x^3} \left(1 - \frac{2x^2}{s^2} - \frac{3x^4}{s^4} \right) \arctan\left(\frac{x}{s} \right) \tag{13}$$

and

$$b_4(s,x) = \frac{3}{2s} - \frac{2}{s^2} - \frac{3s}{2x^2} - \frac{9s^2}{2x^3} \left(1 + \frac{2x^2}{3s^2} - \frac{4x^3}{3} \left(1 + \frac{1}{s^2} \right) - \frac{x^4}{3s^4} \right) \arctan\left(\frac{x}{s} \right). \tag{14}$$

Note that all above functions of s and x are real.

Let us now define Schwarzschild coordinates for the mean metric. We first use angular variables θ and ϕ and put $d\boldsymbol{r}^2$ under the form:

$$d\boldsymbol{r}^2 = dr^2 + r^2 d\Gamma^2 \tag{15}$$

with

$$d\Gamma^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{16}$$

We then use the fact that b_2 (see Eq. (12)) is negative when $r > a$ ($s > x$) and define the new spatial coordinate ρ by:

$$\rho(s,x) = M\sqrt{-b_2(s,x)}\, s. \tag{17}$$

For $x = 0$, $\rho = Ms$ is a strictly increasing function of s and is invertible. Things are more complicated for $x > 0$. Indeed, a minimum of $\rho(s,x)$ appear for $s = s_0(x)$ as soon as $x \neq 0$; the function ρ decreases in s on $(0, s_0(x))$ and increases in s on $(s_0(x), +\infty)$; note that, at fixed x, ρ tends to infinity with s. We thus first restrict the domain of s to the interval $(s_0(x), +\infty)$, over which the function $\rho(\cdot, x)$ is invertible. We then take into account the fact that all preceding results have only been derived under the explicit assumption $a < r$, *i.e.*, $x < s$. Expression (17) shows that $s_0(x)$ behaves as $(3/10)^{1/4} x^{1/2}$ as x approaches zero. A numerical analysis shows that $s_0(x)$ is superior to x as long as x is inferior to a certain critical value $x_c = 0.279$, and that the function ρ does not admit any minimum greater than x when x is superior to x_c. We therefore introduce a new quantity $s_1(x)$ equal to $s_0(x)$ for $x < x_c$ and equal to x otherwise, and we restrict s to the interval $]s_1(x), +\infty)$. On this interval, all results derived above apply (because $s > s_1(x)$ implies $r > a$) and the function ρ is invertible in s (at fixed x); the standard notation ρ^{-1} will be used to designate the inverse of ρ on $]s_1(x), +\infty)$. The mean metric can be rewritten under the form:

$$\left\langle ds^2 \right\rangle = c_1 dt^2 + c_2 d\rho^2 + c_3 dt\, d\rho - \rho^2 d\Gamma^2, \tag{18}$$

with

$$c_1(s,x) = b_1(s,x), \qquad c_2(s,x) = M^2 \left(b_2(s,x) + b_3(s,x) \right) \left(\frac{\partial \rho}{\partial s} \right)^{-2},$$

$$c_3(s,x) = M b_4(s,x) \left(\frac{\partial \rho}{\partial s} \right)^{-1}. \tag{19}$$

We introduce a new time coordinate τ, related to t and ρ by a relation of the form:

$$dt = d\tau - \alpha(\rho) d\rho, \tag{20}$$

where α is an as yet unspecified function. Expressing the mean metric in terms of the coordinates $(\tau, \rho, \theta, \phi)$ yields:

$$\left\langle ds^2 \right\rangle = c_1 d\tau^2 + \left(c_1 \alpha^2 - c_3 \alpha + c_2 \right) d\rho^2 + (c_3 - 2\alpha c_1)\, d\tau\, d\rho - \rho^2 d\Gamma^2. \tag{21}$$

The function α is chosen to enforce $g_{\tau\rho} = 0$; one thus chooses $\alpha(\rho(s, x)) = \frac{c_3(s,x)}{2\,c_1(s,x)}$, for all $x \geq 0$ and $s > s_1(x)$. The mean metric then reads:

$$\langle ds^2 \rangle = F(s, x)d\tau^2 - G(s, x)d\rho^2 - \rho^2 d\Gamma^2, \tag{22}$$

with

$$F(s, x) = c_1(s, x) = 1 - \frac{2}{s} + \frac{3}{2x^2}\left(\left(\frac{s}{x} + \frac{x}{s}\right)\arctan\left(\frac{x}{s}\right) - 1\right) \tag{23}$$

and

$$G(s, x) = \frac{c_3^2(s, x) - 4\,c_1(s, x)\,c_2(s, x)}{4\,c_1(s, x)}. \tag{24}$$

The coordinates $(\tau, \rho, \theta, \phi)$ make evident the static and spherically symmetric character of the mean space–time (Eq. (22)). They are therefore called Schwarzschild coordinates for the mean space–time, by analogy with the unaveraged ERN case.

We conclude this section by giving approximate expressions for $\bar{g}$, ρ, F and G, valid when $a/r = x/s \ll 1$. A direct expansion of (10), along with (11), (12), (13) and (14), in powers of x/s delivers

$$\begin{aligned}\langle ds^2 \rangle = &\left(1 - \frac{2}{s} + \frac{1}{s^2}\left(1 - \frac{x^2}{5s^2}\right)\right)dt^2 + \left(-1 + \frac{2\,x^2}{5s^3} - \frac{x^2}{5s^4}\right)d\boldsymbol{r}^2 \\ &+ \left(-\frac{2}{s} + \frac{1}{s^2}\left(1 + \frac{2\,x^2}{5s^2}\right) - \frac{6\,x^2}{5s^3}\right)\left(\frac{\boldsymbol{r}\cdot d\boldsymbol{r}}{r}\right)^2 + 2\left(\frac{2}{s} - \frac{1}{s^2} + \frac{2\,x^2}{5s^3}\right)\frac{\boldsymbol{r}\cdot d\boldsymbol{r}}{r}dt,\end{aligned} \tag{25}$$

which, of course, reduces to the original ERN metric (1) when $x = a/M$ vanishes. Expression (17) can also expanded (at fixed s) at second order in x/s and one obtains:

$$\rho(s, x) \approx M\left(1 - \frac{x^2}{5s^3} + \frac{x^2}{10\,s^4}\right)s. \tag{26}$$

One finally finds:

$$F(s, x) \approx F_{\text{approx}}(s, x) \equiv 1 - \frac{2}{s} + \frac{1}{s^2}\left(1 - \frac{x^2}{5s^2}\right) \tag{27}$$

and

$$G(s, x) \approx \frac{1}{F_{\text{approx}}(s, x)}\left(1 + \frac{x^2}{5s^4}\right). \tag{28}$$

4. The mean space–time is a black hole of finite temperature

4.1. Horizon of the mean space–time

Expressions (11), (12), (13) and (14) for the Kerr–Schild components of $\bar{g}$ are singular at points $\boldsymbol{r} = 0$. However, these expressions were derived under the assumption $r > a$, and one thus cannot conclude that the line $\boldsymbol{r} = 0$ corresponds to singularities, apparent or not. The behavior of the mean geometry near $\boldsymbol{r} = 0$ will not be further discussed here.

The mean metric describes a black hole if two conditions are fulfilled. The first one is that F admits real positive zeros which are are also singularities of G [28,26]. The second one is that the greatest real positive zero of F which is also a singularity of G, say $s_H(x)$, be superior to the limit value $s_1(x)$ (see the discussion following Eq. (17)). The unaveraged case corresponds to $x = 0$; the radius $s_H(0)$ exists, and equals unity, which is of course larger than $x = 0$. A numerical investigation reveals that $s_H(x)$ still exists as x increases, but gets closer to $s_1(x)$, the limit condition $s_H(x) = s_1(x)$ being realized for $x = 1.379 = x_{\text{max}}$. Thus, the mean metric $\bar{g}$ describes a black hole of horizon radius $\rho_H(x) = \rho(s_H(x), x)$ for all $x < x_{\text{max}}$. We note for further use that $s_H(x)$ is a simple zero of both F and G for all $x < x_{\text{max}}$ (data not shown).

Explicit expressions of $s_H(x)$ and $\rho_H(x)$ can be obtained for $x \ll 1$. These are, at second order in x:

$$s_H(x) = 1 + \frac{x}{\sqrt{5}} - \frac{x^2}{5}, \tag{29}$$

and

$$\rho_H(x) = M\left(1 + \frac{x}{\sqrt{5}} - \frac{3\,x^2}{10}\right). \tag{30}$$

The x-dependance of the horizon radius $\rho_H(x)$ is displayed in Fig. 1.

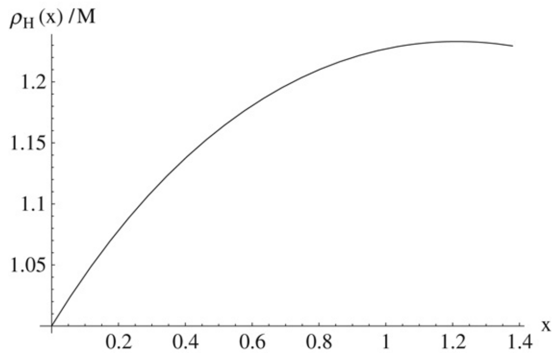

Fig. 1. Evolution of the dimensionless horizon radius $\rho_H(x, M)/M$ versus x.

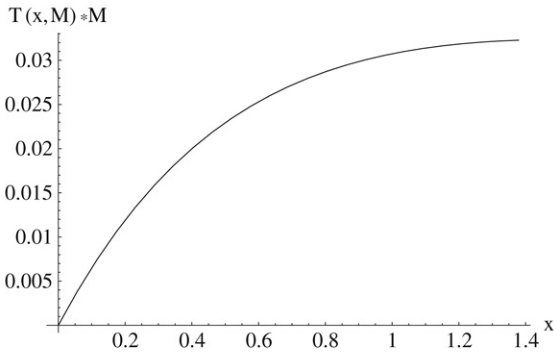

Fig. 2. Evolution of the dimensionless temperature $T(x, M) \times M$ versus x.

4.2. Temperature of the black hole

The temperature of the mean black hole is best obtained by expanding in ρ the Schwarzschild coordinates of the metric $\bar{g}$ near the horizon. Expression (22) leads to:

$$\left\langle \mathrm{d}s^2 \right\rangle \approx \left(\left. \frac{\partial \rho}{\partial s} \right|_{s_H(x),x} \right)^{-1} (\rho - \rho_H(x))\, F'(s_H(x), x)\mathrm{d}\tau^2 - \left. \frac{\partial \rho}{\partial s} \right|_{s_H(x),x} \frac{1}{(\rho - \rho_H(x))\, P'(s_H(x), x)} \mathrm{d}\rho^2 - \rho^2 \mathrm{d}\Gamma^2, \tag{31}$$

where a prime denotes a partial derivative with respect to s and $P(s, x) = 1/G(s, x)$. Changing radial variable to

$$R(\rho, x) = \left. \frac{\partial \rho}{\partial s} \right|_{s_H(x),x} \frac{1}{(\rho - \rho_H(x))\, P'(s_H(x), x)} \ln|\rho - \rho_H(x)| \tag{32}$$

shows that the space–time is naturally periodic in imaginary time, with period

$$\beta(x, M) = \frac{4\pi}{\sqrt{F'(s_H, x)P'(s_H, x)}} \left. \frac{\partial \rho}{\partial s} \right|_{s_H(x),x}. \tag{33}$$

This periodicity is characteristic of a thermal density matrix of temperature T [27,28,12] given by:

$$T(x, M) = \frac{1}{\beta(x, M)} = \frac{1}{4\pi} \sqrt{F'(s_H, x)P'(s_H, x)} \left(\left. \frac{\partial \rho}{\partial s} \right|_{s_H(x),x} \right)^{-1}. \tag{34}$$

The x-dependance of the temperature $T(x, M)$ is displayed in Fig. 2. The second order expression of $T(x, M)$ for $x \ll 1$ reads:

$$T(x, M) \approx \frac{x}{2\sqrt{5}\pi\, M} - \frac{x^2}{5\,\pi\, M}. \tag{35}$$

4.3. Generic case

Let us now prove that the preceding result is generic and that averaging in complex Kerr–Schild 3D space a generic extreme spherical black hole leads to a black hole of finite temperature.

The normal form of the near-horizon metric of a generic extreme spherical black hole [25,26] reads, in Schwarzschild coordinates $(\chi, \xi, \theta, \phi)$:

$$\mathrm{d}s^2 = f(r)\mathrm{d}\chi^2 - \frac{1}{f(r)}\mathrm{d}\xi^2 - r_H^2 \mathrm{d}\Gamma^2, \tag{36}$$

with

$$f(r) = \left(1 - \frac{r}{r_H}\right)^2, \tag{37}$$

where r_H is the horizon radius of the black hole. In Kerr–Schild coordinates, the same line element takes the form:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}\mathbf{r}^2 - (1 - f(r))\left(\mathrm{d}t - \frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}}{r}\right)^2. \tag{38}$$

Let us proceed as above and introduce the ensemble of metrics:

$$\mathrm{d}s_{\omega}^2 = \mathrm{d}t^2 - \mathrm{d}\mathbf{r}^2 - (1 - f(R))\left(\mathrm{d}t - \frac{\boldsymbol{R}\cdot\mathrm{d}\boldsymbol{r}}{R}\right)^2, \tag{39}$$

where $R(\mathbf{r}, \boldsymbol{\omega}) = \mathbf{r} - \mathrm{i}\boldsymbol{\omega}$ and $\boldsymbol{\omega} \in \mathcal{B}_a$. Taking a uniform probability law for $\boldsymbol{\omega}$, the average metric reads, for $a \ll r_H$[2]:

$$\left\langle \mathrm{d}s^2 \right\rangle = b_1 \mathrm{d}t^2 + b_2 \mathrm{d}\boldsymbol{r}^2 + b_3 \left(\frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}}{r}\right)^2 + b_4 \frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}}{r}\mathrm{d}t, \tag{40}$$

with

$$b_1 = \varepsilon^2 - \frac{1}{5}x^2\left(1 + 2\varepsilon - 2\varepsilon^2 + 2\varepsilon^3\right) + O\left(x^4, \varepsilon^4\right), \tag{41}$$

$$b_2 = -1 + \frac{1}{5}x^2\left(1 - 2\varepsilon + 2\varepsilon^2 - 2\varepsilon^3\right) + O\left(x^4, \varepsilon^4\right), \tag{42}$$

$$b_3 = -1 + \varepsilon^2 + \frac{4}{5}x^2\left(-1 + \varepsilon - \varepsilon^2 + \varepsilon^3\right) + O\left(x^4, \varepsilon^4\right), \tag{43}$$

$$b_4 = 2 - 2\varepsilon^2 + \frac{4}{5}x^2 + O\left(x^4, \varepsilon^4\right), \tag{44}$$

where $\varepsilon = s - 1$, $s = r/r_H$ and $x = a/r_H$. The radial Schwarzschild coordinate ρ of the mean metric near the horizon reads, in terms of ε and x:

$$\rho(\varepsilon, x) = r_H(1 + \varepsilon) + \frac{r_H}{10}x^2(-1 + \varepsilon) + O\left(x^4, \varepsilon^4\right). \tag{45}$$

The Schwarzschild components of the mean metric near the horizon are determined by

$$F(\varepsilon, x) = \varepsilon^2 + \frac{1}{5}x^2\left(-1 - 2\varepsilon + 2\varepsilon^2\right) + O\left(x^4, \varepsilon^3\right) \tag{46}$$

$$G(\varepsilon, x) = \frac{1}{F(\varepsilon, x)}\left(4 + \frac{4}{5}x^2\left(1 - 4\varepsilon + 4\varepsilon^2\right) + O\left(x^4, \varepsilon^3\right)\right). \tag{47}$$

The horizon radius of the averaged metric is given by

$$s_H(x) = 1 + \frac{1}{\sqrt{5}}x + \frac{1}{5}x^2 + O\left(x^3\right), \tag{48}$$

or, equivalently,

$$\rho_H(x) = r_H\left(1 + \frac{1}{\sqrt{5}}x + \frac{1}{10}x^2 + O\left(x^3\right)\right). \tag{49}$$

[2] Expression is only valid near the horizon of the extreme black hole and the condition $a \ll r_H$ ensures that $(t, \mathbf{R})$ remains close to this horizon for all $\boldsymbol{\omega} \in \mathcal{B}_a$.

One then finds that the averaged black hole has a finite temperature given by:

$$T(x) = \frac{x}{2\sqrt{5}\pi\, r_H} + O\left(x^2\right). \tag{50}$$

The scale for T is provided by r_H and the dimensionless temperature $r_H T$ measures (in unit r_H) the imaginary extension of the **R** degrees of freedom. Relation (35) now appears, at least at first order, as a particular case of (50), corresponding to $r_H = M$.

5. Energy and charge repartitions around the mean black hole

5.1. Mean stress-energy tensor

The mean metric $\bar{g}$ defines a mean stress-energy tensor $\bar{\mathcal{T}}$ through Einstein equations. The Schwarzschild components of this tensor can be evaluated exactly for every $x < x_{\max}$ and every $s > s_1(x)$; the obtained expressions are extremely complicated and do not warrant reproduction in this article; approximate expressions valid at second order in x/s (at fixed s) read:

$$8\pi\, \bar{\mathcal{T}}_\tau^{\ \tau}(s,x) = \frac{1}{M^2 s^4}\left(1 - \frac{2\,x^2}{5s^4}\right); \tag{51}$$

$$8\pi\, \bar{\mathcal{T}}_\rho^{\ \rho}(s,x) = \frac{1}{M^2 s^4}\left(1 + \frac{4\,x^2}{5s^2} - \frac{8\,x^2}{5s^3} + \frac{2\,x^2}{5s^4}\right); \tag{52}$$

$$8\pi\, \bar{\mathcal{T}}_\theta^{\ \theta}(s,x) = -\frac{1}{M^2 s^4}\left(1 + \frac{8\,x^2}{5s^2} - \frac{28\,x^2}{5s^3} + \frac{12\,x^2}{5s^4}\right); \tag{53}$$

$$8\pi\, \bar{\mathcal{T}}_\phi^{\ \phi}(s,x) = 8\pi\, \bar{\mathcal{T}}_\theta^{\ \theta}(s,x). \tag{54}$$

The component $\bar{\mathcal{T}}_\tau^{\ \tau}$ is the matter energy density $\bar{\varepsilon}$ outside the horizon [18]; it is measured with respect to the volume element $\sqrt{-\det \bar{g}_{\mu\nu}}\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi$, where $\left(\det \bar{g}_{\mu\nu}\right)$ stands for the determinant of the Schwarzschild components of $\bar{g}$. The opposites of $\bar{\mathcal{T}}_\rho^{\ \rho}$, $\bar{\mathcal{T}}_\theta^{\ \theta}$ and $\bar{\mathcal{T}}_\phi^{\ \phi}$ are interpreted as three different pressures p_ρ, p_θ, p_ϕ, oriented respectively in the ρ-direction, θ-direction and ϕ-direction.

5.2. Stress-energy tensor of the mean electromagnetic field

It has been shown in Ref. [8] that the covariant components of the mean 4-potential $\bar{A}$ are: $\bar{A}_\mu = \left\langle A_\mu(\boldsymbol{\omega})\right\rangle$. We do not give here the exact expressions of all components, since some of these are extremely heavy; we remark instead that, by symmetry of the original extreme Kerr–Schild space–time and of $(\Omega, \mathrm{d}p_{\boldsymbol{\omega}})$, the only non vanishing spherical Kerr–Schild components of the averaged potential are $\bar{A}_t$ and $\bar{A}_r$, and that these only depend on x and s, and not on the angular variables θ and ϕ. A direct coordinate transformation then shows that (i) the only non vanishing Schwarzschild components of $\bar{A}$ are $\bar{A}_\tau$ and $\bar{A}_\rho$ (ii) these components depend only on ρ. It follows from this that the only non-vanishing Schwarzschild components of the electromagnetic tensor $\bar{\mathcal{F}}$ associated with $\bar{A}$ in $\bar{g}$ are:

$$\bar{\mathcal{F}}_{\tau\rho} = -\bar{\mathcal{F}}_{\rho\tau} = -\partial_\rho \bar{A}_\tau \tag{55}$$

and that the only non-vanishing Schwarzschild components of the stress-energy tensor $\mathcal{T}(\bar{A},\bar{g})$ associated with $\bar{A}$ in $\bar{g}$ read:

$$\mathcal{T}_\tau^{\ \tau}(\bar{A},\bar{g}) = \mathcal{T}_\rho^{\ \rho}(\bar{A},\bar{g}) = -\mathcal{T}_\theta^{\ \theta}(\bar{A},\bar{g}) = -\mathcal{T}_\phi^{\ \phi}(\bar{A},\bar{g}) = \frac{1}{8\,\pi}\frac{\bar{\mathcal{F}}_{\tau\rho}^2}{F\,G}. \tag{56}$$

A direct coordinate transformation shows that $\bar{A}_\tau(\rho,x) = \bar{A}_t(\rho^{-1}(x,\rho),x)$ and a straightforward computation leads to the exact result

$$\bar{A}_t(s,x) = \frac{1}{s}. \tag{57}$$

This, combined with the above relations and Eqs. (27) and (28), leads to the following approximate expressions for the Schwarzschild components of both $\bar{\mathcal{F}}$ and $\mathcal{T}(\bar{A},\bar{g})$, valid when $x \ll s$:

$$\bar{\mathcal{F}}_{\tau\rho}(s,x) \approx -\frac{1}{M\,s^2}\left(1 + \left(-\frac{2}{5\,s} + \frac{3}{10\,s^2}\right)\frac{x^2}{s^2}\right) \tag{58}$$

and

$$\mathcal{T}_\tau^{\ \tau}(s,x) = \mathcal{T}_\rho^{\ \rho}(s,x) = -\mathcal{T}_\theta^{\ \theta}(s,x) = -\mathcal{T}_\phi^{\ \phi}(s,x) = \frac{1}{8\,\pi\,M^2 s^4}\left(1 - \frac{4\,x^2}{5s^3} + \frac{2\,x^2}{5s^4}\right). \tag{59}$$

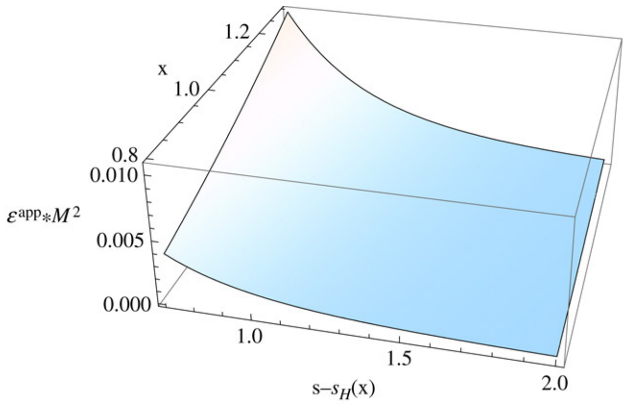

Fig. 3. Evolution of the dimensionless energy density $\varepsilon^{\text{app}} \times M^2$ versus x and $s - s_H$.

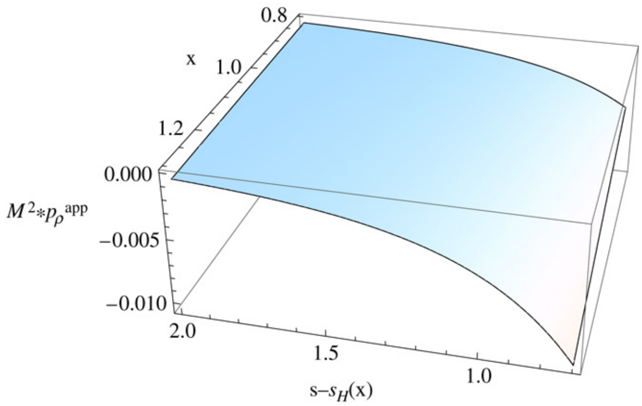

Fig. 4. Evolution of the dimensionless radial pressure $p_\rho^{\text{app}} \times M^2$ versus x and $s - s_H$.

5.3. Stress-energy tensor of the apparent matter

This stress-energy tensor is defined as the difference $\Delta\mathcal{T} = \bar{\mathcal{T}} - \mathcal{T}(\bar{A}, \bar{g})$. Its non-vanishing Schwarzschild components read, for $x \ll s$:

$$8\pi\ \Delta\mathcal{T}^{\ \tau}_{\tau}(s, x) = \frac{4}{5M^2}\left(1 - \frac{1}{s}\right)\frac{x^2}{s^7}; \tag{60}$$

$$8\pi\ \Delta\mathcal{T}^{\ \rho}_{\rho}(s, x) = \frac{4}{5M^2}\left(1 - \frac{1}{s}\right)\frac{x^2}{s^6}; \tag{61}$$

$$8\pi\ \Delta\mathcal{T}^{\ \theta}_{\theta}(s, x) = -\frac{8}{5M^2}\left(1 - \frac{3}{s} + \frac{5}{4s^2}\right)\frac{x^2}{s^6}; \tag{62}$$

$$8\pi\ \bar{\mathcal{T}}^{\phi}_{\phi}(s, x) = 8\pi\ \bar{\mathcal{T}}^{\theta}_{\theta}(s, x). \tag{63}$$

The exact dependance of ε^{app} and p_ρ^{app} on x and s is presented graphically in Fig. 3 and in Fig. 4. Note that ε^{app} is always positive and p_ρ^{app} is always negative; a direct numerical computation reveals that $p_\theta^{\text{app}} = p_\phi^{\text{app}}$ is positive at spatial infinity ($s \to +\infty$).

5.4. Mean current density

The mean current density $\bar{j}$ can be deduced from $\bar{g}$ and $\bar{A}$ via Maxwell equation [27]

$$\bar{g}_{\mu\nu}\bar{j}^{\nu} = -\frac{1}{4\pi}\bar{g}^{\nu\alpha}\bar{\nabla}_{\alpha}\bar{\mathcal{F}}_{\mu\nu}, \tag{64}$$

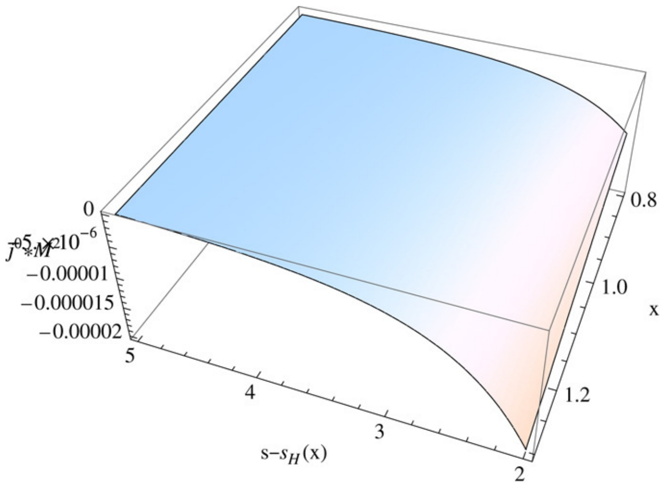

Fig. 5. Evolution of the dimensionless mean current density $\bar{j}^0 \times M^2$ versus x and $s - s_H$..

where $\bar{\nabla}$ stands for the covariant derivative operator associated to $\bar{g}$. The only non-vanishing Schwarzschild component is $\bar{j}^0$, given by:

$$\bar{j}^0(s,x) = \frac{1}{4\pi\, F(s,x)\, G(s,x)} \left(\frac{\partial \rho}{\partial s}\right)^{-1} \left(\partial_\rho \bar{\mathcal{F}}_{\tau\rho} + \bar{\mathcal{F}}_{\tau\rho}\left(\frac{2}{\rho} - \frac{F'(s,x)}{2\,F(s,x)} - \frac{G'(s,x)}{2\,G(s,x)}\right)\right). \tag{65}$$

This gives, for $x \ll s$:

$$\bar{j}^0(s,x) \approx -\frac{3}{5\pi M^2}\left(1 - \frac{2}{3s}\right)\frac{x^2}{s^6}. \tag{66}$$

The component $\bar{j}^0$ represents the spatial charge density outside the horizon; a direct numerical calculation shows that $\bar{j}^0$ has the opposite sign as the total charge M of the unaveraged black hole. The exact dependance of $\bar{j}^0$ in s and x is represented in Fig. 5.

6. Global properties of the mean space–time

6.1. Total mass

One can easily see from (22), (27) and (28) that the mean space–time is asymptotically flat and, therefore, has a finite total mass $\mathcal{E}$. The energy $\mathcal{E}$ can be computed on a 2-sphere ∂S, boundary of a space-like hypersurface S, in the limit where S is located at spatial infinity, that is when $\rho \to \infty$ [27]:

$$\mathcal{E}(x) = -\frac{1}{8\pi} \lim_{\rho\to\infty} \int_{\partial S} \varepsilon_{\mu\nu\alpha\beta} \nabla^\alpha \xi^\beta \mathrm{d}S^{\mu\nu}, \tag{67}$$

which can be rewritten as:

$$\mathcal{E}(x) = \frac{1}{4\pi} \lim_{\rho\to\infty} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta\, \rho^2 n_\alpha \sigma_\beta \nabla^\alpha \xi^\beta. \tag{68}$$

The vector field n is the unit time-like vector field normal to S. Its contravariant Schwarzschild components thus read: $n^\mu = (F(s,x)^{-1/2}, 0, 0, 0)$. The vector field σ is the unit vector normal to ∂S, of contravariant Schwarzschild components: $\sigma^\mu = (0, G(s,x)^{-1/2}, 0, 0)$. Since ρ (Eq. (17)) is an increasing function of s for $s > s_H(x) > s_1(x)$, the limit $\rho \to \infty$ is equivalent to $s \to \infty$. Eq. (68) thus becomes:

$$\mathcal{E}(x) = \lim_{s\to\infty} \frac{\rho^2(s,x) F'(s,x)}{2\sqrt{F(s,x)G(s,x)}}. \tag{69}$$

Using the asymptotic expressions (26), (27) and (28) leads to $\mathcal{E}(x) = M$. Thus, the averaging procedure developed in this paper does change the repartition of energy but does not modify the total energy of the space–time.

6.2. Total charge

The total charge is computed in a similar way. One has:

$$\mathcal{Q}(x) = \frac{1}{8\pi} \lim_{\rho\to\infty} \int_{\partial S} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \mathrm{d}S^{\mu\nu}, \tag{70}$$

which yields:

$$\mathcal{Q}(x) = \frac{1}{4\pi} \lim_{\rho\to\infty} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \,\mathrm{d}\theta \rho^2 \frac{\bar{F}_{\tau\rho}}{\sqrt{F(s,x)G(s,x)}}. \tag{71}$$

We thus get, by proceeding in the same way as in the previous section:

$$\mathcal{Q}(x) = \lim_{s\to\infty} \frac{\rho^2 \bar{F}_{\tau\rho}}{\sqrt{F(s,x)G(s,x)}}. \tag{72}$$

Using the asymptotic expressions (26), (27) and (28) leads to $\mathcal{Q}(x) = M$. The repartition of charge is modified by the averaging procedure, but not the total charge.

7. Summary and discussion

7.1. Summary

A classical extreme black hole has a vanishing temperature. Extreme Reisner–Nordström black holes are exact solutions of the Einstein–Maxwell equations; the black hole described by one of these solutions has a charge equal to the total mass of the space–time and the only matter surrounding it is the electromagnetic field created by this charge. We have introduced new statistical ensembles of classical extreme Reisner–Nordström black holes of identical mass and proved that each of these ensembles describes a black hole of non-vanishing, finite temperature. This result is a particular case of a generic property of extreme spherical black holes. The mean or averaged Reisner–Nordström black hole is surrounded by a mean electromagnetic field and a so-called apparent matter of anisotropic pressure tensor. This matter, which describes averaged-out small scale fluctuations of the gravitational and electromagnetic fields, has also a non-vanishing charge. Thus, the energy and charge repartitions in space–time are changed by statistical averaging. However, the mean black hole has the same total energy and charge as the unaveraged extreme black hole.

7.2. Discussion

The statistical ensembles used in this manuscript can be interpreted in two ways. The first interpretation views the base manifold underlying the space–times in these ensembles as real and the metric as complex. The other interpretation views each space–time of the ensemble as based on a complex manifold. The first interpretation is technically the simplest, but the second one is conceptually more natural, especially in the context of black holes and, perhaps even more so, in the context of black hole thermodynamics. Indeed, the simplest argument delivering Hawking's temperature [12] is based on the topology of the Euclidean analytic continuation of real black hole space–times; independently, general relativistic complex space–times (including black holes) have been considered by various authors [4,15,5] and complex space–times arise naturally in string theory [14,23] and in twistor theory as well [22].

As indicated in the introduction, black hole thermodynamics is currently understood [16] as a reflection that black holes are macroscopic objects built out more fundamental ones describing quantum gravitational degrees of freedom. String theory in particular considers a black hole as an assembly of strings and branes [14] and allows for an explicit computation of the Hawking temperature and the Bekenstein entropy. This article presents an alternative point of view by proving that at least some finite temperature black holes can be viewed as assemblies of classical (as opposed to quantum) objects, namely classical extreme black holes of vanishing temperature. That *all* finite temperature classical black holes can be viewed as assemblies of classical extreme black holes remains an open question which should naturally be investigated. One might also expect the calculations presented in this article to be susceptible of an interpretation in string theory; such an interpretation will probably open new vistas on both classical mean gravitational field theory and string theory itself.

One of the general properties of the mean field theory [8,9,7] used in this article is that the mean space–times it produces are endowed with an apparent matter that traces the averaged out small scale fluctuations of the gravitational and electromagnetic fields. This property is confirmed by the computations presented above. The apparent matter is characterized by a negative radial pressure. A negative pressure is perhaps the most striking feature of the cosmological dark energy [21]. The results presented in this article thus add credibility to the hypothesis that at least part of the cosmological dark energy might be understood as the large scale trace of the small scale fluctuations in the cosmological gravitational field [1,10,3,2,17,19,20,24].

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